

GENERALIZED CONVEXITY AND THE EXISTENCE OF FINITE TIME BLOW-UP SOLUTIONS FOR AN EVOLUTIONARY PROBLEM

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ABSTRACT. In this paper we study a class of nonlinearities for which a non-local parabolic equation with Neumann-Robin boundary conditions, for p -Laplacian, has finite time blow-up solutions.

1. INTRODUCTION

It is a well known fact that convexity plays an important role in the different parts of mathematics, including the study of boundary value problems. The aim of our paper is to introduce a new class of generalized convex functions and to illustrate its usefulness in establishing a sufficient condition for the existence of finite time blow-up solutions for the evolutionary problem

$$(1.1) \quad \begin{cases} u_t - \Delta_p u = f(|u|) - \frac{1}{m(\Omega)} \int_{\Omega} f(|u|) dx & \text{in } \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

with the initial conditions

$$(1.2) \quad u(x, 0) = u_0(x) \text{ on } \Omega, \text{ where } \int_{\Omega} u_0 dx = 0.$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded regular domain of class C^2 , $f : [0, \infty) \mapsto [0, \infty)$ is a locally Lipschitz function, $m(\Omega)$ represents the Lebesgue measure of the domain Ω , and $\Delta_p = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, for $p \geq 2$, is the p -Laplacian operator.

The particular case where $p = 2$ was recently considered by Soufi, Jazar and Monneau [19], and Jazar and Kiwan [11] (under the assumption that f is a power function of the form $f(u) = u^\alpha$, with $\alpha > 1$), and also by the present authors [15] (for f belonging to a larger class of nonlinearities).

The problems of type (1.1) & (1.2) arise naturally in mechanics, biology and population dynamics. See [2], [5], [6], [8] and [10]. For example, if we consider a couple or a mixture of two equations of the above type, the resulting problem describes the temperatures of two substances which constitute a combustible mixture, or represents a model for the behavior of densities of two diffusion biological species which interact each other.

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2. GENERALIZED CONVEXITY OF ORDER α

According to the classical Hermite-Hadamard inequality, the mean value of a continuous convex function $f : [a, b] \rightarrow \mathbb{R}$ lies between the value of f at the midpoint of the interval $[a, b]$ and the arithmetic mean of the values of f at the endpoints of this interval, that is,

$$(HH) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}.$$

Moreover, each side of this double inequality characterizes convexity in the sense that a real-valued continuous function f defined on an interval I is convex if its restriction to each compact subinterval $[a, b] \subset I$ verifies the left hand side of (HH) (equivalently, the right hand side of (HH)). See [1] and [14] for details.

In what follows we will be interested in a class of generalized convex functions motivated by the right hand side of the Hermite-Hadamard inequality.

Definition 1. A real-valued function f defined on an interval $[a, \infty)$ belongs to the class GC_α (for some $\alpha > 0$), if it is continuous, nonnegative, and

$$(2.1) \quad \frac{1}{\alpha+1} f(t) \geq \frac{1}{t-a} \int_a^t f(x) dx \quad \text{for } t \text{ large enough.}$$

Using calculus, one can see easily that the condition (2.1) is equivalent to the fact that the ratio

$$(2.2) \quad \frac{\frac{1}{t-a} \int_a^t f(x) dx}{(t-a)^\alpha}$$

is nondecreasing for t bigger than a suitable value $A \geq a$. In turn, this implies that the mean value $\frac{1}{t-a} \int_a^t f(x) dx$ has a polynomial growth at infinity.

