Weak solutions via bipotentials in mechanics of deformable solids

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Abstract

We consider a displacement-traction boundary values problem for elastic materials, under the small deformations hypothesis, for static processes. The behavior of the material is modelled by a constitutive law involving the subdifferential of a proper, convex, and lower semicontinuous map. The constitutive map and its Fenchel conjugate allow us to construct a bipotential function. Based on this construction, we propose a weak formulation of our mechanical problem. Furthermore, we prove the existence of at least one weak solution and we investigate the uniqueness of the weak solution. We also comment on the relevance of our variational approach, by considering three significant examples.

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1 Introduction

The main purpose of this paper is to prove the weak solvability of the general displacement-traction mechanical model for elastic materials. The behavior of the elastic materials is described by a subdifferential inclusion, with a constitutive map which is proper, convex and lower semicontinuous. The envisaged processes are static and the calculus is performed under the small deformations hypothesis.

In our approach the weak formulation of the model yields a system of two variational inequalities involving a bipotential which is attached to the constitutive map and its Fenchel conjugate; see Problem 2 below. The unknown is the pair consisting of the displacement vector and the Cauchy stress tensor and we seek for it into a Cartesian product between a Hilbert space and a nonempty closed and convex subset of a second Hilbert space. We focus on the existence and uniqueness of the weak solution. However it is worth to mention that our results are suitable to discuss the numerical approximation of this solution (that is, a simultaneous approximation of the displacement field and the Cauchy stress tensor). In the classical approach the displacement field and the Cauchy stress tensor are treated separately.

The presence of the bipotentials in mechanics of solid was noticed quite recently, but the literature covering this subject is fast growing. The construction of several bipotential functions appears in connection with Coulomb's friction law [4] and Cam-Clay models in soil mechanics [14], cyclic plasticity

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[13], [2] and viscoplasticity of metals with non-linear kinematical hardening rule [8], Lemaitre's damage law [1], the coaxial laws [16], [18] etc. See also the overview paper [3]. In the present paper, we illustrate the applicability of bipotentials by providing a new variational formulation for a general model in elastostatics.

Our paper requires a background of mechanics of solid (which can be covered from [6, 17]), and also some familiarity with calculus of variations (see [5, 10]).

In Section 2 we indicate the notation and some preliminaries, including some basic facts of convex analysis. In Section 3 we state the mechanical model and we discuss its weak solvability, more precisely, we prove the existence of at least one weak solution and we comment on the uniqueness of it; see Theorem 2 and Theorem 3 below. In Section 4 we discuss three examples, based on linear constitutive laws, single-valued nonlinear constitutive laws and multi-valued nonlinear constitutive laws respectively. They make clear that all basic facts known nowadays about the existence and uniqueness of the displacement field are covered by our Theorem 2 and Theorem 3.

2 Notation and preliminaries

Throughout this paper $\mathbb{S}^3$ denotes the space of second order symmetric tensors on $\mathbb{R}^3$. Every field in $\mathbb{R}^3$ or $\mathbb{S}^3$ is typeset in boldface. By $\cdot$ and $|\cdot|$ we denote the inner product and the Euclidean norm on $\mathbb{R}^3$ and $\mathbb{S}^3$, respectively. Thus,

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i,\ \ |\boldsymbol{v}| = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2},\ \ \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^3,$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij},\ \ |\boldsymbol{\tau}| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2},\ \ \boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^3.$$

Here and below, the indices i and j run between 1 and 3 and the summation convention over repeated indices is adopted.

Given a bounded domain $\Omega\subset\mathbb{R}^3$ we attach to it the following four functional spaces on Ω :

$$\begin{aligned} H\ &= \{\boldsymbol{u} = (u_i) : u_i \in L^2(\Omega)\}, &\quad \mathcal{H}\ &= \{\boldsymbol{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},\\ H_1 &= \{\boldsymbol{u}\in H : u_{i,j} + u_{j,i} \in L^2(\Omega)\}, &\quad \mathcal{H}_1 &= \{\boldsymbol{\sigma}\in\mathcal{H} : \sigma_{ij,j}\in L^2(\Omega)\}, \end{aligned}$$

where the index following a comma indicates a partial derivative (in weak sense) with respect to the corresponding component of the independent variable.

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the inner products,

$$\begin{aligned} (\boldsymbol{u},\boldsymbol{v})_H &= \int_\Omega u_i v_i\,dx, &\quad (\boldsymbol{u},\boldsymbol{v})_{H_1} &= (\boldsymbol{u},\boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},\\ (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij}\,dx, &\quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma},\mathrm{Div}\,\boldsymbol{\tau})_H, \end{aligned}$$

where $\boldsymbol{\varepsilon} : H_1\to\mathcal{H}$ is a continuous linear operator given by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})),\quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

and $\mathrm{Div} : \mathcal{H}_1\to H$ is given by

$$\mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

We assume that the boundary of Ω, denoted by Γ, is Lipschitz continuous. Thus the unit outward normal vector $\boldsymbol{\nu}$ on the boundary is defined almost everywhere.

