AN EXTENSION OF TWO BASIC RESULTS IN REAL ANALYSIS

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Abstract. Based on the existence of well behaved partitions, we extend the Denjoy-Bourbaki Theorem and Leibniz-Newton Formula to a context where the lack of derivability is supplied by the property of negligible semivariation.

1. Introduction

In what follows $[a,b]$ denotes a nondegenerate compact interval and E denotes a Banach space.

A *subpartition* of $[a,b]$ is a collection $\mathcal{P}=(I_k)_{k=1}^n$ of nonoverlapping closed intervals in $[a,b]$; if $\cup_k I_k=[a,b]$, we say that $\mathcal{P}$ is a *partition*. A *tagged subpartition* of $[a,b]$ is a collection of ordered pairs $(I_k,t_k)_{k=1}^n$ consisting of intervals I_k, that form a subpartition of $[a,b]$, and tags $t_k\in I_k$, for $k=1,...,n$. If δ is a *gauge* (that is, a positive function) on a subset $A\subset[a,b]$ we say that a tagged subpartition $(I_k,t_k)_{k=1}^n$ is (δ,A)-*fine* if all tags t_k belong to A and $I_k\subset(t_k-\delta(t_k),t_k+\delta(t_k))$ for $k=1,...,n$. A result known as Cousin's Lemma asserts the existence of $(\delta,[a,b])$-fine tagged partitions for each $\delta:[a,b]\to(0,\infty)$. See [1], page 11. This result is equivalent to many other basic results such as the Fundamental Lemma of Analysis on $\mathbb{R}$ (see [8]). In what follows we shall need a slightly more general version of Cousin's Lemma:

Lemma 1. *Let δ be a gauge on $[a,b]$ and assume that $\mathcal{A}$ is a family of subintervals $[x',x'']\subset[a,b]$ which satisfies the following two conditions:*

i) for every $z\in[a,b)$ and every $x'\in(z-\delta(z),z]\cap[a,b]$ there exists $x''\in(z,b]$ such that $[x',x'']\in\mathcal{A}$;

ii) for every $x'\in(b-\delta(b),b)\cap[a,b]$, the interval $[x',b]$ belongs to $\mathcal{A}$.

Then there exists a partition of $[a,b]$ consisting of intervals in $\mathcal{A}$.

Proof. Consider the set $\mathcal{C}$ of all points c of $[a,b]$ such that $[a,c]$ admits a partition consisting of intervals in $\mathcal{A}$. Put $z=\sup\mathcal{C}$. According to $i)$, $z>a$. By reductio ad absurdum we infer that actually $z=b$. Then $ii)$ assures that $b\in\mathcal{C}$. □

The original result of Cousin corresponds to the case where $\mathcal{A}$ is the family of all nondegenerate intervals $[x',x'']$ such that

$$[x',x'']\subset(z-\delta(z),z+\delta(z))\cap[a,b] \text{ for some } z\in[a,b].$$

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A related result, also extending Cousin's Lemma, is as follows:

Lemma 2. *Let δ be a gauge on $[a,b]$ and assume that $\mathcal{A}$ is a family of subintervals $[x', x''] \subset [a,b]$ which satisfies the following three conditions:*
i) there is $x'' \in (a,b]$ such that $[a, x''] \in \mathcal{A}$;
ii) for every $z \in (a,b]$ and every $x' \in (z-\delta(z), z) \cap [a,b]$, there is $x'' \in [z,b]$ such that $[x', x''] \in \mathcal{A}$.
iii) for every $[x', x''] \in \mathcal{A}$ with $x'' < b$ there is $y \in (x'', b]$ such that $[x', y] \in \mathcal{A}$.
Then there exists a partition of $[a,b]$ consisting of intervals in $\mathcal{A}$.

The two lemmata above are instrumental in our extension of the following two results in Real Analysis:

- (DB) The Denjoy-Bourbaki Theorem (which in turn is a generalization of the Mean Value Theorem). This theorem was first published in [2], p. 23-24, with an argument adapted from a celebrated paper of A. Denjoy [4], dedicated to the Dini derivatives. A nice account on it is available in [5], Ch. 8, Section 5.2.
- (LN) The Leibniz-Newton Formula for Lebesgue integrable right derivatives. See [7], p. 298-299, or [12].

