Bull. Math. Soc. Sc. Math. Roumanie
Tome 47(95) No. 1-2, 2004, 67–83

Interpolating Newton's Inequalities

by
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Abstract

We discuss the conditions under which a homogeneous symmetric polynomial is non-negative for all non-negative values of its variables.

Key Words: Elementary symmetric functions, Newton's inequalities, semi-algebraic sets.
2000 Mathematics Subject Classification: Primary: 26D05, 14P10, Secondary: 12D05, 51M16, 52A40.

1 Introduction

The *elementary symmetric functions* of n variables are defined by

$$\begin{aligned}
e_0(x_1, x_2, \ldots, x_n) &= 1 \\
e_1(x_1, x_2, \ldots, x_n) &= x_1 + x_2 + \ldots + x_n \\
e_2(x_1, x_2, \ldots, x_n) &= \sum_{i<j} x_i x_j \\
&\ldots \\
e_n(x_1, x_2, \ldots, x_n) &= x_1 x_2 \ldots x_n.
\end{aligned}$$

The different e_k, being of different degrees, are not comparable. However, they are connected by nonlinear inequalities, mostly due to Newton. To state them, it is more convenient to consider their averages,

$$E_k(x_1, x_2, \ldots, x_n) = e_k(x_1, x_2, \ldots, x_n) / \binom{n}{k}$$

and to write E_k for $E_k(x_1, x_2, \ldots, x_n)$ in order to avoid excessively long formulae.

Theorem 1. *(Newton [10] and Maclaurin [7]). Let $\mathcal{F}$ be an n-tuple of non-negative numbers. Then:*

(N) $\quad E_k^2(\mathcal{F}) > E_{k-1}(\mathcal{F}) \cdot E_{k+1}(\mathcal{F}), \quad 1 \leq k \leq n\text{-}1$ *unless all entries of $\mathcal{F}$ coincide;*

(M) $\quad E_1(\mathcal{F}) > E_2^{1/2}(\mathcal{F}) > \ldots > E_n^{1/n}(\mathcal{F})$ *unless all entries of $\mathcal{F}$ coincide.*

Actually, Newton's inequalities (N) work for n-tuples of real, not necessarily positive elements. An analytic proof along Maclaurin's ideas appeared in the famous book of G. H. Hardy, J. E. Littlewood and G. Pólya [4]. The same book contains the following solution to the problem of comparing monomials in $E_1, \ldots, E_n$:

Theorem 2. *(See [4], Theorem 77, page 64). Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be non-negative numbers. Then*

$$E_1^{\alpha_1}(\mathcal{F}) \cdot \ldots \cdot E_n^{\alpha_n}(\mathcal{F}) \leq E_1^{\beta_1}(\mathcal{F}) \cdot \ldots \cdot E_n^{\beta_n}(\mathcal{F})$$

for every n-tuple $\mathcal{F}$ of positive numbers if, and only if,

$$\alpha_m + 2\alpha_{m+1} + \ldots + (n-m+1)\alpha_n \geq \beta_m + 2\beta_{m+1} + \ldots + (n-m+1)\beta_n$$

for $1 \leq m \leq n$, with equality when $m = 1$.

An alternative proof, also based on the Newton Inequalities (N) is given in [8], p. 93, where the final conclusion is derived by a technique from the majorization theory. The essence of Theorem 2 is the log concavity of the functions $k \to E_k$, which we give here in the formulation of [11]:

Theorem 3. *Suppose that $\alpha, \beta \in \mathbb{R}_+$ and j, $k \in \mathbb{N}$ are numbers such that*

$$\alpha + \beta = 1 \quad \text{and} \quad j\alpha + k\beta \in \{0, \ldots, n\}.$$

Then

$$E_{j\alpha+k\beta}(\mathcal{F}) \geq E_j^{\alpha}(\mathcal{F}) \cdot E_k^{\beta}(\mathcal{F}),$$

for every n-tuple $\mathcal{F}$ of non-negative real numbers. Moreover, equality occurs if and only if all entries of $\mathcal{F}$ are equal.

It turned out by S. Rosset [14] and C. P. Niculescu [11] that Newton's inequalities (N) are only the first in a family of sequences of inequalities. For each natural number $n \geq 2$, one can indicate a sequence (N_n) of homogeneous inequalities in terms of E_k, each one being stronger than the previous one. In this respect (N_2) coincides with (N) and (N_3) consists of the inequalities

$$6E_kE_{k+1}E_{k+2}E_{k+3} + 3E_{k+1}^2E_{k+2}^2 \geq 4E_kE_{k+2}^3 + E_k^2E_{k+3}^2 + 4E_{k+1}^3E_{k+3} \qquad (N_3)$$

for $k = 0, 1, 2, \ldots, card\,\mathcal{F} - 3$; as usually, $\mathcal{F}$ denotes the n-tuple for which the different E_j's are computed.

As (N_3) can be rewritten as

$$4(E_{k+2}^2 - E_{k+1}E_{k+3})(E_{k+1}^2 - E_kE_{k+2}) \geq (E_{k+1}E_{k+2} - E_kE_{k+3})^2$$

one can show easily that $(N_3) \Rightarrow (N_2)$.

