A SHORT PROOF OF BURNSIDE’S FORMULA FOR THE GAMMA FUNCTION

CONSTANTIN P. NICULESCU\textsuperscript{1*} AND FLORIN POPOVICI\textsuperscript{2}

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ABSTRACT. We present simple proofs for Burnside’s asymptotic formula and for its extension to positive real numbers.

1. Introduction

Burnside’s asymptotic formula for factorial $n$ asserts that

$$n! \sim \sqrt{2\pi} \left(\frac{n + 1/2}{e}\right)^{n+1/2}, \quad (B)$$

in the sense that the ratio of the two sides tends to 1 as $n \to \infty$. This provides a more efficient estimation of the factorial, comparing to Stirling’s formula,

$$n! \sim \sqrt{2\pi n} n^{n+1/2} e^{-n}. \quad (S)$$

Indeed, for $n = 100$, the exact value of 100! with 24 digits is

$$9.332621544394415268169924 \times 10^{157}.$$  

Burnside’s formula yields the approximation

$$100! \approx 9.336491570312414838264959 \times 10^{157},$$

while Stirling’s formula is less precise, offering only the approximation

$$100! \approx 9.324847625269343247764756 \times 10^{157}.$$
The aim of the present paper is to present a short (and elementary) proof of Burnside’s asymptotic formula and to extend it to positive real numbers. The main ingredients are Wallis’ product formula for $\pi$ and the property of log-convexity of the Gamma function.

2. The proof of Burnside’s formula

The starting point is the following result concerning the monotonicity of the function $(1 + \frac{1}{x})^{x+\alpha}$ on the interval $[1, \infty)$.

Lemma 2.1. (I. Schur [6], Problem 168, page 38). Let $\alpha \in \mathbb{R}$. The sequence $a_\alpha(n) = (1 + \frac{1}{n})^{n+\alpha} \in (0, 1/2)$ arbitrarily fixed, there is a positive integer $N(\alpha)$ such that

$$e < \left(1 + \frac{1}{k}\right)^{k+\alpha}$$

for all $k \geq N(\alpha)$. As a consequence,

$$\prod_{k=n}^{2n} \left(1 + \frac{1}{k}\right)^{k+\alpha} < e^{n+1} < \prod_{k=n}^{2n} \left(1 + \frac{1}{k}\right)^{k+\alpha},$$

for all $n \geq N(\alpha)$, equivalently,

$$\frac{(2n+1)^{2n+\alpha}}{n^{n+\alpha}} \cdot \frac{1}{(n+1) \cdots (2n)} < e^{n+1} < \frac{(2n+1)^{2n+1/2}}{n^{n+1/2}} \cdot \frac{1}{(n+1) \cdots (2n)}.$$}

This can be restated as

$$\frac{2^{2n+\alpha} (n + \frac{1}{2})^{n+1/2} (1 + \frac{1}{2n})^{n+\alpha}}{\sqrt{n + \frac{1}{2}}} \cdot \frac{n!}{(2n)!} < e^{n+1}$$

and

$$< \frac{2^{2n+1/2} (n + \frac{1}{2})^{n+1/2} (1 + \frac{1}{2n})^{n+1/2}}{\sqrt{n + \frac{1}{2}}} \cdot \frac{n!}{(2n)!},$$

whence

$$\frac{1}{\sqrt{2n+1}} \cdot \frac{(2n)!!}{(2n-1)!!} \cdot \frac{2^{n+1/2} (1 + \frac{1}{2n})^{n+\alpha}}{\sqrt{e}} < n! \left(\frac{e}{n + \frac{1}{2}}\right)^{n+1/2}$$

$$< \frac{1}{\sqrt{2n+1}} \cdot \frac{(2n)!!}{(2n-1)!!} \cdot \frac{2 (1 + \frac{1}{2n})^{n+1/2}}{\sqrt{e}}$$

for all $n \geq N(\alpha)$. Here $n!! = n \cdot (n-2) \cdots 4 \cdot 2$ if $n$ is even, and $n \cdot (n-2) \cdots 3 \cdot 1$ if $n$ is odd.
Taking into account Wallis’s formula,

\[
\lim_{n \to \infty} \frac{2 \cdot 2 \cdot 4 \cdot \cdots (2n) \cdot (2n)}{1 \cdot 3 \cdot 5 \cdot 5 \cdot \cdots (2n - 1) \cdot (2n - 1) \cdot (2n + 1)} = \frac{\pi}{2},
\]

that is,

\[
\lim_{n \to \infty} \frac{1}{\sqrt{2n + 1}} \cdot \frac{(2n)!!}{(2n - 1)!!} = \sqrt{\frac{\pi}{2}},
\]

we arrive easily at Burnside’s formula for factorial \( n \):

\[
n! \sim \sqrt{2\pi} \left( \frac{n + 1/2}{e} \right)^{n+1/2}.
\]

3. The extension of Burnside’s formula for the Gamma function

Our next goal is to derive from Burnside’s formula the following asymptotic formula for the Gamma function:

**Theorem 3.1.** (R. J. Wilton [7]). \( \Gamma (x + 1) \sim \sqrt{2\pi} \left( \frac{x+1/2}{e} \right)^{x+1/2} \) as \( x \to \infty \).

