A CONDITION OF UNIFORM EXPONENTIAL STABILITY FOR SEMIGROUPS

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Abstract. The aim of this paper is to prove that the uniform exponential stability of a strongly continuous semigroup \( \{T(t)\}_{t \geq 0} \) (acting on a complex Hilbert space \( H \)) can be derived as a consequence of the well behavior of its numerical range in a suitable Orlicz space. More precisely, assuming that there exists an Orlicz space \( E = (L^\Phi, \rho^\Phi) \) over \( \mathbb{R}_+ \) such that
\[
\liminf_{\alpha \downarrow 0} \alpha \| \exp_{-\alpha} \|_{E^*} = 0
\]
and
\[
\sup_{\|x\| \leq 1} \rho^\Phi(\|\langle \cdot, x \rangle\|) \leq M < \infty,
\]
then the uniform growth bound \( \omega_0 \) of the semigroup verifies an estimate of the form
\[
\omega_0 \leq M \beta := \beta - (2M \|\exp_{-\beta}\|_{E^*})^{-1} < 0
\]
for some positive number \( \beta \). As an application, the well posedness of an abstract infinite time Cauchy problem is discussed.

1. Introduction

Let \( H \) be a complex Hilbert space and let \( 1 \leq p < \infty \). Recall that a semigroup \( T = \{T(t)\}_{t \geq 0} \) on \( H \) is called:

- \textit{weakly-} \( L^p \)-stable if for every \( x, y \in H \) we have
\[
\int_0^\infty \|\langle T(t)x, y \rangle\|^p dt < \infty;
\]
• uniformly exponentially stable if its uniform growth bound is negative, that is

\[ \omega_0(T) := \lim_{t \to \infty} \frac{\ln ||T(t)||}{t} < 0, \]

or, equivalently, if

\[ ||T(t)|| \leq Ne^{-\nu t} \text{ for all } t \geq 0, \]

for some positive constants \( N \) and \( \nu \).

It is clear that each uniformly exponentially semigroup is weakly-\( L^p \)-stable. In 1983 A. J. Pritchard and J. Zabczyk [9] raised the problem whether every weakly-\( L^p \)-stable semigroup is uniformly exponentially stable. The answer is positive and a solution can be found in [3], [11]. In this note we extend their result to the more general framework of Orlicz spaces. In order to formulate our generalization we shall need a preparation on Orlicz spaces. For further details the reader is referred to [4], [5], [6], [1] and references therein.

The Orlicz spaces over \( \mathbb{R}_+ \) are attached to nondecreasing convex functions \( \Phi : [0, \infty) \to [0, \infty] \) such that \( \Phi(0) = \Phi(0+) = 0 \) and \( \Phi \) is not identically 0 or \( \infty \) on \( (0, \infty) \). We denote by \( L^\Phi \) the set of all complex-valued measurable functions \( f \) defined on \( \mathbb{R}_+ \) for which there exists a positive \( \lambda \) such that \( \int_0^\infty \Phi(\lambda|f(t)|)dt < \infty \). Clearly, \( L^\Phi \) is a linear space with respect to the usual operations and we can turn \( L^\Phi \) into an Orlicz space by considering on it the norm \( \rho^\Phi \), where

\[ \rho^\Phi(f) := \inf \{k > 0 : \int_0^\infty \Phi(k^{-1}|f(t)|)dt \leq 1 \}. \]

If \( \Phi \) satisfies the \( \Delta_2 \)-condition i.e., there exists a positive constant \( C \) such that

\[ \Phi(2t) \leq C\Phi(t) \text{ for all } t \geq 0, \]

then the dual space \( (L^\Phi)^* \) is also an Orlicz space. Moreover, in this case \( (L^\Phi)^* \) can be identified with \( L^{\Phi^*} \), where

\[ \Phi^*(t) := \sup_{s \geq 0} (ts - \Phi(s)), \quad t \geq 0 \]

is the Legendre transform of \( \Phi \).