The Sobolev trace operator,

$$\boldsymbol{\gamma} : H_1 \to L^2(\Gamma)^3,$$

is continuous and linear, and for each Lebesgue measurable subset Γ_1 of Γ, of positive measure, we can consider the Hilbert space

$$V = \{\, \boldsymbol{v} \in H_1 : \boldsymbol{\gamma v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\}, \tag{1}$$

endowed with the inner product

$$(\cdot,\cdot)_V : V \times V \to \mathbb{R}, \quad (\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}.$$

The proof that V is indeed a Hilbert space is an easy consequence of Korn's inequality which states the existence of a constant $c_K = c_K(\Omega,\Gamma_1) > 0$ such that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \geq c_K \|\boldsymbol{v}\|_{H_1}, \qquad \text{for all } \boldsymbol{v} \in V.$$

See e.g. [9], p. 79.

We end this section by recalling some elements of convex analysis in Hilbert spaces. The central objects are the functionals $\phi : X \to (-\infty,\infty]$ defined on a Hilbert space X (endowed with the scalar product $(\cdot,\cdot)_X$ and the norm $\|\cdot\|_X$).

The *effective domain* of such a functional ϕ is the set $\text{dom}(\phi) = \{x \in X : \phi(x) < \infty\}$. The *core* of the effective domain, core(dom(ϕ)), is the set of all $x \in \text{dom}(\phi)$ such that for any direction $v \in X$, the vector $x + tv$ lies in dom(ϕ) for all small real t. This set clearly contains the interior of dom(ϕ).

We say that ϕ is *proper* if dom(ϕ) is nonempty, and *convex* if

$$\phi\left((1-\lambda)\,x + \lambda y\right) \leq (1-\lambda)\,\phi\,(x) + \lambda\phi\,(y)\,,$$

for all $x, y \in X$ and $\lambda \in (0,1)$; ϕ is called *strictly convex* if the last inequality is strict whenever $x \neq y$.

We say that ϕ is *lower semicontinuous at* $u \in X$ if

$$\liminf_{n\to\infty} \varphi(u_n) \geq \varphi(u)$$

for each sequence $(u_n)_n$ converging to u in X. The function φ is *lower semicontinuous* if it is lower semicontinuous at every point $u \in X$.

If ϕ is convex, then for every point u in core(dom(ϕ)), the right-hand directional derivative,

$$\phi'_+(u;v) = \lim_{t\to 0+} \frac{\phi(u+tv)-\phi(u)}{t}, \quad v \in X,$$

is everywhere finite and sublinear. This fact is very close to Gâteaux differentiability. Indeed, ϕ is *Gâteaux differentiable* at u if the two-sided limit exists for every v, and the map $\phi'(u) : v \to \lim_{t\to 0} \frac{\phi(u+tv)-\phi(u)}{t}$ defines a continuous linear functional on X. Since X is a Hilbert space, $\phi'(u)$ is necessarily of the form

$$\phi'(u)(v) = (\nabla\phi(u), v)_X, \qquad \text{for all } v \in X,$$

where $\nabla\phi(u) \in X$ represents the *gradient* of ϕ *at* u.

Lemma 1. *Let $\phi : X \to \mathbb{R}$ be a Gâteaux differentiable functional. Then the following statement are equivalent:*

i) ϕ *is a convex functional;*

ii) $\phi(v) - \phi(u) \geq (\nabla\phi(u), v - u)_X$, *for all* $u, v \in X$*;*

iii) $(\nabla\phi(v) - \nabla\phi(u), v - u)_X \geq 0$, *for all* $u, v \in X$.

In the variant of strict convexity, the inequalities in ii) and iii) should be strict for $u \neq v$.

An important property of convex functionals is the existence of a nice substitute for differentiability, the subdifferential. The *subdifferential* of a functional $\phi : X \to (-\infty, +\infty]$ at a point $u \in \mathrm{dom}(\phi)$ is the (possibly empty) set

$$\partial\phi(u) = \{\zeta \in X : \ \phi(v) - \phi(u) \geq (\zeta, v - u)_X, \ \text{ for all } v \in X\}.$$

An interesting remark is that, if $\phi : X \to \mathbb{R}$ is convex and Gâteaux differentiable, then

$$\partial\phi(u) = \{\nabla\phi(u)\} \text{ for all } u \in X. \tag{2}$$

Furthermore, the convex functionals are the only functionals $\phi : X \to (-\infty, +\infty]$ for which $\partial\phi(u)$ is nonempty at any point $u \in \mathrm{dom}(\phi)$. More precisely, the following result holds true.

Lemma 2. *If the subdifferential of* $\phi : X \to (-\infty, \infty]$ *at any point* $u \in \mathrm{dom}(\phi)$ *is nonempty, then* ϕ *is convex, proper and lower semicontinuous.*

The proofs of Lemma 1 and Lemma 2 can be found in [5, 10].

The *Fenchel conjugate* of a functional $\phi : X \to (-\infty, \infty]$ is the functional

$$\phi^* : X \to (-\infty, \infty], \quad \phi^*(x^*) = \sup_{x \in X} \{(x^*, x) - \phi(x)\}.$$

Necessarily, ϕ^* is lower semicontinuous proper and convex, provided that ϕ plays all these properties.

Theorem 1. *Let* $\phi : X \to (-\infty, \infty]$ *be a lower semicontinuous proper convex functional. Then:*

i) for any x, $y \in X$, *we have* $\phi(x) + \phi^*(y) \geq (x, y)_X$*;*

ii) for any x, $y \in X$ *we have the equivalences*

$$y \in \partial\phi(x) \Leftrightarrow x \in \partial\phi^*(y) \Leftrightarrow \phi(x) + \phi^*(y) = (x, y)_X.$$

See [5, 10] for details.

