An Example With Periodic Orbits

11073 [2004, 260]. *Proposed by Vicenţiu Rădulescu, University of Craiova, Romania.* Let a and b be positive real numbers, and let f and g be twice-differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ satisfying the differential equations

$$f'' = -f(1 - f^2 - g^2), \quad g'' = -g(1 - f^2 - g^2)$$

and the initial conditions $f(0) = a$, $f'(0) = 0$, $g(0) = 0$, $g'(0) = b$.
(a) Show that there is a nontrivial polynomial E in two variables such that, for all positive a and b, $E(f^2(t) + g^2(t), f'^2(t) + g'^2(t))$ is independent of t.
(b) Show that if f and g are both periodic in t, with period T, and if $f^2(t) + g^2(t)$ does not reach a local minimum at $t = 0$, then $a \le 1$, $b^2 \le a^2(1 - a^2)$, and $T > 2\pi$.
(c) Give an example of f and g satisfying the premises of part (b).
(d)* Prove that there exist choices of a and b such that the resulting (f, g) is periodic, and $\min(f^2 + g^2) < (1/2)\max(f^2 + g^2)$.

Solution by Richard Stong, Rice University, Houston, TX. For **(a)** multiply the first equation by $2f'$ and the second by $2g'$, add to obtain

$$\frac{d}{dt}\left(\left(f'(t)\right)^2 + \left(g'(t)\right)^2\right) = \frac{d}{dt}\left(\frac{1}{2}\left(f(t)^2 + g(t)^2\right)^2 - \left(f(t)^2 + g(t)^2\right)\right),$$

and then integrate:

$$\left(f'(t)\right)^2 + \left(g'(t)\right)^2 = \frac{1}{2}\left(f(t)^2 + g(t)^2\right)^2 - \left(f(t)^2 + g(t)^2\right) + b^2 + a^2 - \frac{a^4}{2}.$$

Therefore the polynomial $E(x, y) = y^2 + x^2 - x^4/2$ solves **(a)**.

Now let $R = f^2 + g^2$ and $\tan\theta = g/f$; that is, pass to polar coordinates in (f, g) but use the squared length. [We will see that R is never 0. This means that f and g are never simultaneously 0, so there is a continuous choice of θ.] Note that

$$\frac{d}{dt}\left(f(t)g'(t) - g(t)f'(t)\right) = f(t)g''(t) - g(t)f''(t) = 0.$$

Therefore $f(t)g'(t) - g(t)f'(t) = ab$ and

$$\theta' = \frac{f(t)g'(t) - g(t)f'(t)}{f(t)^2 + g(t)^2} = \frac{ab}{R}.$$

Also,

$$\begin{aligned} R'' &= 2\left[(f')^2 + (g')^2\right] + 2ff'' + 2gg'' \\ &= R^2 - 2R + 2b^2 + 2a^2 - a^4 - 2R(1 - R) = 3R^2 - 4R + 2b^2 + 2a^2 - a^4. \end{aligned}$$

Multiply by $2R'$ and integrate to obtain

$$\begin{aligned} (R')^2 &= 2R^3 - 4R^2 + 2(2b^2 + 2a^2 - a^4)R - 4a^2b^2 \\ &= 2(R - a^2)\left(R^2 + (a^2 - 2)R + 2b^2\right). \end{aligned} \tag{1}$$

Suppose now that f and g are periodic in t with period T and that R does not reach a local maximum at $t = 0$. Since $R(0) = a^2$ and $R'(0) = 0$, it follows that $R''(0) = 2(a^4 - a^2 + b^2) \le 0$. If $R''(0) = 0$, then $b^2 = a^2(1 - a^2)$, and the solution is

$$f(t) = a\cos\left(\frac{bt}{a}\right), \quad g(t) = a\sin\left(\frac{bt}{a}\right).$$

However, this gives $R(t) = a^2$ for all t, so that $t = 0$ is a local minimum. Thus we must have $b^2 < a^2(1 - a^2)$. Since $b > 0$, we conclude that $a < 1$.

Suppose now that $b^2 < a^2(1 - a^2)$. The polynomial $p(x) = x^2 + (a^2 - 2)x + 2b^2$ satisfies $p(0) = 2b^2 > 0$ and $p(a^2) = p(2(1 - a^2)) < 0$, ensuring that p has a positive zero, call it d^2, with $d^2 < \min\{a^2, 2(1 - a^2)\}$. From the definition of d, we have $2b^2 = d^2(2 - a^2 - d^2)$. Hence (1) becomes

$$(R')^2 = 2(R - a^2)(R - d^2)(R - 2 + a^2 + d^2),$$

so with the abbreviations $X = a^2 - d^2$, $Y = 2 - a^2 - 2d^2$, and $Z = 2 - a^2 - d^2$, $R(t)$ can be expressed in terms of Jacobi elliptic functions as

$$R(t) = d^2 + X\,\mathrm{sn}^2\left(K\left(\frac{X}{Y}\right) - t\sqrt{1 - d^2 - \frac{a^2}{2}}\;\middle|\;\frac{X}{Y}\right),$$

where $sn(z|m)$ is a Jacobi elliptic function and $K(m)$ is the complete elliptic integral of the first kind. Note that $R(t) \geq d^2$, so R is never 0.

