PROBLEMS AND SOLUTIONS

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before May 31, 2007. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11271. *Proposed by Iliya Bluskov, University of Northern British Columbia, Prince George, BC, Canada.* Let N be a positive even integer. A *placement of queens* on an $N \times N$ chessboard is a set of N squares on the board such that none of these squares lies on either long diagonal, and no two of these squares lie in a single row, column, or diagonal (that is, the queens are non-attacking). A *cover* of an $N \times N$ board is a set of $N-2$ disjoint placements. (Thus on a 4×4 board, the placements $\{(1,3),(2,1),(3,4),(4,2)\}$ and $\{(1,2),(2,4),(3,1),(4,3)\}$ form a cover.)
(**a**) Show that there exists a cover of the $N \times N$ board if $N+1$ is prime.
(**b**) Give an example of an even N for which $N+1$ is not prime and for which there is no cover.
(**c**) Give an example of an even N, and a cover, for which $N+1$ is not prime.

11272. *Proposed by Vasile Mihai, Belleville, ON, Canada.* Let ABC be an acute nonequilateral triangle, and let H be its orthocenter, O its circumcenter, and K its *symmedian point* (defined below). Let R be the circumradius of ABC. Let L be the line through H and parallel to the line segment KO.
(**a**) Show that there are exactly two solution points V on L to the equations

$$\frac{|VA|}{|BC|} = \frac{|VB|}{|CA|} = \frac{|VC|}{|AB|}.$$

(**b**) For the solutions V_1 and V_2 in (a), show that $|HV_1| \cdot |HV_2| = 4R^2$.
(The symmedian point of a triangle is the point of concurrence of its three symmedian lines. The symmedian line through A is the reflection of the line through A and the centroid of ABC across the line bisecting angle BAC, and the symmedian lines through B and C are defined similarly. For more about the symmedian point, see Ross Honsberger's "Episodes in Nineteenth and Twentieth Century Euclidean Geometry," MAA, Washington, D.C., 1995.)

11273. *Proposed by Marian Tetiva, Bîrlad, Romania.* Let $\langle a_n \rangle$ be the sequence such that $a_n = n$ for $n \le 6$ and $a_n = \lfloor (a_1 + \cdots + a_{n-1})/2 \rfloor$ for $n > 6$. Let r_n be the number in $\{0, 1, 2\}$ congruent to $\sum_{k=1}^{n} a_k$ modulo 3. Show that for $n \geq 6$, $\{a_1, \ldots, a_n\}\backslash\{r_n\}$ may be partitioned into three subsets with equal sum. (For example, with $n = 7$, $\{2, 3, 4, 5, 6, 10\} = \{2, 3, 5\} \cup \{4, 6\} \cup \{10\}$.)

11274. *Proposed by Donald Knuth, Stanford, CA.* Prove that for nonnegative integers m and n,

$$\sum_{k=0}^{m} 2^k \binom{2m-k}{m+n} = 4^m - \sum_{j=1}^{n} \binom{2m+1}{m+j}.$$

11275. *Proposed by Michael S. Becker, University of South Carolina at Sumter, Sumter, SC.* Find

$$\int_{y=0}^{\infty} \int_{x=y}^{\infty} \frac{(x-y)^2 \log((x+y)/(x-y))}{xy \sinh(x+y)}\, dx\, dy.$$

11276. *Proposed by Eugene Herman, Grinnell College, Grinnell, IA.* Let $T_1, \ldots, T_n$ be translations in $\mathbb{R}^3$ with translation vectors $\mathbf{t}_1, \ldots, \mathbf{t}_n$, and let R be a rotational linear transformation on $\mathbb{R}^3$ that rotates space through an angle of π/n about an axis parallel to a vector $\mathbf{r}$. Define a transformation C by $C = (RT_n \cdots RT_2RT_1)^2$. Prove that C is a translation, find an explicit formula for its translation vector in terms of $\mathbf{r}$, n, and $\mathbf{t}_1, \ldots, \mathbf{t}_n$, and prove that there is a line ℓ in $\mathbb{R}^3$, independent of $\mathbf{t}_1, \ldots, \mathbf{t}_n$, such that C translates space parallel to ℓ.