A concept that will play an important role in our paper is that of bipotential.

Definition 1. *A bipotential is a function* $B : X \times X \to (-\infty, \infty]$ *with the following three properties:*

i) B *is convex and lower semicontinuous in each argument;*

ii) for any x, $y \in X$, *we have* $B(x, y) \geq (x, y)_X$*;*

iii) for any x, $y \in X$, *we have the equivalences*

$$y \in \partial B(\cdot, y)(x) \Leftrightarrow x \in \partial B(x, \cdot)(y) \Leftrightarrow B(x, y) = (x, y)_X.$$

The bipotentials are related to dissipation. A thorough presentation of their theory can be found in [3].

3 The model and its weak solvability

We consider a body that occupies the bounded domain $\Omega \subset \mathbb{R}^3$, with Lipschitz boundary, partitioned in two measurable parts, Γ_1 and Γ_2, such that the Lebesgue measure of Γ_1 is positive. The unit outward normal to Γ is denoted by $\boldsymbol{\nu}$ and is defined almost everywhere. The body Ω is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface traction of density $\boldsymbol{f}_2$ act on Γ_2. In order to describe the behavior of the materials, we use a constitutive law expressed by the subdifferential of a proper, lower semicontinuous, convex functional. We denote by $\boldsymbol{u} = (u_i)$ the displacement field, by $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ the infinitesimal strain tensor and by $\boldsymbol{\sigma} = (\sigma_{ij})$ the Cauchy stress tensor. The precise statement of our problem is as follows:

Problem 1. *Find* $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ *and* $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$, *such that*

$$\operatorname{Div} \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} \ \textit{in}\ \Omega, \tag{3}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \ \textit{in}\ \Omega, \tag{4}$$

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} \ \textit{on}\ \Gamma_1, \tag{5}$$

$$\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{f}_2(\boldsymbol{x}) \ \textit{on}\ \Gamma_2. \tag{6}$$

We assume that the densities of the volume forces and traction verify

$$\boldsymbol{f}_0 \in H \quad \text{and} \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^3. \tag{7}$$

Concerning the constitutive function ω we assume:

$$\left.\begin{array}{l} \omega : \mathbb{S}^3 \to \mathbb{R} \text{ is a convex, lower semicontinuous functional;} \\ \text{there exists } \alpha > 0 : \ \omega(\boldsymbol{\varepsilon}) \geq \alpha|\boldsymbol{\varepsilon}|^2 \quad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^3; \\ \omega(0_{\mathbb{S}^3}) = 0. \end{array}\right\} \tag{8}$$

The Fenchel conjugate of the function ω,

$$\omega^* : \mathbb{S}^3 \to (-\infty, \infty], \quad \omega^*(\boldsymbol{\tau}) = \sup_{\boldsymbol{\xi} \in \mathbb{S}^3} \{\boldsymbol{\tau} \cdot \boldsymbol{\xi} - \omega(\boldsymbol{\xi})\},$$

is convex, lower semicontinuous and, in addition, $\omega^*(0_{\mathbb{S}^3}) = 0$. Therefore,

$$\omega^*(\boldsymbol{\tau}) \geq 0, \quad \text{for all } \boldsymbol{\tau} \in \mathbb{S}^3. \tag{9}$$

Under the previous hypotheses, (7) and (8), we are interested in the weak solvability of Problem 1. For this, assume that $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a strong solution of Problem 1. Using the Green formula

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) \cdot \boldsymbol{\gamma v}(\boldsymbol{x})\, d\Gamma, \quad \text{for all } \boldsymbol{v} \in H_1, \tag{10}$$

(see [7], p. 145), by taking into account (3), (5) and (6) we obtain

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (\boldsymbol{f}_0, \boldsymbol{v})_H + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma v}(\boldsymbol{x}) d\Gamma, \quad \text{for all } \boldsymbol{v} \in V.$$

By Riesz's representation theorem, we infer the existence of a unique element $\boldsymbol{f} \in V$ such that

$$(\boldsymbol{f}, \boldsymbol{v})_V = (\boldsymbol{f}_0, \boldsymbol{v})_H + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \gamma \boldsymbol{v}(\boldsymbol{x}) d\Gamma \quad \text{for all } \boldsymbol{v}(\boldsymbol{x}) \in V. \tag{11}$$

Thus,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (\boldsymbol{f}, \boldsymbol{v})_V, \quad \text{for all } \boldsymbol{v} \in V.$$

Next, by (4) and Theorem 1 (applied to $X = \mathbb{S}^3$ and $\phi = \omega$), for almost every $\boldsymbol{x} \in \Omega$,

$$\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \Leftrightarrow & \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) \in \partial\omega^*(\boldsymbol{\sigma}(\boldsymbol{x})) \\ \Leftrightarrow & \omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) + \omega^*(\boldsymbol{\sigma}(\boldsymbol{x})) = \boldsymbol{\sigma}(\boldsymbol{x}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})), \end{aligned} \tag{12}$$

and

$$\omega(\boldsymbol{\tau}) + \omega^*(\boldsymbol{\mu}) \geq \boldsymbol{\tau} \cdot \boldsymbol{\mu} \text{ for all } \boldsymbol{\tau}, \boldsymbol{\mu} \in \mathbb{S}^3. \tag{13}$$

