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THE INTEGRAL VERSION OF POPOVICIU'S INEQUALITY

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Dedicated to Josip Pečarić
on his 60th birthday anniversary

Abstract. T. Popoviciu [7] has proved in 1965 an interesting characterization of the convex functions of one real variable, relating the arithmetic mean of its values and the values taken at the barycenters of certain subfamilies of the given family of points. The aim of our paper is to prove an integral analogue.

Most of the people think that passing from a discrete inequality to its integral counterpart is just a snap. In practice, things are not so simple, due to the particularities of discrete probability fields that may hide essential facts. We will discuss here the case of Popoviciu's inequality, a notable discrete inequality, whose statement is as follows:

THEOREM 1. (T. Popoviciu [7]) *If f is a convex function defined on an interval I, then*

$$\sum_{1\leqslant i_1<\cdots<i_p\leqslant n}(\lambda_{i_1}+\cdots+\lambda_{i_p})\,f\left(\frac{\lambda_{i_1}x_{i_1}+\cdots+\lambda_{i_p}x_{i_p}}{\lambda_{i_1}+\cdots+\lambda_{i_p}}\right)$$
$$\leqslant\binom{n-2}{p-2}\left[\frac{n-p}{p-1}\sum_{i=1}^{n}\lambda_i f(x_i)+\left(\sum_{i=1}^{n}\lambda_i\right)f\left(\frac{\lambda_1x_1+\cdots+\lambda_nx_n}{\lambda_1+\cdots+\lambda_n}\right)\right],\qquad (P_{n,p})$$

for all families $x_1,\dots,x_n\in I$, $\lambda_1,\dots,\lambda_n\in[0,\infty)$, $n\geqslant 3$, and all integers $p\in\{2,\dots,n-1\}$.

Actually, the inequality $(P_{3,2})$ is equivalent to the property of convexity, and a simple argument based on mathematical induction yields the implication

$$(P_{3,2})\ \Rightarrow\ (P_{n,p})\text{ for all } n\in\mathbb{N},\ n\geqslant 3,\text{ and all } p\in\{2,\dots,n-1\}.$$

Thus the essence of Theorem 1 is the connection between convexity and $(P_{3,2})$.

Popoviciu's inequality has received a great deal of attention since its discovery in 1965, and appears in a series of monographs such as [2], [3], and [5]. See also [1] (for a higher dimensional analogue) and [4] (for some refinements).

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In [7], only the unweighted case was discussed, but the argument covers the weighted case as well. However no attempt was made until now to write down the integral version of Popoviciu's inequality.

Trying to solve this problem we will start with what seems to be the most plausible candidate for such a version: If μ is a bounded positive Borel measure on an interval $I,$ then for every convex function $f : I \to \mathbb{R},$

$$\begin{aligned} &\frac{1}{\mu(I)}\int_I f(x)d\mu(x) + f\left(\frac{1}{\mu(I)}\int_I x d\mu(x)\right) \qquad (IP) \\ &\qquad \geqslant \frac{2}{(\mu\otimes\mu)(\{(t,x)\in I^2 : t<x\})}\iint_{\{(t,x)\in I^2 : t<x\}} f\left(\frac{x+t}{2}\right)d\mu(t)d\mu(x). \end{aligned}$$

Surprisingly, (IP) is not the right answer and more terms should be added. Theorem 1 offers no guesses, so in order to understand what seems to be the matter, it is instructive to concentrate on the case where μ is the Lebesgue measure on an interval $[a,b]$. In this case the formula (IP) becomes

$$\frac{1}{b-a}\int_a^b f(x)dx + f\left(\frac{a+b}{2}\right) \geqslant \frac{4}{(b-a)^2}\int_a^b\int_a^x f\left(\frac{x+t}{2}\right)dtdx. \qquad (LP)$$

Some remarks are in order:

(1) The inequality (LP) works (as an equality) for every affine function;

(2) The set of all functions $f : [a,b] \to \mathbb{R}$ which verify this inequality is a convex cone;

(3) We can reduce the proof of (LP) to the case of continuous functions, by modifying the values at the endpoints (if necessary);

(4) If a pointwise convergent sequence of convex functions verifies (LP), so does its limit.

Thus we can reduce the proof of the inequality (LP) to the case of piecewise linear convex functions, which can be settled easily via the following result due to Popoviciu:

LEMMA 1. (T. Popoviciu [6]; see [3], p. 34, for details) *Let $f : [a,b] \to \mathbb{R}$ be a piecewise linear convex function. Then f is the sum of an affine function and a linear combination, with positive coefficients, of translates of the absolute value function. In other words, f is of the form*

$$f(x) = \alpha x + \beta + \sum_{k=1}^{n} \gamma_k |x - c_k|$$

for suitable $\alpha, \beta \in \mathbb{R}$ and suitable nonnegative coefficients $\gamma_1, \ldots, \gamma_n$.

