FAN'S INEQUALITY IN THE CONTEXT OF M_p-CONVEXITY

CONSTANTIN P. NICULESCU AND IONEL ROVENTA

ABSTRACT. Several analogues of Fan's inequality are proved in the context of M_p-convexity. As a consequence, a Nash equilibrium theorem is obtained.

1. INTRODUCTION

In what follows we are interested in a class of functions having a nice behavior under the action of means.

The weighted M_p-mean is defined for pairs of positive numbers x, y by the formula

$$M_p(x, y, 1 - \lambda, \lambda) = \begin{cases} ((1 - \lambda)x^p + \lambda y^p)^{1/p}, & \text{if } p \in \mathbb{R} \backslash \{0\} \\ x^{1-\lambda}y^\lambda, & \text{if } p = 0 \\ \min\{x, y\}, & \text{if } p = -\infty \\ \max\{x, y\}, & \text{if } p = \infty, \end{cases}$$

where $\lambda \in [0, 1]$. If p is an odd number, we can extend M_p to pairs of real numbers.

Let E be a linear topological space and assume that C is a nonempty compact and convex subset of E.

Definition 1. *We say that a function $f : C \longrightarrow \mathbb{R}$ is M_p-concave if*

$$f((1 - \lambda)x + \lambda y) \geq M_p(f(x), f(y), 1 - \lambda, \lambda)$$

for all $x, y \in C$ and $\lambda \in (0, 1)$.

Thus the M_1-concave functions are the usual concave functions, while the M_∞-concave functions, are precisely the quasi-concave functions.

A celebrated result due to Ky Fan asserts that any function $f : C \times C \to \mathbb{R}_+$ which is quasi-concave in the first variable and lower semicontinuous in the second variable verifies the inequality

$$(1.1) \qquad \min_{y \in C} \sup_{x \in C} f(x, y) \leq \sup_{z \in C} f(z, z).$$

The aim of this paper is to prove a complementary result, precisely:

Theorem 1. *Suppose that $f : C \times C \to \mathbb{R}_+$ is a function which is M_p-concave and lower-semicontinuous in each variable. Then*

$$\min_{y \in C} \sup_{x \in C} M_p^p(f(x, y), f(y, x), 1 - \lambda, \lambda) \leq \sup_{z \in C} f^p(z, z),$$

for all $\lambda \in (0, 1)$ and $p \in \mathbb{R}$.

Our technics yields also the following fact:

Theorem 2. *Let C be a nonempty compact and convex subset of E, and let f : $C \times C \to \mathbb{R}_+$ be a function which is M_p-concave in the first variable and lower-semicontinuous in the second variable. Then*

$$\min_{y \in C} \max_{x \in C} M_p^p(f(y,y), f(x,x), 1 - \lambda, \lambda) \le \sup_{z \in C} f^p(z,z)$$

for all $\lambda \in (0,1)$ and $p \in \mathbb{R}$.

For p an odd number f is allowed to take negative values.

2. Proof of the main result

We actually prove a much more general result:

Theorem 3. *Assume $f : C \times C \to \mathbb{R}_+$ is a function which is M_p-concave and lower-semicontinuous in each variable and let $g : C \to C$ be a continuous onto function. Then*

$$\min_{y \in C} \sup_{x \in C} M_p^p(f(x,y), f(y,x), 1 - \lambda, \lambda) \le \sup_{z \in C} M_p^p(f(z,g(z)), f(g(z),z), 1 - \lambda, \lambda),$$

for all $\lambda \in (0,1)$ and $p \in \mathbb{R}$.

Theorem 1 represents the particular case where g is the identity of C.

The proof of Theorem 3 is based on the KKM-Theorem, whose statement is recalled here for the convenience of the reader:

Theorem 4. (*Knaster-Kuratowski-Mazurkievicz*). *Suppose that for every point x in a nonempty set $X \subset E$ there is an associated closed subset $M(x) \subset X$ such that*

$$conv \ F \subset \bigcup_{x \in F} M(x)$$

holds for all finite subsets $F \subset X$. Then for any finite subset $F \subset X$ we have

$$\bigcap_{x \in F} M(x) \ne \emptyset.$$

Hence if some subset $M(z)$ is compact, we have

$$\bigcap_{x \in X} M(x) \ne \emptyset.$$

Theorem 4 is one of the many results known to be equivalent to Brouwer's fixed point theorem. See [1].

