Editorial comment. Strictly speaking, the above applies when $x > 0$ and $y > 0$. Other combinations of sign are similar. If exactly one of x or y is 0, then we get a well-known integral of the type $\int_0^{\pi/2} \log(2\sin t)\,dt = 0$. If $x = y = 0$ then the integral in the original problem does not make sense.

Also solved by R. Bagby, D. Beckwith, M. Benito & Ó. Ciaurri & E. Fernández, W. Chu & L. V. DiClaudio, K. Dale, A. Eydelzon, J. Grivaux, C. Hill, G. L. Isaacs, W. Janous, S.-Y. Jeon, G. Lamb, K. D. Lathrop, O. P. Lossers, A. L. Miller, M. A. Prasad, A. Stadler, D. B. Tyler, GCHQ Problem Solving Group, NSA Problems Group, and the proposer.

A Limit Problem

11024 [2003, 543]. *Proposed by Vicenţiu Rădulescu, University of Craiova, Romania.* Consider a continuous function $g\colon (0,\infty) \to (0,\infty)$ such that for some $\alpha > 0$,

$$\lim_{x\to\infty} \frac{g(x)}{x^{1+\alpha}} = \infty.$$

Let $f\colon \mathbb{R} \to (0,\infty)$ be a twice-differentiable function for which there exist $a > 0$ and $x_0 \in \mathbb{R}$ such that for all $x \geq x_0$,

$$f''(x) + f'(x) > ag(f(x)).$$

Prove that $\lim_{x\to\infty} f(x)$ exists and is finite, and evaluate the limit.

Solution by Richard Stong, Rice University, Houston, TX. We prove that the limit is zero. If $f(x)$ is nonincreasing on $[x_0,\infty)$, then $f(x)$ is bounded on $[x_0,\infty)$, since $f(x) > 0$ and $\lim_{x\to\infty} f(x) = C \geq 0$ exists. If $C > 0$, then $f(x) \geq C$ for all x. Let $C_1 = \min_{x\in[C,\infty)} ag(x)$. This value exists and is positive, since $g(x)$ is continuous and tends to ∞ as $x \to \infty$. Now $f''(x) + f'(x) > ag(f(x)) \geq C_1 > 0$, hence $(e^x f'(x))' > C_1 e^x$. Integrating once gives the inequality $f'(x) > C_1 + C_2 e^{-x}$ for some constant C_2. However, this forces $f'(x)$ to be positive for large x since $C_1 > 0$, contradicting our assumption. Thus, in this case $\lim_{x\to\infty} f(x) = 0$.

Otherwise, there must be some $x_1 \geq x_0$ such that $f'(x_1) > 0$. Since $f''(x) + f'(x) > ag(f(x)) > 0$, we have $(e^x f'(x))' > 0$ on $[x_1,\infty)$. Integrating this result gives

$$f'(x) > e^{x_1 - x} f'(x_1) > 0$$

for $x \in [1,\infty)$. Hence, $f(x)$ is increasing on $[x_1,\infty)$. Let $C_3 = \min_{x\in[f(x_1),\infty)} ag(x) > 0$, which exists as earlier. Integrating $f''(x) + f'(x) > C_3$, we see that

$$f'(x) > C_3 + C_4 e^{-x},$$

for some constant C_4. Since $f'(x)$ is bounded away from zero for large x, $f(x)$ tends to ∞. Thus, we must show that this case cannot actually occur.

There is some T such that $g(t) > t^{1+\alpha}/a > 2t/a$ for all $t \geq T$, and there is some $x_2 \geq x_1$ such that $f(x) \geq T$ for $x \geq x_2$. Hence for $x \geq x_2$, we have

$$f''(x) + f'(x) > f^{1+\alpha}(x) > 2f(x).$$

Now using

$$e^{-3x}(e^{2x} f(x))' = (2e^{2x} f(x) + e^{2x} f'(x))e^{-3x} = e^{-x}(2f(x) + f'(x)),$$

the previous inequality can be written in the equivalent form

$$e^x \frac{d}{dx}(e^{-3x}(e^{2x} f(x))') > 0.$$

Integrating this gives $(e^{2x} f(x))' > C_6 e^{3x}$, so $f(x) > \frac{C_6}{3} e^x + C_7 e^{-2x}$ for some constant C_7.

It follows that there is some $x_3 \geq x_2$ such that for $x \geq x_3$, $f(x) > e^{x/2}$. Since $f(x) > 0$, we have then

$$f''(x) + f'(x) + \frac{1}{4} f(x) > f^{1+\alpha}(x).$$

This can be rewritten in the form

$$\frac{d^2}{dx^2}(e^{x/2} f(x)) > e^{x/2} f^{1+\alpha}(x) > (e^{x/2} f(x))^{1+\alpha/2}$$

for $x \geq x_3$, where the last inequality is directly related to $f(x) \geq e^{x/2}$.

Let $h(x) = e^{x/2} f(x)$. From the foregoing, for x sufficiently large, $h'(x) > T/2)e^{x/2}$ and $h''(x) > (h(x))^{1+\alpha/2}$. But in view of the following lemma, this is impossible.

Lemma. *Let x_3, $\beta > 0$, and $K > 0$ be constants. There is no twice-differentiable function $h \colon [x_3, \infty) \to (0, \infty)$ that satisfies $h''(x) > h^{1+\beta}(x)$ and $h'(x) > K$.*

Proof. Suppose that such a function exists. Since $h'(x) > K > 0$, $h(x)$ increases without bound. Integrating the first inequality in the form

$$h'(x)h''(x) > h^{1+\beta}(x)h'(x)$$

gives

$$(h'(x))^2 \geq \frac{2}{2+\beta} h^{2+\beta}(x) + C,$$

for some constant C. Since h increases without bound, there is an x_4 with $x_4 \geq x_3$ such that for $x \geq x_4$ we have

$$(h'(x))^2 \geq \frac{1}{2+\beta} h^{2+\beta}(x),$$

which can be rewritten as

$$\frac{h'(x)}{h^{1+\beta/2}(x)} \geq \frac{1}{\sqrt{2+\beta}} > 0.$$

Integrating again gives the inequality

$$-\frac{2}{\beta h^{\beta/2}(x)} \geq \frac{x}{\sqrt{2+\beta}} + C_6.$$

This is a contradiction because the left- hand side is always negative, but the right-hand side will be positive for x large enough.

Also solved by P. J. Fitzsimmons, J. W. Hagood, O. P. Lossers (The Netherlands), GCHQ Problem Solving Group (U. K.), Univ. Louisiana Lafayette Math Club, and the proposer.