PREDUALS OF BANACH LATTICES, WEAK ORDER UNITS AND THE RADON-NIKODYM PROPERTY

BY

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INTRODUCTION

In [12] R. C. James proved the following assertions for $E$ a Banach space with an unconditional basis $\{e_n\}$:

(WSC) $E$ is weakly sequentially complete if and only if $E$ contains no isomorphic copy of $e_0$;

(R) $E$ is reflexive if and only if $E$ contains no isomorphic copy of $e_0$ and $l_1$;

(RNP) $E'$ is separable if and only if $E$ contains no isomorphic copy of $l_1$.

Later Lozanovski [19], [20] proved (WSC) and (R) for Banach lattices but his approach covers also the case where $E$ embeds into a $\sigma$-complete Banach lattice having order continuous norm. Related results were discussed by Lotz [17], Meyer-Nieberg [21] and Tzafriri [29]. In a conversation, in 1974, H. Lotz showed us a proof that (RNP) holds for every separable Banach lattice.

The first section of our paper is concerned with preduals of Banach lattices. A Banach space $F$ is said to be a predual of the Banach space $E$ if $F'$ is isometric to $E$. The main result (see Theorem 1.1 below, asserts that given an ordered Banach space $E$ which contains no isomorphic copy of $e_0$ then $E'$ is a Banach lattice if and only if $E$ itself is a Banach lattice. Particularly (WSC) and (R) both remain valid in the framework of ordered preduals of Banach lattices. It is proved also that if $E$ does not contain an isomorphic copy of $e_0$ and $E'$ is a Banach lattice then $E$ is the unique (up to isometry) ordered predual of $E'$. That extends the well known fact that a space $L_\infty(\mu)$ has a unique (up to isometry) predual.

In the second section we discuss a geometrical condition in order that the dual of a Banach lattice $E$ having a weak order unit (i.e a total element) fails such a unit : the presence in $E'$ of a lattice isomorph of $l_1(\Gamma)$ for $\Gamma$ an uncountable set of indices. See Corollary 2.5 below. Under additional assumptions this condition is seen to be equivalent to the fact that $E'$ is not weakly compactly generated. See Theorem 2.7 below. That extends an important result due to Rosenthal [25].

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In the third section we establish (RNP) for separable Banach spaces $E$ having local unconditional structure in the sense of Gordon and Lewis [6] i.e. $E''$ is complemented in a Banach lattice. See Theorem 3.3 below. As a consequence we obtain that the recent example (due to R. C. James [13]) of a somewhat reflexive Banach space with a nonseparable dual fails local unconditional structure.

The results presented in the second and the third section have been announced in [22].

1. PREDUALS OF BANACH LATTICE

We shall denote by $L_0$ the class of all Banach spaces $E$ equipped with a closed cone $C$ such that:

$L_1$ ($E$, $C$) has the Riesz decomposition property

$L_2$ $-y \leq x \leq y$ implies $\|x\| \leq \|y\|$

$L_3$ For each $x \in E$, $\|x\| < 1$ there exists a $y \in E$ with $\|y\| < 1$, $y \geq \pm x$.

Here $x > 0$ means precisely that $x \in C$.

By Theorem 1 in [14], page 18, $E'$ is order isometric to a Banach lattice. It is also well known (see [3]) that each ordered Banach space $E$ whose topological dual is a Banach lattice satisfies $L_1 - L_3$ for

$C = \{x \in E; x'(x) > 0 \text{ for all } x' \in E', x' > 0\}$

The class $L_0$ was investigated especially in connection with the study of $L_1(\mu)$-preduals spaces. See [16] for details. Our approach is based on the general theory of $AM$ and $AL$ spaces in the sense of Kakutani.