According to the Hermite-Hadamard inequality, every nonnegative, continuous and convex function $f : [a, \infty) \rightarrow \mathbb{R}$ with $f(a) = 0$ belongs to the class GC_1 . The converse is not true because the membership of a function $f : [a, \infty) \rightarrow \mathbb{R}$ to the class GC_α yields only an asymptotic inequality of the form

$$\frac{1}{\alpha+1} f(t) + \frac{\alpha}{\alpha+1} f(a) \geq \frac{1}{t-a} \int_a^t f(x) dx \quad \text{for } t \text{ large enough.}$$

If $g \in C^1([0, \infty))$ and g is nondecreasing, then the function $f(x) = g(x)(x-a)^\alpha$ belongs to the class $CG_\alpha([0, \infty))$, whenever $\alpha > 0$. In fact,

$$\begin{aligned} \frac{1}{t-a} \int_a^t f(x) dx &= \frac{(t-a)^\alpha}{\alpha+1} g(t) - \frac{1}{t-a} \int_a^t g'(x) \frac{(x-a)^{\alpha+1}}{\alpha+1} dx \\ &\leq \frac{1}{\alpha+1} f(t). \end{aligned}$$

As a consequence, $(x + \sin x)x$ provides an example of function of class GC_1 on $[0, \infty)$ which is not convex.

No positive constant can be a function of class GC_α for any $\alpha > 0$.

Also, the restriction of a function $f : [a, \infty) \rightarrow \mathbb{R}$ of class GC_α to a subinterval $[b, \infty)$ is not necessarily a function of class GC_α .

In the sequel we will describe some other classes of functions of class GC_α .

The following concept of generalized convexity is due to S. Varosanec [20] and generalizes the usual convexity, s -convexity, the Godunova–Levin functions and P -functions.

Definition 2. Suppose that $h : [0, 1] \rightarrow \mathbb{R}$ is a function such that $h(\lambda) + h(1-\lambda) \geq 1$ for all $\lambda \in [0, 1]$. A nonnegative function f defined on an interval I is called h -convex if

$$(2.3) \quad f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y).$$

whenever $\lambda \in [0, 1]$, and $x, y \in I$.

Proposition 1. Suppose that f is a nonnegative continuous function defined on an interval $[a, \infty)$ such that the following two conditions are fulfilled:

- i) $f(a) = 0$;
- ii) f is h -convex with respect to a function h with $\int_0^1 h(\lambda) d\lambda \leq \frac{1}{\alpha+1}$, for some $\alpha > 0$.

Then f belongs to the class GC_α .

Proof. In fact,

$$\begin{aligned} \frac{1}{t-a} \int_a^t f(x) dx &= \int_0^1 f((1-\lambda)a + \lambda t) d\lambda \\ &\leq f(t) \int_0^1 h(\lambda) d\lambda + f(a) \int_0^1 h(1-\lambda) d\lambda \\ &\leq \frac{1}{\alpha+1} f(t). \end{aligned}$$

□

An important class of nonlinearities in partial differential operators theory is that of regularly varying functions, introduced by Karamata in [12].

Definition 3. A positive measurable function f defined on interval $[a, \infty)$ (with $a \geq 0$) is said to be regularly varying at infinity, of index $\sigma \in \mathbb{R}$ (abbreviated, $f \in RV_\infty(\sigma)$), provided that

$$\lim_{x \rightarrow \infty} \frac{f(tx)}{f(x)} = t^\sigma \quad \text{for all } t > 0.$$

All functions of index σ are of the form

$$f(x) = x^\sigma \exp \left(a(x) + \int_0^x \frac{\varepsilon(s)}{s} ds \right),$$

where $a(x)$ and $\varepsilon(x)$ are bounded and measurable, $a(x) \rightarrow \alpha \in \mathbb{R}$ and $\varepsilon(x) \rightarrow 0$ as $x \rightarrow \infty$. In particular, so are

$$x^\sigma \log x, \quad x^\sigma \log \log x, \quad x^\sigma \exp \left(\frac{\log x}{\log \log x} \right), \quad x^\sigma \exp \left((\log x)^{1/3} \left(\cos(\log x)^{1/3} \right) \right).$$

See [4] for details.

Semilinear problems with nonlinearities in the class of regularly varying functions have been studied by many people. See the paper by Cîrstea and Rădulescu [7] and the references therein.