The basic fact (discovered by J. Sylvester [17], [18]) concerns the *semi-algebraic character* of the set of real polynomials with all roots real:

Theorem 4. *For each natural number $n \geq 2$ there exists a set of at most $n-1$ polynomials with integer coefficients,*

$$R_{n,1}(x_1, \ldots, x_n),\ \ldots\ ,\ R_{n,k(n)}(x_1, \ldots, x_n), \tag{R_n}$$

such that the monic real polynomials of order n,

$$P(x) = x^n + a_1x^{n-1} + \ldots + a_n,$$

which have only real roots are precisely those for which

$$R_{n,1}(a_1, \ldots, a_n) \geq 0,\ \ldots\ ,\ R_{n,k(n)}(a_1, \ldots, a_n) \geq 0.$$

The above result can be seen as a generalization of the well known fact that *the roots of a quadratic polynomial $x^2 + a_1x + a_2$ are real if, and only if, its discriminant*

$$D_2(1, a_1, a_2) = a_1^2 - 4a_2, \tag{D_2}$$

is non-negative .

Theorem 4 is built on the Sturm method of counting real roots, taking into account that only the leading coefficients enter the play. It turns out that they are nothing but the principal subresultant coefficients (with convenient signs added), which are determinants extracted from the Sylvester matrix. See [1] or [11] for details.

By evident reasons, we shall call a family $\{R_{n,1}, \ldots, R_{n,k(n)}\}$ (as in the statement of Theorem 4 above) a *discriminant family,* of order n. In Sylvester's approach, $R_{n,1}(a_1, \ldots, a_n)$ equals the *discriminant* D_n of the polynomial $P(x) = x^n + a_1x^{n-1} + \ldots + a_n$ that is,

$$D_n = D_n(1, a_1, \ldots, a_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j)^2$$

where $x_1, \ldots, x_n$ are the roots of $P(x)$; D_n is a polynomial (of weight $n^2 - n$) in $\mathbb{Z}[a_1, \ldots, a_n]$ as being a symmetric and homogeneous (of degree $n^2 - n$) polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$. See, for details, [1] or [6]. Unfortunately, at present no compact formula for D_n is known. According to [15], the number of non-zero coefficients in the expression for the discriminant increases rapidly with the degree; for example, D_9 has 26095 terms!

To give a better understanding how fast the family (N_n) grows up, we shall recall here the form of (N_4), as it was computed in [11]:

$$\begin{cases} -27\,E_k^2E_{k+3}^4 + E_k^3E_{k+4}^3 - 54\,E_kE_{k+2}^3E_{k+3}^2 - 64\,E_{k+1}^3E_{k+3}^3 - 18\,E_k^2E_{k+2}^2E_{k+4}^2 \\ \quad +81\,E_kE_{k+2}^4E_{k+4} - 27\,E_{k+1}^4E_{k+4}^2 + 36\,E_{k+1}^2E_{k+2}^2E_{k+3}^2 \\ \quad +108\,E_kE_{k+1}E_{k+2}E_{k+3}^3 + 108\,E_{k+1}^3E_{k+2}E_{k+4}E_{k+3} - 54\,E_{k+1}^2E_{k+2}^3E_{k+4} \\ \quad -180\,E_kE_{k+1}E_{k+2}^2E_{k+3}E_{k+4} + 54\,E_k^2E_{k+2}E_{k+3}^2E_{k+4} - \\ \quad -6E_kE_{k+1}^2E_{k+3}^2E_{k+4} \\ \quad +54\,E_kE_{k+1}^2E_{k+2}E_{k+4}^2 - 12\,E_k^2E_{k+1}E_{k+3}E_{k+4}^2 \geq 0. \\ 9\,E_k^2E_{k+2}^2 + 4\,E_k^2E_{k+1}E_{k+3} - 24\,E_kE_{k+1}^2E_{k+2} + 12\,E_{k+1}^4 - E_k^3E_{k+4} \geq 0 \\ E_{k+1}^2 - E_kE_{k+2} \geq 0 \end{cases}$$

for $k = 0, 1, 2, \ldots, card\,\mathcal{F} - 4$.

Because of the fast grow of the gaps, not every inequality involving homogeneous polynomials appears the way described above.

In fact, when restricted to the n-tuples of non-negative numbers, the different monomials (of a given weight) in $E_1, E_2, E_3, \ldots$ can be listed via (N_2) as follows:

$$\begin{aligned} & E_1^2 \geq E_2 \\ & E_1^3 \geq E_1E_2 \geq E_3 \\ & E_1^4 \geq E_1^2E_2 \geq E_2^2 \geq \ E_1E_3 \geq E_4 \\ & E_1^5 \geq E_1^3E_2 \geq E_1E_2^2 \geq E_1^2E_3 \geq \max\{E_1E_4,\ E_2E_3\} \geq \\ & \quad \geq \min\{E_1E_4,\ E_2E_3\} \geq E_5 \\ & \quad \ldots \end{aligned}$$

Notice the phenomenon of incomparable monomials (starting with the quintic case).