Observe that $R(t)$ is periodic in t with period

$$\begin{aligned} T_0 &= \sqrt{2}\int_{d^2}^{a^2} \frac{dx}{\sqrt{(a^2 - x)(x - d^2)(Z - x)}} \\ &= 2\sqrt{2}\int_0^{\pi/2} \frac{d\phi}{\sqrt{Y - X\sin^2\phi}} = \frac{2}{\sqrt{1 - d^2 - a^2/2}} K\left(\frac{X}{Y}\right). \end{aligned}$$

Note that $0 < m = X/Y < 1$. Thus $K(m) > \pi/2$, whence $T_0 > \pi$. As t varies over one full period of R, the change in θ is given by

$$\begin{aligned} \Delta\theta &= ab\int_0^{T_0} \frac{dt}{R(t)} = ab\sqrt{2}\int_{d^2}^{a^2} \frac{dx}{x\sqrt{(a^2 - x)(x - d^2)(Z - x)}} \\ &= 2ad\sqrt{Z}\int_0^{\pi/2} \frac{d\phi}{(d^2 + X\sin^2\phi)\sqrt{Y - X\sin^2\phi}} \\ &= \frac{2a\sqrt{Z}}{d\sqrt{2 - 2d^2 - a^2}}\,\Pi\left(-\frac{X}{d^2}\;\middle|\;\frac{X}{Y}\right), \end{aligned} \tag{2}$$

where $\Pi(n|m)$ is the complete elliptic integral of the third kind.

The functions f and g have a common period if and only if θ (modulo 2π) and R have a common period, which happens if and only if $\Delta\theta$ is a rational multiple of 2π. Since $d^2 < 2(1 - a^2)$, and $\Delta\theta$ is a nonconstant analytic function of a and d on the domain $D = \{(a, b) : 0 < a < d < 1, d^2 < 2(1 - a^2)\}$, we conclude that $\Delta\theta$ is not locally constant. Thus the set of (a, d) for which $\Delta\theta$ is a rational multiple of 2π will be dense in D. Since there is a nonempty open set of pairs (a, d) for which $d^2 = \min R(t) < a^2/2 = \max R(t)$, the domain for $\Delta\theta$ has nonempty interior. Thus though we have not found an explicit pair of values for a and b that would meet the conditions of (**d**), they do exist.

It remains only to show that the common period T of f and g satisfies $T > 2\pi$. To see this look at the two factors in (2) in the integral with respect to ϕ. One is an increasing function of ϕ, the other a decreasing function of ϕ. Hence they are negatively correlated, and we get an upper bound by splitting the integral:

$$\Delta\theta \le \frac{4ad\sqrt{Z}}{\pi}\int_0^{\pi/2}\frac{d\phi}{d^2+X\sin^2\phi}\int_0^{\pi/2}\frac{d\phi}{\sqrt{Z-X\sin^2\phi}}$$
$$= 2\sqrt{Z}\int_0^{\pi/2}\frac{d\phi}{\sqrt{Z-X\sin^2\phi}} = T_0\sqrt{1-\frac{a^2+d^2}{2}} < T_0.$$

If $T_0 \le 2\pi$, then evidently $\Delta\theta < 2\pi$, hence at least two periods of R are required before θ repeats modulo 2π. Thus $T \ge 2T_0 > 2\pi$. If $T_0 > 2\pi$, then $T \ge T_0 > 2\pi$. In either case we are done.

All parts also solved by N. Thornber; parts **(a)**–**(c)** also solved by P. Bracken and the proposer.

An Oldie

11098 [2004, 625]. *Proposed by Christopher Hillar and Darren Rhea, University of California, Berkeley, CA.*Let

$$f(n) = \sum_{i=1}^{n}\frac{(-1)^{i+1}}{2^i-1}\binom{n}{i}.$$

Prove that there are constants c and c' such that $c \le f(n)/\log n \le c'$ for sufficiently large n (that is, $f(n) = \Theta(\log n)$).

Solution by Godfrey L. Isaacs, Hollywood, FL. Since $2^i - 1 = 2^i\left(1-(1/2)^i\right)$, we have $f(n) = \sum_{k\ge 1} t(n,k)$, where $t(n,k) = 1-(1-1/2^k)^n$. Since $t(n,k)$ decreases as k increases, we have $I(n) - 1 < f(n) < I(n)$, where

$$I(n) = \int_0^\infty \left[1-\left(1-2^{-x}\right)^n\right]dx.$$

Substituting $w = 1 - 1/2^x$ and multiplying by $\log 2$ gives

$$I(n)\log 2 = \int_0^1 \frac{1-w^n}{1-w}\,dw = \int_0^1 \sum_{j=0}^{n-1} w^j\,dw = \sum_{j=1}^{n}\frac{1}{j} = \log n + \gamma + o(1),$$

where γ is Euler's constant. Thus $f(n)/\log n \to 1/\log 2$, which implies the result.

Editorial comment by Donald Knuth (writing from Texas A & M University): This function $f(n) = u_{n+1} - u_n$ is well known to algorithmic analysts because it arises, for example, in the analysis of "radix-exchange sorting" and also "digital search tries." See my book *Sorting and Searching* (Addison-Wesley, 1973), exercises 5.2.2–48 or 5.2.2–54, for a detailed asymptotic expansion. Generalizations with 2 replaced by m are in exercise 6.3–19. Many other authors have refined these estimates and found alternative proofs since the book came out. It was N. G. de Bruijn who showed me how to solve the problem when I mentioned it to him in the mid-1960s.

Also solved by B. M. Ábrego, Z. Ahmed & M. A. Prasad (India), S. Amghibech (Canada), A. B. Aruffo (Italy), M. R. Avidon, R. Bagby, P. Bracken R. Chapman (U. K.), Y. Dumont (France), J.-P. Grivaux (France), S. Y. Jeon (Korea), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Nijenhuis, Y. Rubinstein & Z. Rubinstein, O. G. Ruehr, N. C. Singer, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), D. B. Tyler, L. Zhou, GCHQ Problem Solving Group (U. K.), NSA Problem Solving Group, and the proposers.