Proof of Theorem 3. We attach to $g : C \to C$ and $\lambda \in [0,1]$ the family of sets $(M(x))_{x \in C}$, where $M(x)$ consists of all $y \in C$ such that

$$M_p^p(f(x,y), f(y,x), 1 - \lambda, \lambda) \le \sup_{z \in C} M_p^p(f(z,g(z)), f(g(z),z), 1 - \lambda, \lambda).$$

We will show that this family satisfies the hypothesis of the KKM-Theorem. In fact, $g(x) \in M(g(x))$ for every $x \in C$ and

$$conv \ F \subset \bigcup_{x \in F} M(g(x))$$

for every finite subset $F \subset C$. For example, if F consists of two elements $g(x_1)$ and $g(x_2)$, we have to show that

$$(2.1) \qquad u = (1 - \alpha)g(x_1) + \alpha g(x_2) \in M(g(x_1)) \cup M(g(x_2))$$

for every $\alpha \in (0, 1)$. Our argument is by reductio ad absurdum.

If (2.1) fails, then for some $\alpha \in (0, 1)$ we have

$$(2.2) \quad M_p^p(f(x_1, u), f(u, x_1), 1 - \lambda, \lambda) > \sup_{z \in C} M_p^p(f(z, g(z)), f(g(z), z), 1 - \lambda, \lambda),$$

and

$$(2.3) \quad M_p^p(f(x_2, u), f(u, x_2), 1 - \lambda, \lambda) > \sup_{z \in C} M_p^p(f(z, g(z)), f(g(z), z), 1 - \lambda, \lambda).$$

The Intermediate Value Theorem yields a $\beta \in [0, 1]$ such that $x_\beta = \beta x_1 + (1 - \beta)x_2 \in C$ verifies

$$g(x_\beta) = u = (1 - \alpha)g(x_1) + \alpha g(x_2).$$

Since f is M_p-concave in each variable it follows that

$$M_p^p(f(x_\beta, g(x_\beta)), f(g(x_\beta), x_\beta), 1 - \lambda, \lambda)$$
$$= (1 - \lambda)f^p((1 - \beta)x_1 + \beta x_2, u) + \lambda f^p(u, (1 - \beta)x_1 + \beta x_2)$$

is not less than

$$(1 - \lambda)((1 - \beta)f^p(x_1, u) + \beta f^p(x_2, u)) + \lambda((1 - \beta)f^p(u, x_1) + \beta f^p(u, x_2))$$
$$= (1 - \beta)((1 - \lambda)f^p(x_1, u) + \lambda f^p(u, x_1)) + \beta((1 - \lambda)f^p(x_2, u) + \lambda f^p(u, x_2))$$
$$= (1 - \beta)M_p^p(f(x_1, u), f(u, x_1), 1 - \lambda, \lambda) + \beta M_p^p(f(x_2, u), f(u, x_2), 1 - \lambda, \lambda)$$
$$> (1 - \beta) \sup_{z \in C} M_p^p(f(z, g(z)), f(g(z), z), 1 - \lambda, \lambda)$$
$$+ \beta \sup_{z \in C} M_p^p(f(z, g(z)), f(g(z), z), 1 - \lambda, \lambda)$$
$$= \sup_{z \in C} M_p^p(f(z, g(z)), f(g(z), z), 1 - \lambda, \lambda),$$

a contradiction. Thus (2.1) follows.

By the KKM-Theorem we infer that

$$\bigcap_{x \in C} M(g(x)) \neq \emptyset,$$

which means the existence of $y \in C$ such that

$$M_p^p(f(x, y), f(y, x), 1 - \lambda, \lambda) \leq \sup_{z \in C} M_p^p(f(z, g(z)), f(g(z), z), 1 - \lambda, \lambda),$$

for every $x \in C$, equivalently,

$$\sup_{x \in C} M_p^p(f(x, y), f(y, x), 1 - \lambda, \lambda) \leq \sup_{z \in C} M_p^p(f(z, g(z)), f(g(z), z), 1 - \lambda, \lambda).$$