SOLUTIONS

Exponential Growth of a Solution

11137 [2005, 181]. *Proposed by Vicenţiu Rădulescu, University of Craiova, Romania.* Let ϕ be a continuous positive function on the open interval (A, ∞), and assume that f is a C^2-function on (A, ∞) satisfying the differential equation

$$f''(t) = (1 + \phi(t)(f^2(t) - 1))f(t).$$

(**a**) Given that there exists $a \in (A, \infty)$ such that $f(a) \geq 1$ and $f'(a) \geq 0$, prove that there is a positive constant K such that $f(x) \geq Ke^x$ whenever $x \geq a$.
(**b**) Given instead that there exists $a \in (A, \infty)$ such that $f'(a) < 0$ and $f(x) > 1$ if $x > a$, prove that there exists a positive constant K such that $f(x) \geq Ke^x$ whenever $x \geq a$.
(**c**) Given that f is bounded on (A, ∞) and that there exists $\alpha > 0$ such that $\phi(x) = O(e^{-(1+\alpha)x})$, prove that $\lim_{x\to\infty} e^x f(x)$ exists and is finite.

Solution by Richard Stong, Rice University, Houston, TX.
(**a**) Let $b = \inf\{t \geq a\colon\ f(t) < 1 \text{ or } f'(t) < 0\}$. If $b < \infty$, then $f(b) \geq 1$ and $f'(b) \geq 0$. Hence the differential equation yields $f''(b) \geq 1$. It follows that $f'(t) > 0$ on some interval $[b, b+\epsilon)$ and hence that $f(t) \geq 1$ on this interval. This contradicts the definition of b. Hence $b = \infty$; i.e., $f(t) \geq 1$ and $f'(t) \geq 0$ for all $t \geq a$. Thus the differential equation gives $f''(t) \geq f(t)$ for all $t \geq a$. Now note that using the weaker inequality $f''(t) \geq 1$ and integrating twice gives $f(t) \geq 1 + (t-a)^2/2$; hence f is unbounded. From the full inequality we get $2f'(t)f''(t) \geq 2f(t)f'(t)$, and integrating gives

$$\left(f'(t)\right)^2 - \left(f'(a)\right)^2 \geq f^2(t) - f^2(a).$$

Hence in particular $(f'(t))^2 \geq f^2(t) - f^2(a)$ for $t \geq a$. Rewriting this in the form $f'(t)/\sqrt{f^2(t) - f^2(a)} \geq 1$ and integrating over $[a, x]$ gives $\cosh^{-1}(f(x)/f(a)) \geq x - a$, or $f(x) \geq f(a)\cosh(x - a) \geq e^x f(a)/(2e^a)$ for $t \geq a$. Hence f satisfies the required inequality with $K = f(a)/(2e^a)$.

(**b**) We prove a stronger result, assuming only that $f(x) \geq 1$ for $x \geq a$. That is, we allow equality and do not require that $f'(a) < 0$. Since $f(x) \geq 1$ for all $x \geq a$, the differential equation gives $f''(t) \geq 1$. Integrating over $[a, c]$ gives $f'(c) \geq c - a + f'(a)$. Hence we can choose $c > a$ with $f'(c) \geq 0$. Applying part (**a**) to $x \geq c$ shows that there is a constant $K_1 > 0$ such that $f(x) \geq K_1 e^x$ for $x \geq c$. Continuity and positivity of f on $[a, c]$ shows there is a constant $K_2 > 0$ such that $f(x) \geq K_2 e^x$ on $[a, c]$ and hence $K = \min(K_1, K_2)$ suffices for $[a, \infty)$.

