ON $L$-$M$ DUALITY IN REAL BANACH SPACES

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Most results in $M$-structure theory are related to the following two order relations that make sense for any real Banach space $E$:

$x \leq_L y$ if and only if $\|y\| = \|x\| + \|y-x\|$.

$x \leq_M y$ if and only if every closed ball containing 0 and $y$ contains also $x$.


It is the purpose of this paper to prove a new characterization of $\leq_M$ by starting with the obvious remark that

$x \leq_M y$ if and only if $\|x+z\| \leq \max \{\|z\|, \|y+z\|\}$ for every $z \in E$.

Particularly that will allow us to explain why numerical range techniques and $M$-theory techniques have the same bulk of applications in the context of $C^*$-algebras.

**Theorem 1.** Let $E$ be a real Banach space and let $x$ and $y$ be two elements of $E$. Then the following assertions are equivalent:

i) $x \leq_M y$;

ii) $\|x+z\| \leq \max \{\|z\|, \|y+z\|\}$ for every $z \in E''$, i.e., $x \leq_M y$ in $E''$;

iii) For every (a certain) $w'-$ dense subspace $\mathcal{H}$ of $E'$,

$$f(x) \leq \sup \{g(y) \mid g \in E', \; g \leq_M f\}, \; \forall f \in \mathcal{H}.$$  

**Proof.** i) $\Rightarrow$ ii). By the principle of local reflexivity (see [4]) for each $\varepsilon > 0$ and each $z \in E''$ there exists a $z_\varepsilon$ in $E$ such that

$$\|x+z\| \leq (1+\varepsilon)\|x+z_\varepsilon\| \leq (1+\varepsilon)\max \{\|z_\varepsilon\|, \|y+z_\varepsilon\|\} \leq$$

$$\leq \frac{1+\varepsilon}{1-3\varepsilon} \max \{\|z\|, \|y+z\|\}$$

so it remains to take the infimum in the right hand over $\varepsilon > 0$.

ii) $\Rightarrow$ iii). Let $f \in \mathcal{H}$, $\varepsilon > 0$ and $n \in \mathbb{N}$ be such that $n > \|y\|$. Then we can consider the sublinear functional $p_n : \mathcal{H} \to \mathbb{R}$ given by

$$p_n(\varphi) = \inf \{n \|h| - h(y) + n\|\varphi - h\| \mid h \in \mathcal{H}\}.$$  

Since $p_n(\varphi) \leq n\|\varphi\|$ for every $\varphi \in \mathcal{H}$, the functional $p_n$ is continuous and thus there exists a $z \in \mathcal{H}'$ such that

$$f(z) = p_n(f)$$

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and

\[ \varphi(z) \leq p_n(\varphi) \text{ for every } \varphi \in \mathcal{H}. \]

The second condition yields \( \|z\| \leq n \) and \( \|z + y\| \leq n \) so by ii),

\[ f(x) + f(z) = f(x + z) \leq \|f\| \cdot \|x + z\| \leq \|f\| \cdot \max\{\|z\|, \|y + z\|\} \leq n\|f\|. \]

Then

\[ f(x) \leq n\|f\| - p_n(f) = n\|f\| - \inf \{n\|h\| - h(y) + n\|\varphi - h\| : h \in \mathcal{H}\} = \sup \{n\|f\| - n\|h\| + h(y) - n\|\varphi - h\| : h \in \mathcal{H}\}. \]

We shall prove that the last term is \(< a + \varepsilon\), where \( a = \sup \{g(y) : g \in \mathcal{B}', g \leq_L f\} \).