Let $E \in L_0$. For each $x \in E$, $x > 0$, we can consider the following vector space:

$E_x = \{y \in E; (\exists) \lambda > 0, \lambda x \geq \pm y\}$

normed by:

$\|y\|_x = \inf \{\lambda > 0; \lambda x \geq \pm y\}$

By Theorem 6 in [14], page 16, $(E_x)'$ is order isometric to a space $L_1(\mu)$, for $\mu$ a suitable positive Radon measure.

If $E$ is supposed to be a Banach lattice then a classical result due to Kakutani yields that $E_x$ is lattice isometric to a $C(S)$ space.

We shall denote by $i_x : E_x \rightarrow E$ the canonical inclusion.

For each $x' \in E'$, $x' > 0$, consider on $E$ the following relation of equivalence:

$x \sim 0 \iff (\forall) \varepsilon > 0 (\exists) y_x \geq \pm x, x'(y_x) \leq \varepsilon$

The completion of $E/\sim$ with respect to the additive norm:

$\|x\|_x' = \inf \{x'(y); y \geq \pm x\}$

will be denoted $E_x'$. Notice that $E_x', x' > 0$.

In [8] [20] the $C(S)$ space has been investigated especially in connection with the study of $AM$ and $AL$ spaces. See [16] for details. Our approach is based on the general theory of $AM$ and $AL$ spaces in the sense of Kakutani.

1.1. Theorem

$E$ is isomorphic to $E_x'$ if and only if $E$ is reflexive.

Proof.

(i) $E$ is reflexive.

(ii) $E'$ is order isometric to a Banach lattice.

On the other hand, the ideal $(E_x)'$ of $E_x'$ is isometric to the closed cone $C$ of $E$ and the ideal $(E_x)'$ of $E_x'$ is isometric to the ideal $(E_x)'$ of $E_x'$.

1.2. Corollary

If $E$ itself is reflexive, then $E$ is isomorphic to $E_x'$. And let $Y$ be a Banach lattice, $\varphi : Y \rightarrow Z$ a surjective linear map with dense kernel.

for every $y \in Y$, $\varphi(y)$ is a Banach lattice.

1.3. Corollary

If $E$ itself is reflexive, then $E$ is isomorphic to $E_x'$. And let $Y$ be a Banach lattice, $\varphi : Y \rightarrow Z$ a surjective linear map with dense kernel.

for every $y \in Y$, $\varphi(y)$ is a Banach lattice.
will be denoted by \( L_1(x') \) and the canonical mapping \( E \to L_1(x') \) by \( j_x' \). Notice that \( (j_x)'' = i_x' \), and \( L_1(x')' \) is order isometric to \((E')_x' \) for each \( x' \in E', x' > 0 \).

In [8] (see also [14]) page 96 Grothendieck remarked that each dual \( C(S) \) space has a unique (up to isometry) predual. This fact together a classical result due to Kakutani shows that each \( L_1(x') \) space is order isometric to an \( L_1(\mu) \) space, for \( \mu \) a suitable positive Radon measure. Another useful remark is that for each \( x \in E, x > 0 \), there exists an order isomorphism \( \rho_x \) from \( L_1(x) \) into \((E_x)' \) satisfying the following two conditions:

\[ \rho_x \circ j_x = (i_x)' \]
\[ \| \rho_x \| = \| (\rho_x)^{-1} \| = 1 \]

Our next result shows that the non lattice \( \mathcal{L} \) —theory requires the presence of \( e_0 \):

1.1. Theorem. Let \( E \in \mathcal{L}_0 \) be a Banach space which contains no isomorphic copy of \( e_0 \). Then:

(i) \( E \) is order isometric to a Banach lattice

(ii) \( E' \) has an unique (up to isometry) ordered predual.

