Birkhoff recurrence theorem and combinatorial properties of abelian semigroups

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ABSTRACT. A well known result due to van der Waerden asserts that given a finite partition of $\mathbf{N}$, one of the subsets contains arbitrarily long finite arithmetic progressions. We shall show that actually all abelian semigroups play a similar combinatorial property. Our approach depends on techniques from topological dynamics and it was inspired by a paper due to Furstenberg [1].

1. INTRODUCTION

In what follows $\mathcal{A}$ will denote an abelian semigroup endowed with a law of multiplication $* : \sum \times \mathcal{A} \to \mathcal{A}$ with elements of a countable semigroup $\sum$, such that, denoting both semigroups additively, the following two conditions are fulfilled:

$$\begin{aligned} 0 * x &= 0 \\ (\alpha + \beta) * x &= \alpha * x + \beta * x. \end{aligned}$$

The simplest example of such a structure on every semigroup $\mathcal{A}$ is the natural multiplication with elements of $\mathbf{N}$,

$$\begin{aligned} 0 * x &= x \\ n * x &= \underbrace{x + ... + x}_{n \text{ times}} \text{ if } n > 0. \end{aligned}$$

At the other extreme there are exotic examples such as $\sum = \prod_{n=1}^{\infty} GL(n, \mathbf{Z})$, $\mathcal{A} = (\mathbf{C}^\star)^{\mathbf{N}^\star}$ and

$$(\alpha * x)(n) = (\det \alpha(n))^{x(n)} \text{ , for } n > 0.$$

We shall call the ordered finite subsets of $\mathcal{A}$ *configurations* in $\mathcal{A}$. A configuration $Q = \{y_1, ..., y_n\}$ *mimics* (modulo $*$) the configuration $P = \{x_1, ..., x_n\}$ provided there exist $\alpha \in \sum \setminus \{0\}$ and $z \in \mathcal{A}$ such that

$$y_k = z + \alpha * x_k \text{ for all k.}$$

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Notice that mimicry is not in general an equivalency relation and could be trivial in the presence of torsion.

If P is an arithmetic progression and Q mimics P, then Q is also an arithmetic progression. Due to that fact, the following theorem extends the van der Waerden aforementioned result:

Theorem 1. *Suppose that both $\mathcal{A}$ and $\sum$ are groups. Then for every configuration P in $\mathcal{A}$ and every finite partition*

$$\mathcal{A} = C_1 \cup ... \cup C_r$$

one of the sets C_k contains a configuration Q that mimics P.

Moreover, the semigroup version is valid provided $\sum$ and $\mathcal{A}$ are both commutative.

Theorem l has been announced (for $*$ the natural multiplication with elements of $\mathbf{N}$) at *The XVth Operator Theory Conference,* Timisoara, June 6 - 10, 1994. See [4], which includes also the remark that Theorem l can fail without the assumption on commutativity of $\mathcal{A}$. It is perhaps worthwhile to mention that the above result depends heavily on the concept of mimicry we work with. For example, α cannot be eliminated by considering only translates. See the two-pieces partition

$$\left(\bigcup_{n=-\infty}^{\infty} \{ z \mid 2n \leq \text{Re } z < 2n+1\} \right) \bigcup \left(\bigcup_{n=-\infty}^{\infty} \{ z \mid 2n+1 \leq \text{ Re } z < 2n+2\} \right)$$

of $\mathbf{C}$ and the configuration $P = \{0,\ 1,\ \mathrm{i},\ 1+\mathrm{i}\}$.

The proof of Theorem l relies on an extension of Birkhoff recurrence theorem, which in the discrete case was first noticed by Furstenberg and Weiss [2] (see also [1]):

Theorem 2. *Let $\Phi_1, \ldots, \Phi_N : G \times X \to X$ be pairwise commuting continuous actions of the recurrent group G on a compact metric space X. Then there exist points x in X and sequences $(s_n)_n$ in $\mathcal{S}$ such that $s_n \to \infty$ and*

$$\Phi_1(s_n,\ x) \to x\ ,\ .\ .\ .\ ,\ \Phi_N(s_n,\ x) \to x$$

simultaneously as $n \to \infty$.