In fact, if the contrary is true, it would exist a sequence \((h_n)_n \subset \mathcal{H}\) such that

\[(*) \quad n\|f\| + h_n(y) \geq n(\|h_n\| + \|f - h_n\|) + a + \varepsilon \]

for all \( n \geq \|y\| \). Consequently

\[ (**) \quad h_n(y) \geq a + \varepsilon \text{ for } n \geq \|y\|, \]

which yields

\[ \lim_{n \to \infty} \|h_n\| \leq \lim_{n \to \infty} \frac{n\|f\| - a}{n - \|y\|} = \|f\|. \]

Particularly the sequence \((h_n)_n\) is bounded and (by passing to a subsequence if necessary) we can assume that \( h_n \rightharpoonup h \). By \((*)\),

\[ \|f\| + \frac{1}{n} h_n(y) \geq \|h_n\| + \|f - h_n\| + \frac{a + \varepsilon}{n} \]

which implies that

\[ \|f\| \geq \|h\| + \|f - h\| \geq \|f\| \]

i.e., \( h \leq_L f \). Then \( h(y) \leq a \), in contradiction with \((**)\).
iii) ⇒ i). Let \( z \in E \) and \( h \in \mathcal{H} \) with \( \|h\| \leq 1 \). Then

\[
\begin{align*}
  h(z) + h(x) &\leq h(z) + \sup \{g(y) \mid g \in E', \ g \ll_L h\} = \\
  &= \sup \{h(z) + g(y) \mid g \in E', \ g \ll_L h\} = \\
  &= \sup \{(h - g)(z) + g(z + y) \mid g \in E', \ g \ll_L h\} \\
  &\leq \sup \{\|h - g\| \|z\| + \|g\| \|z + y\| \mid g \ll_L h\} \\
  &\leq \max \{\|z\|, \|z + y\|\}
\end{align*}
\]

and thus \( \|x + z\| \leq \max \{\|z\|, \|y + z\|\} \) for every \( z \in E \) i.e., \( x \ll_M y \). □

**Corollary 1.** Let \( E \) be a real Banach space. Then:

i) \( x \ll_M y \) in \( E \) if and only if \( f(x) \ll \sup \{g(y) \mid g \in E', \ g \ll_L f\} \) for every \( f \in E' \).

ii) \( f \ll_M g \) in \( E' \) if and only if \( f(x) \ll \sup \{g(y) \mid y \in E'', \ y \ll_L x\} \) for every \( x \in E \).

From Corollary 1 ii) we can infer immediately that all \( \ll_M \)-intervals \([0, f]\) in \( E' \) are \( w' \)-compact and that the adjoint of every operator in the Cunningham algebra of \( E \) belongs to the centralizer of \( E' \).

Let \( \ll \) be one of the order relations \( \ll_L \) and \( \ll_M \). We shall say that \( \ll \) is trivial provided that

\( x \ll y \) if and only if \( x = ax \cdot y \) for some \( a \in [0, 1] \).

For example, \( \ll_L \) is trivial on any strictly convex Banach space.

**Corollary 2.** If \( \ll_L \) is trivial on \( E' \) then \( \ll_M \) is trivial on \( E \).

The duality outlined in Corollary 1 is only one way and an interesting open question is whether the assertion ii) in Corollary 1 could be straighten up to

\[
f \ll_M g \text{ in } E' \text{ if and only if } f(x) \ll \sup \{g(y) \mid y \in E, \ y \ll_L x\}
\]

for every \( x \in E \).

A positive answer to that question would yield the following counterpart of Corollary 2: If \( \ll_L \) is trivial on \( E \) then \( \ll_M \) is trivial on \( E' \).

How thin can be the subsets \( \mathcal{H} \) as in Theorem 1 above? Alfsen and Effros have noticed in [1] that

\[
(M) \ x \ll_M y \text{ in } E \text{ if and only if either } 0 \leq f(x) \leq f(y) \text{ or } f(y) \leq f(x) \leq 0
\]

for every extreme point \( f \) of the closed unit ballk of \( E' \) (i.e., \( f(x) = ax \cdot f(y) \) for a suitable \( a \in [0, 1] \)).

