Lagrange's Identity and Its Developments

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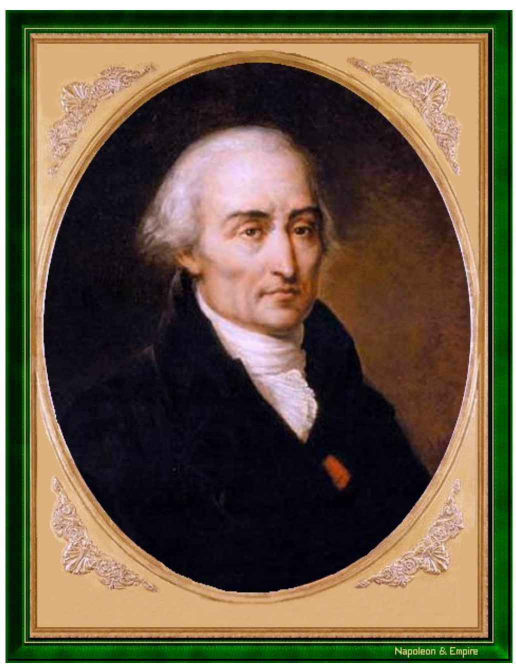
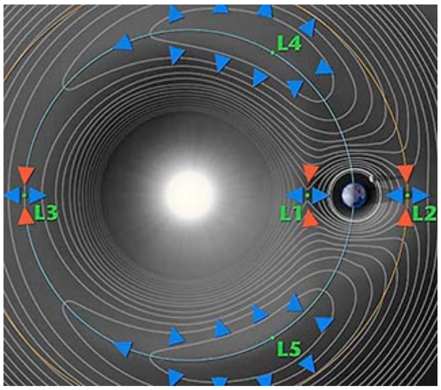


The XVII-th Annual Conference of Romanian Math. Society, Botosani, October 18-20, 2013.

To the memory of Joseph-Louis Lagrange,

one of the greatest mathematicians of all time.

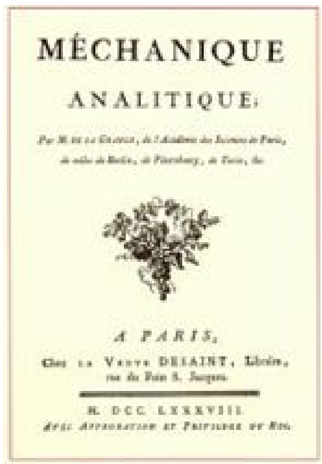
1. Scientific Contribution of Lagrange

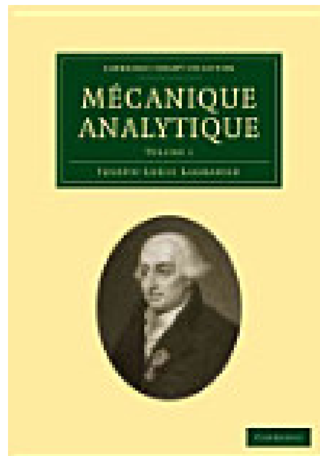


The interplanetary transport network

Lagrange served as president of
the comission who adopted the
base 10 of the metric system.

Lagrangian Mechanics

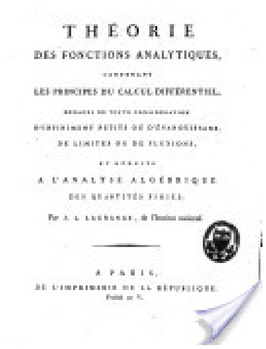


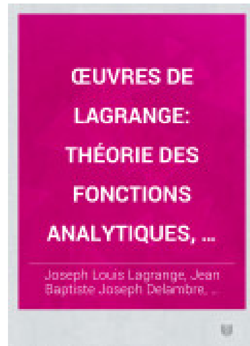


1st Ed. 1788, 2nd Ed. 1811

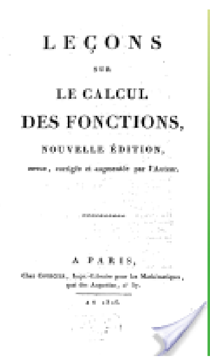
He showed that the subject of mechanics is implicitly included in a single principle, the law of virtual work, from which one can obtain (by the aid of the calculus of variations) general formulae from which the whole of mechanics, both of solids and fluids.