(**c**) Applying part (**a**) and our version of (**b**) to f and $-f$ shows that if f is bounded, then $|f|$ cannot cross from smaller than 1 to at least 1, and cannot remain forever at least 1. Thus there is some d with $|f(x)| < 1$ for $x \geq d$. Choosing d larger, if necessary, we may assume $\phi(x) < 1$ for $x \geq d$.

Now if f is eventually zero, we are done, so assume not. For $r > d$ we claim: if $f(r) > 0$, then $f'(r) < 0$ (and similarly if $f(r) < 0$ then $f'(r) > 0$).

Indeed, suppose there is $r > d$ with $f(r) > 0$ and $f'(r) \geq 0$. Let $s = \inf\{t \geq r : f(t) < 0 \text{ or } f'(t) < 0\}$. If $s < \infty$, then $f(s) \geq f(r) > 0$ and $f'(s) = 0$. The differential equation then gives $f''(s) \geq (1 - \phi(s))f(s) > 0$, so $f'(t) \geq 0$ on some interval $[s, s + \epsilon)$. Also $f(t) \geq 0$ on this interval, which contradicts the definition of s. Therefore $s = \infty$, i.e., $f(t) \geq 0$ and $f'(t) \geq 0$ for all $t \geq r$. It follows from the differential equation that $f''(t) \geq 0$, and hence that f and f' are nondecreasing on $[r, \infty)$. It follows that f, f', and f'' are strictly positive on (r, ∞) and hence f is unbounded, a contradiction.

Thus f and f' always have opposite signs. Since f is not eventually zero, and f can never leave zero, f is never zero.

Without loss of generality, we may assume $f(t) > 0$ and $f'(t) < 0$ on $[d, \infty)$. It follows that $f''(t) > 0$ on $[d, \infty)$, and hence f' is increasing. Thus $\lim_{t\to\infty} f'(t)$ exists; since f is bounded, the limit must be zero. Similarly, $\lim_{t\to\infty} f''(t)$ exists (it equals $\lim_{t\to\infty} f(t)$); since f' is bounded, this limit must be zero.

The differential equation gives $f''(t) = f(t) - \phi(t)f(t)(1 - f^2(t)) < f(t)$. Hence (since $f' < 0$) we have $2f'(t)f''(t) > 2f(t)f'(t)$. Integrating over $[x, \infty)$ gives $(f'(x))^2 < f^2(x)$ or $0 > f'(x) > -f(x)$. Hence $(e^x f(x))' = e^x(f'(x) + f(x)) > 0$. Thus $e^x f(x)$ is an increasing function of x.

Let K be a constant with $\phi(t) < Ke^{-(1+\alpha)t}$. Since $f''(t) > (1 - \phi(t))f(t)$, $2f'(t)f''(t) < 2(1 - \phi(t))f(t)f'(t) < 2\left(1 - Ke^{-(1+\alpha)t}\right) f(t)f'(t)$. Hence

$$\begin{aligned}\left(f'(x)\right)^2 &= -\int_x^\infty 2f'(t)f''(t)\,dt > -\int_x^\infty 2\left(1 - Ke^{-(1+\alpha)t}\right) f(t)f'(t)\,dt \\ &\geq -\left(1 - Ke^{-(1+\alpha)t}\right)\int_x^\infty 2f(t)f'(t)\,dt = \left(1 - Ke^{-(1+\alpha)t}\right) f^2(x).\end{aligned}$$

If $0 < B < 1$, then $\sqrt{1 - B} > 1 - B$, so this gives

$$f'(x) < -f(x) + Ke^{-(1+\alpha)x} f(x) < -f(x) + Ke^{-(1+\alpha)x}.$$

Thus $0 < (e^x f(x))' = e^x(f'(x) + f(x)) < Ke^{-\alpha x}$. Since the upper bound is integrable, we see that $e^x f(x)$ is bounded above. Combined with the result of the previous paragraph, this shows that $\lim_{x\to\infty} e^x f(x)$ exists and is finite.

Also solved by P. Bracken & N. Nadeau, and the proposer.