By Lemma 1 above, the proof of the integral version of Popoviciu's inequality can be reduced to the case of functions of the form

$$f(x) = (x-c)^+,$$

where c is a real parameter. In other words, we have to check the sign of

$$E = \frac{1}{b-a}\int_a^b (x-c)^+ dx + \left(\frac{a+b}{2} - c\right)^+ - \frac{4}{(b-a)^2}\int_a^b \int_a^x \left(\frac{x+t}{2} - c\right)^+ dtdx.$$

Case 1: $a \leqslant c \leqslant (a+b)/2$. Then

$$\begin{aligned}
E &= \frac{1}{b-a}\int_c^b (x-c)dx + \frac{a+b}{2} - c - \frac{4}{(b-a)^2}\int_a^b \int_a^x \left(\frac{x+t}{2} - c\right)^+ dtdx \\
&= \frac{(b-c)^2}{2(b-a)} + \frac{a+b}{2} - c - \frac{4}{(b-a)^2}\int_c^{2c-a} \int_{2c-x}^x \left(\frac{t+x}{2} - c\right) dtdx \\
&\quad - \frac{4}{(b-a)^2}\int_{2c-a}^b \int_a^x \left(\frac{t+x}{2} - c\right) dtdx \\
&= \frac{(b-c)^2}{2(b-a)} + \frac{a+b}{2} - c - \frac{4}{3}\frac{(c-a)^3}{(b-a)^2} + 2c - b - a \\
&= \frac{1}{6}(a-c)^2 \frac{5a+3b-8c}{(a-b)^2}.
\end{aligned}$$

Case 2: $(a+b)/2 \leqslant c \leqslant b$. Then

$$\begin{aligned}
E &= \frac{1}{b-a}\int_c^b (x-c)dx - \frac{4}{(b-a)^2}\int_a^b \int_a^x \left(\frac{x+t}{2} - c\right)^+ dtdx \\
&= \frac{(b-c)^2}{2(b-a)} - \frac{4}{(b-a)^2}\int_c^b \int_{2c-x}^x \left(\frac{x+t}{2} - c\right) dtdx \\
&= \frac{(b-c)^2}{2(b-a)} - \frac{4}{(b-a)^2}\int_c^b (x-c)^2\, dx \\
&= \frac{(b-c)^2}{2(b-a)} - \frac{4}{3\,(b-a)^2}(b-c)^3 \\
&= \frac{1}{6}(b-c)^2 \frac{8c-3a-5b}{(b-a)^2}.
\end{aligned}$$

Consequently

$$E \geqslant 0$$

if and only if $c \in [a, \frac{5a+3b}{8}] \cup [\frac{3a+5b}{8}, b]$, and this conclusion may be summarized as follows:

LEMMA 2. *The formula*

$$\frac{1}{b-a}\int_a^b f(x)dx+f\left(\frac{a+b}{2}\right)\geqslant\frac{4}{(b-a)^2}\int_a^b\int_a^x f\left(\frac{x+t}{2}\right)dtdx$$

holds true for all convex functions $f:[a,b]\to\mathbb{R}$ *whose restriction to the interval* $\left[\frac{5a+3b}{8},\frac{3a+5b}{8}\right]$ *is affine.*

Proof. In fact, the above formula works precisely for all convex functions $f:[a,b]\to\mathbb{R}$ in the closed convex cone generated by the affine functions and the functions of the form $(x+c)^+$ with $c\in[a,\frac{5a+3b}{8}]\cup[\frac{3a+5b}{8},b]$. In the window $\left[\frac{5a+3b}{8},\frac{3a+5b}{8}\right]$ these functions should be affine since the limit of a convergent sequence of affine functions is in turn affine. $\square$

When $f:[a,b]\to\mathbb{R}$ is an arbitrary convex function, we have to take into account the affine function $\omega:[a,b]\to\mathbb{R}$ joining the points $\left(\frac{5a+3b}{8},f\left(\frac{5a+3b}{8}\right)\right)$ and $\left(\frac{3a+5b}{8},f\left(\frac{3a+5b}{8}\right)\right)$. This is given by the formula

$$\omega(x)=f\left(\frac{5a+3b}{8}\right)+4\frac{f\left(\frac{3a+5b}{8}\right)-f\left(\frac{5a+3b}{8}\right)}{b-a}\left(x-\frac{5a+3b}{8}\right).$$