The precise meaning of the term *recurrent group* will be given in the next section. We shall only mention here that all locally compact noncompact abelian groups as well as all discrete countable groups are recurrent. Theorem 2 will be proved by induction, the initial step being a particular case of Poincaré recurrence theorem.

For the sake of completeness, the basic facts on Poincaré recurrence (for actions of locally compact groups) will be detailed in the next section.

It is not very clear wheather Theorem 2 can be in turn derived from Theorem l. However, the later is strong enough to motivate significant cases of recurrence such as the translation on the $n-$ dimensional torus $\mathbf{T}^n$. See section 5 for details.

2. Review on Poincaré recurrence

In what follows X will denote a locally compact Hausdorff space on which acts continuously a $\sigma-$ compact noncompact group $G = (G, \cdot)$, with unit e. Then there is defined a continuous function $\Phi : G \times X \to X$, $\Phi(g, x) = gx$, called a *continuous action*, so that the following conditions are fulfilled:

A1) $ex = x$, $(\forall\ )\ x \in X$

A2) $g(hx) = (gh)x$, $(\ \forall)\ g,\ h \in G$, $(\forall\ )\ x \in X$.

As a consequence, for each g in G, the mapping

$$\Phi_g : X \to X,\ \Phi_g(x) = gx$$

is a homeomorphism.

Letting x in X, the set $Gx = \{gx|\ g \in G\}$ is known as the *orbit* (or trajectory) of x. The actions such that $\overline{Gx} = X$ for every $x \in X$ are said to be *minimal*; particularly, this is the case when the action is *transitive* i.e., when X itself is an orbit. In the sequel we shall be interested in the behavior of orbits when g approaches infinity i.e., eventually outside each compact subset of G.

Given a neighbourhood V of x, we say that *x returns to V* provided for each compact subset K of G there exists a g such that $gx \in G$. A point x is called *recurrent* (with respect to a fixed action) if it returns in each neighbourhood of it i.e., there exists a sequence $(g_n)_n$ in G such that

$$g_n \to \infty \text{ and } g_n \to x.$$

Theorem 3. *(Poincaré recurrence theorem) Assume the existence of a probability Radon measure μ on X, invariant under the action Φ i.e.,*

$$\mu(\Phi_g^{-1}(B)) = \mu(B)$$

for every Borel subset B of X.

Let A be an open subset of X such that $\mu(A) > 0$. Then almost every point of A returns to A.

Proof. For each compact neighbourhood K of e, consider the set

$$A_K = \{x \mid x \in A \text{ and } gx \notin A \text{ for all } g \in G \setminus \mathrm{K}\}\,.$$

Clearly, A_K is closed. We shall show that $\mu(A_K) = 0$. For, given $n \in \mathbf{N}^\star$, choose a family $\{g_1, ..., g_n\}$ of elements of $G \setminus K$ such that

$$g_{k+p} \ ... \ g_{k+1} g_k \in G \setminus K \text{ provided } 1 \leq k \leq ... \leq k + p \leq n.$$

We use here the fact that G is noncompact.

The sets $\Phi^{-1}_{g_k \ ... \ g_1}(A_K)$ are mutually disjoint because

$$\Phi^{-1}_g(A_K) \ \cap \ A_K = \emptyset \text{ for all } g \in G \setminus K.$$

Then

$$1 = \mu(X) \geq \sum_{k\,=\,1}^{n} \ \mu(\Phi^{-1}_{g_k \ ... \ g_1}(A_K)) = n \cdot \ \mu(A_K)$$

which leads to the conclusion that $\mu(A_K) = 0$.