A Quadrature on a Sphere

11159 [2005, 567]. *Proposed by George Lamb, Tucson, AZ.* For $|a| < \pi/2$, evaluate in closed form

$$I(a) = \int_0^{\pi/2}\int_0^{\pi/2} \frac{\cos\psi\, d\psi\, d\varphi}{\cos(a\cos\psi\cos\varphi)}.$$

Solution by J. A. Grzesik, Northrop Grumman Space Technology, Redondo Beach, CA. The answer may be written in these two equivalent forms:

$$I(a) = \frac{\pi}{2a}\log\left\{\tan\left(\frac{a}{2}+\frac{\pi}{4}\right)\right\} = \frac{\pi}{2a}\log\left\{\frac{1+\tan(a/2)}{1-\tan(a/2)}\right\}.$$

Begin with the change of variables $\psi = \pi/2 - \theta$; then

$$I(a) = \int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin\theta\, d\theta\, d\varphi}{\cos(a\sin\theta\cos\varphi)},$$

which is an integration over the positive octant of the unit sphere, with angle θ measured from the z-axis, and then $\sin\theta\cos\varphi$ in the argument of the denominator is the projection on the x-axis of the vector that traces out the given octant of the unit sphere. If we consider instead the polar axis to be the x-axis and θ the angle from the x-axis, then writing $\mu = \cos\theta$ yields $I(a) = \frac{\pi}{2}\int_0^1 d\mu/\cos(a\mu)$. The two logarithmic answers then come from two standard quadratures of the secant.

Editorial comment. Eugene A. Herman (Grinnell College) and M. L. Glasser (Clarkson University), independently, noted the generalization: If f is integrable on [0, 1], then

$$\int_0^{\pi/2}\int_0^{\pi/2} f(\cos\psi\cos\varphi)\cos\psi\, d\psi\, d\varphi = \frac{\pi}{2}\int_0^1 f(t)\, dt.$$

Also solved by R. Bagby, D. Beckwith, B. Bradie, R. Chapman (U. K.), W. Chu (Italy), P. Deiermann, Y. Dumont (France), O. Furdui, T. Jager, M. L. Glasser, G. C. Greubel, J. Grivaux (France), E. A. Herman, K. D. Lathrop, O. P. Lossers (Netherlands), P. Magli (Italy), K. McInturff, R. Richberg (Germany), O. G. Ruehr, V. Schindler (Germany), T. P. Schonbek, H.-J. Seiffert (Germany), J. Singh (India), A. Stadler (Switzerland), R. Stong, N. Thornber, D. B. Tyler, GCHQ Problem Solving Group (U. K.), Microsoft Problem Solving Group, and the proposer.

Series with Sines

11162 [2005, 567]. *Proposed by Paolo Perfetti, University "Tor Vergata", Rome, Italy.*
(**a**) Show that if c is a real number less than 2 then $\sum_{k=1}^{\infty} k^{-c-\sin k}$ diverges.
(**b**) Determine whether $\sum_{k=1}^{\infty} k^{-1-|\sin k|}$ converges.

Solution by Microsoft Research Problems Group, Redmond, WA.
(**a**) If $c \le 0$, then the series diverges by comparison with the harmonic series. If $0 < c < 2$, then let α, β be such that $\sin\alpha = \sin\beta = 1 - c$ and $\pi/2 < \alpha < \beta < 5\pi/2$. Note that $\sin x < 1 - c$ for all x with $\alpha < x < \beta$. Define

$$S = \{k \in \mathbb{N}\colon\ \sin k < 1 - c\} = \mathbb{N} \cap \bigcup_{j\in\mathbb{Z}} \big(2\pi j + \alpha, 2\pi j + \beta\big).$$

Since π is irrational, the sequence $n/(2\pi)$ mod 1 is dense in [0, 1], so $S \ne \emptyset$ and there exist positive integers a, A, b, B with

$$0 < a - 2\pi A < \frac{\beta-\alpha}{2} \quad \text{and} \quad -\frac{\beta-\alpha}{2} < b - 2\pi B < 0.$$

