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A Limit Problem: 11024

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*Editorial comment.* Strictly speaking, the above applies when  $x > 0$  and  $y > 0$ . Other combinations of sign are similar. If exactly one of  $x$  or  $y$  is 0, then we get a well-known integral of the type  $\int_0^{\pi/2} \log(2 \sin t) dt = 0$ . If  $x = y = 0$  then the integral in the original problem does not make sense.

Also solved by R. Bagby, D. Beckwith, M. Benito & Ó. Ciaurri & E. Fernández, W. Chu & L. V. DiClaudio, K. Dale, A. Eydalzon, J. Grivaux, C. Hill, G. L. Isaacs, W. Janous, S.-Y. Jeon, G. Lamb, K. D. Lathrop, O. P. Lossers, A. L. Miller, M. A. Prasad, A. Stadler, D. B. Tyler, GCHQ Problem Solving Group, NSA Problems Group, and the proposer.

### A Limit Problem

**11024** [2003, 543]. *Proposed by Vicențiu Rădulescu, University of Craiova, Romania.* Consider a continuous function  $g: (0, \infty) \rightarrow (0, \infty)$  such that for some  $\alpha > 0$ ,

$$\lim_{x \rightarrow \infty} \frac{g(x)}{x^{1+\alpha}} = \infty.$$

Let  $f: \mathbb{R} \rightarrow (0, \infty)$  be a twice-differentiable function for which there exist  $a > 0$  and  $x_0 \in \mathbb{R}$  such that for all  $x \geq x_0$ ,

$$f''(x) + f'(x) > ag(f(x)).$$

Prove that  $\lim_{x \rightarrow \infty} f(x)$  exists and is finite, and evaluate the limit.

*Solution by Richard Stong, Rice University, Houston, TX.* We prove that the limit is zero. If  $f(x)$  is nonincreasing on  $[x_0, \infty)$ , then  $f(x)$  is bounded on  $[x_0, \infty)$ , since  $f(x) > 0$  and  $\lim_{x \rightarrow \infty} f(x) = C \geq 0$  exists. If  $C > 0$ , then  $f(x) \geq C$  for all  $x$ . Let  $C_1 = \min_{x \in [C, \infty)} ag(x)$ . This value exists and is positive, since  $g(x)$  is continuous and tends to  $\infty$  as  $x \rightarrow \infty$ . Now  $f''(x) + f'(x) > ag(f(x)) \geq C_1 > 0$ , hence  $(e^x f'(x))' > C_1 e^x$ . Integrating once gives the inequality  $f'(x) > C_1 + C_2 e^{-x}$  for some constant  $C_2$ . However, this forces  $f'(x)$  to be positive for large  $x$  since  $C_1 > 0$ , contradicting our assumption. Thus, in this case  $\lim_{x \rightarrow \infty} f(x) = 0$ .

Otherwise, there must be some  $x_1 \geq x_0$  such that  $f'(x_1) > 0$ . Since  $f''(x) + f'(x) > ag(f(x)) > 0$ , we have  $(e^x f'(x))' > 0$  on  $[x_1, \infty)$ . Integrating this result gives

$$f'(x) > e^{x_1-x} f'(x_1) > 0$$

for  $x \in [x_1, \infty)$ . Hence,  $f(x)$  is increasing on  $[x_1, \infty)$ . Let  $C_3 = \min_{x \in [f(x_1), \infty)} ag(x) > 0$ , which exists as earlier. Integrating  $f''(x) + f'(x) > C_3$ , we see that

$$f'(x) > C_3 + C_4 e^{-x},$$

for some constant  $C_4$ . Since  $f'(x)$  is bounded away from zero for large  $x$ ,  $f(x)$  tends to  $\infty$ . Thus, we must show that this case cannot actually occur.

There is some  $T$  such that  $g(t) > t^{1+\alpha}/a > 2t/a$  for all  $t \geq T$ , and there is some  $x_2 \geq x_1$  such that  $f(x) \geq T$  for  $x \geq x_2$ . Hence for  $x \geq x_2$ , we have

$$f''(x) + f'(x) > f^{1+\alpha}(x) > 2f(x).$$

Now using

$$e^{-3x}(e^{2x} f(x))' = (2e^{2x} f(x) + e^{2x} f'(x))e^{-3x} = e^{-x}(2f(x) + f'(x)),$$

the previous inequality can be written in the equivalent form

$$e^x \frac{d}{dx}(e^{-3x}(e^{2x} f(x))') > 0.$$

Integrating this gives  $(e^{2x} f(x))' > C_6 e^{3x}$ , so  $f(x) > \frac{C_6}{3} e^x + C_7 e^{-2x}$  for some constant  $C_7$ .

It follows that there is some  $x_3 \geq x_2$  such that for  $x \geq x_3$ ,  $f(x) > e^{x/2}$ . Since  $f(x) > 0$ , we have then

$$f''(x) + f'(x) + \frac{1}{4}f(x) > f^{1+\alpha}(x).$$

This can be rewritten in the form

$$\frac{d^2}{dx^2}(e^{x/2} f(x)) > e^{x/2} f^{1+\alpha}(x) > (e^{x/2} f(x))^{1+\alpha/2}$$

for  $x \geq x_3$ , where the last inequality is directly related to  $f(x) \geq e^{x/2}$ .

Let  $h(x) = e^{x/2} f(x)$ . From the foregoing, for  $x$  sufficiently large,  $h'(x) > T/2 e^{x/2}$  and  $h''(x) > (h(x))^{1+\alpha/2}$ . But in view of the following lemma, this is impossible.

**Lemma.** Let  $x_3, \beta > 0$ , and  $K > 0$  be constants. There is no twice-differentiable function  $h: [x_3, \infty) \rightarrow (0, \infty)$  that satisfies  $h''(x) > h^{1+\beta}(x)$  and  $h'(x) > K$ .

*Proof.* Suppose that such a function exists. Since  $h'(x) > K > 0$ ,  $h(x)$  increases without bound. Integrating the first inequality in the form

$$h'(x)h''(x) > h^{1+\beta}(x)h'(x)$$

gives

$$(h'(x))^2 \geq \frac{2}{2+\beta} h^{2+\beta}(x) + C,$$

for some constant  $C$ . Since  $h$  increases without bound, there is an  $x_4$  with  $x_4 \geq x_3$  such that for  $x \geq x_4$  we have

$$(h'(x))^2 \geq \frac{1}{2+\beta} h^{2+\beta}(x),$$

which can be rewritten as

$$\frac{h'(x)}{h^{1+\beta/2}(x)} \geq \frac{1}{\sqrt{2+\beta}} > 0.$$

Integrating again gives the inequality

$$-\frac{2}{\beta h^{\beta/2}(x)} \geq \frac{x}{\sqrt{2+\beta}} + C_6.$$

This is a contradiction because the left-hand side is always negative, but the right-hand side will be positive for  $x$  large enough.

Also solved by P. J. Fitzsimmons, J. W. Hagoood, O. P. Lossers (The Netherlands), GCHQ Problem Solving Group (U. K.), Univ. Louisiana Lafayette Math Club, and the proposer.