We are now in a position to associate to the constitutive map ω a new function $B : \mathbb{S}^3 \times \mathbb{S}^3 \to (-\infty, \infty]$ defined by the formula

$$B(\boldsymbol{\tau}, \boldsymbol{\mu}) := \omega(\boldsymbol{\tau}) + \omega^*(\boldsymbol{\mu}), \quad \text{for all } \boldsymbol{\tau}, \boldsymbol{\mu} \in \mathbb{S}^3. \tag{14}$$

Lemma 3. *The function B defined by* (14) *is a bipotential. In addition,*

$$B(\boldsymbol{\tau}, \boldsymbol{\mu}) \geq \alpha |\boldsymbol{\tau}|^2, \quad \textit{for all } \boldsymbol{\tau} \in \mathbb{S}^3. \tag{15}$$

Proof. Taking into account the properties of the functionals ω and ω^*, the function B defined by (14) is convex and lower semicontinuous in each argument. Due to (13),

$$B(\boldsymbol{\tau}, \boldsymbol{\mu}) \geq \boldsymbol{\tau} \cdot \boldsymbol{\mu}, \quad \text{for all } \boldsymbol{\tau}, \boldsymbol{\mu} \in \mathbb{S}^3.$$

Using Theorem 1, the last condition of Definition 1 is also verified. Finally, based on (8) and (9) we get (15). □

Using the bipotential B we define $b : V \times \mathcal{H} \to (-\infty, \infty]$ by the formula

$$b(\boldsymbol{v}, \boldsymbol{\mu}) := \begin{cases} \int_\Omega B(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})), \boldsymbol{\mu}(\boldsymbol{x})) dx, & \text{if } B(\boldsymbol{\varepsilon}(\boldsymbol{v}(\cdot)), \boldsymbol{\mu}(\cdot)) \in L^1(\Omega) \\ \infty & \text{otherwise.} \end{cases} \tag{16}$$

By integrating (over Ω) the equality which appears in (12) we obtain

$$b(\boldsymbol{u}, \boldsymbol{\sigma}) = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}}.$$

Moreover, since B is a bipotential, we get

$$b(\boldsymbol{v}, \boldsymbol{\mu}) \geq (\boldsymbol{\mu}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \text{for all } \boldsymbol{v} \in V, \boldsymbol{\mu} \in \mathcal{H}. \tag{17}$$

In particular,

$$b(\boldsymbol{v}, \boldsymbol{\sigma}) \geq (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \text{for all } \boldsymbol{v} \in V,$$

and thus

$$b(\boldsymbol{v}, \boldsymbol{\sigma}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V, \quad \text{for all } \boldsymbol{v} \in V. \tag{18}$$

Consider now the following subset of $\mathcal{H}$:

$$\Lambda := \{\boldsymbol{\mu} \in \mathcal{H} : (\boldsymbol{\mu}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (\boldsymbol{f}, \boldsymbol{v})_V, \quad \text{for all } \boldsymbol{v} \in V\}.$$

We note that $0_{\mathcal{H}} \notin \Lambda$ but $\boldsymbol{\sigma} \in \Lambda$. Thus, Λ is nonempty. On the other hand, Λ is a convex and closed subset of $\mathcal{H}$. By (17),

$$b(\boldsymbol{v}, \boldsymbol{\mu}) \geq (\boldsymbol{f}, \boldsymbol{v})_V, \quad \text{for all } \boldsymbol{v} \in V, \boldsymbol{\mu} \in \Lambda.$$

In particular,

$$b(\boldsymbol{u}, \boldsymbol{\mu}) \geq (\boldsymbol{f}, \boldsymbol{u})_V, \quad \text{for all } \boldsymbol{\mu} \in \Lambda$$

and

$$b(\boldsymbol{u}, \boldsymbol{\sigma}) = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} = (\boldsymbol{f}, \boldsymbol{u})_V.$$

Consequently,

$$b(\boldsymbol{u}, \boldsymbol{\mu}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) \geq 0, \quad \text{for all } \boldsymbol{\mu} \in \Lambda. \tag{19}$$

Combining (18) and (19) we are led to the following weak formulation of Problem 1.

Problem 2. *Find $\boldsymbol{u} \in V$ and $\boldsymbol{\sigma} \in \Lambda$ such that*

$$\begin{aligned} &b(\boldsymbol{v}, \boldsymbol{\sigma}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V, \quad \textit{for all } \boldsymbol{v} \in V; \\ &b(\boldsymbol{u}, \boldsymbol{\mu}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) \geq 0, \quad \textit{for all } \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

Definition 2. *Any solution $(\boldsymbol{u}, \boldsymbol{\sigma}) \in V \times \Lambda$ of Problem 2 is called a weak solution of Problem 1.*

Theorem 2. (Existence of weak solutions). *Assume (7), (8) and (9). Then, Problem 2 has at least one solution.*

Proof. By the definition of the bipotential B, see (14), since ω and ω^* are convex functions, we infer that the functional b, as defined by (16), is convex. In addition, taking into account that ω and ω^* are lower semicontinuous functions, applying Fatou's Lemma, we conclude that the functional b is also lower semicontinuous. Furthermore, by (8) and (9), we deduce that there exists $C > 0$ such that

$$b(\boldsymbol{v}, \boldsymbol{\mu}) \geq C\|\boldsymbol{v}\|_V^2, \quad \text{for all } \boldsymbol{v} \in V, \boldsymbol{\mu} \in \Lambda. \tag{20}$$