A function $F : [a,b] \to E$ is said to have *negligible variation* on a set $A \subset [a,b]$ (and we write $F \in NV_A([a,b], E)$) if, for every $\varepsilon > 0$ there exists a gauge δ_ε on A such that if $\mathcal{D} = \{([u_k, v_k]), t_k\}_{k=1}^n$ is any (δ_ε, A)-fine tagged subpartition of $[a,b]$, then

$$\operatorname{Var}(F; \mathcal{D}) = \sum_{k=1}^n \|F(v_k) - F(u_k)\| < \varepsilon.$$

Analogously, $F : [a,b] \to E$ is said to have *negligible semivariation* on a set $A \subset [a,b]$ (and we write $F \in NSV_A([a,b], E)$) if, for every $\varepsilon > 0$ there exists a gauge δ_ε on A such that if $\mathcal{D} = \{([u_k, v_k]), t_k\}_{k=1}^n$ is any (δ_ε, A)-fine tagged subpartition of $[a,b]$, then

$$\left\| \sum_{k=1}^n (F(v_k) - F(u_k)) \right\| < \varepsilon.$$

For real-valued functions the two notions agree,

$$NSV_A([a,b], \mathbb{R}) = NV_A([a,b], \mathbb{R}).$$

Clearly, if $F \in NV_A([a,b], E)$, then F is continuous at every point of A. Conversely, if C is a countable set in $[a,b]$ and $F : [a,b] \to E$ is continuous at every point of C, then $F \in NV_C([a,b], E)$. However, when $Z \subset [a,b]$ is a Lebesgue negligible set, there are continuous functions on $[a,b]$ that do not belong to $NSV_Z([a,b], E)$. See [1], page 233, for an example.

Given a scalar function $\varphi : [a,b] \to \mathbb{R}$, one can attach to it the *Dini derivatives.* In what follows we are interested in the *upper right derivative,*

$$D^+\varphi(x) = \limsup_{h \downarrow 0} \frac{\varphi(x+h) - \varphi(x)}{h} \quad \text{for } x \in [a,b)$$

and the *lower left derivative*,

$$D_-\varphi(x) = \liminf_{h \uparrow 0} \frac{\varphi(x+h) - \varphi(x)}{h} \quad \text{for } x \in (a,b].$$

We are now in a position to state our generalization of the Denjoy-Bourbaki Theorem:

Theorem 1. *Let $F : [a,b] \to E$ and $\varphi : [a,b] \to \mathbb{R}$ be two continuous functions which fulfil the following three conditions with respect to a suitable disjoint decomposition $[a,b] = A_1 \cup A_2 \cup A_3$:*
i) F and φ have negligible semivariation on A_1;
ii) F has a right derivative F'_+ at all points of A_2 and $\left\|F'_+\right\| \leq D^+\varphi$ on A_2;
iii) F has a left derivative F'_- at all points of A_3 and $\left\|F'_-\right\| \leq D_-\varphi$ on A_3.
Then

$$\|F(b) - F(a)\| \leq \varphi(b) - \varphi(a).$$

The details will be given in Section 2.

The classical case corresponds to the situation where A_1 is at most countable and both F and φ have a right derivative at all points of $A_2 = [a,b)\backslash A_1$. In that case the condition *i*) is automatically satisfied.

Under the assumption that F and φ are both differentiable outside A_1, Theorem 1 has been proved in [9].

The Dini derivatives take values in $\overline{\mathbb{R}}$. Theorem 1 proves that a continuous function $\varphi : [a,b] \to \mathbb{R}$ cannot have an infinite upper right derivative at all points, even excepting a countable subset (or, more generally, a subset on which φ has negligible variation).

The case where $F = 0$ in Theorem 1 is an improvement of an old criterion of monotonicity mentioned by S. Saks in his monograph [11], p. 204:

Corollary 1. *Let $\varphi : [a,b] \to \mathbb{R}$ be a continuous functions for which there exists a disjoint decomposition $[a,b] = A_1 \cup A_2 \cup A_3$ such that:*
i) φ has negligible variation on A_1;
ii) $D^+\varphi \geq 0$ on A_2;
iii) $D_-\varphi \geq 0$ on A_3.
Then φ is nondecreasing.