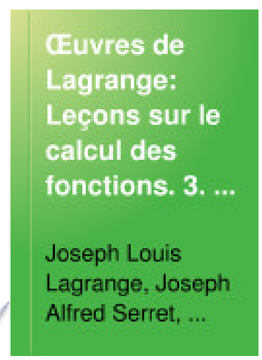
Differential Calculus and Calculus of Variations





Théorie des fonctions analytiques 1797.





Leçons sur le calcul des fonctions (1804, 2nd ed. in 1806). It is in this book that Lagrange formulated his celebrated method of Lagrange multipliers, in the context of problems of variational calculus with integral constraints. These works devoted to differential calculus and calculus of variations may be considered as the starting point for the researches of Cauchy, Jacobi, and Weierstrass.

2. Prizes and Distinctions

Member of the Berlin Academy, elected on 2 September 1756.

Fellow of the Royal Society of Edinburgh in 1790.

Fellow of the Royal Society and a foreign member of the Royal Swedish Academy of Sciences in 1806.

In 1808, Napoleon made Lagrange a Grand Officer of the Legion of Honour and a Comte of the Empire. He was awarded the Grand Croix of the Ordre Impérial de la Réunion in 1813, a week before his death in Paris.

Lagrange was awarded the 1764 prize of the French Academy of Sciences for his memoir on the libration of the Moon. In 1766 the Academy proposed a problem of the motion of the satellites of Jupiter, and the prize again was awarded to Lagrange. He also shared or won the prizes of 1772, 1774, and 1778.

3. A Gallery of Fame

E. Halley

L. Euler

D 'Alembert

G. Monge

S. D. Poisson

J. B. J. Fourier

4. Lagrange's Algebraic Identity

1773, *Quelques problémes sur les pyramides triangulaires*, p. 663, lines 6-8:

$$\left(\sum_{i=1}^{3} a_i^2\right)\left(\sum_{i=1}^{3} b_i^2\right) = \left(\sum_{i=1}^{3} a_i b_i\right)^2 + \sum_{1\le i<j\le 3} (a_i b_j - a_j b_i)^2.$$

In other words

$$\|u\|^2 \, \|v\|^2 = |\langle u, v\rangle|^2 + \|u \times v\|^2 \quad \text{for all } u, v \in \mathbb{R}^3.$$

Fibonacci's *Book of Squares* (*Liber Quadratorum*, in the original Latin):

$$\left(a_1^2 + a_2^2\right)\left(b_1^2 + b_2^2\right) = (a_1 b_1 + a_2 b_2)^2 + (a_1 b_2 - a_2 b_1)^2 .$$

Complex number multiplication,

$$|a_1 + i a_2|^2 \, |b_1 + i b_2|^2 = |(a_1 + i a_2)\,(b_1 + i b_2)|^2 .$$

5. Lagrange's Barycentric Identity

1783, *Sur une nouvelle proprieté du centre de gravité*:

$$(L)\ \frac{1}{M}\sum_{k=1}^{n} m_k \|z - x_k\|^2$$

$$= \left\| z - \frac{1}{M}\sum_{k=1}^{n} m_k x_k \right\|^2 + \frac{1}{M^2}\sum_{1\le i<j\le n} m_i m_j \|x_i - x_j\|^2.$$

For $z = 0$, $m_k = p_k a_k^2$ and $x_k = y_k/a_k$ in (L) :

$$\left(\sum_{k=1}^{n} p_k a_k^2\right)\left(\sum_{k=1}^{n} p_k \|y_k\|^2\right)$$

$$= \left\|\sum_{k=1}^{n} p_k a_k y_k\right\|^2 + \sum_{1\le i<j\le n} p_i p_j \|a_j y_i - a_i y_j\|^2.$$

Consequence:

$$\left\|\sum_{k=1}^{n} p_k a_k y_k\right\|^2 \le \left(\sum_{k=1}^{n} p_k a_k^2\right)\left(\sum_{k=1}^{n} p_k \|y_k\|^2\right).$$

6. Augustin-Louis Cauchy, 1821

COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

PAR M. AUGUSTIN-LOUIS CAUCHY,

Ingénieur des Ponts et Chaussées, Professeur d'Analyse à l'École polytechnique, Membre de l'Académie des sciences, Chevalier de la Légion d'honneur.