Let $C = \max(a, b)$. Given $k \in S$, we claim that there exists $m \in S$ with $k < m \leq k + C$. Indeed, there exists $j \in \mathbb{Z}$ with $\alpha < k - 2\pi j < \beta$. Compare $k - 2\pi j$ to $(\alpha + \beta)/2$. If $\alpha < k - 2\pi j \leq (\alpha + \beta)/2$, then $\alpha < (k + a) - 2\pi(j + A) < \beta$, so we can use $m = k + a$. On the other hand, if $(\alpha + \beta)/2 < k - 2\pi j < \beta$, then $\alpha < (k + b) - 2\pi(j + B) < \beta$, so we can use $m = k + b$.

Thus S is infinite. Index it in increasing order, $k_1 < k_2 < \cdots$. Now $k_j \leq k_1 + C(j - 1)$, so

$$\sum_{k=1}^{k_n} k^{-c-\sin k} \geq \sum_{\substack{1\leq k\leq k_n \\ k\in S}} k^{-c-\sin k} \geq \sum_{\substack{1\leq k\leq k_n \\ k\in S}} \frac{1}{k} = \sum_{j=1}^{n} \frac{1}{k_j} \geq \sum_{j=1}^{n} \frac{1}{k_1 + C(j-1)},$$

which diverges as $n \to \infty$ by the integral test.

(**b**) This series also diverges. For positive integer n, define

$$A_n = [0, 2^n) \cap \left\{k \in \mathbb{N} \colon \ |\sin k| < \tfrac{1}{n}\right\}, \qquad B_n = [2^{n-1}, 2^n) \cap A_n.$$

If $k \in B_n$, then $k^{-1-|\sin k|} > (2^n)^{-1-1/n} = 2^{-n-1}$. If $n > 1$, then A_n is contained in the disjoint union of A_{n-1} and B_n, so $|B_n| \geq |A_n| - |A_{n-1}|$. To estimate $|A_n|$, partition the unit circle into $7n$ arcs, each with angle $2\pi/(7n)$. Of the values e^{ik} for $0 \leq k < 2^n$, at least $2^n/(7n)$ lie in the same arc by the Pigeonhole Principle. If e^{ik_1} and e^{ik_2} lie in the same arc, then

$$|\sin(k_1 - k_2)| \leq \left|e^{i(k_1-k_2)} - 1\right| = \left|e^{ik_1} - e^{ik_2}\right| < \frac{2\pi}{7n} < \frac{1}{n}$$

and $|k_1 - k_2| \in A_n$. Subtracting the smallest k from the others (and itself), we find that $|A_n| \geq 2^n/(7n)$. Now if $N \geq 2$, then

$$\begin{aligned}
\sum_{k=2}^{2^N-1} k^{-1-|\sin k|} &= \sum_{n=2}^{N} \sum_{k=2^{n-1}}^{2^n-1} k^{-1-|\sin k|} \geq \sum_{n=2}^{N} \sum_{k\in B_n} k^{-1-|\sin k|} \geq \sum_{n=2}^{N} \frac{|B_n|}{2^{n+1}} \\
&\geq \sum_{n=2}^{N} \frac{|A_n| - |A_{n-1}|}{2^{n+1}} = \sum_{n=2}^{N} \left(\left(\frac{|A_n|}{2^{n+2}} - \frac{|A_{n-1}|}{2^{n+1}} \right) + \frac{|A_n|}{2^{n+2}} \right) \\
&= \frac{A_N}{2^{N+2}} - \frac{|A_1|}{8} + \sum_{n=2}^{N} \frac{|A_n|}{2^{n+2}} \\
&\geq -\frac{|A_1|}{8} + \sum_{n=2}^{N} \frac{2^n/(7n)}{2^{n+2}} = -\frac{|A_1|}{8} + \sum_{n=2}^{N} \frac{1}{28n}.
\end{aligned}$$

This grows without bound as $N \to \infty$.

