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A NOTE ON THE DENJOY-BOURBAKI THEOREM

Abstract

We prove the following extension of the Mean Value Theorem. *Let E be a Banach space and let $F : [a, b] \to E$ and $\varphi : [a, b] \to \mathbb{R}$ be two functions for which there exists a subset $A \subset [a, b]$ such that:*

i) *F and φ have negligible variation on A,*

ii) *F and φ are differentiable on $[a, b] \setminus A$ and $\|F'\| \le \varphi'$ on $[a, b] \setminus A$.*

Then $\|F(b) - F(a)\| \le \varphi(b) - \varphi(a)$.

Several applications are included.

1 Introduction

In what follows $I = [a, b]$ denotes a nondegenerate compact interval and E denotes a Banach space.

A *subpartition* of I is a collection $\mathcal{P} = (I_j)_{j=1}^s$ of nonoverlapping closed intervals in I; if $\cup_j I_j = I$, we say that $\mathcal{P}$ is a *partition.* A *tagged subpartition* of I is a collection of ordered pairs $(I_j, t_j)_{j=1}^s$ consisting of intervals I_j, that form a *subpartition* of I, and tags $t_j \in I_j$, for $j = 1, \dots, s$. If δ is a *gauge* (i.e., a positive function) on a subset $A \subset I$, we say that a tagged subpartition $(I_j, t_j)_{j=1}^s$ is (δ, A)-*fine* if all tags t_j belong to A and $I_j \subset [t_j - \delta(t_j), t_j + \delta(t_j)]$ for $j = 1, \dots, s$. A result (usually ascribed to P. Cousin) asserts the existence of (δ, I)-fine partitions for each $\delta : I \to (0, \infty)$. See [1], page 11.

A function $F : I \to E$ is said to have *negligible variation* on a set $A \subset I$ (and we write $F \in NV_I(A, E)$) if, for every $\varepsilon > 0$ there exists a gauge δ_ε on A

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such that if $\mathcal{D} = \{([u_j, v_j]), t_j\}_{j=1}^{s}$ is any (δ_ε, A)-fine subpartition of I, then

$$\operatorname{Var}(F; \mathcal{D}) = \sum_{j=1}^{s} \|F(v_j) - F(u_j)\| < \varepsilon.$$

As is well known (see [1]), if $F \in NV_I(A, E)$, then F is continuous at every point of A. Conversely, if C is a countable set in I and $F : I \to E$ is continuous at every point of C, then $F \in NV_I(C, E)$. However, when $Z \subset I$ is a null set, not every continuous function on I belongs to $NV_I(Z, E)$. See [1], page 233, for an example.

The aim of this paper is to prove the following generalization of the classical Denjoy-Bourbaki theorem.

Theorem 1. *Let $F : [a, b] \to E$ and $\varphi : [a, b] \to \mathbb{R}$ be two functions for which there exists a subset $A \subset [a, b]$ such that:*

i) F and φ have negligible variation on A,

ii) F and φ are differentiable on $[a, b] \setminus A$ and $\|F'\| \le \varphi'$ on $[a, b] \setminus A$.

Then $\|F(b) - F(a)\| \le \varphi(b) - \varphi(a)$.

The details of the proof are given in Section 2.

The classical case corresponds to the situation when A is at most countable. It was published in [2], pp. 23–24, with an argument adapted from a celebrated paper of A. Denjoy [3], dedicated to the Dini derivates. In that case, it is usual to reformulate the assumption *i)* by requiring the continuity of both F and φ on $[a, b]$. See [4], Ch. 8, Section 5. An immediate consequence is the integral representation of continuous convex functions on compact intervals. If $F : [a, b] \to \mathbb{R}$ is such a function, then

$$F(x) = F(c) + \int_c^x F'_+(t)\, dt,$$

for every $c \in (a, b)$ and every $x \in [a, b]$.

According to Theorem 1 we can enlarge the concept of a primitive function as follows. Given a function $f : [a, b] \to E$, by a *primitive* of f we mean any continuous function $F : [a, b] \to E$ which is differentiable except for a null subset $A \subset [a, b]$, on which F has negligible variation, and $F' = f$ on $[a, b] \setminus A$. By Theorem 1 above, every two primitives (of a same function) differ by a constant. Letting

$$\int_a^b f(t)\, dt = F(b) - F(a), \text{ if } F \text{ is a primitive of } f,$$

we arrive at a concept of integral which, in the scalar case, is equivalent to the Denjoy integral.