All other inequalities constituting the different families (N_n) with $n \geq 3$, interpolate in the appropriate row of inequalities displayed above. For example, the inequalities (N_3) interpolate in the following sequence of homogeneous monomials of degree $4k + 6$:

$$E_{k+1}^2E_{k+2}^2 \geq E_kE_{k+2}^3,\ E_{k+1}^3E_{k+3},\ E_k^2E_{k+3}^2 \geq E_kE_{k+1}E_{k+2}E_{k+3}.$$

A moment's reflection shows that the process of listing the different families (N_n) missed an infinity of inequalities, which are to be recovered by interpolation. An example is the cubic inequality (first noticed by G. Peano),

$$3E_1^3 + E_3 \geq 4E_1E_2, \tag{C}$$

which for triplets reduces to the following inequality:

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x + y) + yz(y + z) + zx(z + x) \tag{C'}$$

for every $x, y, z \geq 0$. The inequality (C') represents the case $\alpha = 1$ of the following classical inequality:

Lemma 1. (*See* [4], *Theorem* 80, *page* 64). *Suppose that* $x, y, z > 0$ *and* $\alpha \in \mathbb{R}$. *Then*

$$\sum x^\alpha(x - y)(x - z) > 0$$

unless $x = y = z$.

Then (C) can be recovered from (C') via mathematical induction. See Lemma 4 below. A Calculus based approach of (C) can be built on the following result due to N. Sato [16]:

Lemma 2. *Let $a_1, \ldots, a_n$ be real numbers. Then the polynomial*

$$P(x_1, \ldots, x_n) = \sum_{k=1}^{n} a_k E_1^{n-k} E_k$$

attaints its infimum over the compact set

$$K = \{(x_1, \ldots, x_n) \in [0, \infty)^n;\ x_1 + \ldots + x_n = 1\}$$

at a point $Q_k = (\underbrace{1/k, \ldots, 1/k}_{k\ times}, 0, \ldots, 0)$, *for some* $k \in \{1, \ldots, n\}$.

Lemma 2 yields easily an algorithm to check the positivity of the polynomials P as in Lemma 2 above:

$$P \geq 0 \text{ on } K \text{ if and only if } P(Q_k) \geq 0 \text{ for all points } Q_k.$$

The cubic case reads as follows:

Proposition 1. *The inequality*

$$aE_1^3 + bE_1E_2 + cE_3 \geq 0$$

holds for every n-tuple $\mathcal{F}$ of non-negative numbers with $n \geq 3$ if, and only if,

$$a \geq 0,\ 4a + 3b \geq 0 \ \text{and}\ a + b + c \geq 0.$$

Proof: The Necessity is immediate, checking the inequality for the families

$$\{1, 0, 0\}, \{1/2, 1/2, 0\} \text{ and } \{1/3, 1/3, 1/3\}.$$

The Sufficiency. According to Lemma 2, it suffices to verify the inequality $aE_1^3 + bE_3 + cE_1E_2 \geq 0$ for all n-tuples of the form $(1/k, \ldots, 1/k, 0, \ldots, 0)$, where $1/k$ repeats k times ($k \in \{1, \ldots, n\}$). □

Corollary 1. *For every n-tuple $\mathcal{F}$ of non-negative numbers ($n \geq 3$), we have*

$$\frac{3}{4}E_1^3 + \frac{1}{4}E_3 \geq E_1E_2$$

and $\lambda = 3/4$ is the smallest value of λ in $[0, 1]$ such that

$$\lambda E_1^3 + (1 - \lambda)E_3 \geq E_1E_2$$

holds for all $\mathcal{F}$ as above.

Proof: The optimality of $\lambda = 3/4$ follows by considering the triplet $\{1, 1, 0\}$. □

In section 3 we shall indicate a third proof of Proposition 1, based on a downward induction argument and the quadratic analogue of Proposition 1:

Lemma 3. *The inequality*

$$aE_1^2 + bE_2 \geq 0$$

holds for all n-tuples of non-negative numbers $(n \geq 2)$ *if and only if*

$$a \geq 0 \text{ and } a + b \geq 0.$$

At this point it is natural to ask whether the entire problem of displaying all positively homogeneous symmetric real polynomials of nth degree can be settled in terms of finite sets of universal inequalities. The delicate question is the *nature* of these inequalities. The aim of this paper is to give support for the following conjecture we presented to *The Fifth Congress of Romanian Mathematicians*, Piteşti, June 22-28, 2003 [12]:

Conjecture 1. *For each natural number* $n \geq 1$, *the set of all positively homogeneous symmetric real polynomials* $P(x_1, \ldots, x_n)$, *of degree* n, *which take non-negative values for all* $x_1, \ldots, x_n \geq 0$, *has a semi-algebraic character. In other words, there is a finite set* $Q_{n,1}, \ldots, Q_{n,k(n)}$ *of real polynomials with integer coefficients, such that a positively homogeneous symmetric real polynomial*

$$P(x_1, \ldots, x_n) = aE_1^n + bE_1^{n-2}E_2 + \ldots,$$

is positive for all $x_1, \ldots, x_n \geq 0$ *if, and only if,*

$$Q_{n,1}(a, b, \ldots) \geq 0, \; \ldots, \; Q_{n,k(n)}(a, b, \ldots) \geq 0.$$

Attached to this Conjecture is the problem to find out an algorithm for generating the polynomials $Q_{n,1}, \ldots, Q_{n,k(n)}$ for each $n \geq 2$. Lemma 3 shows that $k(2) = 2$ and

$$Q_{2,1}(a, b) = a \text{ and } Q_{2,2}(a, b) = a + b,$$

while Proposition 1 shows that $k(3) = 3$ in the cubic case and the corresponding test family consists of

$$Q_{3,1}(a, b, c) = a, \; Q_{3,2}(a, b, c) = 4a + 3b \text{ and } Q_{3,3}(a, b, c) = a + b + c.$$

In Section 4 we shall prove the validity of this conjecture in the quartic case. This will be done by applying a descending technique that reduces the whole business to the case of three variables, where a previous result due to O. Bottema and J. T. Groenman [2] can be applied. The process of writing down a concrete test family is based on the quartic case of another open problem:

Conjecture 2. *Consider the set A of all $(n+1)$-tuples $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$, with the following property:*

$$a_0 + a_1 x + \ldots + a_n x^n \geq 0 \text{ for every } x \geq 0.$$

Then there exists a finite family of polynomials $P_{i,j} \in \mathbb{Z}[X_0, X_1, \ldots, X_n]$ such that

$$A = \cup_i \cap_j \{(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}; P_{i,j}(a_0, a_1, \ldots, a_n) \succ 0\}.$$

Here $\succ$ can be either $\geq$ or $>$.

Letting $P(x) = a_0 + a_1 x + \ldots + a_n x^n$, it is clear that

$$P(x) \geq 0 \quad \text{for all } x \geq 0$$

if and only if $a_0 = P(0) \geq 0$ and $P(x)$ is non-negative at all non-negative zeros of its derivative $P'(x)$. The problem is to prove the semi-algebraic character of this solution that is, to embed the inequality $a_0 = P(0) \geq 0$ into a full test family $(Q_{i,j})_{i,j}$.

A solution to Conjecture 2 will solve in principle the problem of finding an algorithm for *all* metric inequalities in a triangle that can be expressed in terms of s (the semiperimeter), r (the radius of the inscribed circle) and R (the radius of the circumscribed circle).

The last, but not the least, our results extends mutatis mutandis to the context of Banach lattices. For example, using the well known functional calculus with elements in a Banach lattice (which works for positively homogeneous functions of degree 1; see [5], Vol. 2), Proposition 1 can be extended as follows:

Theorem 5. *If $a \geq 0$, $4a + 3b \geq 0$ and $a + b + c \geq 0$ then*

$$a\left(\frac{1}{n}\sum x_k^{1/3}\right)^3 + b\left(\frac{1}{n}\sum x_k^{1/3}\right)\left(\frac{2}{n(n-1)}\sum_{j<k} x_j^{1/3} x_k^{1/3}\right)$$
$$+c\left(\frac{6}{n(n-1)(n-2)}\sum_{j<k<l} x_j^{1/3} x_k^{1/3} x_l^{1/3}\right) \geq 0 \tag{1}$$

for every family $x_1, \ldots, x_n$ $(n \geq 3)$ of positive elements in an arbitrary Banach lattice.

2 Preliminaries on the functions E_k

According to Rolle's Theorem, if all roots of a polynomial $P \in \mathbb{R}[X]$ are real (respectively, real and distinct), then the same is true for its derivative P'. Given an n-tuple $\mathcal{F} = (x_1, \ldots, x_n)$, we shall attach to it the polynomial

$$P_{\mathcal{F}}(x) = (x - x_1) \ldots (x - x_n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E_k(x_1, \ldots, x_n)\, x^{n-1-k}.$$

The $(n-1)$-tuple $\mathcal{F}' = \{y_1, \ldots, y_{n-1}\}$, consisting of all roots of the derivative of $P_{\mathcal{F}}(x)$ will be called the *derived n-tuple* of $\mathcal{F}$. Because

$$(x - y_1) \ldots (x - y_{n-1}) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} E_k(y_1, \ldots, y_{n-1})\, x^{n-k}$$

and

$$\begin{aligned}(x - y_1) \ldots (x - y_{n-1}) &= \frac{1}{n} \cdot \frac{dP_{\mathcal{F}}}{dx}(x) \\ &= \sum_{k=0}^{n} (-1)^k \, \frac{n-k}{n} \binom{n}{k} E_k(x_1, \ldots, x_n)\, x^{n-k-1} \\ &= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} E_k(x_1, \ldots, x_n)\, x^{n-1-k}\end{aligned}$$

we are led to the following result, which enables us to reduce the number of variables when dealing with symmetric functions:

Lemma 4. $E_k(\mathcal{F}) = E_k(\mathcal{F}')$ for every $k \in \{0, \ldots, |\mathcal{F}| - 1\}$. *Moreover, if $\mathcal{F}$ consists of non-negative numbers, so does $\mathcal{F}'$.*

Another useful remark, which can be checked by a simple computation, is as follows:

Lemma 5. *Suppose that $\mathcal{F}$ is an n-tuple of real numbers and $0 \notin \mathcal{F}$. Put $\mathcal{F}^{-1} = \{1/a \mid a \in \mathcal{F}\}$. Then*

$$E_k(\mathcal{F}^{-1}) = E_{n-k}(\mathcal{F}) \,/\, E_n(\mathcal{F})$$

for every $k \in \{0, \ldots, n\}$.