In conclusion,

$$\min_{y \in C} \sup_x M_p^p(f(x, y), f(y, x), 1 - \lambda, \lambda) \leq \sup_{z \in C} M_p^p(f(z, g(z)), f(g(z), z), 1 - \lambda, \lambda). \qquad \blacksquare$$

3. A NONSYMMETRIC EXTENSION OF FAN'S THEOREM

The aim of this section is to prove the following result:

Theorem 5. *Let D and F be two two nonempty, compact and convex subsets of E and let g be a continuous onto function $g : D \to F$. Then for every function $f : D \times F \to \mathbb{R}_+$ which is quasi-concave in the first variable and lower-semicontinuous in the second variable the following inequality holds:*

$$\min_{y \in F} \sup_{x \in D} f(x, y) \leq \sup_{z \in D} f(z, g(z)).$$

This result extends Fan's Theorem (which corresponds to the case where $D = F$ and $g(z) = z$ for all $z \in D$).

Proof. Consider the family of sets

$$M(g(x)) = \{y \in F : f(x, y) \leq \sup_{z \in D} f(z, g(z))\}, \quad \text{for } x \in D.$$

We will show that this family verifies the assumptions of the KKM-Theorem. In fact, it is easy to see that $g(x) \in M(g(x))$, for $x \in D$. Let A be a nonempty finite subset of F, say $A = \{g(x_1), g(x_2)\}$ for simplicity. We have to prove that

(3.1)
$$\text{conv } A \subset \bigcup_{g(x) \in A} M(g(x)),$$

that is,

$$(1 - \lambda)g(x_1) + \lambda g(x_2) \in M(g(x_1)) \cup M(g(x_2)),$$

for all $\lambda \in (0, 1)$.

Indeed, if the contrary is true, then for a suitable $\lambda \in (0, 1)$ we have

$$f(x_1, (1 - \lambda)g(x_1) + \lambda g(x_2)) > \sup_{z \in D} f(z, g(z))$$

and

$$f(x_2, (1 - \lambda)g(x_1) + \lambda g(x_2)) > \sup_{z \in D} f(z, g(z)).$$

The intermediate Value Theorem yields an $\alpha \in [0, 1]$ such that $x_\lambda = \alpha x_1 + (1 - \alpha)x_2 \in D$ verifies

$$(1 - \lambda)g(x_1) + \lambda g(x_2) = g(x_\lambda).$$

Then

$$
\begin{aligned}
f(x_\lambda, g(x_\lambda)) &= f(\alpha x_1 + (1 - \alpha)x_2, (1 - \lambda)g(x_1) + \lambda g(x_2)) \\
&\geq \min \{f(x_1, (1 - \lambda)g(x_1) + \lambda g(x_2)), f(x_2, (1 - \lambda)g(x_1) + \lambda g(x_2))\} \\
&> \sup_{z \in D} f(z, g(z)),
\end{aligned}
$$

a contradiction that shows that (3.1) works. By the KKM-Theorem, $\cap_{x \in D} M(g(x)) \neq \emptyset$, and this fact assures the existence of a $y_0 \in F$ such that

$$f(x, y_0) \leq \sup_{z \in D} f(z, g(z)) \quad \text{for every } x \in D.$$

Consequently $\sup_{x \in D} f(x, y_0) \leq \sup_{z \in D} f(z, g(z))$, which yields

$$\min_{y \in F} \sup_{x \in D} f(x, y_0) \leq \sup_{z \in D} f(z, g(z)). \quad \blacksquare$$

A similar argument yields the following theorem in the case of M_p-convex functions:

Theorem 6. *Let C be a nonempty, compact and convex subset of E and let g be a continuous onto function $g : C \to C$. Then for every M_p-convex function $f : C \times C \to \mathbb{R}_+$ which is upper-semicontinuous with respect to each variable we have*

$$\max_{x \in C} \inf_{y \in C} M_p^p(f(x,y), f(y,x), 1 - \lambda, \lambda) \geq \inf_{z \in C} M_p^p(f(z, g(z)), f(g(z), z), 1 - \lambda, \lambda),$$

for all $\lambda \in (0, 1)$.