An immediate consequence of Theorem 1 (for $\varphi(x) = M(x-a)$) is the following:

Corollary 2. *Let $F : [a,b] \to E$ be a continuous function for which there exists a subset $A \subset [a,b]$ such that:*
i) F has negligible semivariation on A;
ii) F has a right derivative F'_+ at all points of $[a,b)\backslash A$ and $\left\|F'_+\right\| \leq M$ on $[a,b)\backslash A$.
Then

$$\|F(b) - F(a)\| \leq M(b-a).$$

Corollary 1 allows us to retrieve the following classical result due to L. Scheefer:

Proposition 1. *Suppose that $F : [a,b] \to \mathbb{R}$ and $G : [a,b] \to \mathbb{R}$ are two continuous functions which admit finite upper right derivatives except on a countable subset C and $D^+F = D^+G$ at all points of $[a,b]\backslash C$. Then $F - G$ is a constant function.*

Proof. In fact, from $G = (G-F) + F$ we infer that

$$D^+G \leq D^+(G-F) + D^+F,$$

so by our hypothesis we get $D^+(G-F) \geq 0$ on $[a,b]\backslash C$. As C is countable, $G - F$ has negligible semivariation on C and thus $G - F$ is nondecreasing by Corollary 1. Changing the role of F and G we conclude that $F - G$ is constant. □

The discussion above suggests us to consider the following generalization of the concept of a primitive function:

Definition 1. *Given a function $f : [a, b] \to E$, by a right primitive of f we mean any continuous function $F : [a, b] \to E$ which verifies the following two conditions*:

i) F has a right derivative F'_+ at all points of $[a, b]$ except for a Lebesgue negligible subset A on which F has negligible semivariation;

ii) $F'_+ = f$ on $[a, b]\backslash A$.

The concept of a left primitive can be introduced in the same manner.

By using Lemma 1 one can prove that any two right primitives of a function differ by a constant.

The importance of Definition 1 above is outlined by the following generalization of the classical Leibniz-Newton Formula:

Theorem 2. *Let $f : [a, b] \to E$ be a function which is integrable in the sense of Henstock and Kurzweil and admits right primitives. Then*

$$\int_a^b f(t)\, dt = F(b) - F(a)$$

for every right primitive F of f.

Recall that a function $f : [a, b] \to E$ is said to be *integrable in the sense of Henstock and Kurzweil* if there exists a vector $I \in E$ such that for every $\varepsilon > 0$ one can find a gauge $\delta : [a, b] \to (0, \infty)$ such that for every $(\delta, [a, b])$-fine tagged partition $\{([u_k, v_k]), t_k\}_{k=1}^n$ of $[a, b]$, we have

$$\left\| I - \sum_{k=1}^{n} f(t_k)(v_k - u_k) \right\| < \varepsilon.$$

The vector I is unique with the above properties. It represents the integral of f over $[a, b]$, usually denoted by $\int_a^b f(t)\, dt$.

In the context of Lebesgue integrability, a special case of Theorem 2 has been proved by E. Hewitt and K. Stromberg [7]. See also [12] for a simple proof. A nice application is the fact that

$$\int_a^b f'_+(t) dt = f(b) - f(a)$$

for every continuous convex function $f : [a, b] \to \mathbb{R}$.

Theorem 2 yields Corollary 2. This is clear in the case where $E = \mathbb{R}$. In the general case, notice that we may restrict to the case of real Banach spaces and then use the formula

$$(h \circ F)'_+ = h \circ F'_+ \quad \text{for every } h \in E'.$$

In Section 3 we shall prove a result which extends Theorem 2.

Finally, it is worth noticing that the entire theory above can be extended to the framework of relative derivatives. Given a function $F : [a, b] \to E$, a subset $A \subset [a, b]$ and a point $z \in [a, b]$ (assumed to be a limit point of A), we define the *derivative of F at z relative to A* by the formula

$$F'(z; A) = \lim_{\substack{x \to z \\ x \in A}} \frac{F(x) - F(z)}{x - z}$$

provided that the limit exists. In a similar manner one can define the relative Dini derivatives $D^+F(z;A)$, $D_+F(z;A)$, $D^-F(z;A)$ and $D_-F(z;A)$. The details concerning the extension of Theorems 1 and 2 to this framework will be presented elsewhere.

2. Proof of Theorem 1

Suppose there is given $\varepsilon > 0$.

By the assumption $i)$, there exists a gauge $\delta : A_1 \to (0, \infty)$ such that for every (δ, A_1)-fine subpartition $([u_k, v_k])_{k=1}^n$ we have

$$\left\| \sum_{k=1}^{n} (F(v_k) - F(u_k)) \right\| \le \varepsilon/4 \quad \text{and} \quad \sum_{k=1}^{n} |\varphi(v_k) - \varphi(u_k)| < \varepsilon/4. \tag{2.1}$$

We shall denote by $\mathcal{A}_1$ the family of all subintervals $[x', x'']$ of $[a, b]$ such that

$$[x', x''] \subset (y - \delta(y), y + \delta(y))$$

for suitable $y \in [x', x''] \cap A_1$.