Clearly, all Lebesgue spaces \( L^p(\mathbb{R}_+) \) (for \( 1 \leq p < \infty \)) are examples of Orlicz spaces which satisfy the \( \Delta_2 \)-condition.

We can now state our main result:

**Theorem 1.** Let \( T = \{T(t)\}_{t \geq 0} \) be a strongly continuous semigroup acting on a complex Hilbert space \( H \). Then \( T \) is uniformly exponentially
stable if (and only if) it verifies the following condition

\[(1.1) \quad M = \sup_{\|x\| \leq 1} \rho^\Phi (|\langle T(t)x, x \rangle|) < \infty,\]

with respect to an Orlicz space \(E = (L^\Phi, \rho^\Phi)\) whose dual space \(E^*\) has the property that

\[(1.2) \quad \liminf_{\alpha \downarrow 0} \|\alpha\| \exp_{\alpha} \|E^*\| = 0.\]

The necessity of the condition (1.1) is straightforward. In fact, if the semigroup \(T\) is uniformly exponentially stable, then (1.1) works for all Orlicz spaces. The sufficiency part is detailed in the next section.

As shows the case where \(T\) is the left translation semigroup on \(H = L^2(\mathbb{R})\) and \(E = L^1(\mathbb{R}^+),\) the condition (1.2) is essential for the validity of Theorem 1.

In the special case where \(\Phi(t) = t^p\) (for \(1 \leq p < \infty\)), the result of Theorem 1 was first proved by G. Weiss [11]. Clearly, in that case the condition (1.2) is automatically fulfilled. Our result covers more general Orlicz functions \(\Phi\) which satisfy the \(\Delta_2\)-condition and \(\lim_{t \to 0^+} t \rho^{\Phi^*}( \exp_{-t} ) = 0\) such as \(\Phi(t) = e^t - t - 1.\) In this case

\[\Phi^*(t) = (t + 1) \ln(t + 1) - t\]

and

\[
\rho^{\Phi^*}(\exp_{-\alpha}) = \inf \{ k > 0 : \int_0^\infty \Phi^*(e^{-\alpha t}) dt \leq 1 \} \\
= \inf \{ k > 0 : \frac{1}{\alpha} \int_0^{1/k} \frac{u + 1}{u} \ln(u + 1) du - \frac{1}{k\alpha} \leq 1 \} \\
= \sup \{ b > 0 : \int_0^b \frac{u + 1}{u} \ln(u + 1) du \leq b + \alpha \} = b_0,
\]

where \(b_0\) is the unique solution of the following equation (in variable \(x\)),

\[\int_0^x \frac{u + 1}{u} \ln(u + 1) du = x + \alpha.\]

The map \(\alpha : x \to \int_0^x \frac{u + 1}{u} \ln(u + 1) du - x\) (from \([0, \infty)\) into \([0, \infty))\) is surjective and also increasing, so that its inverse is continuous. Consequently \(\alpha^{-1}\) is bounded on \([0, 1]\), which yields \(\lim_{t \to 0^+} t \rho^{\Phi^*}(\exp_{-t}) = 0\).

J.M.A.M. van Neerven [7] has noticed that any bounded strongly continuous semigroup (acting on a complex Hilbert space \(H\)) is uniformly exponentially stable if there exists a nondecreasing function
& : & \mathbb{R}_+ \to \mathbb{R}_+ & & \text{such that } & & \varphi(t) > 0 & & \text{for } t > 0 & & \text{and} & & \\
& & \int_0^\infty \varphi(\|\langle T(t)x,y\rangle\|)dt < \infty, & & \text{for all } x, y \in H. & & \\

We leave open the question whether the boundedness condition can be dropped.