Also solved by M. R. Avidon, P. Budney, Y. Dumont (France), J. H. Lindsey II, A. Stadler (Switzerland), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

Ginzburg–Landau Energy

11167 [2005, 654]. *Proposed by Vicenţiu Rădulescu, University of Craiova,Craiova, Romania.* Let Ω be the set of all complex numbers z satisfying $0 < |z| < 1$. Fix a positive integer n and, for all distinct elements $z_1, \ldots, z_n$ in Ω, define the function

$$f(z_1, \ldots, z_n) = \frac{\prod_{j=1}^{n} |z_j|^2(1 - |z_j|^2) \cdot \prod_{1\leq j<k\leq n} |z_j| \cdot |z_k| \cdot |z_j - z_k|^2 \cdot}{\prod_{1\leq j<k\leq n} |z_j| \cdot |z_k| \cdot \left[|z_j - z_k|^2 + (1 - |z_j|^2)(1 - |z_k|^2)\right]}.$$

(**a**) Prove that if $n = 2$ then the maximum of f is attained for a unique configuration (up to a rotation) that consists of two points symmetric with respect to the origin.
(**b**) Prove that if $n = 3$ then the maximal configuration for f is also unique (up to a rotation) and consists of three points arranged as the vertices of an equilateral triangle centered at the origin.

Solution by Microsoft Research Problems Group, Redmond, WA. Observe $|z_j - z_k|^2 + (1 - |z_j|^2)(1 - |z_k|^2) = 1 - \overline{z_j}z_k - \overline{z_k}z_j + |z_j|^2|z_k|^2 = |1 - \overline{z_j}z_k|^2$. We can rewrite

$$f(z_1, \ldots, z_n) = \prod_{j=1}^{n} |z_j|^{2n}\left(1 - |z_j|^2\right) \cdot \prod_{1 \le j < k \le n} |z_j - z_k|^2 \, |1 - \overline{z_j}z_k|^2.$$

Write $r_j = |z_j|$ for $1 \le j \le n$ and $R = \sqrt{(r_1^2 + \cdots + r_n^2)/n}$. Then $0 < r_j < 1$ and $0 < R < 1$. By the arithmetic-geometric mean inequality, we have $r_1 \cdots r_n \le R^n$ and $\prod_{j=1}^{n}(1 - r_j^2) \le (1 - R^2)^n$. Equality holds only if $r_1 = r_2 = \cdots = r_n = R$. Also

$$\begin{aligned}\sum_{1 \le j < k \le n} |z_j - z_k|^2 &= \sum_{1 \le j < k \le n} (r_j^2 + r_k^2 - \overline{z_j}z_k - z_j\overline{z_k}) \\ &= n\left(r_1^2 + \cdots + r_n^2\right) - \left|z_1 + \cdots + z_n\right|^2 \le n^2R^2.\end{aligned}$$

Equality holds only if $z_1 + \cdots + z_n = 0$. Again applying the arithmetic-geometric mean inequality, we get

$$\prod_{1 \le j < k \le n} |z_j - z_k|^2 \le \left(\frac{n^2R^2}{n(n-1)/2}\right)^{n(n-1)/2} = \left(\frac{2nR^2}{n-1}\right)^{n(n-1)/2}.$$

Equality holds only if all $|z_j - z_k|^2$ for $j < k$ are equal. Similarly,

$$\begin{aligned}\sum_{1 \le j < k \le n} \left|1 - \overline{z_j}z_k\right|^2 &= \sum_{1 \le j < k \le n} \left(1 + r_j^2r_k^2 - \overline{z_j}z_k - z_j\overline{z_k}\right) \\ &= \frac{n(n-1)}{2} + \sum_{1 \le j < k \le n} r_j^2r_k^2 + \sum_{j=1}^{n} r_j^2 - \left|z_1 + \cdots + z_n\right|^2 \\ &\le \frac{n(n-1)}{2} + nR^2 + \sum_{1 \le j < k \le n}\left(r_j^2r_k^2 + \frac{(r_j^2 - r_k^2)^2}{2n}\right) \\ &= \frac{n(n-1)}{2} + nR^2 + \frac{n-1}{2n}(r_1^2 + \cdots + r_n^2)^2 \\ &= \frac{n(n-1)}{2}\left(1 + \frac{2R^2}{n-1} + R^4\right)\end{aligned}$$

and

$$\prod_{1 \le j < k \le n} |1 - \overline{z_j}z_k|^2 \le \left(1 + \frac{2R^2}{n-1} + R^4\right)^{n(n-1)/2}.$$

