Popoviciu's Inequality, Fifty Years Later

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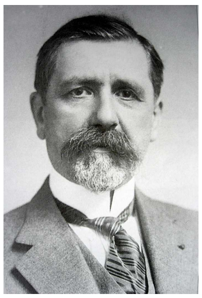
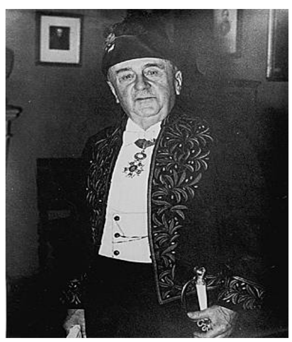
Tiberiu Popoviciu (1906-1975)

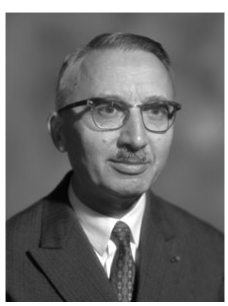
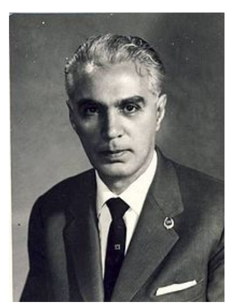
Ph. D.: *Sur quelques propriétés des fonctions d'une ou de deux variables réelles,* Université Paris IV-Sorbonne 1933 (under the supervision of Paul Montel). Following E. Hopf, he developed the theory of n-convex functions.

Montel himself tackled with the algebraic variants of convexity in *Sur les functions convexes et les fonctions sousharmoniques*, Journal de Math. (**9**), **7** (1928), 29-60.

A Branch of the Mathematics Genealogy Tree of Paul Montel

Borel and LebesgueSupervisors

 Paul Montel

Three other students of Montel: Henri Cartan, Jean Dieudonné and Miron Nicolescu

1. Popoviciu's Inequality [29]

If $f : I \to \mathbb{R}$ *is a continuous convex function, then*

$$\sum_{1 \le i_1 < \cdots < i_p \le n} (\lambda_{i_1} + \cdots + \lambda_{i_p})\, f\left(\frac{\lambda_{i_1} x_{i_1} + \cdots + \lambda_{i_p} x_{i_p}}{\lambda_{i_1} + \cdots + \lambda_{i_p}}\right)$$
$$\le \binom{n-2}{p-2} \left[\frac{n-p}{p-1} \sum_{i=1}^{n} \lambda_i\, f(x_i) + \left(\sum_{i=1}^{n} \lambda_i\right) f\left(\frac{\lambda_1 x_1 + \cdots + \lambda_n x_n}{\lambda_1 + \cdots + \lambda_n}\right)\right].$$

Here $x_1, ..., x_n \in I$, $n \ge 3$, $p \in \{2, ..., n-1\}$, and $\lambda_1, ..., \lambda_n$ are positive numbers (representing weights).

Basic case: *Let* $f : I \to \mathbb{R}$ *be a continuous function. Then* f *is convex if, and only if,*

$$\frac{f(x_1) + f(x_2) + f(x_3)}{3} + f\left(\frac{x_1 + x_2 + x_3}{3}\right)$$
$$\ge \frac{2}{3}\left[f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right) + f\left(\frac{x_3 + x_1}{2}\right)\right]$$

for all $x_1, x_2, x_3 \in I$.

Proof (of the basic case). We may assume the ordering $x \le y \le z$. If $y \le (x+y+z)/3$, then $(x+y+z)/3 \le (x+z)/2 \le z$ and $(x+y+z)/3 \le (y+z)/2 \le z$. This yields $s, t \in [0,1]$ such that

$$\frac{x+z}{2} = s \cdot \frac{x+y+z}{3} + (1-s) \cdot z$$
$$\frac{y+z}{2} = t \cdot \frac{x+y+z}{3} + (1-t) \cdot z,$$

whence $(x+y-2z)(s+t-3/2) = 0$.