2. Proof of Theorem 1

Proof. We already noticed that only the sufficiency part needs an argument. For this, we need the remark that the condition of boundedness (1.1) yields

\[ N = \sup_{\|x\|,\|y\| \leq 1} \rho^\Phi(\|\langle T(\cdot)x,y\rangle\|) \leq 2M < \infty, \]

as a consequence of the polarization identity

\[ \langle T(t)x,y \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \langle T(t)(x + i^k y),x + i^k y \rangle. \]

The next step is to motivate the existence of the improper integral

\[ \int_0^\infty u^*(t)T(t)xdt := \lim_{s \to \infty} \int_0^s u^*(t)T(t)xdt, \]

for all \( u^* \in E^* \) and \( x \in H \). In terms of series, this limit means the convergence of

\[ \sum_{n=0}^\infty \int_{s_n}^{s_{n+1}} u^*(t)T(t)xdt \]

for all positive sequences \((s_n)_n\), with \( s_0 = 0 \), which are increasing to \( \infty \). This can be derived from a classical result due to Orlicz-Pettis, which asserts that every weakly unconditionally convergent series (in a Banach space) is also unconditionally convergent. In fact,

\[ \sum_{n=0}^N \left| \int_{s_n}^{s_{n+1}} u^*(t)T(t)xdt, y \right| = \sum_{n=0}^N e^{i\lambda_n} \left| \int_{s_n}^{s_{n+1}} u^*(t)T(t)xdt, y \right| \]

\[ = \left( \sum_{n=0}^N \int_{s_n}^{s_{n+1}} e^{i\lambda_n} u^*(t)T(t)xdt, y \right) \]

\[ = \left( \int_0^{s_{N+1}} \left( \sum_{n=0}^N e^{i\lambda_n} \chi_{[s_n,s_{n+1})}(t) \right) u^*(t)T(t)xdt, y \right) \]

\[ \leq M\|u^*\|_{E^*}\|x\|\|y\|, \]
for all $x, y \in H$ and $N \in \mathbb{N}$, which yields the weak unconditional convergence of the series (2.1).

Since

$$\left\| \int_0^s u^*(t)T(t)x\,dt \right\| = \sup_{\|y\| \leq 1} \left| \left\langle \int_0^s u^*(t)T(t)x\,dt, y \right\rangle \right|$$

we get also the inequality

$$\left\| \int_0^\infty u^*(t)T(t)x\,dt \right\| \leq M\|x\|\|u^*\|_{E^*}.$$

As well known, the dual space of any Orlicz space is a rearrangement invariant Banach function space which contains the space $L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. See [1], [5], [6]. Thus for each $\beta > 0$ the function $\exp_{-\beta}$ belongs to $E^*$. Moreover, if $\lambda \in \mathbb{C}$ and $\text{Re}\lambda > 0$, then the improper integral $\int_0^\infty e^{-\lambda t}T(t)x\,dt$ exists for all $x \in X$; necessarily, every such $\lambda$ belongs to $\rho(A)$ and the formula $R(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)x\,dt$ holds.

By our hypothesis (1.2), we can choose a $z_0 \in \mathbb{C}$ such that $\beta = \text{Re} z_0 > 0$ and

$$M_{z_0} := \beta - (2M\|\exp_{-\beta}\|_{E^*})^{-1} < 0.$$

Then for every $\lambda \in \mathbb{C}$ with $M_{z_0} < \text{Re}\lambda < 0$ the point $\lambda_0 = \text{Re} z_0 + i\text{Im}\lambda$ belongs to $\rho(A)$. Since

$$|\lambda - \lambda_0| = \text{Re}\lambda_0 - \text{Re}\lambda < (2M\|\exp_{-\beta}\|_{E^*})^{-1} \leq \frac{1}{2\|R(\lambda_0, A)\|} < \frac{1}{\|R(\lambda_0, A)\|},$$

this yields that $\lambda$ also belongs to $\rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{\|R(\lambda_0, A)\|}{1 - |\lambda - \lambda_0|\|R(\lambda_0, A)\|} \leq 2M\|\exp_{-\beta}\|_{E^*}.$$

Finally, the Gearhart-Prüss Theorem (see [2], [10]) allows us to conclude that $\omega_0(T) \leq M_{z_0} < 0$. 