The proof of the above theorem will be done by estimating the function

\[
f(x) = \Gamma (x + 1) \left( \frac{e}{x + 1/2} \right)^{x+1/2},
\]

for large values of \( x \). We shall need the following double inequality:

**Lemma 3.2.** \( [x]! \cdot x^{\{x\}} \leq \Gamma (x + 1) \leq [x]! \cdot (\lfloor x \rfloor + 1)^{\{x\}} \) for all \( x \geq 1 \).

Here \( [x] \) denotes the largest integer less than or equal to \( x \) and \( \{x\} = x - [x] \).

**Proof.** Our argument is based on the property of log-convexity of the Gamma function:

\[
\Gamma((1 - \lambda)x + \lambda y) \leq \Gamma(x)^{1-\lambda} \Gamma(y)^{\lambda},
\]

for all \( x, y > 0 \) and \( \lambda \in [0, 1] \). See [5], Theorem 2.2.4, pp. 69-70.

If \( x \) is a positive number, then \( [x] + 1 \leq x + 1 < [x] + 2 \), which yields

\[
x + 1 = (1 - \{x\}) ([x] + 1) + \{x\} ([x] + 2).
\]

Therefore,

\[
\Gamma(x + 1) \leq \Gamma([x] + 1)^{1-\{x\}} \Gamma([x] + 2)^{\{x\}}
\]

\[
= [x]^{1-\{x\}} ([x] + 1)^{\{x\}}
\]

\[
\leq [x]! ([x] + 1)^{x-[x]}.
\]

In a similar way, taking into account that \( [x] + 1 = \{x\} x + (1 - \{x\}) (x + 1) \),

we obtain

\[
[x]! = \Gamma([x] + 1) \leq \Gamma(x)^{\{x\}} \Gamma(x + 1)^{1-\{x\}} = \frac{\Gamma(x + 1)}{x^{x-[x]}},
\]

whence \( [x]! x^{x-[x]} \leq \Gamma(x + 1) \). The proof is done. \( \square \)
According to Lemma 2,

\[ f(x) \geq [x]! \left\{ x \right\} \cdot \frac{e^{x+1/2}}{(x+1/2)^{x+1/2}} \]

\[ = \Gamma([x]+1) \cdot \frac{e^{[x]+1/2}}{([x]+1/2)^{[x]+1/2}} \cdot \frac{e^{\{x\}}(x+1/2)^{x+1/2}}{(x+1/2)^{x+1/2}} \]

\[ = f([x]) \cdot \left( \frac{[x]+1/2}{x+1/2} \right)^{[x]+1/2} \cdot e^{\{x\}} \cdot \left( \frac{x}{x+1/2} \right)^{\{x\}} \]

\[ \geq f([x]) \cdot \left( \frac{x}{x+1/2} \right)^{\{x\}}. \quad (LW) \]

Similarly,

\[ f(x) = \Gamma(x+1) \left( \frac{e}{x+1/2} \right)^{x+1/2} \]

\[ \leq [x]! \left( \frac{e}{x+1/2} \right)^{x+1/2} (x+1)^{\{x\}} \]

\[ = f([x]) \left( \frac{[x]+1/2}{x+1/2} \right)^{[x]+1/2} \left( \frac{[x]+1}{x+1/2} \right)^{\{x\}} e^{\{x\}} \]

\[ = f([x]) \left( \frac{e}{1 + \frac{\{x\}}{[x]+1/2}} \right)^{\{x\}} \left( \frac{[x]+1}{x+1/2} \right)^{\{x\}}. \quad (RW) \]

The formulas (LW) and (RW) show that

\[ \lim_{x \to \infty} f(x) = \lim_{n \to \infty} f(n), \]

and this fact combined with Burnside’s formula (B) allows us to conclude that the limit of $f$ at infinity is $\sqrt{2\pi}$, that is,

\[ \lim_{x \to \infty} \Gamma(x+1) \left( \frac{e}{x+1/2} \right)^{x+1/2} = \sqrt{2\pi}. \]

This ends the proof of Wilton’s asymptotic formula.

It seems very likely that the above technique can be adapted to cover more accurate asymptotic formulas such as that of Gosper [4],

\[ n! \sim \sqrt{2\pi \left( n + \frac{1}{6} \right)} \left( \frac{n}{e} \right)^n, \]

and of its extension to real numbers. This is also supported by our joint paper with D. E. Dutkay [3].
Additional information concerning the approximation of the Gamma function may be found in the recent paper of G. D. Anderson, M. Vuorinen and X. Zhang [1].

REFERENCES

7. J.R. Wilton, *A proof of Burnside’s formula for log Γ(x + 1) and certain allied properties of the Riemann ζ-function*, Messenger Math. 52 (1922), 90–93.

1 The Academy of Romanian Scientists, Splaiul Independentei No. 54, Bucharest, RO-050094 Romania.
E-mail address: cpniculescu@gmail.com

2 College Grigore Moisil, Brasov, Romania.
E-mail address: popovici.florin@yahoo.com