If $s + t = 3/2$, sum up the inequalities

$$f\left(\frac{x+z}{2}\right) \le s \cdot f\left(\frac{x+y+z}{3}\right) + (1-s) \cdot f(z)$$
$$f\left(\frac{y+z}{2}\right) \le t \cdot f\left(\frac{x+y+z}{3}\right) + (1-t) \cdot f(z)$$
$$f\left(\frac{x+y}{2}\right) \le \frac{1}{2} \cdot f(x) + \frac{1}{2} \cdot f(y).$$

to conclude that

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)$$
$$\le \frac{1}{2} \cdot (f(x) + f(y) + f(z)) + \frac{3}{2} \cdot f\left(\frac{x+y+z}{3}\right). \blacksquare$$

General case by mathematical induction.

Other form of the basic case:

$$\frac{f(x_1)+f(x_2)+f(x_3)}{3} - f\left(\frac{x_1+x_2+x_3}{3}\right)$$

$$\geq 2\left[\frac{f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_3+x_1}{2}\right)}{3} - f\left(\frac{x_1+x_2+x_3}{3}\right)\right].$$

Second proof (the argument of Popoviciu)

By an approximation argument we may reduce to the case of piecewise linear convex functions. Their general form is

$$f(x) = \alpha x + \beta + \sum_{k=1}^{m-1} c_k |x - x_k|,$$

so the proof ends by taking into account Hlawka's inequality:

$$|x|+|y|+|z|+|x+y+z| \geq |x+y|+|y+z|+|z+x|$$

Third proof (using majorization)

Theorem 1. (*The Hardy–Littlewood–Pólya inequality*) *Suppose that f is a convex function on an interval I and consider two families $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of points in I such that*

$$\sum_{k=1}^{m} x_k \leq \sum_{k=1}^{m} y_k \quad \text{for } m \in \{1, \ldots, n\}$$

and

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k.$$

If $x_1 \geq \cdots \geq x_n$, then

$$\sum_{k=1}^{n} f(x_k) \leq \sum_{k=1}^{n} f(y_k),$$

while if $y_1 \leq \cdots \leq y_n$ this inequality works in the reverse direction.

Proof (of the basic case of Popoviciu's inequality) Consider the ordering $x \geq y \geq z$. Then

$$\frac{x+y}{2} \geq \frac{z+x}{2} \geq \frac{y+z}{2}$$

and $\quad x \geq (x+y+z)/3 \geq z.$

Case 1: $x \geq (x+y+z)/3 \geq y \geq z.$ Consider the families

$$x_1 = x, \quad x_2 = x_3 = x_4 = \frac{x+y+z}{3}, \quad x_5 = y, \quad x_6 = z,$$
$$y_1 = y_2 = \frac{x+y}{2}, \quad y_3 = y_4 = \frac{x+z}{2}, \quad y_5 = y_6 = \frac{y+z}{2}$$

Case 2: $x \geq y \geq (x+y+z)/3 \geq z.$ Consider the families

$$x_1 = x, \quad x_2 = y, \quad x_3 = x_4 = x_5 = \frac{x+y+z}{3}, \quad x_6 = z,$$
$$y_1 = y_2 = \frac{x+y}{2}, \; y_3 = y_4 = \frac{x+z}{2}, \quad y_5 = y_6 = \frac{y+z}{2}. \; \blacksquare$$

2. Simple Remarks

Cf. Mihai and Mitroi-Symeonidis [**15**].

Proposition 1. *Suppose that* $f \in C^2([a,b])$ *and put*

$$m = \inf\left\{f''(x) : x \in [a,b]\right\}$$
$$M = \sup\left\{f''(x) : x \in [a,b]\right\}.$$

Then

$$\frac{M}{36}\left((x-y)^2 + (y-z)^2 + (z-x)^2\right) \geq \frac{f(x)+f(y)+f(z)}{3} + f\left(\frac{x+y+z}{3}\right) - \frac{2}{3}\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right) \geq \frac{m}{36}\left((x-y)^2 + (y-z)^2 + (z-x)^2\right)$$

for all $x, y, z \in [a,b]$.

