



---

An Example With Periodic Orbits: 11073

Author(s): Vicențiu Rădulescu and Richard Stong

Source: *The American Mathematical Monthly*, Vol. 113, No. 2 (Feb., 2006), pp. 184-186

Published by: Mathematical Association of America

Stable URL: <http://www.jstor.org/stable/27641881>

Accessed: 17/01/2010 09:40

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=maa>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).



*Mathematical Association of America* is collaborating with JSTOR to digitize, preserve and extend access to *The American Mathematical Monthly*.

<http://www.jstor.org>

### An Example With Periodic Orbits

**11073** [2004, 260]. *Proposed by Vicențiu Rădulescu, University of Craiova, Romania.* Let  $a$  and  $b$  be positive real numbers, and let  $f$  and  $g$  be twice-differentiable functions from  $\mathbb{R}$  to  $\mathbb{R}$  satisfying the differential equations

$$f'' = -f(1 - f^2 - g^2), \quad g'' = -g(1 - f^2 - g^2)$$

and the initial conditions  $f(0) = a$ ,  $f'(0) = 0$ ,  $g(0) = 0$ ,  $g'(0) = b$ .

(a) Show that there is a nontrivial polynomial  $E$  in two variables such that, for all positive  $a$  and  $b$ ,  $E(f^2(t) + g^2(t), f'^2(t) + g'^2(t))$  is independent of  $t$ .

(b) Show that if  $f$  and  $g$  are both periodic in  $t$ , with period  $T$ , and if  $f^2(t) + g^2(t)$  does not reach a local minimum at  $t = 0$ , then  $a \leq 1$ ,  $b^2 \leq a^2(1 - a^2)$ , and  $T > 2\pi$ .

(c) Give an example of  $f$  and  $g$  satisfying the premises of part (b).

(d)\* Prove that there exist choices of  $a$  and  $b$  such that the resulting  $(f, g)$  is periodic, and  $\min(f^2 + g^2) < (1/2) \max(f^2 + g^2)$ .

*Solution by Richard Stong, Rice University, Houston, TX.* For (a) multiply the first equation by  $2f'$  and the second by  $2g'$ , add to obtain

$$\frac{d}{dt} \left( (f'(t))^2 + (g'(t))^2 \right) = \frac{d}{dt} \left( \frac{1}{2} (f(t)^2 + g(t)^2)^2 - (f(t)^2 + g(t)^2) \right),$$

and then integrate:

$$(f'(t))^2 + (g'(t))^2 = \frac{1}{2} (f(t)^2 + g(t)^2)^2 - (f(t)^2 + g(t)^2) + b^2 + a^2 - \frac{a^4}{2}.$$

Therefore the polynomial  $E(x, y) = y^2 + x^2 - x^4/2$  solves (a).

Now let  $R = f^2 + g^2$  and  $\tan \theta = g/f$ ; that is, pass to polar coordinates in  $(f, g)$  but use the squared length. [We will see that  $R$  is never 0. This means that  $f$  and  $g$  are never simultaneously 0, so there is a continuous choice of  $\theta$ .] Note that

$$\frac{d}{dt} (f(t)g'(t) - g(t)f'(t)) = f(t)g''(t) - g(t)f''(t) = 0.$$

Therefore  $f(t)g'(t) - g(t)f'(t) = ab$  and

$$\theta' = \frac{f(t)g'(t) - g(t)f'(t)}{f(t)^2 + g(t)^2} = \frac{ab}{R}.$$

Also,

$$\begin{aligned} R'' &= 2[(f')^2 + (g')^2] + 2ff'' + 2gg'' \\ &= R^2 - 2R + 2b^2 + 2a^2 - a^4 - 2R(1 - R) = 3R^2 - 4R + 2b^2 + 2a^2 - a^4. \end{aligned}$$

Multiply by  $2R'$  and integrate to obtain

$$\begin{aligned} (R')^2 &= 2R^3 - 4R^2 + 2(2b^2 + 2a^2 - a^4)R - 4a^2b^2 \\ &= 2(R - a^2)(R^2 + (a^2 - 2)R + 2b^2). \end{aligned} \tag{1}$$

Suppose now that  $f$  and  $g$  are periodic in  $t$  with period  $T$  and that  $R$  does not reach a local maximum at  $t = 0$ . Since  $R(0) = a^2$  and  $R'(0) = 0$ , it follows that  $R''(0) = 2(a^4 - a^2 + b^2) \leq 0$ . If  $R''(0) = 0$ , then  $b^2 = a^2(1 - a^2)$ , and the solution is

$$f(t) = a \cos\left(\frac{bt}{a}\right), \quad g(t) = a \sin\left(\frac{bt}{a}\right).$$

However, this gives  $R(t) = a^2$  for all  $t$ , so that  $t = 0$  is a local minimum. Thus we must have  $b^2 < a^2(1 - a^2)$ . Since  $b > 0$ , we conclude that  $a < 1$ .