The two lemmas above were used in [11] to prove Theorem 3 and other classical results.

3 The descending technique in the cubic case

The aim of this section is to indicate a new proof of Proposition 1. Our approach is based on a technique of reducing the numbers of variables, up to a point where the solution is a consequence of the semi-algebraic character of certain sets of real polynomials in one variable.

According to Lemma 4, we may restrict ourselves to the case of triplets $\mathcal{F} = \{x, y, z\}$ of non-negative numbers. Also, we may assume that

$$z = \lambda(x + y)$$

for a suitable $\lambda \geq 0$ (that is, one variable will be replaced by a parameter). Then the symmetric functions E_1, E_2, E_3 corresponding to $\mathcal{F}$ can be expressed by the symmetric functions $\widetilde{E}_1, \widetilde{E}_2$ corresponding to $\mathcal{F} = \{x, y\}$, as follows:

$$\begin{aligned} E_1 &= \frac{x+y+\lambda(x+y)}{3} = \frac{2(1+\lambda)}{3}\widetilde{E}_1 \\ E_2 &= \frac{xy+\lambda(x+y)^2}{3} = \frac{1}{3}\widetilde{E}_2 + \frac{4\lambda}{3}\widetilde{E}_1^2 \\ E_3 &= \lambda xy(x+y) = 2\lambda \widetilde{E}_1 \widetilde{E}_2. \end{aligned}$$

An easy computation yields

$$aE_1^3 + bE_1E_2 + cE_3 = \left(\widetilde{a}\widetilde{E}_1^2 + \widetilde{b}\widetilde{E}_2\right)\widetilde{E}_1,$$

where

$$\widetilde{a} = \frac{1}{27}\left[8a(1+\lambda)^3 + 24b\lambda(1+\lambda)\right]$$

and

$$\widetilde{b} = \frac{1}{27}\left[6b(1+\lambda) + 54c\lambda\right].$$

According to Lemma 3,

$$aE_1^3 + bE_1E_2 + cE_3 \geq 0$$

for every $x, y, z \geq 0$ if and only if

$$\widetilde{a} \geq 0 \text{ and } \widetilde{a} + \widetilde{b} \geq 0$$

for all $\lambda \geq 0$, which (after suitable simplification) means the conjunction of the following two inequalities:

$$a(1+\lambda)^2 + 3b\lambda \geq 0 \tag{2}$$

and

$$4a\lambda^3 + 12(a+b)\lambda^2 + (12a+15b+27c)\lambda + 4a + 3b \geq 0 \tag{3}$$

for all $\lambda \geq 0$.

This leads us to the semi-algebraic problem of deciding when a quadratic or a cubic real polynomial $P(x)$ verifies the condition

$$P(x) \geq 0 \text{ for all } x \geq 0.$$

However, we shall prefer a straightforward argument, noticing that 3 yields (respectively for $\lambda = 0$, $1/2$ and ∞) the following set of necessary conditions:

$$4a + 3b \geq 0, \;\; a + b + c \geq 0 \text{ and } a \geq 0.$$

It turns out that these conditions are also sufficient for the validity of both 2 and 3. In fact,

$$a(1+\lambda)^2 + 3b\lambda = a(\lambda-1)^2 + (4a+3b)\,\lambda \geq 0$$

and the left hand side of 3 can be rewritten as

$$a\lambda\left(4\lambda^2 - 4\lambda + 1\right) + 27(a+b+c)\lambda + (4a+3b)\left(4\lambda^2 - 4\lambda + 1\right),$$

which is a sum of non-negative numbers.

4 Proof of Conjectures 1 and 2 in the Quartic Case

We start with some preparation concerning the case of quartic polynomials of three variables. Our first goal is to restate a result due to O. Bottema and J. T. Groenman [2] (see [13] for a simple proof) in order to prove the following fact:

Lemma 6. *The inequality*

$$aE_1^4 + bE_1^2E_2 + cE_2^2 + dE_1E_3 \geq 0$$

holds for every triplet of non-negative numbers if, and only if,

$$\begin{gathered} a \geq 0,\ 16a + 12b + 9c \geq 0,\ a+b+c+d \geq 0 \text{ and} \\ 5a + 3b \geq -\sqrt{a(16a+12b+9c)}. \end{gathered} \tag{RQ}$$

Notice that the last inequality can be restated as follows:

$$5a + 3b \geq 0, \text{ or } 5a+3b \leq 0 \text{ and } (a+b)^2 - ac \leq 0.$$

Proof: The basic idea is to switch to another basis of the space of all positively homogeneous quartic polynomials of three variables), constituted by the Muirhead polynomials:

$$\begin{aligned}
T(4,0,0) &= 2\sum x^4 = 162E_1^4 - 216E_1^2E_2 + 36E_2^2 + 24E_1E_3 \\
T(3,1,0) &= \sum x^3(y+z) = 27E_1^2E_2 - 18E_2^2 - 3E_1E_3 \\
T(2,2,0) &= \sum x^2y^2 = 18E_2^2 - 12E_1E_3 \\
T(2,1,1) &= 2xyz\sum x = 6E_1E_3.
\end{aligned}$$