Indeed, both functions $\frac{M}{2}x^2 - f(x)$ and $f(x) - \frac{m}{2}x^2$ are convex.

The variant for uniformly convex functions (the functions f such that $f - \frac{C}{2}x^2$ is convex for suitable $C > 0$) :

$$\frac{f(x)+f(y)+f(z)}{3} + f\left(\frac{x+y+z}{3}\right)$$

$$- \frac{C}{36}\left((x-y)^2 + (y-z)^2 + (z-x)^2\right)$$

$$\geq \frac{2}{3}\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right).$$

Since $e^x \geq \frac{1}{2}x^2$ for $x \geq 0$, this fact yields the inequality

$$\frac{a+b+c}{3} + \sqrt[3]{abc} - \frac{2}{3}\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right)$$
$$\geq \frac{1}{36}\left(\log^2 \frac{a}{b} + \log^2 \frac{a}{b} + \log^2 \frac{a}{b}\right),$$

for all $a, b, c \geq 1$.

Popoviciu's inequality also works in the case of convex functions with values in a regularly ordered Banach space (in the sense of Davis [**7**]).

3. Convexity according to means [15]

Quasi-arithmetic mean (associated to a strictly monotonic function φ) :

$$\mathfrak{M}_{\varphi}(x_1, ..., x_n; \lambda_1, ..., \lambda_n) = \varphi^{-1}\left(\sum_{k=1}^{n} \lambda_k \varphi(x_k)\right).$$

Other means:

$$L(a, b) = \frac{a-b}{\ln a - \ln b} \qquad \text{(logarithmic mean)}$$

$$I(a, b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \qquad \text{(identric mean).}$$

Definition 1. *(G. Aumann [**3**]). Given two means M and N defined respectively on the intervals I and J, a function $f : I \to J$ is* (M, N)-convex *if it is continuous on I and*

$$f(M(x_1, \ldots, x_n)) \le N(f(x_1), \ldots, f(x_n))$$

See Montel [**16**].

Theorem 2. *Suppose that* $f : I \to J$ *is* $\left(\mathfrak{M}_{\varphi}, \mathfrak{M}_{\psi}\right)$*-convex. If* ψ *strictly increasing,*

$$\begin{aligned}&\mathfrak{M}_{\psi}\left(\mathfrak{M}_{\psi}\left(f(x), f(y), f(z)\right), f\left(\mathfrak{M}_{\varphi}\left(x, y, z\right)\right)\right)\\ &\geq \mathfrak{M}_{\psi}\left(f\left(\mathfrak{M}_{\varphi}\left(x, y\right)\right), f\left(\mathfrak{M}_{\varphi}\left(y, z\right)\right), f\left(\mathfrak{M}_{\varphi}\left(z, x\right)\right)\right)\end{aligned}$$

for all $x, y, z \in I$*. The inequality works in the reverse way when* ψ *is strictly decreasing.*

A counterexample. The log-convex functions are also (A, L)-convex, because

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\\ &\leq \exp\left(\frac{1}{b-a}\int_a^b \log\, f(x)\,\mathrm{d}x\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\\ &\qquad\qquad \leq L(f(a), f(b)) \leq \frac{f(a)+f(b)}{2}.\end{aligned}$$

Logarithmic mean for triplets

$$L(a,b,c) = \frac{2a}{\log\frac{a}{b}\log\frac{a}{c}} + \frac{2b}{\log\frac{b}{a}\log\frac{b}{c}} + \frac{2c}{\log\frac{c}{a}\log\frac{c}{b}}.$$

Thus the analogue of Popoviciu's inequality in the case of (A, L)-convex functions should be

$$\frac{L\left(f(x), f(y), f(z)\right) - f\left(\frac{x+y+z}{3}\right)}{\log L\left(f(x), f(y), f(z)\right) - \log f\left(\frac{x+y+z}{3}\right)} \geq L\left(f\left(\frac{x+y}{2}\right), f\left(\frac{y+z}{2}\right), f\left(\frac{z+x}{2}\right)\right),$$

This does not work in the case of Gamma function.