Suppose now that  $b^2 < a^2(1 - a^2)$ . The polynomial  $p(x) = x^2 + (a^2 - 2)x + 2b^2$  satisfies  $p(0) = 2b^2 > 0$  and  $p(a^2) = p(2(1 - a^2)) < 0$ , ensuring that  $p$  has a positive zero, call it  $d^2$ , with  $d^2 < \min\{a^2, 2(1 - a^2)\}$ . From the definition of  $d$ , we have  $2b^2 = d^2(2 - a^2 - d^2)$ . Hence (1) becomes

$$(R')^2 = 2(R - a^2)(R - d^2)(R - 2 + a^2 + d^2),$$

so with the abbreviations  $X = a^2 - d^2$ ,  $Y = 2 - a^2 - 2d^2$ , and  $Z = 2 - a^2 - d^2$ ,  $R(t)$  can be expressed in terms of Jacobi elliptic functions as

$$R(t) = d^2 + X \operatorname{sn}^2 \left( K \left( \frac{X}{Y} \right) - t \sqrt{1 - d^2 - \frac{a^2}{2}} \mid \frac{X}{Y} \right),$$

where  $\operatorname{sn}(z|m)$  is a Jacobi elliptic function and  $K(m)$  is the complete elliptic integral of the first kind. Note that  $R(t) \geq d^2$ , so  $R$  is never 0.

Observe that  $R(t)$  is periodic in  $t$  with period

$$\begin{aligned} T_0 &= \sqrt{2} \int_{d^2}^{a^2} \frac{dx}{\sqrt{(a^2 - x)(x - d^2)(Z - x)}} \\ &= 2\sqrt{2} \int_0^{\pi/2} \frac{d\phi}{\sqrt{Y - X \sin^2 \phi}} = \frac{2}{\sqrt{1 - d^2 - a^2/2}} K \left( \frac{X}{Y} \right). \end{aligned}$$

Note that  $0 < m = X/Y < 1$ . Thus  $K(m) > \pi/2$ , whence  $T_0 > \pi$ . As  $t$  varies over one full period of  $R$ , the change in  $\theta$  is given by

$$\begin{aligned} \Delta\theta &= ab \int_0^{T_0} \frac{dt}{R(t)} = ab\sqrt{2} \int_{d^2}^{a^2} \frac{dx}{x\sqrt{(a^2 - x)(x - d^2)(Z - x)}} \\ &= 2ad\sqrt{Z} \int_0^{\pi/2} \frac{d\phi}{(d^2 + X \sin^2 \phi)\sqrt{Y - X \sin^2 \phi}} \\ &= \frac{2a\sqrt{Z}}{d\sqrt{2 - 2d^2 - a^2}} \Pi \left( -\frac{X}{d^2} \mid \frac{X}{Y} \right), \end{aligned} \tag{2}$$

where  $\Pi(n|m)$  is the complete elliptic integral of the third kind.

The functions  $f$  and  $g$  have a common period if and only if  $\theta$  (modulo  $2\pi$ ) and  $R$  have a common period, which happens if and only if  $\Delta\theta$  is a rational multiple of  $2\pi$ . Since  $d^2 < 2(1 - a^2)$ , and  $\Delta\theta$  is a nonconstant analytic function of  $a$  and  $d$  on the domain  $D = \{(a, b) : 0 < a < d < 1, d^2 < 2(1 - a^2)\}$ , we conclude that  $\Delta\theta$  is not locally constant. Thus the set of  $(a, d)$  for which  $\Delta\theta$  is a rational multiple of  $2\pi$  will be dense in  $D$ . Since there is a nonempty open set of pairs  $(a, d)$  for which  $d^2 = \min R(t) < a^2/2 = \max R(t)$ , the domain for  $\Delta\theta$  has nonempty interior. Thus though we have not found an explicit pair of values for  $a$  and  $b$  that would meet the conditions of **(d)**, they do exist.