Clearly, the function $(f-\omega)^+$ vanishes on $\left[\frac{5a+3b}{8},\frac{3a+5b}{8}\right]$ and verifies the hypotheses of Lemma 2. Since

$$\begin{aligned}\frac{1}{b-a}\int_a^b(f-\omega)^+(x)dx=\frac{1}{b-a}\int_a^b f(x)dx-\frac{3}{8}\left[f\left(\frac{5a+3b}{8}\right)+f\left(\frac{3a+5b}{8}\right)\right]\\ -\frac{1}{b-a}\int_{(5a+3b)/8}^{(3a+5b)/8}f(x)dx,\end{aligned}$$

we are thus led to the following integral analogue of Popoviciu's inequality,

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx-\frac{3}{8}\left[f\left(\frac{5a+3b}{8}\right)+f\left(\frac{3a+5b}{8}\right)\right]-\frac{1}{b-a}\int_{(5a+3b)/8}^{(3a+5b)/8}f(x)dx\\ \geqslant\frac{4}{(b-a)^2}\int_a^b\int_a^x(f-\omega)^+\left(\frac{x+t}{2}\right)dtdx,\end{aligned}$$

that works for *all* convex functions $f:[a,b]\to\mathbb{R}$.

A simple (though tedious) computation allows us to eliminate completely the presence of ω :

$$\begin{aligned}&\frac{4}{(b-a)^2}\int_a^b\int_a^x(f-\omega)^+\left(\frac{x+t}{2}\right)dtdx\\ &\quad=\frac{4}{(b-a)^2}\int_a^{(5a+3b)/8}\int_a^x f\left(\frac{x+t}{2}\right)dtdx\\ &\quad\quad+\frac{4}{(b-a)^2}\int_{(5a+3b)/8}^{(a+3b)/4}\int_a^{(5a+3b)/4-x}f\left(\frac{x+t}{2}\right)dtdx\end{aligned}$$

$$
\begin{aligned}
&+\frac{4}{(b-a)^2}\int_{(3a+5b)/8}^{b}\int_{(3a+5b)/4-x}^{x} f\left(\frac{x+t}{2}\right)dtdx \\
&-\frac{9}{16}f\left(\frac{5a+3b}{8}\right)-\frac{9}{16}f\left(\frac{3a+5b}{8}\right).
\end{aligned}
$$

Thus the integral analogue of Popoviciu's inequality (put in final form) reads as follows:

THEOREM 2. *For every convex function $f:[a,b]\to\mathbb{R}$,*

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)dx &+ \frac{3}{16}f\left(\frac{5a+3b}{8}\right)+\frac{3}{16}f\left(\frac{3a+5b}{8}\right)-\frac{1}{b-a}\int_{(5a+3b)/8}^{(3a+5b)/8} f(x)dx \\
&\geqslant \frac{4}{(b-a)^2}\int_a^{(5a+3b)/8}\int_a^x f\left(\frac{x+t}{2}\right)dtdx \\
&\quad+\frac{4}{(b-a)^2}\int_{(5a+3b)/8}^{(a+3b)/4}\int_a^{(5a+3b)/4-x} f\left(\frac{x+t}{2}\right)dtdx \\
&\quad+\frac{4}{(b-a)^2}\int_{(3a+5b)/8}^{b}\int_{(3a+5b)/4-x}^{x} f\left(\frac{x+t}{2}\right)dtdx.
\end{aligned}
$$

For concave functions, the inequality should be reversed.

Needless to say, Theorem 2 leads to rather curious inequalities even in the most simple cases. For example, when f is the restriction of the exponential function to the interval $[\log x^8, \log y^8]$, we infer from Theorem 2 that

$$
\frac{2y^8-2x^8+4x^3y^3\left(y^2-x^2\right)}{\ln y-\ln x}+3x^3y^3\left(y^2+x^2\right)\geqslant\frac{2(x^8+y^8-x^3y^3(x^2+y^2))}{(\ln y-\ln x)^2},
$$

for all $0<x<y$; notice the presence of the logarithmic mean $L(x,y)=\frac{x-y}{\ln x-\ln y}$.

In principle, the technique described above works well for all positive Borel measures with compact support. Popoviciu's original result corresponds to the case of the discrete measure $\mu=\lambda_1\delta_{x_1}+\lambda_2\delta_{x_2}+\lambda_3\delta_{x_3}$. The fact that the integral formula (IP) works in this case for all convex functions is explained by Hlawka's inequality, which asserts that

$$
|x_1|+|x_2|+|x_3|+|x_1+x_2+x_3|\geqslant|x_1+x_2|+|x_2+x_3|+|x_3+x_1|,
$$

for all $x_1,x_2,x_3\in\mathbb{R}$. See [3], p. 100. Indeed, Hlawka's inequality assures the validity of (IP) for all convex functions of the form

$$
f(x)=|x-c|,
$$

where c is a real parameter.

We end this note by mentioning that an integral version of Hlawka's inequality was obtained by S.-E. Takahasi, Y. Takahashi, and S. Wada [8], [9], but their result has nothing to do with convexity.

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