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1821

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) = \left(\sum_{i=1}^{n} a_i b_i\right)^2 + \sum_{1\le i<j\le n} (a_i b_j - a_j b_i)^2.$$

(page 456, formula (31))

Poincaré's Inequality (1890): If μ is a probability measure on a space Ω and f and g are two real random variables belonging to the space $L^2(\mu)$:

$$\left(\int_\Omega f^2 d\mu\right)\left(\int_\Omega g^2 d\mu\right) - \left(\int_\Omega fg d\mu\right)^2$$
$$= \frac{1}{2}\int_\Omega\int_\Omega \left(f(x)g(y) - f(y)g(x)\right)^2 d\mu(x)d\mu(y).$$

Now consider smooth functions $f : [0,1] \to \mathbb{R}$ that verify the condition $\int_0^1 f dx = 0$. Then

$$\int_0^1 f^2 dx = \frac{1}{2}\int_0^1\int_0^1 \left(f(x) - f(y)\right)^2 dxdy.$$

By taking into account Lagrange's mean value theorem (with integral remainder) we infer that

$$f(x) = f(y) + (x-y)\int_0^1 f'(tx + (1-t)y)dt,$$

whence so one can easily conclude the existence of a positive constant k (that does not depend on f) such that

$$\int_0^1 f^2 dx \le \frac{1}{2}\int_0^1 f'^2(s)ds.$$

The uncertainty principle shows that one can not jointly localize a signal in time and frequency arbitrarily well; either one has poor frequency localization or poor time localization.

Suppose that $f(t)$ is a finite energy signal with Fourier transform $F(\omega)$. Let

$$E = \int_{\mathbb{R}} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} |F(\omega)|^2 \, d\omega,$$

$d^2 = \dfrac{1}{E} \displaystyle\int_{\mathbb{R}} t^2 \, |f(t)|^2 \, dt$ and $D^2 = \dfrac{1}{2\pi E} \displaystyle\int_{\mathbb{R}} \omega^2 \, |F(\omega)|^2 \, dt.$

If $\sqrt{|t|} f(t) \to 0$ as $|t| \to \infty$, then

$$Dd \geq \frac{1}{2},$$

and equality holds only if $f(t)$ has the form

$$f(t) = Ce^{-\alpha t^2}.$$

(From Hermann Weyl, *Theory of groups and Quantum Mechanics*, Dover, 1950)

Proof. Suppose that f is real. Notice that

$$\left|\int_{-\infty}^{\infty} tf(t)f'(t)dt\right|^2 \leq \int_{-\infty}^{\infty} t^2 f^2(t)dt \int_{-\infty}^{\infty} f'^2(t)dt$$

and

$$\int_{-\infty}^{\infty} tf(t)f'(t)dt = t\frac{f^2(t)}{2}\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{2}f^2(t)dt = -\frac{1}{2}E.$$

By Parseval's Theorem,

$$\int_{-\infty}^{\infty} f'^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 \left|F(\omega)\right|^2 d\omega.$$

Therefore

$$\frac{1}{4}E^2 \leq d^2E \times D^2E,$$

that is,

$$\frac{1}{2} \leq dD.$$

7. Mechanical Interpretation

1673 Christiaan Huygens: the parallel axis theorem,

(PAT) $$I_d = I_{com} + Md^2.$$

1765 Leonhard Euler, *Theoria motus corporum solidorum sev rigidorum*

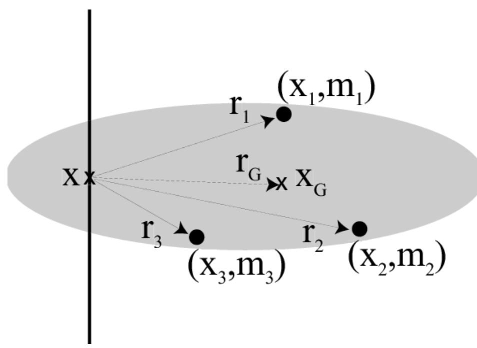


Figure 1. The Huygens-Steiner theorem

The identity provided by (PAT) is equivalent to (L):

$$\frac{1}{M}\sum_{k=1}^{n} m_k \|z - x_k\|^2$$
$$= ||z-\frac{1}{M}\sum_{k=1}^{n} m_k x_k||^2+\frac{1}{M}\sum_{k=1}^{n} m_k\|x_k-\frac{1}{M}\sum_{j=1}^{n} m_j x_j\|^2.$$