Consider now the functional $\mathcal{L} : V \times \Lambda \to (-\infty, \infty]$ defined by the formula

$$\mathcal{L}(\boldsymbol{v}, \boldsymbol{\mu}) := b(\boldsymbol{v}, \boldsymbol{\mu}) - (\boldsymbol{f}, \boldsymbol{v})_V.$$

Since the functional b is proper, convex and lower semicontinuous, the map

$$V \times \Lambda \ni (\boldsymbol{v}, \boldsymbol{\mu}) \to \mathcal{L}(\boldsymbol{v}, \boldsymbol{\mu}) \in (-\infty, \infty]$$

is proper, convex and lower semicontinuous, too. As a consequence of (20), $\mathcal{L}$ is also coercive.

Notice that $V \times \Lambda$ is a nonempty, closed, convex subset of the space $V \times \mathcal{H}$.

Therefore, there exists at least one pair $(\boldsymbol{u}^*, \boldsymbol{\sigma}^*)$ such that

$$\mathcal{L}(\boldsymbol{u}^*, \boldsymbol{\sigma}^*) = \min_{(\boldsymbol{v}, \boldsymbol{\mu}) \in V \times \Lambda} \mathcal{L}(\boldsymbol{v}, \boldsymbol{\mu}). \tag{21}$$

The functional $\mathcal{L}$ allows us to reformulate Problem 2 as follows: find $(\boldsymbol{u},\boldsymbol{\sigma}) \in V \times \Lambda$ such that

$$\left.\begin{array}{c}\mathcal{L}(\boldsymbol{u},\boldsymbol{\sigma}) \leq \mathcal{L}(\boldsymbol{v},\boldsymbol{\sigma}) \ \text{ for all } \boldsymbol{v} \in V \\ \\ \text{and} \\ \\ \mathcal{L}(\boldsymbol{u},\boldsymbol{\sigma}) \leq \mathcal{L}(\boldsymbol{u},\boldsymbol{\mu}) \ \text{ for all } \boldsymbol{\mu} \in \Lambda.\end{array}\right\} \tag{22}$$

It is straightforward to observe that any solution of the minimization problem (21) is a solution of the problem (22). Thus, any minimizing pair $(\boldsymbol{u}^*,\boldsymbol{\sigma}^*)$ is a solution of Problem 2. □

We note that

$$b(\boldsymbol{v},\boldsymbol{\mu}) - (\boldsymbol{f},\boldsymbol{v})_V \geq 0, \qquad \text{for all } \boldsymbol{v} \in V, \text{ for all } \boldsymbol{\mu} \in \Lambda. \tag{23}$$

Indeed, let $\boldsymbol{v} \in V$ and $\boldsymbol{\mu} \in \Lambda$. If $B(\boldsymbol{\varepsilon}(\boldsymbol{v}(\cdot)),\boldsymbol{\mu}(\cdot)) \in L^1(\Omega)$ then

$$\begin{aligned} b(\boldsymbol{v},\boldsymbol{\mu}) &= \int_\Omega B(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})),\boldsymbol{\mu}(\boldsymbol{x}))dx \\ &\geq \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})) \cdot \boldsymbol{\mu}(\boldsymbol{x})dx \\ &= (\boldsymbol{f},\boldsymbol{v})_V. \end{aligned}$$

Otherwise, (23) is clearly satisfied.

Consequently,

$$\min_{(\boldsymbol{v},\boldsymbol{\mu}) \in V \times \Lambda} \mathcal{L}(\boldsymbol{v},\boldsymbol{\mu}) \geq 0.$$

Let us comment now on the uniqueness of the weak solution.

If ω is strictly convex, then the first component of the solution of Problem 2 is uniquely determined. Indeed, let us consider the functional $J : V \to (-\infty,\infty]$ defined by the formula

$$J(\boldsymbol{v}) = \begin{cases} \int_\Omega \omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))dx - (\boldsymbol{f},\boldsymbol{v})_V, & \text{if } \omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\cdot))) \in L^1(\Omega) \\ \\ \infty, & \text{otherwise.} \end{cases} \tag{24}$$

Obviously, J is a proper, strictly convex, lower semicontinuous and coercive functional. We note that, taking into account (16) and (14), the inequality

$$b(\boldsymbol{v},\boldsymbol{\sigma}^*) - b(\boldsymbol{u}^*,\boldsymbol{\sigma}^*) \geq (\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}^*)_V, \qquad \text{for all } \boldsymbol{v} \in V$$

yields

$$J(\boldsymbol{v}) - J(\boldsymbol{u}^*) \geq 0, \qquad \text{for all } \boldsymbol{v} \in V.$$

Thus, $\boldsymbol{u}^*$ is the unique minimizer of the functional J.