According to $ii)$, for each $z \in A_2$,

$$\liminf_{x \to z+} \left(\left\| \frac{F(x) - F(z)}{x - z} \right\| - \frac{\varphi(x) - \varphi(z)}{x - z} \right) = \left\| F'_+(z) \right\| - D^+\varphi(z) \le 0,$$

which yields an $y \in (z, b]$ such that

$$\left\| \frac{F(y) - F(z)}{y - z} \right\| - \frac{\varphi(y) - \varphi(z)}{y - z} < \frac{\varepsilon}{2(b - a)},$$

equivalently,

$$\alpha = \frac{\varepsilon}{2(b - a)}(y - z) - \|F(y) - F(z)\| + (\varphi(y) - \varphi(z)) > 0.$$

Since the functions F and φ are continuous at z, there exists a positive number $\delta_1(z)$ such that for every $x' \in (z - \delta_1(z), z] \cap [a, b]$ we have

$$\|F(x') - F(z)\| < \alpha/4 \quad \text{and} \quad |\varphi(x') - \varphi(z)| < \alpha/4$$

and for every $x'' \in [y, y + \delta_1(z)) \cap [a, b]$ we have

$$\|F(x'') - F(y)\| < \alpha/4 \quad \text{and} \quad |\varphi(x'') - \varphi(y)| < \alpha/4.$$

Then

$$\frac{\varepsilon}{2(b - a)}(x'' - x') - \|F(x'') - F(x')\| + (\varphi(x'') - \varphi(x')) > \alpha - 4 \cdot \alpha/4 = 0,$$

that is,

$$\|F(x'') - F(x')\| - (\varphi(x'') - \varphi(x')) < \frac{\varepsilon}{2(b - a)}(x'' - x'). \tag{2.2}$$

We denote by $\mathcal{A}_2$ be the family of all intervals $[x', x'']$ which appear this way.

Similarly, for every $z \in A_3$,

$$\limsup_{x \to z-} \left(\left\| \frac{F(x) - F(z)}{x - z} \right\| - \frac{\varphi(x) - \varphi(z)}{x - z} \right) = \left\| F'_-(z) \right\| - D_-\varphi(z) \le 0,$$

and thus there exists a positive number $\delta_1(z)$ such that for every $x' \in (z - \delta_1(z), z) \cap [a, b]$, we have

$$\|F(z) - F(x')\| - (\varphi(z) - \varphi(x')) < \frac{\varepsilon}{2(b - a)}(z - x').$$

Then

$$\beta = \frac{\varepsilon}{2(b-a)}(z-x') - \|F(z)-F(x')\| + (\varphi(z)-\varphi(x')) > 0.$$

Since F and φ are continuous on $[a,b]$, we can find a positive number δ_2 such that

$$\|F(x'')-F(z)\| < \beta/2 \quad \text{and} \quad |\varphi(x'')-\varphi(z)| < \beta/2$$

for every $x'' \in [z, z+\delta_2) \cap [a,b]$. Then

$$\frac{\varepsilon}{2(b-a)}(x''-x') - \|F(x'')-F(x')\| + (\varphi(x'')-\varphi(x')) > \beta - 2\cdot\beta/2 = 0$$

that is,

$$\|F(x'')-F(x')\| - (\varphi(x'')-\varphi(x')) < \frac{\varepsilon}{2(b-a)}(x''-x'). \tag{2.3}$$

This reasoning yields a new family $\mathcal{A}_3$ of subintervals of $[a,b]$.

The family

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$$

verifies the hypothesis of Lemma 2 and thus there exists a partition $\mathcal{D} = ([x_i, x_{i+1}])_{i=0}^{n-1}$ of $[a,b]$ into subintervals of $\mathcal{A}$.