An immediate consequence of Theorem 1 (for $\varphi(x) = M(x-a)$) is as follows.

Theorem 2. *Let $F : [a,b] \to E$ be a function for which there exists a subset $A \subset [a,b]$ such that:*

i) F has negligible variation on A,

ii) F is differentiable at all points of $[a,b] \setminus A$ and $\|F'\| \leq M$ on $[a,b] \setminus A$.

Then $\|F(b) - F(a)\| \leq M(b-a)$.

Theorem 1 can be used to improve upon the usual criterion of differentiation of the limit of differentiable functions (as formulated in [4], Theorem 8.6.4).

Theorem 3. *Assume there are given for each $n \in \mathbb{N}$ a pair of functions $F_n, f_n : [a,b] \to E$, and a subset $A_n \subset [a,b]$, such that:*

i) F_n has negligible variation on A_n,

ii) Except at points of A_n, f_n is the derivative of F_n,

iii) There is at least one point $\xi \in [a,b]$ such that the sequence $(F_n(\xi))_n$ is convergent,

iv) For each $x \in [a,b]$ there is a neighborhood U_x on which the sequence $(f_n)_n$ converges uniformly.

Then the sequence $(F_n)_n$ converges uniformly on each U_x and, letting $F(x) = \lim_{n\to\infty} F_n(x)$ and $f(x) = \lim_{n\to\infty} f_n(x)$, the function F is differentiable at each $x \in [a,b] \setminus \cup_{n=1}^{\infty} A_n$ and $F'(x) = f(x)$.

The proof is essentially the same as in the classical case and therefore it will be omitted.

Theorem 1 can be used to derive some classical inequalities such as the Steffensen and Iyengar inequalities. This will be discussed in Section 4 below.

2 Proof of Theorem 1

Suppose there is given $\varepsilon > 0$. By the assumption *ii*), for every $x \in [a,b] \setminus A$,

$$\lim_{z\to x} \left(\left\| \frac{F(z) - F(x)}{z - x} \right\| - \frac{\varphi(z) - \varphi(x)}{z - x} \right) \leq 0,$$

so that for every $x \in [a, b] \setminus A$, there is a $\delta_\varepsilon(x) > 0$ for which

$$0 < |z - x| < \delta_\varepsilon(x) \text{ in } [a, b]$$

implies

$$\left\| \frac{F(z) - F(x)}{z - x} \right\| - \frac{\varphi(z) - \varphi(x)}{z - x} < \frac{\varepsilon}{2(b - a)}. \tag{2.1}$$

Consequently, for every $x', x'' \in [a, b]$ with $x' \leq x \leq x''$ and

$$[x', x''] \subset (x - \delta_\varepsilon(x), x + \delta_\varepsilon(x))$$

we have

$$\|F(x'') - F(x')\| - (\varphi(x'') - \varphi(x')) \leq \frac{\varepsilon(x'' - x')}{2(b - a)}.$$

By the assumption *i*), F and φ both have negligible variation on A. Then there are gauges $\delta_\varepsilon^{'}, \delta_\varepsilon^{''} : A \to (0, \infty)$ such that

$$\operatorname{Var}(F; \mathcal{D}') < \varepsilon/4 \tag{2.2}$$

for every $(\delta_\varepsilon^{'}, A)$-fine tagged subpartition $\mathcal{D}'$, and

$$\operatorname{Var}(\varphi; \mathcal{D}'') < \varepsilon/4 \tag{2.3}$$

for every $(\delta_\varepsilon^{''}, A)$-fine tagged subpartition $\mathcal{D}^{''}$. This allows us to extend the function $\delta_\varepsilon : x \to \delta_\varepsilon(x)$ to the whole interval $[a, b]$, by letting

$$\delta_\varepsilon(x) = \inf \left\{ \delta_\varepsilon^{'}(x), \delta_\varepsilon^{''}(x) \right\} \text{ for } x \in A.$$