Combining all these bounds, we get

$$f(z_1, \cdots, z_n) \le (R^n)^{2n}(1-R^2)^n \left(\frac{2nR^2}{n-1}\right)^{n(n-1)/2} \left(1 + \frac{2R^2}{n-1} + R^4\right)^{n(n-1)/2}$$
$$= \left(\left(\frac{2n}{n-1}\right)^{(n-1)/2} R^{3n-1}(1-R^2)\left(1+\frac{2R^2}{n-1}+R^4\right)^{(n-1)/2}\right)^n.$$

Equality can occur only if $r_1 = \cdots = r_n = R$, $z_1 + \cdots + z_n = 0$, and all $|z_j - z_k|^2$ for $j \ne k$ are equal.

For (**a**), where $n = 2$, the bound simplifies to

$$f(z_1, z_2)^{1/2} \le 2R^5(1-R^2)(1+2R^2+R^4)^{1/2} = 2R^5(1-R^4).$$

The right side attains its maximum at $R = R_{\max} := (5/9)^{1/4}$. Equality implies $r_1 = r_2 = R_{\max}$ and $z_1 + z_2 = 0$. The maximum value $2^6 5^{5/2}/3^9$ of f is achieved when $|z_1| = R_{\max}$ and $z_2 = -z_1$.

For (**b**), where $n = 3$, the bound simplifies to

$$f(z_1, z_2, z_3)^{1/3} \le 3R^8(1-R^2)(1+R^2+R^4) = 3R^8(1-R^6).$$

The right side attains its maximum at $R = R_{\max} = (4/7)^{1/6}$. The maximum value $2^8 3^6/7^7$ of f is achieved when z_1, z_2, z_3 form an equilateral triangle of side $R_{\max}\sqrt{3}$ centered at the origin—such a triple is in the domain of f. Any maximum requires $z_1 + z_2 + z_3 = 0$, so these are the only maxima.

Editorial comment. The proposer notes that $W(z_1, \cdots, z_n) = -\pi \log f(z_1, \cdots, z_n)$ is the *renormalized Ginzburg–Landau energy* corresponding to the open set $\Omega = \{z \in \mathbb{C}\colon\ 0 < |z| < 1\}$. See F. Bethuel, H. Brezis, F. Hélein, *Ginzburg–Landau Vortices,* Birkhäuser, Boston, 1994.

Also solved by D. R. Bridges, L. Zhou, and the proposer.

A Functional Equation with Exponential Solutions

11180 [2005, 839]. *Proposed by Suat Namli, Louisiana State University, Baton Rouge, LA.* Find all real-valued functions f defined on some interval I about the origin by a power series having all coefficients nonzero and possessing the property that for all real s and t there exist constants $A_{s,t}$, $B_{s,t}$, and $C_{s,t}$ such that whenever sx, tx, and $(s+t)x$ lie in I,

$$f(sx)f(tx) = A_{s,t}f(sx) + B_{s,t}f(tx) + C_{s,t}f((s+t)x).$$

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. We use only the assumptions that f has a power series around $x = 0$, $f(0) \ne 0$, $f'(0) \ne 0$ and that the functional equation holds when $s = t = 1$:

$$f^2(x) = af(x) + bf(2x).$$

Now $b \ne 0$, for otherwise f would be a constant function. Taking $x = 0$, we have $f(0)\big(f(0) - a - b\big) = 0$, so $f(0) = a + b$. Differentiating and setting $x = 0$, we get $f'(0)\,(2f(0) - a - 2b) = 0$, so $a = 0$ and $f(0) = b$. The functional equation is $f^2(x) = bf(2x)$. Let $h(x) = \log\big(f(x)/b\big)$. Since $f(0) = b$, also h has a power series expansion around $x = 0$, and it satisfies $2h(x) = h(2x)$. From the power series it follows that $2h^{(n)}(0) = 2^n h^{(n)}(0)$ for all n. Therefore $h^{(n)}(0) = 0$ for all $n \neq 1$, so $h(x) = ux$ for some constant u. Therefore $f(x) = b\exp(ux)$.