Proof. (i) Let \( x \in E, x > 0 \). Because \( E \) contains no isomorphic copy of \( e_0 \), a result due to Lindenstrauss and Tzafriri (see [16], page 184) implies that the mapping \( i_x \) is weakly compact and thus \( (i_x)'' (E_x)' \subset E \). On the other hand \( (i_x)'' = (j_x)'' \circ (\rho_x)' \) and \( (\rho_x)' \) is onto. Then:

\[ \text{Im} (j_x)' \subset \overline{\text{Im} (i_x)''} \subset E \]

the closure being considered in the norm topology of \( E \). In other words the ideal \((E'')_x \), generated by \( x \) in \( E'' \), is contained in \( E \) and thus the modulus (calculated in \( E'' \)) of each \( x \in E \) belongs also to \( E \).

(ii) Let \( Z \) be the closed subspace of all order continuous functionals \( x'' \in E'' \) i.e.

\[ x'' \downarrow 0 \text{ (in order) implies } x''(x'_0) \to 0 \]

and let \( Y \) be an ordered predual of \( E' \). Then there exists an isometry \( \varphi : Y \to Z \) given by:

\[ \varphi(y)(x') = \langle y, x' \rangle \]

for every \( y \in Y, x' \in E' \). By (i) and Proposition 2.4 (d) in [17] it follows that \( E = Z \), which in turn implies that \( Y \) is isometric to \( E \), q.e.d.

1.2. Corollary. Let \( E \) be an ordered Banach space which contains no isomorphic copy of \( e_0 \). Then \( E' \) is isometric to a Banach lattice if and only if \( E \) itself is isometric to a Banach lattice.
After this paper has been accepted for publication, Professor Lacey has informed us of the following simple proof of Corollary 1.2: Let $x \in E$ and look at all upper bounds of $\{x, 0\}$; this collection is downwards directed by the interpolatory property. If it does not converge then there is an $\varepsilon > 0$ and a decreasing sequence $\{x_n\}_{n=1}^{\infty}$, such that $\|x_{n+1} - x_n\| \geq \varepsilon$ for all $n$. It follows that $\overline{\text{Sp}} \{x_{n+1} - x_n\} \sim e_0$, q.e.d.

The Banach lattice $Jh = (\sum \oplus I_\infty(n))_1$ is weakly sequentially complete and thus, by our Theorem 1.1 (ii) above, $Jh$ is the only ordered predual of $(Jh)' = (\sum \oplus I_1(n))_\infty$. However $(Jh)'$ contains a complemented copy of $I_1$ (see W. B. Johnson, Israel J. Math. 13 (1972), 301–310), and, $I_1$ fails an unique predual. In connection with this example we ask the following:

1.3. Problem. Does there exist a Banach lattice $E$ such that $E$ contains an isomorph of $e_0$ and $E'$ has a unique (up to isometry) predual?

2. WEAK ORDER UNITS

In this section we discuss a geometrical condition for the existence of a weak order unit in the dual of a separable Banach lattice; the non existence of a lattice isomorph of a space $I_1(\Gamma)$ for $\Gamma$ an uncountable set.

We need a preliminary result which works for all separable spaces in $\mathcal{L}$ and also for all Banach lattice having a weak order unit. Here $\mathcal{L}$ denotes the class of all preduals of Banach lattices.

2.1. Lemma. Let $E \in \mathcal{L}$ such that for a suitable $v \in E''$ we have:

$$x \in E, \ |x| \wedge |v| = 0 \text{ implies } x = 0.$$  

Then there exists an order complete Banach lattice $M(E)$ with a weak order unit and a lattice isometry $i : E' \to M(E)'$ such that:

a) $i(E')$ is complemented in $M(E)'$,

b) $i(E')$ is formed by order continuous functionals on $M(E)$.

Thus $M(E)$ plays the same role as $L_\infty[0,1]$ for $E = C[0,1]$ and $M(E)$ extends a well known lemma due to Dini.

Proof. We shall consider for $M(E)$ the band generated by $v$ in $E''$. Then a lattice isometry $i : E' \to M(E)'$ is given by:

$$i(x')(e) = \langle x', e \rangle$$

for every $x' \in E'$, $e \in M(E)$. Let $j : E \to M(E)$ the canonical inclusion. Then

$$(j' \circ i)(x')(x) = \langle i(x'), j(x) \rangle = \langle x, x' \rangle$$

for every $x \in E$, $x' \in E'$, which implies the existence of a positive projection $P : M(E)' \to i(E')$.