According to Cousin's principle, there exists a $(\delta_\varepsilon, [a, b])$-fine partition

$$\{([x_j, x_{j+1}]), t_j\}_{j=0}^{n-1}$$

of $[a, b]$. Then

$$\begin{aligned}
&\|F(b) - F(a)\| - (\varphi(b) - \varphi(a)) \\
\leq &\sum_{j=0}^{n-1} (\|F(x_{j+1}) - F(x_j)\| - (\varphi(x_{j+1}) - \varphi(x_j))) \\
\leq &\sum_{\{j;\ t_j \notin A\}} (\|F(x_{j+1}) - F(x_j)\| - (\varphi(x_{j+1}) - \varphi(x_j))) \\
&+ \sum_{\{j;\ t_j \in A\}} \|F(x_{j+1}) - F(x_j)\| + \sum_{\{j;\ z_j \in A\}} |\varphi(x_{j+1}) - \varphi(x_j)| \\
< &\frac{\varepsilon}{2(b - a)} \sum_{\{j;\ z_j \notin A\}} (x_{j+1} - x_j) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \varepsilon,
\end{aligned}$$

by (2.1), (2.2) and respectively (2.3). As $\varepsilon > 0$ was fixed arbitrarily, we conclude that $\|F(b) - F(a)\| - (\varphi(b) - \varphi(a)) \leq 0$. $\square$

3 The Case of Absolutely Continuous Functions

Negligible variation is related to generalized absolute continuity. A function $F : [a, b] \to E$ is said to be *absolutely continuous* on a set A if for every $\varepsilon > 0$ there is some $\eta > 0$ such that

$$\sum_{i=1}^{N} \|F(x_i) - F(y_i)\| < \varepsilon \tag{AC}$$

for all finite sets of disjoint open intervals $\{(x_i, y_i)\}_{i=1}^{N}$ with endpoints in A and $\sum_{i=1}^{N}(y_i - x_i) < \eta$. F is said to be *absolutely continuous in the restricted sense* on A if instead we have

$$\sum_{i=1}^{N} \sup_{x,y\in[x_i,y_i]} \|F(x) - F(y)\| < \varepsilon \tag{AC$_\star$}$$

under the same conditions as for (AC). And, F is *generalized absolutely continuous in the restricted sense on* A (i.e., $F \in AC_\star G_{[a,b]}(A, E)$) if F is continuous and A is the countable union of sets on each of which F is $AC_\star$. Notice that among continuous functions, the $AC_\star G$ functions on $[a, b]$ are properly contained in the class of functions that are differentiable almost everywhere and they properly contain the class of functions that are differentiable nearly everywhere (differentiable except perhaps on a countable set). See [5].

A function $f : [a, b] \to \mathbb{R}$ is Henstock-Kurzweil integrable if and only if there is a function $F \in AC_\star G_{[a,b]}([a, b], \mathbb{R})$ with $F' = f$ almost everywhere. In this case, $F(x) - F(a) = \int_a^x f(t)\, dt$. See [5].

Lemma 1. *If A is a null subset of $[a, b]$, then $AC_\star G_{[a,b]}(A, E) \subset NV_{[a,b]}(A, E)$.*

The proof is straightforward and we shall omit it. By combining Theorem 1 with Lemma 1 we obtain the following result.

Theorem 4. *Let $F : [a, b] \to E$ and $\varphi : [a, b] \to \mathbb{R}$ be two functions and let A be a null subset of $[a, b]$ such that:*

i) F and φ are generalized absolutely continuous in the restricted sense on $[a, b]$,

ii) F and φ are differentiable on $[a, b] \setminus A$ and $\|F'\| \leq \varphi'$ on $[a, b] \setminus A$.

Then $\|F(b) - F(a)\| \leq \varphi(b) - \varphi(a)$.

4 Application to Inequalities

We need the following easy consequence of Theorem 4.

Theorem 5. *Let $\varphi : [a, b] \to E$ be a continuous function and let A be a null subset of $[a, b]$ such that:*

i) φ is generalized absolutely continuous in the restricted sense on $[a, b]$;

ii) φ is differentiable on $[a, b] \setminus A$ and $\varphi' \geq 0$ on $[a, b] \setminus A$.

Then φ is nondecreasing.