By (2.1), (2.2) and (2.3), we get

$$\begin{aligned}
&\|F(b)-F(a)\| - (\varphi(b)-\varphi(a)) \le \\
&\le \left\| \sum_{[x_i,x_{i+1}]\in\mathcal{A}_1} (F(x_{i+1})-F(x_i)) \right\| + \sum_{[x_i,x_{i+1}]\in\mathcal{A}_1} |\varphi(x_{i+1})-\varphi(x_i)| \\
&+ \sum_{[x_i,x_{i+1}]\in\mathcal{A}\setminus\mathcal{A}_1} (\|F(x_{i+1})-F(x_i)\| - (\varphi(x_{i+1})-\varphi(x_i))) \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2(b-a)} \sum_{[x_i,x_{i+1}]\in\mathcal{A}\setminus\mathcal{A}_1} (x_{i+1}-x_i) \le \varepsilon,
\end{aligned}$$

which means that $\|F(b)-F(a)\| - (\varphi(b)-\varphi(a)) < \varepsilon$. As $\varepsilon > 0$ was fixed arbitrary, we conclude that$\|F(b)-F(a)\| - (\varphi(b)-\varphi(a)) \le 0$.

3. A General Leibniz-Newton Formula

The aim of this section is to prove the following generalization of Theorem 2:

Theorem 3. *Let $F : [a,b] \to E$ and $f : [a,b] \to \mathbb{R}$ be two functions for which there exists a disjoint decomposition $[a,b] = A_1 \cup A_2 \cup A_3$ such that:*

i) F is continuous on $[a,b]$ and has negligible semivariation on A_1;

ii) F has a right derivative F'_+ at all points of A_2 and a left derivative F'_- at all points of A_3;

iii) f is integrable in the sense of Henstock-Kurzweil and

$$f(x) = \begin{cases} 0 & \text{if } x \in A_1 \\ F'_+(x) & \text{if } x \in A_2 \\ F'_-(x) & \text{if } x \in A_3. \end{cases}$$

Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

When A_1 is Lebesgue negligible, the condition $f = 0$ on A_1 can be removed.

Proof. Let $\varepsilon > 0$ be arbitrarily fixed. Since the function f is integrable, there is a gauge $\delta_1 : [a,b] \to (0,\infty)$ such that for every δ_1-fine tagged partition $\mathcal{D} = \{([x_{i-1}, x_i]), t_i\}_{i=1}^m$ of $[a,b]$, we have

$$\left\| \int_a^b f(x)dx - \sum_{i=1}^m f(t_i)(x_i - x_{i-1}) \right\| < \frac{\varepsilon}{2}. \tag{3.1}$$

Since $F \in NSV_{A_1}([a,b],E)$, we can choose a gauge $\delta_2 : [a,b] \to (0,\infty)$ such that $\delta_2 \le \delta_1$ on A_1 and for any (δ_2, A_1)-fine tagged subpartition $\mathcal{D} = \{([x_i', x_i'']), s_i\}_{i=1}^n$ of $[a,b]$, we have

$$\left\| \sum_{i=1}^n (F(x_i'') - F(x_i')) \right\| < \frac{\varepsilon}{4}. \tag{3.2}$$

We shall denote by $\mathcal{A}_1$ the family of all subintervals of $[a,b]$ for which there are points $z \in [x', x''] \cap A_1$ such that

$$[x', x''] \subset (z - \delta_2(z), z + \delta_2(z)).$$

Clearly, $\mathcal{A}_1$ consists of δ_1-fine intervals.

Suppose that $z \in A_2$. By *ii*), we can choose a number $\delta_2(z) \in (0, \delta_1(z)]$ such that

$$y \in (z, z + \delta_2(z)) \cap [a,b] \text{ implies } \left\| \frac{F(y) - F(z)}{y - z} - f(z) \right\| < \frac{\varepsilon}{4(b-a)}.$$

The last inequality says that

$$\alpha = \frac{\varepsilon}{4(b-a)}(y - z) - \|F(y) - F(z) - f(z)(y - z)\| > 0,$$

so that by the continuity of F we may choose a number

$$\delta_3(z,y) \in \left(0, \min\left\{ z + \delta_1(z) - y, \frac{\alpha}{4(1 + \|f(z)\|)} \right\}\right)$$

for which

$$x' \in (z - \delta_3(z,y), z] \cap [a,b] \text{ implies } \|F(x') - F(z)\| < \frac{\alpha}{4}$$

and

$$x'' \in [y, y - \delta_3(z,y)) \cap [a,b] \text{ implies } \|F(x'') - F(y)\| < \frac{\alpha}{4}.$$

Therefore for all $x' \in (z - \delta_3(z,y), z] \cap [a,b]$ and all $x'' \in [y, y + \delta_3(z,y)) \cap [a,b]$ we have

$$\frac{\varepsilon}{4(b-a)}(x'' - x') - \|F(x'') - F(x') - f(z)(x'' - x')\| > \alpha - 4 \cdot \frac{\alpha}{4} = 0$$

and thus

$$\|F(x'') - F(x') - f(z)(x'' - x')\| < \frac{\varepsilon}{4(b-a)}(x'' - x'). \tag{3.3}$$

We shall denote by $\mathcal{A}_2$ the set of all intervals $[x', x'']$ that appear by the preceding reasoning.