Consider the points $1.40; 1.46; 1.47$

4. The case of h-convex functions

I is an interval and $h : (0,1) \to (0,\infty)$ is a function such that

$$(h1) \qquad h(1-\lambda) + h(\lambda) \geq 1 \text{ for all } \lambda \in (0,1).$$

Definition 2. *(Varošanec* [**30**]*) A function $f : I \to \mathbb{R}$ is called h-convex if*

$$f\left((1-\lambda)x + \lambda y\right) \leq h(1-\lambda)f(x) + h(\lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in (0,1)$.

The role of the condition $(h1)$ is to assure that the function identically 1 is h-convex.

Special cases:

$i)$ usual convex functions (when h is the identity function).

$ii)$ s-*convex functions* in the sense of Breckner [**6**] ($h(\lambda) = \lambda^s$ for some $s \in (0,1]$); see Hudzik and Maligranda [**12**] and Pinheiro [**24**]. An example of an s-convex function (for $0 < s < 1$):

$$f(t) = \begin{cases} a & \text{if } t = 0 \\ bt^s + c & \text{if } t > 0 \end{cases}$$

where $b \geq 0$ and $0 \leq c \leq a$. In particular, the function t^s is s-convex on $[0, \infty)$ if $0 < s < 1$.

iii) the *convex functions in the sense of Godunova-Levin* [**9**] (when $h(\lambda) = \frac{1}{\lambda}$). They verify the inequality

$$f\left((1-\lambda)\, x + \lambda y\right) \leq \frac{f(x)}{1-\lambda} + \frac{f(y)}{\lambda}$$

for all $x, y \in I$ and $\lambda \in (0,1)$. Every nonnegative monotonic function (as well as every nonnegative convex function) is convex in the sense of Godunova-Levin.

iv) The h-convex functions corresponding to the case $h(\lambda) \equiv 1$ are the P-*convex functions* in the sense of Dragomir, Pečarić and Persson [**8**]. They verify inequalities of the form

$$f\left((1-\lambda)\, x + \lambda y\right) \leq f(x) + f(y)$$

for all $x, y \in I$ and $\lambda \in (0,1)$.

Theorem 3. (*Mihai and Mitroi-Symeonidis* [**15**]) *If h is concave, then every positive h-convex function f verifies the inequality*

$$\max\left\{h\left(\frac{1}{2}\right), 2h(1/4)\right\}(f(x)+f(y)+f(z)) \\ + 2h(3/4)f\left(\frac{x+y+z}{3}\right) \\ \geq f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)$$

for all $x, y, z \in I$.

Application to the case $f(t) = t^{1/2}$ (which is s-convex for $s = 1/2$). In this case $h(t) = t^{1/2}$, $2h(3/4) = \sqrt{3}$ and $\max\left\{h\left(\frac{1}{2}\right), 2h(1/4)\right\} = 1$. Therefore

$$x^{1/2}+y^{1/2}+z^{1/2}+\sqrt{3}\left(\frac{x+y+z}{3}\right)^{1/2} \\ \geq \left(\frac{x+y}{2}\right)^{1/2}+\left(\frac{y+z}{2}\right)^{1/2}+\left(\frac{z+x}{2}\right)^{1/2}$$

for all $x, y, z \geq 0$.

5. An Elementary Problem

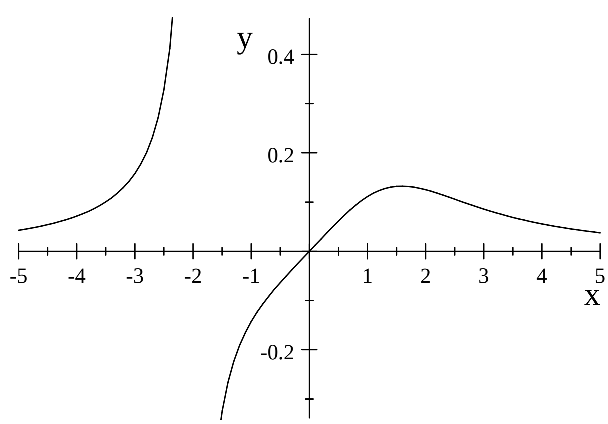
Let $a, b, c, d \geq 0$ and $a+b+c+d=4$. Show that