Proposition 2. If $f \in RV_\infty(\sigma)$ with $\sigma > 0$, then

$$\lim_{x \rightarrow \infty} \frac{F(x)}{xf(x)} = \frac{1}{\sigma+1},$$

where

$$(2.4) \quad F(x) := \int_0^x f(s) ds.$$

As a consequence, if f is also continuous, then f is of class GC_α , whenever $\alpha \in (0, \sigma)$.

Proof. To prove this, consider the change of variable $s = tx$ which yields

$$F(x) = \int_0^x f(s) ds = \int_0^1 xf(tx) dt.$$

The continuity of f and the fact that $f \in RV_\infty(\sigma)$ assure the existence of a $\delta > 0$ such that for every $x > \delta$ we have

$$\frac{f(tx)}{f(x)} \leq t^\sigma + 1,$$

whence the integrability of the function $t \rightarrow \frac{f(tx)}{f(x)}$ on $[0, 1]$. Then

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{F(x)}{xf(x)} &= \lim_{x \rightarrow \infty} \int_0^1 \frac{f(tx)}{f(x)} dt \\ &= \int_0^1 \lim_{x \rightarrow \infty} \frac{f(tx)}{f(x)} dt = \int_0^1 t^\sigma dt = \frac{1}{\sigma + 1}. \end{aligned}$$

The commutation of the limit with the integral is motivated by the Lebesgue dominated convergence theorem. \square

Another important class of nonlinearities which appear in connection with the study of boundary blow-up problems for elliptic equations is the class of functions satisfying the Keller-Osserman condition. See [16], [9], [18] and [15].

Definition 4. A nonnegative and nondecreasing function $f \in C^1([0, \infty))$ with $f(0) = 0$ satisfies the generalized Keller-Osserman condition of order $p > 1$ if

$$(2.5) \quad \int_1^\infty \frac{1}{(F(t))^{1/p}} dt < \infty,$$

where F is the primitive of f given by the formula (2.4).

If $f \in RV_\infty(\sigma+1)$ with $\sigma+2 > p > 1$ is a nondecreasing and continuous function, then $F \in RV_\infty(\sigma+2)$ and $F^{-1/p} \in RV_\infty((-\sigma-2)/p)$. Since $(-\sigma-2)/p < -1$, we infer that $F^{-1/p} \in L^1([1, \infty))$ and thus f satisfies the generalized Keller-Osserman condition.

It is worth to notice that the function $\exp(t)$ is not regularly varying at infinity though satisfies the generalized Keller-Osserman condition and belongs also to any class GC_α with $\alpha > 0$.

Necessarily, if a function f satisfies the generalized Keller-Osserman condition of order $p > 1$, then

$$(2.6) \quad \lim_{t \rightarrow \infty} \frac{F(t)}{t^p} = \infty,$$

while $\frac{F(t)}{t^p}$ may be (or may be not) a monotonic function.

If $\frac{F(t)}{t^p}$ is nondecreasing for some $p > 2$, then the function f belongs to the class GC_{p-1} . In particular, this is the case of the function $f(t) = pt^{p-1} \log(t+1) + \frac{t^p}{t+1}$

(whose primitive is $F(t) = t^p \log(t+1)$). Notice that this function does not satisfy the generalized Keller-Osserman condition of order p .

We end this section by discussing the connection Definition 1 with a class of functions due to W. Orlicz.

Definition 5. An N -function is any function $M : [0, \infty) \rightarrow \mathbb{R}$ of the form

$$M(x) = \int_0^x p(t) dt,$$

where p is nondecreasing and right continuous, $p(0) = 0$, $p(t) > 0$ for $t > 0$, and $\lim_{t \rightarrow \infty} p(t) = \infty$.

An N -function M satisfies the Δ_2 -condition if there exist constants $k > 0$ and $x_0 \geq 0$ such that

$$M(2x) \leq kM(x) \quad \text{for all } x \geq x_0.$$

Any N -function M is convex and plays the following properties:

- N1) $M(0) = 0$ and $M(x) > 0$ for $x > 0$;
- N2) $\frac{M(x)}{x} \rightarrow 0$ as $x \rightarrow 0$ and $\frac{M(x)}{x} \rightarrow \infty$ as $x \rightarrow \infty$.