3. Applications

In this section we consider a linear operator $A : D(A) \subset H \to H$ acting on the complex Hilbert space $H$, that generates a strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t \geq 0}$.

**Theorem 2.** Under the above assumptions on $A$, if moreover

(i) $\Phi$ verifies the $\Delta_2$-condition and $M = \sup_{\|x\| \leq 1} \rho^\Phi(\|T(\cdot)x, x\|) < \infty;$

Then...
\(\text{(ii) the corresponding dual function } \Phi^* \text{ is strictly increasing on } [0, \infty) \text{ and }\\ \hspace{1cm} \lim_{\alpha \downarrow 0} \inf \left\{ \alpha \left| \exp_{-\alpha} \right|_{L^{\Phi^*}} \right\} = 0,\\ \text{then for each } b \in H \text{ and each } u^*(\cdot) \text{ in } L^{\Phi^*}, \text{ the following infinite time Cauchy Problem}\\ \hspace{1cm} (A, b, -\infty, 0) : \left\{ \begin{array}{ll} \dot{x}(t) = A x(t) + b u^*(-t) & \text{for } t \leq 0 \\ x(-\infty) = \lim_{t \to -\infty} x(t) = 0, \end{array} \right.\\ \hspace{1cm} \text{has a unique solution on } (-\infty, 0].\)

\textbf{Proof.} First we shall prove that the function } \phi \text{ given by the improper integral\\ \hspace{1cm} \phi(t) = \int_{-\infty}^{t} T(t - \tau) u^*(-\tau) bd\tau = \lim_{s \to -\infty} \int_{s}^{t} T(t - \tau) u^*(-\tau) bd\tau,\\ \text{is correctly defined on } (-\infty, 0]. \text{ In fact, using the H"{o}lder inequality, for}\\ \hspace{1cm} \text{all } t_1 < t_2 \text{ in } (-\infty, t], \text{ we get}\\ \hspace{1cm} \left\| \int_{t_1}^{t_2} T(t - \tau) u^*(-\tau) bd\tau \right\| \leq \sup_{||y|| \leq 1} \int_{t_1}^{t_2} |\langle T(t - \tau)b, y \rangle| \cdot |u^*(-\tau)| d\tau\\ \hspace{1cm} \leq \sup_{||y|| \leq 1} \int_{t-t_1}^{t-t_2} |\langle T(\rho)b, y \rangle| \cdot |u^*(\rho - t)| d\rho\\ \hspace{1cm} \leq \sup_{||y|| \leq 1} \int_{0}^{\infty} \left[ \int_{t-t_2}^{t-t_1} (\rho) |\langle T(\rho)b, y \rangle| \cdot |u^*(\rho - t)| d\rho \right] \cdot |u^*(\cdot - t)| d\rho\\ \hspace{1cm} \leq 2M ||b|| \rho^{\Phi^*}(1_{[t-t_2,t-t_1]}(\cdot)|u^*(\cdot - t)|).\\ \text{Taking into account that } L^{\Phi^*} \text{ is rearrangement invariant, we have the relations}\\ \hspace{1cm} \rho^{\Phi^*}(1_{[t-t_2,t-t_1]}(\cdot)|u^*(\cdot - t)|) = \rho^{\Phi^*}(1_{[-t_2,-t_1]}(t + \cdot)|u^*(\cdot)|)\\ \hspace{1cm} = \rho^{\Phi^*}(1_{[-t-t_2,-t-t_1]}(\cdot)|u^*(\cdot)|).\\ \text{Put } s_1 = -t - t_2 \text{ and } s_2 = -t - t_1. \text{ Then } 0 \leq -t \leq s_1 < s_2 < \infty \text{ and,}\\ \hspace{1cm} \text{conversely, all such pairs } s_1, s_2 \text{ come this way.}\\ \text{Given } 0 < \eta \leq 1, \text{ the function } \frac{1}{\eta} u^*(\cdot) \text{ belongs to } L^{\Phi^*}, \text{ which yields}\\ \hspace{1cm} \int_{0}^{\infty} \Phi^* \left( \frac{1}{\eta} |u^*(\tau)| \right) d\tau < \infty. \text{ Therefore there exists } \delta > 0 \text{ such that for all}\\ \hspace{1cm} \delta \leq s_1 < s_2 < \infty \text{ we have}\\ \hspace{1cm} \int_{s_1}^{s_2} \Phi^* \left( \frac{1}{\eta} |u^*(\tau)| \right) d\tau = \int_{0}^{\infty} \Phi^* \left( \frac{1}{\eta s_{1,s_2}}(\tau) \frac{1}{\eta} |u^*(\tau)| \right) d\tau \leq \eta \leq 1.\\ \text{This gives us}\\ \hspace{1cm} \rho^{\Phi^*}(1_{[-t-t_2,-t-t_1]}(\cdot)|u^*(\cdot)|) \leq \eta;
whenever \( t_1 < t_2 < -\delta \). In fact,