It remains only to show that the common period  $T$  of  $f$  and  $g$  satisfies  $T > 2\pi$ . To see this look at the two factors in (2) in the integral with respect to  $\phi$ . One is an increasing function of  $\phi$ , the other a decreasing function of  $\phi$ . Hence they are negatively

correlated, and we get an upper bound by splitting the integral:

$$\begin{aligned} \Delta\theta &\leq \frac{4ad\sqrt{Z}}{\pi} \int_0^{\pi/2} \frac{d\phi}{d^2 + X \sin^2 \phi} \int_0^{\pi/2} \frac{d\phi}{\sqrt{Z - X \sin^2 \phi}} \\ &= 2\sqrt{Z} \int_0^{\pi/2} \frac{d\phi}{\sqrt{Z - X \sin^2 \phi}} = T_0 \sqrt{1 - \frac{a^2 + d^2}{2}} < T_0. \end{aligned}$$

If  $T_0 \leq 2\pi$ , then evidently  $\Delta\theta < 2\pi$ , hence at least two periods of  $R$  are required before  $\theta$  repeats modulo  $2\pi$ . Thus  $T \geq 2T_0 > 2\pi$ . If  $T_0 > 2\pi$ , then  $T \geq T_0 > 2\pi$ . In either case we are done.

All parts also solved by N. Thornber; parts (a)–(c) also solved by P. Bracken and the proposer.

### An Oldie

**11098** [2004, 625]. *Proposed by Christopher Hillar and Darren Rhea, University of California, Berkeley, CA.* Let

$$f(n) = \sum_{i=1}^n \frac{(-1)^{i+1}}{2^i - 1} \binom{n}{i}.$$

Prove that there are constants  $c$  and  $c'$  such that  $c \leq f(n)/\log n \leq c'$  for sufficiently large  $n$  (that is,  $f(n) = \Theta(\log n)$ ).

*Solution by Godfrey L. Isaacs, Hollywood, FL.* Since  $2^i - 1 = 2^i(1 - (1/2)^i)$ , we have  $f(n) = \sum_{k \geq 1} t(n, k)$ , where  $t(n, k) = 1 - (1 - 1/2^k)^n$ . Since  $t(n, k)$  decreases as  $k$  increases, we have  $I(n) - 1 < f(n) < I(n)$ , where

$$I(n) = \int_0^\infty [1 - (1 - 2^{-x})^n] dx.$$

Substituting  $w = 1 - 1/2^x$  and multiplying by  $\log 2$  gives

$$I(n) \log 2 = \int_0^1 \frac{1 - w^n}{1 - w} dw = \int_0^1 \sum_{j=0}^{n-1} w^j dw = \sum_{j=1}^n \frac{1}{j} = \log n + \gamma + o(1),$$

where  $\gamma$  is Euler's constant. Thus  $f(n)/\log n \rightarrow 1/\log 2$ , which implies the result.

*Editorial comment* by Donald Knuth (writing from Texas A & M University): This function  $f(n) = u_{n+1} - u_n$  is well known to algorithmic analysts because it arises, for example, in the analysis of “radix-exchange sorting” and also “digital search tries.” See my book *Sorting and Searching* (Addison-Wesley, 1973), exercises 5.2.2–48 or 5.2.2–54, for a detailed asymptotic expansion. Generalizations with 2 replaced by  $m$  are in exercise 6.3–19. Many other authors have refined these estimates and found alternative proofs since the book came out. It was N. G. de Bruijn who showed me how to solve the problem when I mentioned it to him in the mid-1960s.

Also solved by B. M. Ábrego, Z. Ahmed & M. A. Prasad (India), S. Amghibech (Canada), A. B. Aruffo (Italy), M. R. Avidon, R. Bagby, P. Bracken R. Chapman (U. K.), Y. Dumont (France), J.-P. Grivaux (France), S. Y. Jeon (Korea), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Nijenhuis, Y. Rubinstein & Z. Rubinstein, O. G. Ruehr, N. C. Singer, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), D. B. Tyler, L. Zhou, GCHQ Problem Solving Group (U. K.), NSA Problem Solving Group, and the proposers.