The proof ends by considering an exhaustion of G by an increasing set of compact subsets K_n (such that $K_n \subset \text{Int } K_n$ for all n). In fact, every point of $A \setminus \cup_{n\,=\,0}^{\infty} A_{K_n}$ returns to A. □

Corollary 4. *Suppose in addition that X is a separable metric space. Then almost every point of X is recurrent.*

Proof. Cover X by countable many open balls of radius $\varepsilon/2$ and apply the result of Theorem 3 above to each of them. We obtain that almost every point of X returns to within ε to itself. Since $\varepsilon > 0$ is arbitrary, we conclude that almost every point of X is recurrent. □

A classical result due to Kakutani and Markov guarantees the existence of invariant probability Radon measures for each continuous action of a locally compact abelian group on a compact metric space. See [5], p. 59. Since the action of every locally compact abelian semigroup can be embeded into the action of a locally compact abelian group, from Theorem 3 and the discussion above we infer the following

Corollary 5. *(Birkhoff recurrence theorem). Suppose Φ is a continuous action of a σ - compact noncompact abelian semigroup $\mathcal{S}$ on a compact metric space X. Then there exist sequences $(s_n)_n$ of elements of $\mathcal{S}$ and points $x \in X$ such that*

$$s_n \to \infty \text{ and } s_n x \to x$$

as $n \to \infty$.

The result of Corolary 5 applies primarily to the dynamical systems associated to hamiltonians,

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\partial H}{\partial y} \\ \frac{dy}{dt} &= \frac{\partial H}{\partial x} \ . \end{aligned}$$

The fact that Lebesgue measure is invariant under the action of such a system is known as Liouville's theorem.

Clearly, the condition of abelianity in Corollary 5 above can be relaxed up to amenability and even beyond that. Think at the case of a discrete group (e.g. $SL(2,$ **Z**$)$) that contains a subgroup isomorphic to **Z.** In this connection it seems worthwhile to put things in an abstract setting:

Definition 6. *A σ - compact noncompact semigroup $\mathcal{S}$ is said to be recurrent provided the assertion of Birkhoff recurrence theorem is valid for all continuous actions of $\mathcal{S}$ on any compact metric space.*

Recurrence can be derived (even in a stronger form) in pure topological terms when the action is exerted by a metrizable group.

Proposition 7. *Suppose that G is a $\sigma-$ compact noncompact metrizable group and $\Phi : G \times X \to X$, $\Phi(g, x) = gx$, is a continuous action of G on a compact metrizable space.*

Then Φ admits uniformly recurrent points (i.e. points $z \in X$ such that for any $\varepsilon > 0$ there exists $R > 0$ such that in any open ball in G with diameter $\geq R$ there exists a g with $d(gz, z) < \varepsilon$).

Proof. Assume first that Φ is minimal. We shall show that each point of X is uniformly recurrent.

In fact, let U be a nonempty open subset of X. Then the set $\cup_{g \in G} g^{-1}U$ is open, nonempty and invariant, so is complement is closed and invariant. Because Φ is minimal, it follows that

$$X = \cup_{g \in G} g^{-1}U.$$

Taking into account the compactness of X we can find $g_1, \ldots , g_N$ in G such that

$$X = \cup_{k\,=\,0}^{N} g_k^{-1}U. \tag{1}$$

Since G is metrizable, we can assume the topology of G is given by a right invariant metric d. See [3], Theorem 8.3. Then choose

$$R = \max \{d(e, g_k) \mid 0 \leq k \leq N\}.$$

If $x \in X$ and $g \in G$, then by (1) we infer that one of the points $g_0 gx$, ... , $g_N gx$ belongs to U. Or,

$$g_k g \in B_R(g) \text{ for } k = 0, ..., N.$$

When Φ is not minimal, an easy application of Zorn ' s Lemma yields that the restriction of Φ to a certain nonempty closed subset of X is minimal. □

Proposition 7 allows us to include in the list of recurrent groups examples such as $\prod_{n=1}^{\infty} U(n)$, $SL(n, \mathbf{Z})$, $S(\infty) = \varinjlim S(n)$ (the infinite symmetric group) and $GL(\infty, \mathbf{F}_q)$.

3. Multiple recurrence

The aim of this section is to extend Birkhoff recurrence theorem to families of pairwise commuting actions.Our approach was inspired by previous work due to Furstenberg and Weiss, who treated a special case. See [1] and [2] for details.