The second assertion is an easy consequence of the following result due to Riesz: if $Z$ is a Banach lattice and $f_n \downarrow 0$ in $Z'$ then $f_n(z) \to 0$ for every $z \in Z$, q.e.d.
If \( Z \) is an order complete Banach lattice and \( A \subset Z \) is a closed subspace, we shall denote by \( \Sigma(A) \) an order complete closed sublattice of the band generated by \( A \) such that \( A \subset \Sigma(A) \).

2.2. Theorem. Let \( E \in \mathcal{S} \) be a Banach space which is contained in the band generated by a positive \( v \in E'' \). If \( A \) is a closed subspace of \( E' \) then either:

(i) \( \Sigma(A) \) has a weak order unit; or
(ii) \( A \) contains an isomorph of \( \mathbb{I}_1(\Gamma) \) (for \( \Gamma \) an uncountable set) which is complemented in \( \Sigma(A) \) and \( \Sigma(A) \) contains a lattice isomorph of \( \mathbb{I}_1(\Gamma) \).

Proof. By Lemma 2.1 above we can assume that \( E \) is an order complete Banach lattice with a weak order unit \( u \) and \( A \) is formed by order continuous functionals. The subspace \( E'_0 \) of all order \( \sigma \)-continuous functionals \( x' \in E' \) constitutes a band (see [28] page 74) and thus \( \Sigma(A) \subset E'_0 \).

By Zorn's lemma there exists a family \( \{u'_i\}_{i \in I} \) of pairwise disjoint normalized elements of \( \Sigma(A) \) such that:

\[
x' \in \Sigma(A), \sup(|x'| \wedge |u'_i|) = 0 \text{ implies } x' = 0.
\]

Put:

\[
H = \{i \in I ; (3) a_i \in A, \|a_i\| = 1, [u'_i]a_i \neq 0 \}
\]

and for each \( i \in H \), choose an \( a_i \in A \) with \( \|a_i\| = 1 \) and \( [u'_i]a_i \neq 0 \). Here \([z]\) denotes the band projection generated by \( z \).

Notice that \( x = \sup(x \wedge nu) \) for each \( x \in E, x > 0 \), and \( [u'_i]a_i \in E'_0 \) for each \( i \in H \). Then for each \( i \in H \) we can find an \( e_i \in [0, u] \) with \( ([u'_i]a_i)e_i \neq 0 \).

The following two possibilities occur:

(1) \( \text{Card } H \leq \aleph_0 \) and thus by identifying \( H \) as a subset of \( \mathbb{N} \) we shall denote:

\[
u' = \sum_{n \in H} 2^{-n}|u'_n|.
\]

Then \( u' \in \Sigma(A) \) and:

\[
x' \in A, |x'| \wedge u' = 0 \text{ implies } x' = 0.
\]

Because \( \Sigma(A) \subset A^{**} \), it follows that \( u' \) is a weak order unit for \( \Sigma(A) \).

(2) \( \text{Card } H > \aleph_0 \) and in this case we shall consider the operator \( T \in L(\Sigma(A), \mathbb{I}_1(H)) \) given by:

\[
T(x') = \{([u'_i]x')e_i\}_{i \in H}
\]

for every \( x' \in \Sigma(A) \). Notice that \( \|T\| \leq \|u\| \) and \( T(a_i) \neq 0, i \in H \). Put:

\[
H_n = \{i \in H ; |T(a_i)(i)| \leq 1/n\}
\]