As

$$\begin{aligned}
E_1E_3 &= \frac{1}{6}T(2,1,1) \\
E_2^2 &= \frac{1}{18}T(2,2,0) + \frac{1}{9}T(2,1,1) \\
E_1^2E_2 &= \frac{1}{27}T(3,1,0) + \frac{1}{27}T(2,2,0) + \frac{5}{54}T(2,1,1) \\
E_1^4 &= \frac{1}{162}T(4,0,0) + \frac{4}{81}T(3,1,0) + \frac{1}{27}T(2,2,0) + \frac{1}{5}T(2,1,1),
\end{aligned}$$

the Muirhead polynomials form really a basis. Muirhead noticed that

$$T(4,0,0) \geq T(3,1,0) \geq T(2,2,0) \geq T(2,1,1)$$

which represents part of the inequalities involving the normalized elementary functions E_k :

$$\begin{aligned} 6E_1^4 + 2E_2^2 + E_1E_3 &\geq 9E_1^2E_2 \\ 3E_1^2E_2 + E_1E_3 &\geq 4E_2^2 \\ E_2^2 &\geq E_1E_3. \end{aligned}$$

We now consider the polynomials:

$$\begin{aligned} A &= \sum x^4 - \sum x^3(y+z) + xyz\sum x = \frac{1}{2}T(4,0,0) - T(3,1,0) + \frac{1}{2}T(2,1,1) \\ B &= \sum x^3(y+z) - 2\sum x^2y^2 = T(3,1,0) - T(2,2,0) \\ C &= \sum x^2y^2 - xyz\sum x = \frac{1}{2}T(2,2,0) - \frac{1}{2}T(2,1,1). \end{aligned}$$

Then:

$$\begin{aligned} \lambda A + \mu B + \nu C &= \frac{\lambda}{2}T(4,0,0) + (\mu - \lambda)T(3,1,0) \\ &\quad + \frac{\nu - 2\mu}{2}T(2,2,0) + \frac{\lambda - \nu}{2}T(2,1,1) \\ &= 81\lambda E_1^4 + (27\mu - 135\lambda)E_1^2E_2 \\ &\quad + (36\lambda - 36\mu + 9\nu)E_2^2 + (18\lambda + 9\mu - 3\nu)E_1E_3 \end{aligned}$$

which shows that

$$aT(4,0,0) + bT(3,1,0) + cT(2,2,0) - (a+b+c)T(2,1,1)$$

equals $2aA + (2a+b)B + (4a+2b+2c)C$.

A result of O. Bottema and J. T. Groenman [2], as restated by J. F. Rigby [13], asserts that

$$\lambda A + \mu B + \nu C \geq 0 \text{ for positive } x, y, z \Leftrightarrow \lambda, \nu \geq 0 \text{ and } \mu \geq -\sqrt{\lambda\nu},$$

equivalently,

$$aT(4,0,0) + bT(3,1,0) + cT(2,2,0) - (a+b+c)T(2,1,1) \geq 0$$

for positive x, y, z if and only if

$$a \geq 0,\ 2a + b + c \geq 0 \text{ and } 2a + b \geq -2\sqrt{a(2a+b+c)}.$$

Finally, an easy computation yields the desired conclusion:

$$aE_1^4 + bE_1^2E_2 + cE_2^2 - (a+b+c)E_1E_3 \geq 0 \text{ for every } x, y, z \geq 0$$

if and only if

$$a \geq 0,\ 16a + 12b + 9c \geq 0 \text{ and}$$
$$5a + 3b \geq -\sqrt{a(16a + 12b + 9c)}.$$

□

Our next goal is to settle the quartic case of Conjecture 2:

Proposition 2. *The set*

$$\left\{(\alpha, \beta, \gamma, \delta, \varepsilon) \in \mathbb{R}^5;\ \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon \geq 0 \text{ for every } x \geq 0\right\}$$

has a semi-algebraic character.

Proof: We shall consider here only the subset of all $(\alpha, \beta, \gamma, \delta, \varepsilon)$ with $\alpha > 0$. Put

$$P(x) = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon.$$

As $P(0) \geq 0$, we infer that $\varepsilon \geq 0$. The derivative of $P(x)$ can have one or three real roots, depending on the sign of the discriminant of $P'(x)$. More precisely, $P'(x)$ has only one real root, say u, if and only if the discriminant D of $4\alpha x^3 + 3\beta x^3 + 2\gamma x + \delta$ verifies the inequality

$$D > 0$$

which has a semi-algebraic character.

When $D > 0$, and $u \leq 0$, the polynomial $P(x)$ is increasing on $[0, \infty)$, which yields $P(x) > P(0)$ for $x > 0$.