Two examples of N -functions which satisfy the Δ_2 -condition are $\frac{x^p}{p}$ (for $p \geq 1$) and $t(\log t)^+$.

The N -functions which satisfy the Δ_2 -condition are instrumental in the theory of Orlicz spaces (which extend the $L^p(\mu)$ spaces). Their theory is available in many books, such as [13] and [17], and has important applications to interpolation theory [3] and Fourier analysis [21].

According to [13], page 23, the constant k which appears in the formulation of Δ_2 -condition is always greater than or equal to 2.

Proposition 3. Every N -function $M : [0, \infty) \rightarrow \mathbb{R}$ which satisfies the Δ_2 -condition belongs to the class GC_α , whenever $\alpha \in (0, 2 \log_2 k)$.

Proof. Since M is nondecreasing,

$$M(tx) = M(2^{\log_2 t} x) \leq M(2^{[\log_2 t]+1} x),$$

and taking into account the Δ_2 -condition we infer that

$$\begin{aligned} M(tx) &\leq M(x)k^{[\log_2 t]+1} \leq M(x)k^{\log_2 t+1} \\ &\leq M(x)t^{2 \log_2 k}, \end{aligned}$$

for x big enough and $t \geq 2$. Hence,

$$\begin{aligned} \int_0^t M(x) dx &= \int_0^1 tM(ts) ds \\ &\leq \int_0^1 tM(t)s^{2 \log_2 k} ds = \frac{1}{2 \log_2 k + 1} tM(t) \end{aligned}$$

and the proof is done. \square

3. AN APPLICATION TO THE EXISTENCE OF FINITE TIME BLOW-UP SOLUTIONS

This section is devoted to the existence of finite time blow-up solutions of the evolutionary p -Laplacian problem

$$(3.1) \quad u_t - \Delta_p u = f(|u|) - \frac{1}{m(\Omega)} \int_{\Omega} f(|u|) dx \quad \text{in } \Omega$$

with Neumann-Robin boundary values,

$$(3.2) \quad |\nabla u|^{p-2} \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

and the initial conditions

$$(3.3) \quad u(x, 0) = u_0(x) \text{ on } \Omega, \text{ where } \int_{\Omega} u_0 dx = 0.$$

As was mentioned in the introduction, we restrict ourselves to the case where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain of class C^2 , and $f : [0, \infty) \mapsto [0, \infty)$ is a locally Lipschitz function; $m(\Omega)$ represents the Lebesgue measure of the domain Ω , and Δ_p , for $p \geq 2$, is the p -Laplacian operator.

The purpose of this section, is to extend a natural energetic criterion for the blow-up in finite time of solutions of (3.1) – (3.3). Our proof relies on the same idea used by Jazar and Kiwan [11] in the case where $p = 2$ and f is a power function.

We start by noticing that each solution u of the problem above has the property

$$\int_{\Omega} u dx = 0$$

because the integral in the right hand side of (3.1) is 0 and

$$\begin{aligned} \frac{d}{dt} \left(\int_{\Omega} u dx \right) &= \int_{\Omega} u_t dx = \int_{\Omega} \Delta_p u dx \\ &= \int_{\Omega} \operatorname{div}(|\nabla u|^{p-2} \nabla u) dx = 0. \end{aligned}$$

Hence, by the initial condition (3.3), we have $\int_{\Omega} u dx = 0$.

Next, it is easy to see that for $p > 1$ the energy

$$E(u(t)) = \int_{\Omega} \left(\frac{1}{p} |\nabla u|^p - \int_0^u f(|\tau|) d\tau \right) dx,$$

of any solution u of our evolutionary problem is nonincreasing in time. In fact,

$$\begin{aligned} \frac{dE(u(t))}{dt} &= \int_{\Omega} (|\nabla u|^{p-2} \nabla u_t \nabla u - u_t f(|u|)) dx \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial n} |\nabla u|^{p-2} u_t d\sigma - \int_{\Omega} u_t \Delta_p u dx - \int_{\Omega} u_t f(|u|) dx \\ &= - \int_{\Omega} u_t (\Delta_p u + f(|u|)) dx = - \int_{\Omega} u_t^2 dx, \end{aligned}$$

and by integrating both sides over $[0, t]$ we obtain the formula

$$(3.4) \quad E(u(t)) = E(u_0) - \int_0^t \int_{\Omega} u_t^2 dx dt, \quad \text{for all } t > 0.$$