for \( n = 1, 2, \ldots \). Then \( H = \cup H_n \) and there exists an \( n_0 \) with \( \text{Card } H_{n_0} > \aleph_0 \). By Lemma 1.1 in [25], \( A \) contains a non separable \( \mathbb{I}_1(\Gamma) \) space which is complemented in \( \Sigma(A) \).
Notice that \( T([u'_i]_a) \neq 0 \) for each \( i \in H \) and the elements \([u'_i]_a\) are pairwise disjoint. Then the proof of Lemma 1.1 in [25] easily yields the existence of an uncountable family

\[
\{z_\omega\} \in \text{Abco} \{[u'_i]_a ; i \in H\}
\]

which is equivalent to the vector unit basis of \( l_1(\Omega) \). Here \( \text{Abco} Z \) means the absolute convex hull of \( Z \). Put:

\[
J = \{i \in H ; (3) \omega \in \Omega, |z_\omega| \cdot [u'_i]_a \neq 0\}
\]

Then \( \text{Card} J > \aleph_0 \) (otherwise \( \{z_\omega\} \) would be contained in a separable space) and because each \( z_\omega \) is a finite combination of the elements \([u'_i]_a\), there exists an uncountable subset of \( \{z_\omega\} \) formed by pairwise disjoint elements, q.e.d.

2.3. REMARK. The two possibilities of Theorem 2.2 are mutually exclusive. In fact, by Lemma 2.1 it suffices to consider the case where \( E \) is an order complete Banach lattice with a weak order unit \( u > 0 \) and \( u' \) is a positive element of \( E'_0 \) such that \( [u'] \) contains a pairwise disjoint-normalized family \( \{e'_\gamma\} \in \Gamma \) which is equivalent to the unit vector basis of a non separable \( l_1(\Gamma) \). Since \( u' \in E'_0 \), \( (|u'| \wedge |e'_\gamma|)u \neq 0 \) for each \( \gamma \in \Gamma \). Then there exists \( n_0 \in \mathbb{N} \) and an uncountable subset \( \Gamma_0 < \Gamma \) with \( (u' \wedge |e'_\gamma|)u \geq 1/n_0 \) for each \( \gamma \in \Gamma_0 \). Consequently:

\[
u'(u) = \sup \{\sum_{\gamma \in \Gamma} (u' \Delta |e'_\gamma|)u ; \text{Card} F < \infty\} = \infty
\]

contradiction.

2.4. REMARK. It is possible that \( E \) and \( E' \) both have a weak order unit and \( E' \) contains a complemented isomorph (but not a lattice isomorph) of \( l_1(2^\mathbb{N}) \). For example, consider \( E = (\Sigma \oplus l_0(u))_1 \). See [11] for details.

2.5. COROLLARY. Let \( E \) be an order complete Banach lattice with a weak order unit. Then either \( E' \) contains a weak order unit or \( E' \) contains a lattice isomorph of a non separable \( l_1(\Gamma) \) space.

Professor R. G. Bartle has kindly informed us that H. P. Lotz and H. Rosenthal (Urbana) have obtained results related to Corollary 2.5 above.

2.6. COROLLARY. Let \( E \) be an order complete Banach lattice such that \( E' \) has a weak order unit. Then either \( E \) contains a weak order unit or \( E \) contains a lattice isomorph of a non separable \( l_1(\Gamma) \) space.

A special case of our Theorem 2.2 is the following:

2.7. THEOREM. Let \( E \in \mathcal{E} \) be a Banach space which is contained in the band generated by a suitable \( v \in E'' \). If \( E \) contains no complemented copy of \( l_1 \) and \( A \) is a closed subspace of \( E' \) then either:

(i) \( A \) is contained in a weakly compactly generated sublattice of \( E' \) having a weak order unit; or,

(ii) \( A \) contains an isomorph of a non separable \( l_1(\Gamma) \) space that is complemented in \( E' \).
Proof. By [1], $E'$ contains no isomorphic copy of $e_0$ and thus by Propositions 2.1 and 2.4 in [17] the order intervals of $E'$ are relatively weakly compact and the topology of $E'$ is order continuous i.e. $x'_n \downarrow 0$ (in order) implies $\|x'_n\| \to 0$.