Suppose that $z \in A_3$. By *iii*), we can choose a number $\delta_2(z) \in (0, \delta_1(z)]$ such that

$$y \in (z - \delta_2(z), z) \cap [a,b] \text{ implies } \left\| \frac{F(y) - F(z)}{y - z} - f(z) \right\| < \frac{\varepsilon}{4(b-a)}.$$

The last inequality says that

$$\beta = \frac{\varepsilon}{4\,(b-a)}\,(z-y) - \|F(z) - F(y) - f(z)\,(z-y)\| > 0,$$

so that by the continuity of F we may choose for each $x' \in (z - \delta_2(z), z) \cap [a,b]$ a number

$$\delta_3(z,x') \in \left(0, \min\left\{\delta_2(z), \frac{\beta}{2\,(1+\|f(z)\|)}\right\}\right)$$

for which

$$x'' \in [z, z+\delta_3(z,x')] \cap [a,b] \text{ implies } \|F(x'') - F(z)\| < \frac{\beta}{2}.$$

Therefore for all $x'' \in [z, z+\delta_3(z,x')] \cap [a,b]$ we have

$$\frac{\varepsilon}{4\,(b-a)}\,(x''-x') - \|F(x'') - F(x') - f(z)\,(x''-x')\| > \beta - 2\cdot\frac{\beta}{2} = 0$$

and thus

$$\|F(x'') - F(x') - f(z)\,(x''-x')\| < \frac{\varepsilon}{4\,(b-a)}\,(x''-x')\,. \tag{3.4}$$

We shall denote by $\mathcal{A}_3$ the new set of intervals $[x', x'']$ that appear by the last reasoning.

The family of intervals

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$$

fulfils the hypotheses of Lemma 2 and thus there is a partition $\mathcal{D} = ([x_i, x_{i+1}])_{i=0}^{n-1}$ of $[a,b]$ consisting of intervals of $\mathcal{A}$. Clearly, $\mathcal{D}$ is δ_1-fine. By the relation (4) we get

$$\begin{aligned}
&\left\| F(b) - F(a) - \int_a^b f(x)dx \right\| \\
&\le \left\| \sum_{i=0}^{n-1} [F(x_{i+1}) - F(x_i) - f(z_i)\,(x_{i+1}-x_i)] \right\| + \left\| \sum_{i=0}^{n-1} f(z_i)\,(x_{i+1}-x_i) - \int_a^b f(x)dx \right\| \\
&\qquad < \left\| \sum_{i=0}^{n-1} [F(x_{i+1}) - F(x_i) - f(z_i)\,(x_{i+1}-x_i)] \right\| + \frac{\varepsilon}{2}.
\end{aligned}$$

On the other hand, by (5)-(7) and the fact that $f|_{A_1} = 0$, we get

$$\begin{aligned}
&\left\| \sum_{i=0}^{n-1} [F(x_{i+1}) - F(x_i) - f(z_i)\,(x_{i+1}-x_i)] \right\| \\
&\qquad \le \left\| \sum_{\{i \mid [x_i,x_{i+1}] \in \mathcal{A}_1\}} (F(x_{i+1}) - F(x_i)) \right\| \\
&\qquad + \sum_{\{i \mid [x_i,x_{i+1}] \in \mathcal{A}\setminus\mathcal{A}_1\}} \|F(x_{i+1}) - F(x_i) - f(z_i)\,(x_{i+1}-x_i)\| \\
&\qquad < \frac{\varepsilon}{4} + \frac{\varepsilon}{4(b-a)} \sum_{i=1}^{n-1} (x_{i+1}-x_i) = \frac{\varepsilon}{2}
\end{aligned}$$

and the proof ends by noticing that $\varepsilon > 0$ was arbitrarily fixed. □

Letting $A_3 = \emptyset$ in Theorem 3 we get the assertion of Theorem 2. Actually, Theorem 2 can be proved via a direct argument based on Lemma 1.

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