According to this formula, if the initial energy $E(u_0)$ is nonpositive, then $E(u(t))$ is nonpositive for all $t > 0$. In the case of generalized convex functions of order α , with $\alpha > \frac{1}{p-1}$ we have

$$(3.5) \quad C \int_{\Omega} u f(|u|) dx \geq \int_{\Omega} \int_0^u f(|t|) dt dx \geq \frac{1}{p} \int_{\Omega} |\nabla u|^p,$$

where $C = \frac{1}{1+\alpha} \in (0, \frac{1}{p(p-1)})$.

Theorem 1. (*The energetic criterion for blow-up in finite time, case $p \geq 2$) Assume that $f : [0, \infty) \mapsto [0, \infty)$ is a locally Lipschitz function belonging to the class GC_α , with $\alpha > p^2 - p - 1$, and let u be a solution of the problem (3.1) – (3.3) corresponding to an initial data $u_0 \in C(\bar{\Omega})$, u_0 not identically zero.*

If $E(u_0) \leq 0$, then u , as a function of t , cannot be in $L^\infty((0, T); L^2(\Omega))$ for all $T > 0$. In other word, there is $T > 0$ such that

$$(3.6) \quad \limsup_{t \rightarrow T^-} \|u(t)\|_{L^2} = \infty.$$

Notice that the condition $E(u_0) \leq 0$ in Theorem 1 is also necessary for the blow-up in finite time (of the L^2 norm of $u(t)$). In fact, (3.6) forces that

$$\inf \{E(u(t)) : 0 < t < T\} = -\infty.$$

This can be argued by contradiction. If $E(u(t)) \geq -C_0$, for some $C_0 > 0$, then the function

$$h(t) := \frac{1}{2} \int_{\Omega} u^2(x, t) dx$$

verifies the condition

$$\begin{aligned} \frac{1}{2} h'(t) &= \int_{\Omega} u u_t dx \leq \frac{1}{2} \int_{\Omega} (u^2 + u_t^2) dx \\ &= \frac{1}{2} (h(t) - E'(u(t))), \end{aligned}$$

which yields

$$(h(t) + E(u(t)) + C_0)' \leq h(t) \leq h(t) + E(u(t)) + C_0.$$

Therefore

$$h(t) \leq h(t) + E(u(t)) + C_0 \leq (h(0) + E(u_0) + C_0)e^t, \text{ for all } t \in (0, T),$$

and thus the L^2 -norm of $u(t)$ is bounded.

The proof of Theorem 1 needs a preparation.

Lemma 1. *Under the assumptions of Theorem 1, for $C = 1/(1 + \alpha)$, the two auxiliary functions*

$$h(t) := \frac{1}{2} \int_{\Omega} u^2(x, t) dx \quad \text{and} \quad H(t) := \int_0^t h(s) ds$$

verify the following three conditions:

$$(3.7) \quad h'(t) \geq \frac{1}{C} \int_0^t \int_{\Omega} u_t^2 dt;$$

$$(3.8) \quad h'(t) \geq 2 \left(\frac{1}{Cp} - p + 1 \right) \lambda h(t), \text{ for some } \lambda > 0;$$

$$(3.9) \quad \frac{1}{2C} \left(H'(t) - H'(0) \right)^2 \leq H(t) H''(t),$$

Proof. In fact,

$$\begin{aligned}
h'(t) &= \int_{\Omega} u_t u \, dx = \int_{\Omega} u(\Delta_p u + f(|u|)) \, dx \\
&\geq \int_{\Omega} \left(-(p-1)|\nabla u|^p + \frac{1}{C} \int_0^u f(|t|) \, dt \right) dx \\
&= -\frac{1}{C} \int_{\Omega} \left(\frac{1}{p} |\nabla u|^p - \int_0^u f(|t|) \, dt \right) dx + \left(\frac{1}{Cp} - p + 1 \right) \int_{\Omega} |\nabla u|^p \, dx.
\end{aligned}$$