8. Weighted Least Squares

Given a family of points $x_1, ..., x_n$ *in* $\mathbb{R}^N$ *and real weights* $m_1, ..., m_n \in \mathbb{R}$ with $M = \sum_{k=1}^{n} m_k > 0$, *then*

$$\min_{x \in \mathbb{R}^N} \sum_{k=1}^{n} m_k \|x - x_k\|^2 = \frac{1}{M} \cdot \sum_{i<j} m_i m_j \|x_i - x_j\|^2.$$

The minimum is attained at one point,

$$x_G = \frac{1}{M} \sum_{k=1}^{n} m_k x_k.$$

Giulio Carlo Fagnano: the existence of a point P in the plane of a triangle ABC that minimizes the sum $PA^2 + PB^2 + PC^2$.

Carl Friedrich Gauss: the foundations of the least-squares analysis in 1795.

1809 Theory of motion of the celestial bodies moving in conic sections around the Sun.

Applications to Metric Geometry

a, b, c the side lengths of a triangle ΔABC

G the centroid

O the center of the circumscribed circle (radius R)

I the center of the inscribed circle (radius r).

Leibniz: formula for the radius of circumscribed circle,

$$R^2 = OG^2 + \frac{1}{9}(a^2 + b^2 + c^2).$$

W. Chapple (1746) and L. Euler (1765):

$$R^2 = OI^2 + 2Rr.$$

If R is the radius of the smallest ball containing a finite family of points $x_1, \ldots, x_n \in \mathbb{R}^N$, then

$$\frac{1}{n}\left(\sum_{1\le i<j\le n} \|x_i - x_j\|^2\right)^{1/2} \le R.$$

A GREAT discovery solves a great problem, but there is a grain of discovery in the solution of any problem. Your problem may be modest, but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.

Look around when you have got your first mushroom or made your first discovery: they grow in clusters.

George Pólya 

9. Understanding, Learning and Teaching Problem Solving

A set of heuristics from *How to Solve It*, Princeton, 1945:

1. Understand the Problem (What is the unknown? What are the data?)

2. Devise a Plan (Do you know a related problem? Did you use all the data?)

3. Carry out the Plan (Can you see/prove that each step is correct?)

4. Look Back at the Solution (Can you check the result? Can you use the result for some other problem?)

10. Positive Polynomials and SoS

Every nonnegative polynomial of a *single* variable can be expressed as a sum of squares (sos) of polynomials.

Basic idea:

$$c^2 \prod_{j=1}^{r} (t - t_j)^{2m_j} \prod_{k=1}^{s} (t - (\alpha_k + i\beta_k))\,(t - (\alpha_k - i\beta_k))$$

$$= Q^2(t) \prod_{k=1}^{s} ((t - \alpha_k)^2 + \beta_k^2) = R^2(t) + S^2(t),$$

via Fibonacci's identity.

The several variables case

Special cases when nonnegative polynomials are sums of squares

Hilbert (1888): quadratic polynomials in any number of variables; quartic polynomial in 2 variables.

A. Hurwitz (1891):

$$\frac{x_1^{2n}+x_2^{2n}+\cdots+x_n^{2n}}{n} - x_1^2 x_2^2 \cdots x_{2n}^2 = \text{sum of squares.}$$

Concrete examples:

$$\frac{x_1^2+x_2^2}{2} - x_1x_2 = \frac{(x_1-x_2)^2}{2}$$

$$\begin{aligned}&\frac{x_1^3+x_2^3+x_3^3}{3} - x_1x_2x_3 \\ &= \frac{(x_1+x_2+x_3)}{6}\Big[(x_1-x_2)^2+(x_1-x_3)^2+(x_2-x_3)^2\Big]\end{aligned}$$

$$\begin{aligned}&\frac{x_1^4+x_2^4+x_3^4+x_4^4}{4} - x_1x_2x_3x_4 \\ &\qquad = \frac{\left(x_1^2-x_2^2\right)^2+(x_3^2-x_4^2)^2}{4} + \frac{(x_1x_2-x_3x_4)^2}{2}.\end{aligned}$$

Other inequalities from identities:

P. E. Frenkel and P. Horvath, *Minkowski's inequality and sums of squares*, ArXiv 1206.5783v2 /4 January 2013

11. A Surprise

The positive polynomial

$$P(x,y) = 1 + x^4y^2 + x^2y^4 - 3x^2y^2$$

is *not* a sum of squares. Proof by reductio ad absurdum.