Then $[z] = z^{-1} = \text{Span} [0, z]$ for each $z \in E'$, $z > 0$, and our result follows from Theorem 2.2 above, q.e.d.

2.8. Remark. If $E$ is a Banach lattice and $u'$ is a positive element of $E'$ then $[u']$ contains precisely those functionals $x' \in E'$ which are absolutely continuous with respect to $u'$, i.e.

$|y| < x, \quad u'(|y|) < \delta$ implies $|x'(y)| < \varepsilon$.

See [2] for details. Consequently for $E$ a $C(\delta)$ space our Theorem 2.7 above implies Lemma 1.3 in [25].

3. THE RADON-NIKODYM PROPERTY

In 1975, at the Kent University conference on the Radon-Nikodym property, H. Lotz proved the following result:

3.1. Theorem. Let $E$ be a Banach lattice such that $E'$ has a weak order unit and let $A$ be a closed subspace of $E$. Then either:

(i) $A$ contains an isomorphic copy of $e_0$; or

(ii) $A'$ is weakly compactly generated.

We next present similar results in the setting of Banach spaces having local unconditional structure.

The unconditional basis constant $\chi(E)$ of a Banach space $E$ is the least constant $\lambda$ having the following property: there exists a basis $\{e_i\}_{i \in I}$ for $E$ such that $\|\sum a_i e_i\| \leq \lambda$ whenever $\sum a_i e_i \in E$ has norm one and $|a_i| \leq 1, i \in I$. If not, such $\lambda$ exists, $\chi(E) = \infty$.

3.2. Definition. A Banach space $E$ is said to have local unconditional structure (l.u.st.) in the sense of Gordon and Lewis [6] if $E$ satisfies one of the following equivalent conditions:

(i) There exists a $\lambda > 0$ such that for any finite dimensional subspace $F \subset E$ one can find a space $U$ and operators $\alpha \in \mathcal{L}(F, U)$, $\beta \in \mathcal{L}(U, E)$ such that $\beta \circ \alpha$ is the identity on $F$ and

$$\|\alpha\| \|\beta\| \chi(U) \leq \lambda,$$

(ii) $E''$ is complemented in a Banach lattice

(iii) There exists an isomorphism $h$ from $E$ into a Banach lattice $L$ and $\varphi \in \mathcal{L}(E', L')$ with $h' \circ \varphi = 1_E$.

The equivalence (i) $\iff$ (ii) was remarked in [5] while (ii) $\iff$ (iii) is immediate.

A stronger concept of l.u.st. was introduced by Dubinski, Pelczynski and Rosenthal. See [5] for details.
3.3. THEOREM. For a separable Banach space $E$ having l.u.s.t. the following assertions are equivalent:

(i) $E'$ is separable
(ii) $E$ does not contain a copy of $l_1$
(iii) $E'$ is weakly compactly generated.

Proof. (i) $\Rightarrow$ (ii). In fact, if a Banach space $Z$ contains an isomorphic copy of $l_1$, then $Z'$ has a quotient space isomorphic to $l_\infty$ which in turn implies that $Z'$ is nonseparable.

(ii) $\Rightarrow$ (iii). Let $h$, $\varphi$ and $L$ be as in Definition 3.2 (iii) above. Since $E$ is separable, we may assume that $L$ is separable.

Since $E$ is separable and contains no isomorphic copy of $l_1$, $E'$ contains no isomorphic copy of $l_1(\Gamma)$ for $\Gamma$ an uncountable set (see [23]), so by our Theorem 2.2 above it follows that $\varphi(E')$ is contained in the band generated by a positive $w' \in L'$.