When $D > 0$, and $u > 0$, we have to impose the condition $P(u) \geq 0$. This is a semi-algebraic condition as

$$P = P'Q + R$$

yields $P(u) = R(u)$ and R is a quadratic polynomial.

When $D \leq 0$, we have to look at the largest root of $P''(x)$,

$$v = \frac{-3\beta + \sqrt{9\beta^2 - 24\alpha\gamma}}{12\alpha}.$$

Clearly, $P(x)$ is increasing for $x \geq v$. If $v \leq 0$, then everything is OK. If $v > 0$, then we have to impose the condition

$$P(v) \geq 0.$$

The proof ends by noticing that this is a semi-algebraic restriction. □

Coming back to the proof of Conjecture 1 in the case $n = 4$, we start with the remark (motivated by Lemma 4) that we may restrict ourselves to the case of where $\mathcal{F} = \{x, y, z, t\}$. Clearly, we may also assume that

$$t = \lambda(x + y + z)$$

for a suitable $\lambda \geq 0$ (that is, one variable will be replaced by a parameter). Then the symmetric functions E_1, E_2, E_3, E_4, corresponding to $\mathcal{F}$, can be expressed by the symmetric functions $\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3$, corresponding to $\mathcal{F} = \{x, y, z\}$, as follows:

$$\begin{aligned}
E_1 &= \frac{x + y + z + \lambda(x + y + z)}{4} = \frac{3(1+\lambda)}{4}\widetilde{E}_1 \\
E_2 &= \frac{xy + yz + zx + \lambda(x + y + z)^2}{6} = \frac{1}{2}\widetilde{E}_2 + \frac{3\lambda}{2}\widetilde{E}_1^2 \\
E_3 &= \frac{xyz + \lambda(x + y + z)\,(xy + yz + zx)}{4} \\
&= \frac{1}{4}\widetilde{E}_3 + \frac{9\lambda}{4}\widetilde{E}_1\widetilde{E}_2. \\
E_4 &= \lambda xyz(x + y + z) = 3\lambda\widetilde{E}_1\widetilde{E}_3
\end{aligned}$$

An easy computation yields

$$aE_1^4 + bE_1^2E_2 + cE_2^2 + dE_1E_3 + eE_4 = \widetilde{a}\widetilde{E}_1^4 + \widetilde{b}\widetilde{E}_1^2\widetilde{E}_2 + \widetilde{c}\widetilde{E}_2^2 + \widetilde{d}\widetilde{E}_1\widetilde{E}_3,$$

where

$$\begin{aligned}
\widetilde{a} &= \frac{9}{256}\left[9a(1+\lambda)^4 + 24b\lambda(1+\lambda)^2 + 64c\lambda^2\right] \\
\widetilde{b} &= \frac{3}{32}\left[3b(1+\lambda)^2 + 16c\lambda + 18\lambda(1+\lambda)d\right] \\
\widetilde{c} &= \frac{c}{4} \\
\widetilde{d} &= \frac{3}{16}\left[d(1+\lambda) + 16\lambda e\right].
\end{aligned}$$

Now the truth of Conjecture 1 in the quartic case is a consequence of Lemma 6 and Proposition 2.

5 Interpolation Inequalities in the Quartic Case

While we know the existence of a test family in the quartic case, the problem of writing down a concrete family remains open.

By Lemma 6, the inequality

$$aE_1^4 + bE_1^2E_2 + cE_2^2 + dE_1E_3 \geq 0$$

holds for every n-tuple of non-negative numbers ($n \geq 4$) if, and only if,

$$a \geq 0,\ 9a + 6b + 4c \geq 0,\ a + b + c + d \geq 0 \text{ and}$$
$$15a + 8b \geq -4\sqrt{a(9a + 6b + 4c)}.$$

This remark is complemented by Lemma 2, which yields the following fact:

Proposition 3. *The inequality*

$$aE_1^4 + bE_1^2E_2 + dE_1E_3 + eE_4 \geq 0$$

holds for every n-tuple $\mathcal{F}$ of non-negative numbers ($n \geq 4$) if, and only if, the coefficients verify the following set of conditions:

$$a \geq 0, \quad 3a + 2b \geq 0, \quad 27a + 24b + 16d \geq 0, \quad a + b + d + e \geq 0.$$

The above discussion leaves out important inequalities such as

$$12\, E_1^4 - 24\, E_1^2E_2 + 9\, E_2^2 + 4\, E_1E_3 - E_4 \geq 0,$$

which is a component of the family (N_4), for $k = 0$, mentioned in the Introduction.