(1) $P(x,y) = Q_1^2(x,y) + Q_2^2(x,y) + \cdots + Q_n^2(x,y)$

For $y = 0$:

$$1 = P(x,0) = Q_1^2(x,0) + Q_2^2(x,0) + \cdots + Q_n^2(x,0)$$

For $x = 0$:

$$1 = P(0,y) = Q_1^2(0,y) + Q_2^2(0,y) + \cdots + Q_n^2(0,y)$$

Conclusion: Every $Q_k(x,y)$ has the form

$$Q_k(x,y) = a_k + b_kxy + c_kx^2y + d_kxy^2,$$

so by equating the coefficients of x^2y^2 in (1) we get $\sum b_k^2 = -3$, a contradiction.

12. Hilbert's Seventeenth Problem

Given a multivariate polynomial that takes only nonnegative values over the reals, can it be represented as a sum of squares of rational functions?

Yes, Emil Artin (1927)

Motzkin (1966) :

$$
\begin{aligned}
&1 + x^4y^2 + x^2y^4 - 3x^2y^2 \\
&\quad = \left(\frac{x^2y(x^2+y^2-2)}{x^2+y^2}\right)^2 + \left(\frac{xy^2(x^2+y^2-2)}{x^2+y^2}\right)^2 \\
&\qquad\qquad + \left(\frac{xy(x^2+y^2-2)}{x^2+y^2}\right)^2 + \left(\frac{x^2-y^2}{x^2+y^2}\right)^2.
\end{aligned}
$$

13. Two Open Problems

Is every inequality the consequence of an identity?

Find new identities and their associated inequalities.

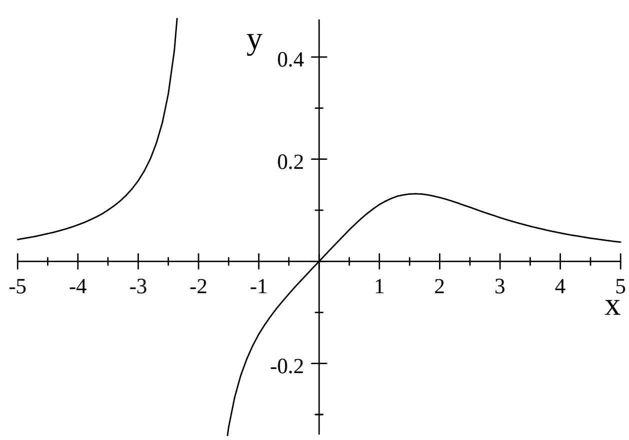
14. A Math. Olympiad Problem

Let $a, b, c, d \geq 0$ and $a+b+c+d=4$. Show

$$\sum \frac{a}{a^3+8} \leq \frac{4}{9}.$$

Partial solution. The function $f(x) = \frac{x}{x^3+8}$ is concave for $x \in [0,2]$. According to Jensen's inequality, for $a, b, c, d \in [0,2]$ and $a+b+c+d=4$, we have

$$\frac{1}{4}\left(\sum \frac{a}{a^3+8}\right) \leq \frac{\frac{a+b+c+d}{4}}{\left(\frac{a+b+c+d}{4}\right)^3+8} = \frac{1}{9}.$$



15. A First Generalization of Lagrange's Identity

I an interval of $\mathbb{R}$ endowed with a discrete measure

$$\mu = \sum_{i=1}^{n} p_i \delta_{x_i}$$

whose weights p_i are all nonzero and sum up to 1. The *barycenter* of μ,

$$b_\mu = \sum_{i=1}^{n} p_i x_i,$$

is supposed to be in $I \setminus \{x_1, ..., x_n\}$.

Theorem 1. (N&Stephan [**12**], [**13**]) *Under the above assumptions on I and μ, every function $f : I \longrightarrow \mathbb{R}$ verifies the following extension of Lagrange's identity,*

$$\sum_{i=1}^{n} p_i f(x_i) = f\left(b_\mu\right) + \sum_{1 \leq i < j \leq n} p_i p_j \left(s(x_i) - s(x_j)\right)\left(x_i - x_j\right)$$

where

$$s(x) = \frac{f(x) - f(b_\mu)}{x - b_\mu} \quad \textit{for } x \in I \setminus \{b_\mu\}.$$