\[
\eta \in \{ k > 0 : \int_{s_1}^{s_2} \Phi^* \left( \frac{1}{k} |u^*(\tau)| \right) d\tau = \int_0^\infty \Phi^* \left( \frac{1}{|s_1, s_2|} \frac{1}{k} |u^*(\tau)| \right) d\tau \leq 1 \}.
\]

Clearly, \( \phi \) verifies the integral equation

\[
x(t) = T(t-s)x(s) + \int_s^t T(t-\tau)u^*(-\tau)bd\tau, \quad s \leq t \leq 0.
\]

Moreover, for each \( t < 0 \) we have

\[
\|\phi(t)\| = \left\| \int_{-\infty}^t T(t-\tau)u^*(-\tau)bd\tau \right\|
\]

\[
= \sup_{\|y\| \leq 1} \int_0^\infty |\langle T(\rho)b, y \rangle| \cdot |u^*(\rho-t)|d\rho
\]

\[
\leq 2M\|b\|\rho^\Phi^*(|u^*(\cdot-t)|) = \rho^\Phi^*(1_{[-t,\infty)}(\cdot)|u^*(\cdot)|).
\]

On the other hand \( \rho^\Phi^*(1_{[-t,\infty)}(\cdot)|u^*(\cdot)|) \to 0 \) as \( t \to -\infty \). Indeed, for \( 1 \geq \varepsilon > 0 \) arbitrarily fixed and \( t < 0 \) sufficiently small, we have

\[
\int_0^\infty \Phi^* \left( 1_{[-t,\infty)}(s) \frac{|u^*(s)|}{\varepsilon} \right) ds = \int_{-t}^\infty \Phi^* \left( \frac{1}{\varepsilon} |u^*(s)| \right) ds < \varepsilon.
\]

Then \( \lim_{t \to -\infty} \phi(t) = 0 \), which ends the proof of the fact that \( \phi \) is a solution of the problem \((A, b, -\infty, 0)\). \( \square \)

**Theorem 3.** Assume that \( \Phi \) satisfies the condition (1.2). If for each \( b \in H \) and each \( u^*(\cdot) \in (L^\Phi)^* \) the infinite time Cauchy Problem \((A, b, -\infty, 0)\) has a unique solution, then the semigroup generated by \( A \) is uniformly exponentially stable.