Instead, it is easy to write down the interpolating inequalities associated to each triplet of quartic monomials in the following list:

$$E_1^4 \geq E_1^2E_2 \geq E_2^2 \geq E_1E_3 \geq E_4. \tag{4}$$

Indeed, for $\lambda \in [0,1]$, the following statements hold true:

1. $\lambda E_1^4 + (1-\lambda)E_1E_3 \geq E_1^2E_2$ for all n-tuples of non-negative numbers ($n \geq 3$) if and only if $\lambda \in [3/4, 1]$;

2. $\lambda E_1^4 + (1-\lambda)E_4 \geq E_1^2E_2$ for all n-tuples of non-negative numbers ($n \geq 4$) if and only if $\lambda \in [8/9, 1]$;

3. $\lambda\, E_1^4 + (1-\lambda)E_4 \geq E_1E_3$ for all n-tuples of non-negative numbers ($n \geq 4$) if and only if $\lambda \in [16/27, 1]$;

4. $\lambda\, E_1^2E_2 + (1-\lambda)E_4 \geq E_1E_3$ for all n-tuples of non-negative numbers ($n \geq 4$) if and only if $\lambda \in [2/3, 1]$;

5. $\lambda E_1^4 + (1-\lambda)E_2^2 \geq E_1^2E_2$ for all n-tuples of non-negative numbers ($n \geq 2$) if and only if $\lambda \in [1/2, 1]$;

6. $\lambda E_1^4 + (1-\lambda)E_1E_3 \geq E_2^2$ for all n-tuples of non-negative numbers ($n \geq 3$) if and only if $\lambda \in [9/16, 1]$;

7. $\lambda\, E_1^4 + (1-\lambda)E_4 \geq E_2^2$ for all n-tuples of non-negative numbers ($n \geq 4$) if and only if $\lambda \geq 64/81$;

8. $\lambda E_1^2 E_2 + (1-\lambda) E_1 E_3 \geq E_2^2$ for all n-tuples of non-negative numbers ($n \geq 3$) if and only if $\lambda \geq 3/4$;

9. $\lambda E_1^2 E_2 + (1-\lambda) E_4 \geq E_2^2$ for all n-tuples of non-negative numbers ($n \geq 4$) if and only if $\lambda \geq 8/9$;

10. $\lambda E_2^2 + (1-\lambda) E_4 \geq E_1 E_3$ for all n-tuples of non-negative numbers ($n \geq 4$) if and only if $\lambda \geq 3/4$.

Most of these statements admit straightforward arguments. For example, the inequality (5) can be put in the form

$$\left(\lambda E_1^2 - (1-\lambda) E_2\right) \left(E_1^2 - E_2\right) \geq 0$$

and thus it works for $\lambda \geq 1/2$. On the other hand, the case $\mathcal{F} = \{1, 1, 0\}$ makes this condition also necessary.

Also simple is inequality (8). It can be restated in the form

$$\lambda \left(\sum xy(x^2+y^2) - 2\sum x^2y^2\right) + (4\lambda - 3)\left(\sum x^2y^2 - xzy\sum x\right) \geq 0$$

which represents a linear combination of two non-negative expressions. The optimality of the value $\lambda = 3/4$ can be checked by considering the triplet $\mathcal{F} = (1, 1, 0)$.

The inequality (6) is a bit more involving. The case$\mathcal{F} = \{1, 1, 0\}$ yields $\lambda \geq 9/16$. For $\lambda = 9/16$, this inequality is equivalent to

$$\sum x^4 + 4\sum x^3(y+z) + xyz\sum x - 10\sum x^2y^2 \geq 0 \text{ for all } x, y, z \geq 0.$$

Or, the left hand side equals

$$\left[\sum x^4 + xyz\sum x - \sum x^3(y+z)\right] + 5\left[\sum x^3(y+z) - 2\sum x^2y^2\right]$$

that is,

$$\sum x^2(x-y)(x-z) + 5\sum xy(x-y)^2$$

and the conclusion follows from Lemma 1, applied for $\alpha = 2$. If $\lambda \in [9/16, 1]$, then

$$\begin{aligned} \lambda E_1^4 + (1-\lambda) E_1 E_3 - E_2^2 &= \left[9/16 \cdot E_1^4 + 7/16 \cdot E_1 E_3 - E_2^2\right] \\ &\quad + (\lambda - 9/16)\left[E_1^4 - E_1 E_3\right] \end{aligned}$$

is a combination of non-negative expressions and the proof of (6) is done.

The proof of the inequality (7) follows a different route. For $\lambda = 64/81$ this inequality is equivalent to

$$(x+y+z+t)^4 + 68xyzt - 9\left(xy + xz + xt + yz + yt + zt\right)^2 \geq 0 \text{ for all } x, y, z, t \geq 0,$$

a fact which can be established via the Lagrange multiplier method. The optimality of the value $64/81$ can be checked by considering the 4-tuple $\mathcal{F} = \{1, 1, 1, 0\}$.

The inequalities (1)-(10) above can be combined. For example, from the inequalities (1) and (6) we infer the following result:

Proposition 4. *For every n-tuple $\mathcal{F}$ of non-negative numbers $(n \geq 3)$, and every $\lambda \in [0,1]$ we have*

$$\frac{3\lambda+9}{16}E_1^4+\frac{7-3\lambda}{16}E_1E_3 \geq \lambda E_1^2E_2+(1-\lambda)E_2^2.$$

Moreover, if $\lambda, \mu \in [0,1]$ are such that

$$\mu E_1^4+(1-\mu)E_1E_3 \geq \lambda E_1^2E_2+(1-\lambda)E_2^2$$

for every $\mathcal{F}$ as above, then $\mu \geq (3\lambda+9)/16$.

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Received: 19 June 2004

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