Corollary 1. *If all p_i are nonnegative and $s(x)$ is decreasing, then*

$$\sum_{i=1}^{n} p_i f(x_i) \leq f\left(b_\mu\right).$$

Solution to the Math. Olympiad Problem: Apply Corollary for $n = 4$, and

$$x_1 = a < x_2 = b < x_3 = c < x_4 = d.$$

The barycenter of $\mu = \frac{1}{4}\sum_{i=1}^{4} \delta_{x_i}$ is $b_\mu = 1$.The slope function

$$s(x) = \frac{f(x) - f(1)}{x - 1} = \frac{8 - x - x^2}{9(8 + x^3)}$$

is decreasing, so by (SGL),

$$\frac{1}{4}\sum_{i=1}^{4} f(x_i) - f(1) = \frac{1}{16}\sum_{i<j} \frac{s(x_i) - s(x_j)}{x_i - x_j} \leq 0.$$

Polya's Fourth Principle: Review/extend

To gain more insight, extend you results to several variables!

16. The Several Variables Case

First Step: Adapt to several variables what we did in $\mathbb{R}$.

C a subset of the Euclidean space $\mathbb{R}^N$ endowed with a real measure $\mu = \sum_{i=1}^{n} p_i \delta_{x_i}$ whose weights p_i are all nonzero and sum up to 1. The *barycenter* of μ,

$$b_\mu = \sum_{i=1}^{n} p_i x_i,$$

is supposed to be in $C \setminus \{x_1, ..., x_n\}$.

Theorem 2. (N&Stephan [**12**], [**13**]) *Under the above assumptions on C and μ, every function $f : C \to \mathbb{R}$ verifies the following extension of Lagrange's identity*:

(GL)

$$\sum_{i=1}^{n} p_i f(x_i) = f(b_\mu) + \sum_{i<j} p_i p_j \left\langle s(x_i) - s(x_j), x_i - x_j \right\rangle,$$

where

$$s(x) = \frac{f(x) - f(b_\mu)}{\|x - b_\mu\|} \cdot \frac{x - b_\mu}{\|x - b_\mu\|} \quad \textit{for } x \in C \setminus \{b_\mu\}.$$

When f is a continuously differentiable function defined on a convex subset C of $\mathbb{R}^N$, one can state the identity (GL) in terms of gradients:

$$(SGL)\ \ \sum_{i=1}^{n} p_i f(x_i) = f\left(b_\mu\right)$$
$$+\sum_{i<j} p_i p_j \int_0^1 \langle \nabla f(P_i(t)) - \nabla f(P_j(t)), x_i - x_j\rangle dt,$$

where $P_i(t) = tx_i + (1-t)b_\mu$.

Note: (L) is the particular case where $f(x) = \frac{1}{2}\|x\|^2$, $x \in \mathbb{R}^N$. Indeed,

$$\nabla f(x) = x \quad \text{and} \quad \nabla^2 f(x) = I_n.$$

Corollary 2. *If f is continuously differentiable, then*

$$|\mathcal{E}(f;\mu) - f\left(b_\mu\right)| \leq \|\nabla f\|_{Lip}\ \sigma_\mu^2,$$

where $\mathcal{E}(f;\mu)$ is the expectation of f,

$$\|\nabla f\|_{Lip} = \sup\left\{\frac{\|\nabla f(x) - \nabla f(y)\|}{\|x-y\|} : x, y \in C,\ x \neq y\right\}$$

is the Lipschitz constant of ∇f, and σ_μ^2 is the variance of μ.

17. Two Examples

Example 1. Consider $f(x) = 1/x$ on an interval $[m, M]$ $(m > 0)$, and positive weights $p_1, ..., p_n$ that sum to unity. Theorem 1 relates the (weighted) arithmetic mean $A = \sum_{i=1}^{n} p_i x_i$ to the (weighted) harmonic mean $H = \left(\sum_{i=1}^{n} \frac{p_i}{x_i}\right)^{-1}$ as follows:

$$\frac{A}{H} - 1 = \sum_{1 \le i < j \le n} p_i p_j \frac{(x_i - x_j)^2}{x_i x_j}.$$

Bounds for the variance $\sigma_\mu^2 = \sum_{1 \le i < j \le n} p_i p_j (x_i - x_j)^2$ of $\mu = \sum_{i=1}^{n} p_i \delta_{x_i}$

$$m^2 \left(\frac{A}{H} - 1\right) \le \sigma_\mu^2 \le M^2 \left(\frac{A}{H} - 1\right).$$