On the other hand, if ω^* is strictly convex and coercive, then $\boldsymbol{\sigma}^*$ is the unique minimizer of the functional $\tilde{J} : \Lambda \to (-\infty,\infty]$,

$$\tilde{J}(\boldsymbol{\tau}) = \begin{cases} \int_\Omega \omega^*(\boldsymbol{\tau}(\boldsymbol{x}))dx, & \text{if } \omega^*(\boldsymbol{\tau}(\cdot)) \in L^1(\Omega) \\ \\ \infty, & \text{otherwise.} \end{cases}$$

The above discussion yields to the following uniqueness result:

Theorem 3. (A uniqueness result) *Assume* (7)*,* (8) *and* (9)*. If, in addition,* ω^* *is coercive and* ω*,* ω^* *are both strictly convex, then Problem* 2 *has a unique solution* $(\boldsymbol{u}^*, \boldsymbol{\sigma}^*) \in V \times \Lambda$*.*

An important case when ω and ω^* are both strictly convex is outlined in [12], Theorem 11.13, p. 483. Its essence is the duality (under the Fenchel conjugation) between differentiability and strict convexity.

4 The relevance of our approach

In this section we discuss three examples based on linear constitutive laws, single-valued nonlinear constitutive laws and multi-valued nonlinear constitutive laws respectively.

Example 4.1. Let us consider

$$\omega : \Omega \times \mathbb{S}^3 \to \mathbb{R}, \quad \omega(\boldsymbol{x}, \boldsymbol{\tau}) := \frac{1}{2}\mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \tag{25}$$

where $\mathcal{E} : \mathbb{S}^3 \to \mathbb{S}^3$ is a fourth order tensor with the following two properties:

$$\begin{cases} \mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{E}\boldsymbol{\tau}, \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^3, \text{ a.e. in } \Omega; \\ \text{There exists } M > 0 : \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq M|\boldsymbol{\tau}|^2 \text{ for all } \boldsymbol{\tau} \in \mathbb{S}^3, \text{ a.e. in } \Omega. \end{cases} \tag{26}$$

An example of such a tensor $\mathcal{E} = (\mathcal{E}_{ijkl})$ is

$$\mathcal{E}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad 1 \leq i, j, k, l \leq 3,$$

where λ and μ are positive constants.

Obviously, for this example (8) is verified and the constitutive law (4) reduces to the well known linear elastic constitutive law,

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}).$$

Problem 1 can be rewritten as follows,

$$(\mathbf{L}) \quad \begin{cases} \text{Find } \ \boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3 & \text{and} & \boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3, & \text{such that} \\ \operatorname{Div} \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) & = & \mathbf{0} & \text{in } \Omega, \\ \boldsymbol{\sigma}(\boldsymbol{x}) & = & \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) & \text{in } \Omega, \\ \boldsymbol{u}(\boldsymbol{x}) & = & \mathbf{0} & \text{on } \Gamma_1, \\ \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) & = & \boldsymbol{f}_2(\boldsymbol{x}) & \text{on } \Gamma_2. \end{cases}$$

Using the space V defined by (1), the element $\boldsymbol{f}$, defined by (11), and the Green formula (10), we obtain the following weak formulation in displacements:

$$(\mathbf{wL}) : \text{Find } \boldsymbol{u} \in V \text{ such that } a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_V, \quad \text{for all } \boldsymbol{v} \in V,$$

where $a : V \times V \to \mathbb{R}$ is the bilinear, continuous, $V-$elliptic, symmetric form

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))dx.$$

Due to Lax-Milgram Theorem, the problem $(\mathbf{wL})$ has a unique solution.

Definition 3. *Any solution $\boldsymbol{u} \in V$ of the problem* **(wL)** *is called a weak solution of the problem* **(L)**.

Proposition 1. *Let $(\boldsymbol{u}, \boldsymbol{\sigma})$ be a weak solution of Problem 1 with the constitutive function ω given by the formula (25). Then its first component $\boldsymbol{u}$ is the unique weak solution of the problem* **(L)**, *while the second component $\boldsymbol{\sigma}$ verifies the elastic constitutive law in the following weak form,*

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \textit{for all } \boldsymbol{v} \in V. \tag{27}$$

Proof. Let $(\boldsymbol{u}, \boldsymbol{\sigma})$ be a weak solution of Problem 1. Then

$$b(\boldsymbol{w}, \boldsymbol{\sigma}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) \geq (\boldsymbol{f}, \boldsymbol{w} - \boldsymbol{u})_V, \quad \text{for all } \boldsymbol{w} \in V.$$

According to (16) and (14), we obtain

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} \geq 2(\boldsymbol{f}, \boldsymbol{w} - \boldsymbol{u})_V, \quad \text{for all } \boldsymbol{w} \in V.$$

Let $t > 0$ and let $\boldsymbol{v} \in V$ be arbitrarily fixed. Putting in the previous inequality $\boldsymbol{w} = \boldsymbol{u} \pm t\boldsymbol{v}$, and taking into account that

$$\lim_{t \to 0^+} \frac{(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u} \pm t\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{u} \pm t\boldsymbol{v}))_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}}}{t} = 2(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$$

we infer

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \geq \pm(\boldsymbol{f}, \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V.$$

Therefore, $\boldsymbol{u}$ is the weak solution of the problem **(L)**. On the other hand, since $\boldsymbol{\sigma} \in \Lambda$, (27) is verified. □

Example 4.2. Consider the constitutive function

$$\omega : \Omega \times \mathbb{S}^3 \to \mathbb{R}, \ w(\boldsymbol{x}, \boldsymbol{\tau}) := \frac{1}{2}\mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} + \frac{\beta}{2}|\boldsymbol{\tau} - P_{\mathcal{K}}\boldsymbol{\tau}|^2, \tag{28}$$

where $\mathcal{E} : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3$ verifies (26), $\beta > 0$ is a constant coefficient of the material, $\mathcal{K} \subset \mathbb{S}^3$ is a nonempty, closed and convex set and $P_{\mathcal{K}} : \mathbb{S}^3 \to \mathcal{K}$ represents the projection operator on $\mathcal{K}$.