Editorial comment. Stephen Gagola proved a variant: If the functional equation holds only for all real nonzero s and t satisfying $s \neq t$, then we can have solutions of the form $f(x) = C(1-kx)^{-1}$ as well as solutions of the form $f(x) = Ce^{kx}$.

Also solved by D. Beckwith, M. Bello-Hernandez & M. Benito (Spain), R. Chapman (U. K.), K. Dale (Norway), S. Gagola, E. Herman, T. L. McCoy (Taiwan), R. Stong, X. Wang, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), Microsoft Research Problems Group, and the proposer.

Apply Hölder

11202 [2006, 179]. *Proposed by Grahame Bennett, Indiana University, Bloomington, IN.* Prove that if $\langle a_n \rangle$ is a sequence of positive numbers with $\sum_{n=1}^{\infty} a_n < \infty$, then for all p in (0, 1)

$$\lim_{n\to\infty} n^{1-1/p}\left(a_1^p + \cdots + a_n^p\right)^{1/p} = 0.$$

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA. Define $\langle x_n \rangle$ by $x_n = (1/n)^{1-p}(a_1^p + \cdots + a_n^p)$. Since

$$n^{1-1/p}(a_1^p + \cdots + a_n^p)^{1/p} = ((1/n)^{1-p}(a_1^p + \cdots + a_n^p))^{1/p} = x_n^{1/p},$$

it suffices to show that $x_n \to 0$. The Hölder inequality will be applied as follows:

$$\sum_k \left(\frac{1}{n}\right)^{1-p} a_k^p \leq \left(\sum_k \frac{1}{n}\right)^{1-p} \left(\sum_k a_k\right)^p.$$

Given any $\epsilon > 0$, choose N such that $\sum_{n=N}^{\infty} a_n < (\epsilon/2)^{1/p}$. For any n larger than both N and $((a_1^p + \cdots + a_N^p)2/\epsilon)^{1/(1-p)}$, we have

$$\begin{aligned}
x_n &= \left(\frac{1}{n}\right)^{1-p}(a_1^p + \cdots + a_N^p) + \left(\frac{1}{n}\right)^{1-p}\sum_{k=N+1}^{n} a_k^p \\
&< \frac{\epsilon}{2} + \left(\sum_{k=N+1}^{n}\frac{1}{n}\right)^{1-p}\left(\sum_{k=N+1}^{n} a_k\right)^p = \frac{\epsilon}{2} + \left(\frac{n-N}{n}\right)^{1-p}\left(\sum_{k=N+1}^{n} a_k\right)^p \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Editorial comment. The sequence $\{1/n\}$ can be replaced by any sequence $\{b_n\}$ of positive numbers such that $\{nb_n\}$ is bounded; the same proof applies.

Also solved by A. Alt, S. Amghibech (Canada), R. Bagby, D. Borwein, P. Budney, R. Chapman (U. K.), P. P. Dályay (Hungary), P. J. Fitzsimmons, J. Hagood, C. S. Holroyd, R. B. Israel (Canada), Y.-J. Kuo (Japan), J. H. Lindsey II , R. Mortini (France), G. Nika & H. To, I. Olkin, P. Perfetti (Italy), R. C. Raimundo (Australia), J. Rooin & A. Karam-Shafie (Iran), B. Schmuland (Canada), A. Stenger, A. Tissier (France), S. Vagi, E. I. Verriest, L. Zhou, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.