Our theory also covers the upper bound of Bhatia and Davis [**2**],

$$\sigma_\mu^2 \le (M - A)(A - m).$$

Discrepancy between the harmonic mean and the arithmetic mean

$$1 + \frac{\sigma_\mu^2}{M^2} \le \frac{A}{H} \le 1 + \frac{\sigma_\mu^2}{m^2}.$$

Example 2. In the Euclidean space:

$$
\begin{aligned}
6\left(\|x_1\|^2+\|x_2\|^2+\|x_3\|^2\right)&+2\,\|x_1+x_2+x_3\|^2 \\
&=3\left(\|x_1+x_2\|^2+\|x_2+x_3\|^2+\|x_3+x_1\|^2\right) \\
&\qquad+\sum_{1\le i<j\le 3}\left\|x_i-x_j\right\|^2 .
\end{aligned}
$$

Apply twice (GL) as follows:

$$
\begin{aligned}
&\frac{\|x_1\|^2+\|x_2\|^2+\|x_3\|^2}{3}+\left\|\frac{x_1+x_2+x_3}{3}\right\|^2 \\
&\quad-\frac{2}{3}\left(\left\|\frac{x_1+x_2}{2}\right\|^2+\left\|\frac{x_2+x_3}{2}\right\|^2+\left\|\frac{x_3+x_1}{2}\right\|^2\right) \\
&\quad=\frac{\|x_1\|^2+\|x_2\|^2+\|x_3\|^2}{3}-\left\|\frac{x_1+x_2+x_3}{3}\right\|^2 \\
&\quad-\frac{2}{3}\left(\left\|\frac{x_1+x_2}{2}\right\|^2+\left\|\frac{x_2+x_3}{2}\right\|^2+\left\|\frac{x_3+x_1}{2}\right\|^2\right) \\
&\qquad+\left\|\frac{x_1+x_2+x_3}{3}\right\|^2 \\
&\qquad\qquad=\frac{1}{18}\sum_{1\le i<j\le 3}\left\|x_i-x_j\right\|^2 .
\end{aligned}
$$

A consequence is the inequality

$$\frac{\|x_1\|^2+\|x_2\|^2+\|x_3\|^2}{3}+\left\|\frac{x_1+x_2+x_3}{3}\right\|^2$$
$$\geq \frac{2}{3}\left(\left\|\frac{x_1+x_2}{2}\right\|^2+\left\|\frac{x_2+x_3}{2}\right\|^2+\left\|\frac{x_3+x_1}{2}\right\|^2\right),$$

which illustrates the phenomenon of $(2D)$-*convexity* as developed by M. Bencze, C. P. Niculescu and Florin Popovici [**1**].

18. A Second Generalization

Embedding Jensen's Inequality into an identity:

Theorem 3. (N&Stephan [**13**]). *Suppose that K is a Borel measurable convex subset of $\mathbb{R}^N$* (*or more generally of a real Hilbert space*)*, endowed with a real Borel measure μ such that $\mu(K) = 1$ and $b_\mu \in K$. Then for every function $f : K \to \mathbb{R}$ of class C^2 we have the identity*

$$f(b_\mu) + \frac{1}{2}\int_K \int_K \langle \nabla f(x) - \nabla f(y), x - y\rangle d\mu(x) d\mu(y)$$

$$= \int_K f(x) d\mu(x)$$

$$+ \int_K \int_0^1 (1-t) \langle \nabla^2 f(x + t(b_\mu - x)) (b_\mu - x), b_\mu - x\rangle dt d\mu(x),$$

provided that all integrals are legitimate.

In the particular case where $f(x) = \frac{1}{2}\|x\|^2$, $x \in \mathbb{R}^N$, we recover the identity (L).

Theorem 4. (N&Stephan [**13**]). *Suppose that K is a compact convex subset of the Euclidean space $\mathbb{R}^N$, endowed with a Borel probability measure μ, and f is a convex function of class C^1, defined on a neighborhood of K. Then*

$$\frac{1}{2}\int_K\int_K \langle \nabla f(x) - \nabla f(y), x - y\rangle d\mu(x)d\mu(y)$$
$$\geq \int_U f(x)d\mu(x) - f(b_\mu) \geq 0.$$

This provides a converse for each instance of Jensen's inequality.

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Thank You!