$$\sum \frac{a}{a^3+8} \leq \frac{4}{9}.$$

Partial solution: $f(x) = \frac{x}{x^3+8}$ is concave for $x \in [0,2]$. By Jensen's inequality,

$$\frac{1}{4}\left(\sum \frac{a}{a^3+8}\right) \leq \frac{\frac{a+b+c+d}{4}}{\left(\frac{a+b+c+d}{4}\right)^3+8} = \frac{1}{9}.$$

for all $a, b, c, d \in [0,2]$ with $a+b+c+d=4$. The general case is covered by the theory of relative convexity [**20**].



6. Convexity Relative to a Subset

f is a real-valued continuous function defined on the compact convex set K.

Definition 3. (Niculescu and Roventa [**20**]) *A point a in K is a point of convexity of the function f if there exists an infinite convex subset V of K (called neighborhood of convexity) such that $a \in V$ and*

$$(J) \qquad f(a) \leq \sum_{k=1}^{n} \lambda_k f(x_k),$$

for every family of points $x_1, ..., x_n$ in V and every family of positive weights $\lambda_1, ..., \lambda_n$ with $\sum_{k=1}^{n} \lambda_k = 1$ and $\sum_{k=1}^{n} \lambda_k x_k = a$.

If a is a point of convexity of f relative to the compact convex neighborhood V, then

$$f(a) \leq \int_V f(x) d\mu(x)$$

for every Borel probability measure μ on V whose barycenter is a.

Theorem 4. (Niculescu and Roventa [**20**]) *Suppose that f is a real-valued function defined on an interval I. If a, b, c belong to I and $\frac{a+b}{2}, \frac{a+c}{2}$ and $\frac{b+c}{2}$ are points of convexity of f relative to the entire interval I, then*

$$\frac{f(a)+f(b)+f(c)}{3}+f\left(\frac{a+b+c}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{a+b}{2}\right)+f\left(\frac{a+c}{2}\right)+f\left(\frac{b+c}{2}\right)\right].$$

The proof is done via an extension of Hardy-Littlewood-Pólya inequality of majorization.

7. The integral version

The only known result:

Theorem 5. (Niculescu [**19**]) *For every convex function f on* $[a, b]$,

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)dx + \frac{3}{16}f\left(\frac{5a+3b}{8}\right) \\
&\quad + \frac{3}{16}f\left(\frac{3a+5b}{8}\right) - \frac{1}{b-a}\int_{(5a+3b)/8}^{(3a+5b)/8} f(x)dx \\
&\qquad \geq \frac{4}{(b-a)^2}\int_a^{(5a+3b)/8}\int_a^x f\left(\frac{x+t}{2}\right)dtdx \\
&+ \frac{4}{(b-a)^2}\int_{(5a+3b)/8}^{(a+3b)/4}\int_a^{(5a+3b)/4-x} f\left(\frac{x+t}{2}\right)dtdx \\
&+ \frac{4}{(b-a)^2}\int_{(3a+5b)/8}^{b}\int_{(3a+5b)/4-x}^{x} f\left(\frac{x+t}{2}\right)dtdx.
\end{aligned}$$

The proof is based on Popoviciu's idea to consider first the case of piecewise linear convex functions.

8. Some open problems

- The real significance of Popoviciu's inequality (related to the concentration of points with the same barycenter) is still unclear, though some facts suggests to look at generalized entropies.

- Popoviciu's inequality in the case of an arbitrary Borel positive measure is still open.

- Popoviciu's inequality does **not** work in the general setting of convex functions of several variables (because it implies a stronger concept of convexity, called 2D-convexity). See the paper of Bencze, Niculescu and Popovici [**4**] from 2010.

  However, the theory of 2D-convex functions is still in infancy and need considerable more attention.

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