The functional ω is Gâteaux differentiable at any $\boldsymbol{\tau} \in \mathbb{S}_3$. Indeed,

$$\lim_{t \to 0} \frac{\omega(\boldsymbol{\tau} + t\boldsymbol{\xi}) - \omega(\boldsymbol{\tau})}{t} = \nabla\omega(\boldsymbol{\tau}) \cdot \boldsymbol{\xi} \quad \text{for all } \boldsymbol{\xi} \in \mathbb{S}^3,$$

where

$$\nabla\omega(\boldsymbol{\tau}) := \mathcal{E}\boldsymbol{\tau} + \beta(\boldsymbol{\tau} - P_{\mathcal{K}}\boldsymbol{\tau}),$$

see [11], Example d), pp. 8-9.

Moreover, it can be verified that

$$\omega(\boldsymbol{\tau}) - \omega(\boldsymbol{\varepsilon}) \geq \nabla\omega(\boldsymbol{\varepsilon}) \cdot (\boldsymbol{\tau} - \boldsymbol{\varepsilon}), \quad \text{for all } \boldsymbol{\tau}, \boldsymbol{\varepsilon} \in \mathbb{S}^3.$$

Using Lemma 1 we conclude that the functional ω is convex. On the other hand, since ω is convex and Gâteaux differentiable, by (2) we get

$$\partial\omega(\boldsymbol{\tau}) = \{\mathcal{E}\boldsymbol{\tau} + \beta(\boldsymbol{\tau} - P_{\mathcal{K}}\boldsymbol{\tau})\} \text{ for all } \boldsymbol{\tau} \in \mathbb{S}^3.$$

In addition, by Lemma 2 we conclude that w is lower semicontinuous. Notice that

$$\omega(\boldsymbol{\tau}) \geq M|\boldsymbol{\tau}|^2, \quad \text{for all } \boldsymbol{\tau} \in \mathbb{S}^3$$

and

$$\omega(0_{\mathbb{S}^3}) = 0.$$

Therefore, (8) and (9) are verified for this second example too.

In this situation, the constitutive law (4) reduces to the following piecewise linear constitutive law

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_{\mathcal{K}}\boldsymbol{\varepsilon}(\boldsymbol{u})),$$

which is discussed for example in [7] p. 124 and [17], p. 14. Thus, for this second example, Problem 1 can be rewritten as follows.

$$(\mathbf{PL}) \quad \begin{cases} \text{Find } \boldsymbol{u}:\bar{\Omega}\to\mathbb{R}^3 \text{ and } \boldsymbol{\sigma}:\bar{\Omega}\to\mathbb{S}^3, \text{ such that} \\ \operatorname{Div}\boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \mathbf{0} \text{ in } \Omega, \\ \boldsymbol{\sigma}(\boldsymbol{x}) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) + \beta(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) - P_{\mathcal{K}}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \text{ in } \Omega, \\ \boldsymbol{u}(\boldsymbol{x}) = \mathbf{0} \text{ on } \Gamma_1, \\ \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{f}_2(\boldsymbol{x}) \text{ on } \Gamma_2. \end{cases}$$

For this problem we can introduce the following weak formulation,

$$(\mathbf{wPL}):\ \text{Find } \boldsymbol{u}\in V \text{ such that } A\boldsymbol{u} = \boldsymbol{f},$$

where the operator $A: V \to V$ is defined as follows: for any $\boldsymbol{u} \in V$, $A\boldsymbol{u}$ is the element of V that satisfies

$$(A\boldsymbol{u},\boldsymbol{v})_V = \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))dx + \beta\int_\Omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) - P_{\mathcal{K}}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})))\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))dx$$

for all $\boldsymbol{v} \in V$. Taking into account that the projector operator is nonexpansive, it can be verified that the operator A is a strongly monotone and Lipschitz continuous operator. We infer that the problem (**wPL**) has a unique solution; see, for example, [19], p. 173.

Definition 4. *Any solution $\boldsymbol{u} \in V$ of the problem* **(wPL)** *is called a weak solution of the problem* **(PL)**.

