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Convex Functions and their Applications
A Contemporary Approach

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Preface

It seems to me that the notion of convex function is just as fundamental as positive function or increasing function. If I am not mistaken in this, the notion ought to find its place in elementary expositions of the theory of real functions.

J. L. W. V. Jensen

Convexity is a simple and natural notion which can be traced back to Archimedes (circa 250 B.C.), in connection with his famous estimate of the value of π (by using inscribed and circumscribed regular polygons). He noticed the important fact that the perimeter of a convex figure is smaller than the perimeter of any other convex figure surrounding it.

As a matter of fact, we experience convexity all the time and in many ways. The most prosaic example is our upright position, which is secured as long as the vertical projection of our center of gravity lies inside the convex envelope of our feet. Also, convexity has a great impact on our everyday life through numerous applications in industry, business, medicine, and art. So do the problems of optimum allocation of resources and equilibrium of noncooperative games.

The theory of convex functions is part of the general subject of convexity, since a convex function is one whose epigraph is a convex set. Nonetheless it is an important theory per se, which touches almost all branches of mathematics. Graphical analysis is one of the first topics in mathematics which requires the concept of convexity. Calculus gives us a powerful tool in recognizing convexity, the second-derivative test. Miraculously, this has a natural generalization for the several variables case, the Hessian test. Motivated by some deep problems in optimization and control theory, convex function theory has been extended to the framework of infinite dimensional Banach spaces (and even further).

The recognition of the subject of convex functions as one that deserves to be studied in its own right is generally ascribed to J. L. W. V. Jensen [114], [115]. However he was not the first to deal with such functions. Among his predecessors we should recall here Ch. Hermite [102], O. Hölder [106] and O. Stolz [233]. During the twentieth century, there was intense research activity and significant results were obtained in geometric functional analysis, mathematical economics, convex analysis, and nonlinear optimization. A classic book by G. H. Hardy, J. E. Littlewood and G. Pólya [99] played a large role in the popularization of the subject of convex functions.

Roughly speaking, there are two basic properties of convex functions that make them so widely used in theoretical and applied mathematics:

- *The maximum is attained at a boundary point.*
- *Any local minimum is a global one. Moreover, a strictly convex function admits at most one minimum.*

The modern viewpoint on convex functions entails a powerful and elegant interaction between analysis and geometry. In a memorable paper dedicated to the Brunn–Minkowski inequality, R. J. Gardner [88, p. 358], described this reality in beautiful phrases: [convexity] "appears like an octopus, tentacles reaching far and wide, its shape and color changing as it roams from one area to the next. It is quite clear that research opportunities abound."

Over the years a number of notable books dedicated to the theory and applications of convex functions appeared. We mention here: L. Hörmander [108], M. A. Krasnosel'skii and Ya. B. Rutickii [132], J. E. Pečarić, F. Proschan and Y. C. Tong [196], R. R. Phelps [199], [200] and A. W. Roberts and D. E. Varberg [212]. The references at the end of this book include many other fine books dedicated to one aspect or another of the theory.

The title of the book by L. Hörmander, *Notions of Convexity,* is very suggestive for the present state of art. In fact, nowadays the study of convex functions has evolved into a larger theory about functions which are adapted to other geometries of the domain and/or obey other laws of comparison of means. Examples are log-convex functions, multiplicatively convex functions, subharmonic functions, and functions which are convex with respect to a subgroup of the linear group.

Our book aims to be a thorough introduction to contemporary convex function theory. It covers a large variety of subjects, from the one real variable case to the infinite dimensional case, including Jensen's inequality and its ramifications, the Hardy–Littlewood–Pólya theory of majorization, the theory of gamma and beta functions, the Borell–Brascamp–Lieb form of the Prékopa–Leindler inequality (as well as the connection with isoperimetric inequalities), Alexandrov's well-known result on the second differentiability of convex functions, the highlights of Choquet's theory, a brief account on the recent solution to Horn's conjecture, and many more. It is certainly a book where inequalities play a central role but in no case a book on inequalities. Many results are new, and the whole book reflects our own experiences, both in teaching and research.

This book may serve many purposes, ranging from a one-semester graduate course on Convex Functions and Applications to additional bibliographic material. In a course for first year graduate students, we used the following route:

- *Background*: Sections 1.1–1.3, 1.5, 1.7, 1.8, 1.10.
- *The beta and gamma functions*: Section 2.2.

- *Convex functions of several variables*: Sections 3.1–3.12.
- *The variational approach of partial differential equations*: Appendix C.

The necessary background is advanced calculus and linear algebra. This can be covered from many sources, for example, from *Analysis I* and *II* by S. Lang [137], [138]. A thorough presentation of the fundamentals of measure theory is also available in L. C. Evans and R. F. Gariepy [74]. For further reading we recommend the classical texts by F. H. Clarke [56] and I. Ekeland and R. Temam [70].

Our book is not meant to be read from cover to cover. For example, Section 1.9, which deals with the Hermite–Hadamard inequality, offers a good starting point for Choquet's theory. Then the reader may continue with Chapter 4, where this theory is presented in a slightly more general form, to allow the presence of certain signed measures. We recommend this chapter to be studied in parallel with the *Lectures on Choquet's theory* by R. R. Phelps [200]. For the reader's convenience, we collected in Appendix A all the necessary material on the separation of convex sets in locally convex Hausdorff spaces (as well as a proof of the Krein–Milman theorem).

Appendix B may be seen both as an illustration of convex function theory and an introduction to an important topic in real algebraic geometry: the theory of semi-algebraic sets.

Sections 3.11 and 3.12 offer all necessary background on a further study of convex geometric analysis, a fast-growing topic which relates many important branches of mathematics.

To help the reader in understanding the theory presented, each section ends with exercises (accompanied by hints). Also, each chapter ends with comments covering supplementary material and historical information. The primary sources we have relied upon for this book are listed in the references.

In order to avoid any confusion relative to our notation, a symbol index was added for the convenience of the reader. Notice that our book deals only with *real* linear spaces and all Borel measures under attention are assumed to be *regular*.

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In order to keep in touch with our readers, a web page for this book will be made available at http://www.inf.ucv.ro/~niculescu/Convex_Functions.html

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