As in the precedent section, G will denote a $\sigma-$ compact noncompact group and $\Phi : G \times X \to X$, $\Phi(g, x) = gx$, a continuous action of G on a compact metric space X.

A closed subset A of X is called *recurrent* provided for any $\varepsilon > 0$ and any compact subset H of G there exist points x and y in A and g in $G \setminus H$ such that

$$d\ (gx,\ y) < \varepsilon.$$

A natural question is the following: When the points of a recurrent set are recurrent ?

Let us call a subset A of X *homogeneous* (respectively *weakly homogeneous),* with respect to Φ, provided there exists a group $\mathcal{G}$ of homeomorphisms of X, commuting with Φ (i.e., $\Phi(g, \cdot\ ) \circ S = S \circ \Phi(g, \cdot\ )$ for every $g \in G$ and every $S \in \mathcal{G}$),which leave A invariant and act transitively (respectively minimally) on A.

Proposition 8. *Every weakly homogeneous recurrent set contains a recurrent point.*

Because every homeomorphism $S \in \mathcal{G}$ maps recurrent points into recurrent points, from Proposition 8 we infer immediately the following :

Theorem 9. *If A is a homogeneous recurrent set, then every point of A is recurrent. If A is weakly homogeneous and recurrent, then A contains a dense subset of recurrent points.*

In order to prove Proposition 8 we need a technical lemma:

Lemma 10. *Suppose A is a weakly homogeneous recurrent set. Then for any $\varepsilon > 0$ and any compact subset H of G there exist points $z \in A$ and $g \in G \setminus H$ such that*

$$d(gz, z) < \varepsilon.$$

Proof. Our argument needs two steps :

Step 1. Under the above hypotheses, for any $\varepsilon > 0$, any y $\in A$ and any compact subset H of G there exist $x \in A$ and $g \in G \setminus H$ such that

$$d(gx,\, y) < \varepsilon.$$

In fact, because A is weakly homogeneous, there exist sequences $(x_n)_n$ and $(y_n)_n$ in A and sequences $(g_n)_n$ in G such that

$$g_n \longrightarrow \infty \text{ and } d(g_n x_n\ ,\, y_n) \longrightarrow 0$$

as $n \longrightarrow \infty$.

Because A is compact, we can assume (by passing to a subsequence if necessary) that $(y_n)_n$ is convergent, say to u. Then u has the property that for every $\varepsilon > 0$ and every compact subset H of G there exist $x \in A$ and $g \in G \setminus H$ such that $d(gx,\, u) < \varepsilon$. The set of all such u is closed and invariant under $\mathcal{G}$. Because $\mathcal{G}$ acts minimally on A, the aforementioned property holds for every point of A.

Step 2. Let H be a compact subset of G and let $\varepsilon > 0$. We shall exhibit inductively a sequence $(z_n)_n$ of elements of A, one of which will satisfy the required condition for z.

Set $\varepsilon_1 = \varepsilon/2$ and choose arbitrarily z_0 in A. By Step 1, there exist $z_1 \in A$ and $g_1 \in G \setminus H$ such that

$$d(g_1 z_1\ ,\, z_0) < \varepsilon_1.$$

Now choose $\varepsilon_2 \in (0,\, \infty)$ such that

$$d(z,\ z_1) < \varepsilon_2 \text{ in } A \text{ implies } d(g_1 z,\ z_0) < \varepsilon_1.$$

Again by Step 1, we can choose $z_2 \in A$ and $g_2 \in G \setminus (H \cup g^{-1}H)$ such that

$$d(g_2 z_2\ , z_1) < \varepsilon_2.$$

Then we shall have $g_1 g_2 \notin H$ and

$$d(g_1 g_2 z_2\ ,\ z_0) < \varepsilon_1.$$

Proceeding this way, succesive ε_k, z_k, g_k are chosen so that

$$g_i g_{i+1} \dots g_j \notin H$$

$$d(g_i g_{i+1} \dots g_j z_j,\ z_i) < \varepsilon_{j+1}$$

$$\varepsilon/2 = \varepsilon_1 > \varepsilon_2 > \ \dots > \varepsilon_i > \ \dots > \varepsilon_j$$

whenever $i < j$.