Their argument depends upon Choquet's theory. We can offer a (somewhat) simpler argument via Corollary 1 above.
Suppose that \( x \leq_M y \) in \( E \). If \( f \) is an extreme point of \( K \), then \( g \leq_L f \) in \( E' \) yields \( g = \alpha \cdot f \) for a suitable \( \alpha \in [0, 1] \). See [1], p. 106. Then by Corollary 1 i),

\[
f(x) \leq \sup \{ \alpha \cdot f(y) \mid \alpha \in [0, 1] \}.
\]

Since \(-f\) is also an extreme point, we can restrict ourselves to the case where \( f(x) > 0 \). Then the inequality above yields \( 0 < f(x) < f(y) \).

Conversely, let \( \sum_{k=1}^{n} \lambda_k f_k \) be a convex combination of extreme points of \( K \). By hypotheses, for each \( k \) there exists an \( \alpha_k \in [0, 1] \) such that \( f_k(x) = \alpha_k \cdot f_k(y) \). Suppose that \( B_k(x) \) is a closed ball in \( E \) containing 0 and \( y \). Then

\[
\left| \sum_{k=1}^{n} \lambda_k f_k(x - z) \right| \leq \sum_{k=1}^{n} \lambda_k \alpha_k |f_k(y - z)| + \\
+ \sum_{k=1}^{n} \lambda_k (1 - \alpha_k) |f_k(z)| \leq r
\]

so by Krein-Milman Theorem we can conclude that \( x \in B_k(z) \) too.

**Theorem 2.** Suppose that \( E \) is an \( M \)-ideal of \( E'' \). Then \( x \leq_M y \) in \( E'' \) if and only if and only if for each extreme point \( f \) of the closed unit ball \( K \) of \( E' \) either \( 0 < f(x) < f(y) \) or \( f(y) < f(x) \leq 0 \).

**Proof.** We can proceed as in the case of the assertion \((M)\), by noticing the fact that every extreme point of \( K \) extends uniquely to an extreme point of the closed unit ball \( K_0 \) of \( E'' \). In fact, since \( E \) is an \( M \)-ideal of \( E'' \), then every functional \( f \in E' \) has a unique extension \( g \in E'' \) such that \( \|g\| = \|f\| \). See [2], p. 35.

Theorem 2 reveals an interesting connection between \( \leq_M \) and numerical range in the case of \( C^* \)-algebras. To avoid some technicalities, we shall restrict ourselves to a special case.

Let \( H \) be a complex Hilbert space and let \( \mathcal{A}(H) \) be the self-adjoint part of \( L(H, H) \). It is known that \( \mathcal{A}(H) \) is the second dual of the Banach space \( E \) of all self-adjoint compact operators on \( H \) and the dual of the Banach space of all self-adjoint nuclear operators on \( H \); the natural pairing of \( E' \) and \( E'' \) is given by \( (A, B) \rightarrow \text{Trace} AB \). See [3] for details. The extreme points of the closed unit ball of \( E' \) are of the form \( \langle x, x \rangle x \), where \( x \) runs over unit sphere of \( H \). \( E \) is an \( M \)-ideal of \( E'' \) by [1], p. 167. Then by Theorem 2 above we can conclude that

\( A \leq_M B \) in \( \mathcal{A}(H) \) if and only if for each \( x \in H \) there exists an \( \alpha \in [0, 1] \) such that \( \langle Ax, x \rangle = \alpha \cdot \langle Bx, x \rangle \).

Some comments are in order. Let \( A, B \in \mathcal{A}(H) \).

If \( B \geq 0 \), then \( A \leq_M B \) if and only if \( 0 \leq A \leq B \).

If \( AB = BA \), then \( A \leq_M B \) if and only if \( A^- \leq B^- \) and \( A^+ \leq B^+ \).

Particularly, \( -A^-, A^+ \leq_M A \).

If \( A \leq_M B \), then \( C^*AC \leq_M C^*BC \) for every \( C \in L(H, H) \).
REFERENCES


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