Hence,

$$\begin{aligned}
h'(t) &\geq -\frac{1}{C} E(u) + \left(\frac{1}{Cp} - p + 1 \right) \int_{\Omega} |\nabla u|^p \, dx \\
&\geq -\frac{1}{C} E(u) \\
&= -\frac{1}{C} E(u_0) + \frac{1}{C} \int_0^t \int_{\Omega} u_t^2 \, dx dt \\
&\geq \frac{1}{C} \int_0^t \int_{\Omega} u_t^2 \, dx dt.
\end{aligned}$$

On the other hand, by the Poincaré inequality, we have

$$\begin{aligned}
h'(t) &\geq \left(\frac{1}{Cp} - p + 1 \right) \int_{\Omega} |\nabla u|^2 \, dx \\
&\geq \left(\frac{1}{Cp} - p + 1 \right) \lambda \int_{\Omega} u^2 \, dx \\
&= 2 \left(\frac{1}{Cp} - p + 1 \right) \lambda h(t),
\end{aligned}$$

where λ is a suitable positive constant.

We pass now to the proof of (3.9). Since

$$\begin{aligned}
H'(t) - H'(0) &= \int_0^t h'(s) \, ds = \int_0^t \int_{\Omega} u u_t \, dx dt \\
&\leq \left(\int_0^t \int_{\Omega} u^2 \, dx dt \right)^{1/2} \left(\int_0^t \int_{\Omega} u_t^2 \, dx dt \right)^{1/2} \\
&\leq (2H(t))^{1/2} (Ch'(t))^{1/2} = (2CH(t)H''(t))^{1/2},
\end{aligned}$$

by (3.7) we infer that

$$H'(t) - H'(0) = \int_0^t h'(s) \, ds \geq 0,$$

and thus

$$\frac{1}{2C} \left(H'(t) - H'(0) \right)^2 \leq H(t)H''(t).$$

□

Proof of Theorem 1. Suppose, by *reduction ad absurdum*, that the solution $u(x, \cdot)$ exists in

$$L^\infty((0, T); L^2(\Omega))$$

for all $T > 0$. By (3.8),

$$(3.10) \quad \lim_{t \rightarrow \infty} H'(t) = \lim_{t \rightarrow \infty} h(t) = \infty,$$

which yields, for each $\beta \in (0, 1/C)$, the existence of a number $T_0 > 0$ such that for all $t > T_0$,

$$\beta H'(t)^2 \leq \frac{1}{C} \left(H'(t) - H'(0) \right)^2.$$

Now, by (3.9) we obtain

$$\beta H'(t)^2 \leq 2H(t)H''(t).$$

We will show, by considering the function $G(t) = H(t)^{-q}$, for a suitable $q > 0$, that the last inequality leads to a contradiction. In fact,

$$\begin{aligned} G''(t) &= qH(t)^{-q-2} \left((q+1)(H'(t))^2 - H(t)H''(t) \right) \\ &\leq qH(t)^{-q-2} \left(\frac{2(q+1)}{\beta} - 1 \right) H(t)H''(t), \end{aligned}$$

for all $t \geq T_0$, so that for $\beta \in (0, 1/C)$ and $q \in (0, 1/(2C) - 1)$ with $2(q+1) < \beta < 1/C$, the corresponding function $G(t)$ is concave.

By (3.10), $\lim_{t \rightarrow \infty} H(t) = \infty$, whence $\lim_{t \rightarrow \infty} G(t) = 0$. Thus G provides an example of a concave and strictly positive function which tends to 0 at infinity, a fact which is not possible. Consequently u , as a function of t , cannot be in $L^\infty((0, T); L^2(\Omega))$ for all $T > 0$. The proof of Theorem 1 is done.

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