Because A is compact, for some $i < j$ we shall have $d(z_i\ ,\ z_j) < \varepsilon_1$ so that

$$d(g_i g_{i+1} \ . \ . \ . \ g_j z_j\ ,\ z_j) < \varepsilon_{j+1} + \varepsilon_1 < \varepsilon.$$

The proof ends by letting $g = g_i g_{i+1} \dots g_j$ and $z = z_j$ for the corresponding pair $(i,\ j)$. $\square$

Proof of Proposition 8. Let A be a weakly homogeneous recurrent set for Φ and put

$$F(x) = \liminf_{g\to\infty} d(gx,\ x),\ x \in X.$$

Clearly, a point x is recurrent if $F(x) = 0$, so we shall be looking for points vanishing F. In this setting, let us remark that F is upper semicontinuous, so it has a point of continuity (say u) when restricted to A. We shall show that $F(u) = 0$.

In fact, if $F(u) > 0$, then we can find a relatively open nonempty subset V of A and a $\delta > 0$ such that

$$F(x) > \delta \text{ for every } x \in V.$$

Let $\mathcal{G}$ be as in the definition of a weakly homogeneous subset. Then

$$\bigcup_{S\in\mathcal{G}} S(V)$$

is a $\mathcal{G}$ - invariant open subset of A for which, by the minimality and compactness of A , there exists a finite subset $\mathcal{G}_0$ of $\mathcal{G}$ such that

$$A = \bigcup_{S\in\mathcal{G}_0} S(V)\ .$$

For each $S \in \mathcal{G}_0$,

$$\begin{aligned} F(S^{-1}x) &= \liminf_{g\to\infty}\ d(gS^{-1}x,\ S^{-1}x) = \\ &= \liminf_{g\to\infty}\ d(S^{-1}gx,\ S^{-1}x) \end{aligned}$$

and thus for $\delta > 0$ given above we can choose a $\varepsilon > 0$ such that

$$F(x) < \varepsilon \text{ implies } F(S^{-1}x) < \delta.$$

Letting $\tilde{\varepsilon}$ as the minimum of the corresponding ε' s, we shall have $F(x) \geq \tilde{\varepsilon}$ for all $x \in A$. In fact, then $F(y) = F(S_0^{-1}x) < \delta$, in contradiction with the choice of δ.

However, by Lemma 10 above can't be such a $\tilde{\varepsilon}$ and thus we conclude that $F(u) = 0$. □

We can prove now the following generalization of Birkhoff recurrence theorem:

Theorem 11. *Let Φ_1 , . . . , Φ_N be continuous actions of the recurrent group G on the compact metric space X, such that*

$$\Phi_j(g, \Phi_k(h, \cdot)) = \Phi_k(g, \Phi_j(h, \cdot))$$

for all j,k $\in \{1, ..., N\}$ and all g,h $\in G$. Then there exist points x in X and sequences $(g_n)_n$ in G such that $g_n \to \infty$ and

$$\Phi_1(g_n, x) \to x, ..., \Phi_N(g_n, x) \to x$$

as n$\to \infty$.

Proof. This is done by induction on N. For $N = 1$, the assertion follows from Corollary 5.

Suppose the result has been established for all families of N - 1 commuting actions and we are given N commutting actions $\Phi_1, ..., \Phi_N$. We can attach to them the action

$$\Phi : G \times X^N \to X^N, \ \Phi(g, x_1, ..., x_N) = (\Phi_1(g, x_1), ..., \Phi_N(g, x_N)).$$

Let $\mathcal{G}_0$ be the group generated by all homeomorphisms $\Phi_i(g, \cdot)$, where $i \in \{1, ..., N\}$ and $g \in G$. Without loss of generality we can assume that $\mathcal{G}_0$ acts minimally on X . Then the group

$$\mathcal{G} = \{ \ T \times ... \times T \mid T \in \mathcal{G}_0\}$$

acts minimally on the diagonal Δ of X^N and commutes with Φ.