**Proof.** Let \( E \) the set of all \( H \)-valued bounded and continuous functions \( g \) defined on \((-\infty, 0]\). Endowed with the norm \( |g|_E := \sup_{t \leq 0} |g(t)| \), the set \( E \) becomes a Banach space. Let \( b \in H \) and \( h > 0 \) be fixed and denote by \( x_{u^*} \) the unique solution of \((A, b, -\infty, 0)\). We will consider the bounded linear operators \( P \) and \( Q \), defined by:

\[
u^* \mapsto Qu^* := x_{u^*} : (L^\Phi)^* \to E \quad \text{and} \quad g \mapsto Pg := g(0) : E \to H.
\]

Since \( PQ \) is bounded we infer the existence of a positive constant \( K_b \) such that

\[
\| \int_{-\infty}^0 T(-\tau)u^*(-\tau)d\tau \| \leq K_b\|u^*\|_{(L^\Phi)^*} \quad \text{for all} \ u^* \in (L^\Phi)^*.
\]
Then for each $u^* \in (L^\Phi)^*$ with $u^*(s) = 0$ for all $s > h$, we have that
\[
\left| \int_0^T \langle T(\tau)b, y \rangle u^*(\tau) d\tau \right| \leq K_b \|u^*\|_{(L^\Phi)^*} \text{ for all } y \in H, \|y\| \leq 1,
\]
and because $(L^\Phi)^*$ is a Banach function space, the previous inequality actually works for all $u^* \in (L^\Phi)^*$. Equivalently,
\[
\left| \int_0^\infty 1_{[0,h]}(\tau) \langle T(\tau)b, y \rangle u^*(\tau) d\tau \right| \leq K_b \|u^*\|_{(L^\Phi)^*},
\]
for all $y \in H, \|y\| \leq 1$, and all $u^* \in (L^\Phi)^*$. Now it is easy to see that
\[
\rho^\Phi(1_{[0,h]}(\cdot) \langle T(\cdot)b, y \rangle) \leq K_b \text{ for all } y \in H, \|y\| \leq 1.
\]
Therefore
\[
\rho^\Phi(\|T(\cdot)b, y\|) \leq K_b \text{ for all } y \in H, \|y\| \leq 1,
\]
and from Theorem 1 we can conclude that the semigroup $T$ is uniformly exponentially stable.

Assume that for each $x, y \in H$ the map $\langle T(\cdot)x, y \rangle$ defines an element of $L^\Phi$. Then the map given by the formula
\[
(x, y) \mapsto \langle T(\cdot)x, y \rangle : H \times H \to L^\Phi
\]
is a continuous sesquilinear function (linear in the first variable and anti-linear in the second one). By the Closed Graph Theorem we get the existence of a positive constant $M$ such that
\[
\rho^\Phi(\|T(\cdot)x, y\|) \leq M \|x\| \cdot \|y\| \text{ for all } x, y \in H.
\]
This shows that the condition (1.1) can be replaced by the following one,
\[
(3.1) \quad \int_0^\infty \Phi(\|T(t)x, y\|) dt < \infty, \text{ for all } x, y \in H.
\]

We conclude our paper with an example.

Let $H = L^2[0, \pi]$ and $A : D(A) \subset H \to H$ given by $Ax = \frac{d^2x}{dx^2}$, where the domain $D(A)$ consists of all absolutely continuous functions $x(\cdot)$ defined on $[0, \pi]$, which verify the following three conditions: $i)$ $x(0) = x(\pi) = 0$; $ii)$ the first derivative $\frac{dx}{dx}$ is absolutely continuous on $[0, \pi]$; $iii)$ the second derivative $\frac{d^2x}{dx^2}$ belongs to $H$. With the above notations, for each $u^*(\cdot) \in (L^\Phi)^*$ and each $b(\cdot) \in H$, the infinite time
Cauchy Problem

$$\frac{\partial y(t,\xi)}{\partial t} = \frac{\partial^2 y(t,\xi)}{\partial \xi^2} + u^*(-t)b(\xi) \quad \text{for } t \in (-\infty, 0], \; \xi \in (0, \pi)$$

$$\lim_{t \to -\infty} \int_0^\pi |y(t, \xi)|^2 d\xi = 0$$

has a unique solution. Indeed, the uniform growth bound $\omega_0(T)$ of the semigroup $T$ generated by $A$ is equal to $-1$ and condition (3.1) applies (due to the fact that $\Phi$ is a convex function).

References


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