In the same manner we can prove the following result for the quasi-convex functions:

Theorem 7. *Let D and F be two nonempty, compact and convex subsets and let g be a continuous onto function $g : D \to F$. Let $f : D \times F \to \mathbb{R}_+$ be a quasi-convex function in the second variable and upper-semicontinuous in the first variable. Then we have*

$$\max_{x \in D} \inf_{y \in F} f(x, y) \geq \inf_{z \in D} f(z, g(z)).$$

Remark 1. *We impose the condition that g is an onto function in order to assure the applicability of KKM-Theorem.*

4. Further results

An important application of Theorem 5 is the existence of a g-equilibrium, a fact that generalizes the existence of a Nash equilibrium.

Theorem 8. [1]*Let $C = C_1 \times C_2 \times ... \times C_n$, where C_i, $i = 1, \ldots, n$ are nonempty, compact and convex subsets of E, let $g = (g_1, g_2, ..., g_n) : C \to C$ be a continuous onto function and let $f_1, \ldots, f_n : C \to C$ be lower-semicontinuous functions such that $x \to \sum_{i=1}^n f_i(y_1, ..., g(x_i), ...y_n)$ is quasi convex for each $y \in C$. Then there exists an $y \in C$ such that*

$$f_i(y) \leq f_i(y_1, ..., g(x_i), ..., y_n)),$$

for every $x_i \in C_i$, $i = 1, ..., n$.

Proof. Let $f(x, y) = \sum_{i=1}^n (f_i(y) - f_i(y_1, ..., g(x_i), ..., y_n))$. It is easy to see that f satisfies the assumptions of Theorem 5. This yields an $y \in C$ such that

$$\sup_{x \in C} f(x, y) \leq \sup_{z \in C} f(z, g(z)) = 0.$$

Letting $x = (y_1, y_2, ..., x_i, ...y_n)$ $(i = 1, ..., n)$ in the last inequality we conclude that

$$f_i(y) - f_i(y_1, ..., g(x_i), ..., y_n)) \leq 0$$

for every $x_i \in C_i$, $i = 1, ..., n$. ∎

The following result due to M. Sion [5] has important applications in convex analysis and games theory:

Theorem 9. *Let D and F be two nonempty, compact and convex subsets and let $f : D \times F \to \mathbb{R}_+$ be a function which is upper-semicontinuous and quasi-concave function in the first variable, and lower-semicontinuous and quasi-convex in the second variable. Then*

$$\min_{y \in F} \max_{x \in D} f(x, y) = \max_{x \in D} \min_{y \in F} f(x, y).$$

---

[1]Corrected, February 12, 2009. The authors thank Professor S. Park for calling their attention to a mistake in the original version of this theorem.

This result can be derived via Theorems 5 and 7 above provided that

$$(4.1) \qquad \inf_{z \in D} f(z, g_1(z)) = \sup_{z \in D} f(z, g_2(z)).$$

for suitably chosen continuous onto functions g_1 and g_2. In fact, the hypothesis of Sion's theorem make possible to apply Theorems 5 and 7 and thus

$$\inf_{z \in D} f(z, g_1(z)) \le \max_{x \in D} \min_{y \in F} f(x, y) \le \min_{y \in F} \max_{x \in D} f(x, y) \le \sup_{z \in D} f(z, g_2(z)),$$

for all continuous onto functions $g_1, g_2 : D \to F$. While the topological condition (4.1) can be easily verified in a number of particular cases, we do not know how general is it. Results of this type, concerning the existence of continuous onto maps relating compact convex sets, may be found in [2].

References

[1] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization*, Springer-Verlag, Berlin, 2000.
[2] J. G. Hocking and G. S. Young, *Topology*, Dover, 1988.
[3] J. Kindler, Über Spiele auf konvexen Mengen, in *Operations Res. Vehrfahren*, Hain, Meisenheim/Glan, (1977), 695-704.
[4] J. Kindler, A Simple proof of Sion's Minimax Theorem, *The American Mathematical Monthly*, **112** (2005), 356-358.
[5] M. Sion, On general minimax theorems, *Pacific J. Math*, **8** (1958), 171-176.

Department of Mathematics, University of Craiova, Craiova RO-200585, Roman
E-mail address: cniculescu@central.ucv.ro

Department of Mathematics, University of Craiova, Craiova RO-200585, Romania
E-mail address: roventaionel@yahoo.com