Δ is also a recurrent set for Φ. In fact, by using the induction hypotheses for the N - 1 actions

$$\Phi_1(g, \ \Phi_N(g^{-1}, x)), \ . \ . \ . \ , \Phi_{N-1}(g, \ \Phi_N(g^{-1}, x))$$

we obtain a sequence $(g_n)_n$ of elements of G and a point z of X such that $g_n \to \infty$ and

$$\Phi_1(g_n, \ \Phi_N(g_n^{-1}, x)) \ \to x, \ . \ . \ . \ , \Phi_{N-1}(g_n, \ \Phi_N(g_n^{-1}, x)) \ \to x.$$

Then, for n sufficiently large,

$$d(\Phi(g_n, u),\ v) < \varepsilon$$

where $u = (\Phi_N(g_n^{-1}, x),\ .\ .\ .\ ,\ \Phi_N(g_n^{-1}, x))$ and $v = (z,\ .\ .\ .\ ,\ z)$. Because u and v both belongs to Δ, it follows that Δ is a weakly homogeneous recurrent set for Φ and Proposition 8 applies. □

Theorem 2 can now be derived easily from Theorem 11, due to the fact that all continuous actions of abelian semigroups extend to continuous actions of abelian groups and abelian groups are recurrent.

4. Proof of Theorem 1

We consider $\sum$ endowed with the discrete topology. Let $P = \{x_1, ..., x_p\}$ be a configuration in $\mathcal{A}$ and let

$$\mathcal{A} = \ C_1 \cup ... \cup C_r$$

a finite partition of $\mathcal{A}$. Consider the compact Hausdorff space

$$\Omega = \{1,\ ...\ ,\ r\}^{\mathcal{A}}$$

endowed with the product topologgy and let ξ be the point of Ω given by

$$\xi(x) = k \text{ if and only if } x \in C_k.$$

Then the actions $\Phi_k : \sum \times \Omega \to \Omega$ $(k \in \{1, ..., p\})$ given by

$$\Phi_k(\alpha, \omega) = \omega(x + \alpha * x_k)$$

are continuous and mutually commuting. We use here the commutativity of $\mathcal{A}$.

The smallest closed subset X of Ω, containing ξ and invariant under the action of $\Phi_1, ..., \Phi_p$, is precisely the closure of the sequence of translates of ξ i.e.

$$X = \overline{\left\{\xi(x + \sum_{i\,=\,l}^{p} \alpha_k * x_k) \mid \alpha_k \in \sum\ ,\ 1 \le k \le p\right\}}\ .$$

Consequently, X is compact and separable and thus metrisable. By Theorem 3, there exist $\eta \in X$ and $\alpha \in \sum \setminus \{0\}$ such that

$$\Phi_1(\alpha, \eta)(0) = \ ... = \Phi_p(\alpha, \eta)(0)$$

i.e. $\eta(\alpha * x_1) = ... = \eta(\alpha * x_p)$. Due to the definition of X , one can find a $z \in \mathcal{A}$ such that

$$\xi(z + \alpha * x_1) = ... = \xi(z + \alpha * x_p)$$

so letting k the common value, the later means that $z + \alpha * x_1$,..., $z + \alpha * x_p$ belong to C_k . □

5. Groups extensions and recurrence

Let $\Phi : G \times X \to X$, $\Phi(g, x) = gx$, a continuous action (of a $\sigma-$ compact noncompact group on a compact metric space) and let K be a compact metric group, with unit 1; as in section 2, the group operations will be denoted multiplicatively and e will designate the unit of G.

By a ($K-$ valuated) *cocycle* of Φ we shall mean any continuous function $\alpha : G \times X \to K$ that satisfies the following two conditions:

C1) $\alpha(e, x) = 1$

C2) $\alpha(gh, x) = \alpha(g, hx) \cdot \alpha(h, x)$.