Proposition 2. *Let $(\boldsymbol{u},\boldsymbol{\sigma})$ be a weak solution of Problem* 1 *for the constitutive function ω given by* (28)*. Then its first component $\boldsymbol{u}$ is the unique weak solution of the problem* **(PL)***, and the second component $\boldsymbol{\sigma}$ verifies the piecewise linear constitutive law in the following weak form,*

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_{\mathcal{K}}\boldsymbol{\varepsilon}(\boldsymbol{u})),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \textit{for all } \boldsymbol{v}\in V. \tag{29}$$

Proof. Let $(\boldsymbol{u},\boldsymbol{\sigma})$ be a weak solution of Problem 1. Then,

$$b(\boldsymbol{w},\boldsymbol{\sigma}) - b(\boldsymbol{u},\boldsymbol{\sigma}) \geq (\boldsymbol{f},\boldsymbol{w}-\boldsymbol{u})_V \quad \text{for all } \boldsymbol{w}\in V.$$

By (16), (14) and (28), we obtain

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{w}),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + \beta\int_\Omega \Big(|\boldsymbol{\varepsilon}(\boldsymbol{w}(\boldsymbol{x})) - P_{\mathcal{K}}(\boldsymbol{\varepsilon}(\boldsymbol{w}(\boldsymbol{x})))|^2$$
$$- |\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) - P_{\mathcal{K}}(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})))|^2\Big)dx \geq (\boldsymbol{f}, \boldsymbol{w}-\boldsymbol{u})_V \quad \text{for all } \boldsymbol{w}\in V.$$

Let $t>0$ and let $\boldsymbol{v}\in V$ be arbitrarily fixed. Putting in the previous inequality $\boldsymbol{w}=\boldsymbol{u}\pm t\boldsymbol{v}$, and taking into account the fact that

$$\begin{aligned}\lim_{t\to 0^+}\int_\Omega \Big[&\frac{\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})\pm t\boldsymbol{v}(\boldsymbol{x}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})\pm t\boldsymbol{v}(\boldsymbol{x})) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))}{2t}\\
&+\frac{\beta(|\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})\pm t\boldsymbol{v}(\boldsymbol{x})) - P_{\mathcal{K}}(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})\pm t\boldsymbol{v}(\boldsymbol{x})))|^2 - |\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) - P_{\mathcal{K}}(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})))|^2)}{t}\Big]dx\\
&= (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_{\mathcal{K}}\boldsymbol{\varepsilon}(\boldsymbol{u})),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},\end{aligned}$$

we infer that

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \beta P_{\mathcal{K}}(\boldsymbol{\varepsilon}(\boldsymbol{u}))),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \geq \pm(\boldsymbol{f},\boldsymbol{v})_V \quad \text{for all } \boldsymbol{v}\in V.$$

This last inequality allows us to conclude that $\boldsymbol{u}$ is the unique weak solution of the problem **(PL)**. On the other hand, since $\boldsymbol{\sigma}\in\Lambda$, we obtain (29). □

Both examples presented before involve constitutive maps leading to single-valued constitutive laws. Below we will discuss a more general example leading to possibly multi-valued constitutive laws.

Example 4.3. Assume now that ω is a constitutive map satisfying (8) such that (4) is a possibly multi-valued constitutive law. In this situation, using again the space V and the element $\boldsymbol{f}$, by applying Green's formula (10), we obtain for Problem 1 the following weak formulation in displacements:

$$(\mathbf{wM}): \ \text{Find } \boldsymbol{u}\in V \text{ such that } W(\boldsymbol{v}) - W(\boldsymbol{u}) \geq (\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_V \quad \text{for all } \boldsymbol{v}\in V,$$

where $W: V\to(-\infty,\infty]$ is defined by the formula

$$W(\boldsymbol{v}) = \begin{cases} \int_\Omega \omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))dx & \text{if } \omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\cdot)))\in L^1(\Omega)\\ \infty & \text{otherwise.}\end{cases}$$

Obviously, this weak formulation is equivalent with the following problem of minimization: find $\boldsymbol{u}\in V$ such that

$$J(\boldsymbol{u}) = \min_{\boldsymbol{v}\in V} J(\boldsymbol{v}),$$

where $J: V\to(-\infty,\infty]$ was defined by the formula (24). Since J is a proper, convex, lower semicontinuous, coercive functional, the problem **(wM)** has at least one solution $\boldsymbol{u}\in V$.

Proposition 3. *Let $(\boldsymbol{u},\boldsymbol{\sigma})$ be a weak solution of Problem 1 with a constitutive function ω that satisfies (8). Then, its first component $\boldsymbol{u}$ is a solution of the problem* **(wM)**. *In addition, the second component $\boldsymbol{\sigma}$ verifies the possibly multi-valued nonlinear constitutive law in the following weak form,*

$$\int_\Omega \omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))dx - \int_\Omega \omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})))dx \geq \int_\Omega \boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})) - \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})))dx \tag{30}$$

for each $\boldsymbol{v}\in V$ such that $\omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\cdot)))\in L^1(\Omega)$.

Proof. Let $(\boldsymbol{u},\boldsymbol{\sigma})$ be a weak solution of Problem 1. Then,

$$b(\boldsymbol{v},\boldsymbol{\sigma})-b(\boldsymbol{u},\boldsymbol{\sigma})\geq(\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_V\quad\text{ for all }\boldsymbol{v}\in V.$$

Therefore, taking into account the definition of W, by (14) and (16), we deduce

$$W(\boldsymbol{v})-W(\boldsymbol{u})\geq(\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_V\quad\text{for all }\boldsymbol{v}\in V.$$

Thus, $\boldsymbol{u}$ is a solution of the problem **(wM)**. Moreover, since $\boldsymbol{\sigma}\in\Lambda$, we get

$$W(\boldsymbol{v})-W(\boldsymbol{u})\geq\int_\Omega\boldsymbol{\sigma}(\boldsymbol{x})\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))-\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})))dx\quad\text{for all }\boldsymbol{v}\in V,$$

and from this inequality, taking into account the definition of W, it is straightforward to obtain (30). $\square$

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