Corollary 1. (*Steffensen's Inequalities [7], Theorem 6.25*). *Let $f : [a, b] \to \mathbb{R}$ be a nondecreasing function and let $g : [a, b] \to [0, \infty)$ be a Lebesgue integrable function such that*

$$\int_a^x g(t)\,dt \leq x - a \text{ and } \int_x^b g(t)\,dt \leq b - x$$

for every $x \in [a, b]$. Then

$$\int_a^{a+\lambda} f(t)\,dt \leq \int_a^b f(t)g(t)\,dt \leq \int_{b-\lambda}^b f(t)\,dt,$$

where $\lambda = \int_a^b g(t)\,dt$.

Proof. Here we shall prove the left hand inequality; the other one can be obtained in a similar manner. For this we put

$$F(x) = \int_a^x f(t)\,dt,\ G(x) = a + \int_a^x g(t)\,dt, \text{ and } H(x) = \int_a^x f(t)g(t)\,dt.$$

Then $H - F \circ G$ is absolutely continuous and $(H - F \circ G)' \geq 0$ almost everywhere. Consequently, $H(b) - F(G(b)) \geq H(a) - F(G(a)) = 0$; i.e.,

$$\int_a^b f(t)g(t)\,dt - \int_a^{a+\lambda} f(t)\,dt \geq 0. \qquad \square$$

The hypotheses on g are fulfilled by all integrable functions g such that $0 \leq g \leq 1$ (and also by some other functions, outside this range of values).

As is well known, if $F : [a, b] \to \mathbb{R}$ is a convex function (which admits finite derivatives at the endpoints), then

$$\lambda F'(a) \leq F(a + \lambda) - F(a) \text{ and } F(b) - F(b - \lambda) \leq \lambda F'(b)$$

for every $\lambda \in [0, b-a]$. These inequalities are complemented by Steffensen's Inequalities as follows

$$F(a+\lambda) - F(a) \leq \inf\left\{\int_a^b F'(t)g(t)\,dt; g \in L^1[a,b],\ 0 \leq g \leq 1, \int_a^b g(t)\,dt = \lambda\right\}$$

$$F(b) - F(b-\lambda) \geq \sup\left\{\int_a^b F'(t)g(t)\,dt; g \in L^1[a,b],\ 0 \leq g \leq 1, \int_a^b g(t)\,dt = \lambda\right\}.$$

Corollary 1 allows us to derive the following extension of the Iyengar inequality [6].

Proposition 1. *Consider a Riemann integrable function $f : [a,b] \to \mathbb{R}$ such that the slopes of the lines AC and CB, joining the endpoints $A\,(a, f(a))$ and $B\,(b, f(b))$ of the graph of f to the other points $C(x, f(x))$ of the graph, vary between $-M$ and M. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \leq \frac{M}{4}(b-a) - \frac{(f(b)-f(a))^2}{4M(b-a)}.$$

PROOF. According to the trapezoidal approximation, it suffices to consider the case where f is piecewise linear. In that case f is absolutely continuous and it satisfies the inequalities

$$0 \leq \int_a^x \frac{f'(t)+M}{2M}\,dt = \frac{f(x)-f(a)+M(x-a)}{2M} \leq x-a$$

and

$$0 \leq \int_x^b \frac{f'(t)+M}{2M}\,dt = \frac{f(b)-f(x)+M(b-x)}{2M} \leq b-x$$

for every $x \in [a,b]$. The proof ends by applying Corollary 1 to $(f'+M)/(2M)$. $\square$

References

[1] R. G. Bartle, *A Modern Theory of Integration,* Graduate Studies in Mathematics, **32**, Amer. Math. Soc., Providence, R. I., 2001.

[2] N. Bourbaki, *Fonctions d'une variable réelle* (Théorie élémentaire), Hermann, Paris, 1949.

[3] A. Denjoy, *Mémoire sur les nombres derivés des fonctions continues*, Journal de Mathématiques Pures et Appliquées, 7th Series, **1** (1915), 105–240.

[4] J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, New York and London, 1960.

[5] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, Graduate Studies in Mathematics, **4**, Amer. Math. Soc., Providence, R. I., 1994.

[6] K. S. K. Iyengar, *Note on an inequality*, Math. Student, **6** (1938), 75–76.

[7] J. Pečarić, F. Proschan and Y. L. Tong, *Convex Functions, Partial Ordering and Statistical Applications*, Academic Press, 1992.