One can attach to each cocycle α of Φ a group extension $\Phi \rtimes_\alpha K : G \times X \times K \to X \times K$ (*the skew product* of Φ by K), where

$$(\Phi \rtimes_\alpha K)(g,\ (x, k)) = (gx,\ \alpha(g, x)k).$$

Lemma 12. *A point (x, k) $\in X \times K$ is recurrent for $\Phi \rtimes_\alpha K$ if and only if x is recurrent for Φ.*

Proof. Notice first that the sets $\{x\} \times K$ are homogeneous. In fact, a family of homeomorphisms of $X \times K$, commuting with $\Phi \rtimes_\alpha K$, consists of the mappings

$$(x,\ k) \to (x,\ kk^{'}),$$

where $k^{'}$ runs over K. It remains to apply Theorem 9 in conjunction with the fact that $\{x\} \times K$ is recurrent for $\Phi \rtimes_\alpha K$ if and only if x is recurrent for Φ. □

Corollary 13. *Let G and K be as above and let $\chi : G \to K$ a morphism of continuous groups. Then there exists a sequence $(g_n)_n$ of elements of G such that $g_n \to \infty$ and $\chi(g_n) \to 1$.*

Consequently, letting fixed two such morphisms χ_1 and χ_2, each point of K is recurrent with respect to the action

$$\Phi : G \times K \to K,\ \Phi(g, k) = \chi_1(g)\ \chi_2(g).$$

Proof. It suffices to notice that Φ is essentially a group extension of the constant action $G \times \{1\} \to \{1\}$ via the cocycle

$$\alpha(g, 1, k) = (1, \chi_1(g)\chi_2(g)).\ \square$$

For finite groups, Corollary 13 asserts their periodicity.

The group morphisms $\chi : \mathbf{Z} \to K$ are of the form $\chi(n) = k_0^n$ (for a suitable k_0)and any cocycle α of $\Phi(n,k) = k_1^n\ k\ k_2^{-n}$ is associated to a pair of continuous functions f_l , $f_2 : G \to K$ by the formulas

$$\begin{aligned} \alpha(0,k) &= 1 \\ \alpha(1,k) &= f_1(k)f_2(k) \\ \alpha(2,k) &= f_1(k_0k)f_1(k)f_2(kk_0) \\ &\dots \end{aligned}$$

A very special case of the above discussion is as follows:

Proposition 14. *Let K be a compact metrizable group and let $k_1, \dots, k_p$ be elements of K. Then there exist sequences $(m_n)_n$ of natural numbers such that $m_n \to \infty$ and*

$$k_1^{m_n} \to k_1 \ , \ \dots \ , \ k_p^{m_n} \to k_p$$

simultaneously as $n \to \infty$.

The restriction on compactness in Proposition 14 above appears to be essential. Think at the abelian group

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a,\ b \in \mathbf{R}^\star \right\} \ .$$

Proposition 14 was previously known for $K = \mathbf{T}$. For $K = S^3$ (viewed as the group of unitary quaternions) and $K = U(n)$ it yields non-commutative counterparts that extend (in different ways) the case of $\mathbf{T}$.

Not surprisingly, Proposition 14 can be derived from Theorem 1. In fact, consider the natural multiplication $\alpha : \mathbf{N}\times K \to K$, $\alpha(n,k) = k^n$, and let d be an invariant metric on K. As a configuration in K we take $\{1, k_1, \dots, k_p\}$. Then by considering succesive partitions of diameters $< 1 / 2^n$, we get a sequence $(m_n)_n$ of naturals such that 1, $k_1^{m_n}, \dots$, $k_p^{m_n}$ are within $1 / 2^n$ of each other. Consequently,

$$k_1^{m_n} \to 1 \ , \ \dots \ , \ k_p^{m_n} \to 1$$

as $n \to \infty$ and thus

$$k_1^{m_n + 1} \to k_1 \ , \ \dots \ , \ k_p^{m_n + 1} \to k_p$$

as $n \to \infty$.

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