A result due to Bessaga and Pečynski [1] implies that $E'$ contains no isomorphic copy of $e_t$ and thus the composition $h' = h' \circ i_{x}: (L')_x \rightarrow E'$ is weakly compact for each $x' \in L'$, $x' > 0$. See [26], Theorem 3.7.

Since $(L')_x = (I_{(x')})'$ has the Dunford-Pettis property, $h_x$ maps decreasing sequences of positive elements of $(L')_x$ into converging sequences of elements of $E'$. See Proposition 1 and Theorem 1 in [7]. On the other hand $h' = (j_x\circ h')$ is $w'$-continuous and thus $h_x$ is order $\sigma$-continuous for each $x' > 0$. Consequently:

$$E' = \overline{\text{span}} h'[0, u'] = \overline{\text{span}} (h')[0, u']$$

and we already remarked that $h_x$ is weakly compact.

The implication (iii) $\Rightarrow$ (i) is an easy consequence of the following result due to Davis, Figiel, Johnson and Pečynski [4]: a Banach space $Z$ is weakly compactly generated if there exists a reflexive space $R$ and a one to one operator $T \in \mathcal{L}(R, Z)$ with $T(R)$ dense in $Z$, q.e.d.

3.4. COROLLARY. Let $E$ be a Banach space with l.u.s.t. Then the following assertions are equivalent:

(i) $E$ contains no isomorphic copy of $l_1$
(ii) $E'$ has the Radon-Nikodym property i.e. every absolutely summing operator from a space $C(S)$ into $E'$ is nuclear.

Proof. (i) $\Rightarrow$ (ii) If $T \in \mathcal{L}(C(S), E')$ is absolutely summing then $T$ admits a factorization $C(S) \rightarrow l_1(\mu) \rightarrow E'$ where $\mu$ is a positive Radon measure on $S$ and $i$ denotes the canonical inclusion. See Proposition 2.3.4 in [24]. Clearly $i$ is weakly compact and $T'' = (i' \circ U'|E')$. By Theorem 1 in [7] it follows that $i': l_\infty(\mu) \rightarrow C(S)'$ maps weak Cauchy sequences into converging sequences. Because $E$ contains no isomorphic copy of $l_1$, each bounded sequence in $E$ has a weak Cauchy subsequence (see [27]) and thus $i' \circ U'|E$ and $T$ are compact operators. Consequently we have only to prove that each separable subspace of $E'$ has the Radon-Nikodym property.

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It is easy to show that each separable subspace of $E'$ embeds into a space $F'$ for $F$ a suitable separable sub-space of $E$. It was noted in [9], page 134, that each separable dual space has the Radon-Nikodym property and thus in order to prove (ii) it suffices to prove that each separable subspace of $E$ has a separable dual. On the other hand if $Z$ has $l.u.s.t.$ and $Y$ is a separable subspace of $Z$ then there exists a separable Banach space $X$ with $l.u.s.t.$ such that $Y \subset X \subset Z$. Use Definition 3.2 (ii) above. Then we can assume that $E$ itself is separable and our result follows from Theorem 3.3.

(ii) $\Rightarrow$ (i). Since the canonical inclusion $C[0,1] \rightarrow L_1[0,1]$ is not nuclear but absolutely summing, $E'$ contains no isomorphic copy of $L_1[0,1]$, which in turn implies (see [23]) that $E$ contains no isomorphic copy of $I_1$, q.e.d.

3.5. REMARK. A Banach space $E$ with local unconditional structure is reflexive if (and only if) $I$ does not isomorphically embed in $E$ and $E'$.

Consequently, a separable Banach space $E$ with local unconditional structure is reflexive if $E'$ contains no isomorph of $I_1$, if $E''$ is separable.

Proof. Since $E$ contains no isomorph of $I$, then $E'$ contains no isomorph of $e_0$. See [1] for details. Then Theorem 1 in [30] and Theorem 13 in [29] together imply that $E'$ is weakly sequentially complete. By [27] any bounded subset of $E'$ is weakly sequentially precompact, and our result follows.

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