PROBLEMS AND SOLUTIONS

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Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before February 28, 2015. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11796. *Proposed by Gleb Glebov, Simon Fraser University, Burnaby, Canada.* Find

$$\int_0^\infty \frac{\sin((2n+1)x)}{\sin x} e^{-\alpha x} x^{m-1}\, dx$$

in terms of α, m, and n, when $\alpha > 0$, $m \geq 1$, and n is a nonnegative integer.

11797. *Proposed by Zhang Yun, Xi'an, Shaanxi Province, China.* Let A_1, A_2, A_3, and A_4 be the vertices of a tetrahedron. Let h_k be the length of the altitude from A_k to the plane of the opposite face, and let r be the radius of the inscribed sphere. Prove that

$$\sum_{k=1}^{4} \frac{h_k - r}{h_k + r} \geq \frac{12}{5}.$$

11798. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* For positive integers n, let f_n be the polynomial given by

$$f_n(x) = \sum_{r=0}^{n} \binom{n}{r} x^{\lfloor r/2 \rfloor}.$$

(**a**) Prove that if $n+1$ is prime, then f_n is irreducible over $\mathbb{Q}$.
(**b**) Prove that for all n (whether $n+1$ is prime or not),

$$f_n(1+x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{n-2k} x^k.$$

http://dx.doi.org/10.4169/amer.math.monthly.121.08.738

11799. *Proposed by Vicenţiu Rădulescu, King Abdulaziz University, Jeddah, Saudi Arabia.* Let a, b, and c be positive.
(**a**) Prove that there is a unique continuously differentiable function f from $[0, \infty)$ into $\mathbb{R}$ such that $f(0) = 0$ and, for all $x \geq 0$,

$$f'(x)\left(1 + a|f(x)|^b\right)^c = 1.$$

(**b**) Find, in terms of a, b, and c, the largest θ such that $f(x) = O(x^\theta)$ as $x \to \infty$.

11800. *Proposed by Oleksiy Klurman, University of Montreal, Montreal, Canada.* Let f be a continuous function from $[0, 1]$ into $\mathbb{R}^+$. Prove that

$$\int_0^1 f(x)\,dx - \exp\left[\int_0^1 \log f(x)\,dx\right] \leq \max_{0\leq x,y\leq 1}\left(\sqrt{f(x)} - \sqrt{f(y)}\right)^2.$$

11801. *Proposed by David Carter, Nahant, MA.* Let f be a polynomial in one variable with rational coefficients that has no nonnegative real root. Show that there is a nonzero polynomial g with rational coefficients such that the coefficients of fg are positive.

11802. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Let $H_{n,2} = \sum_{k=1}^n k^{-2}$, and let $D_n = n!\sum_{k=0}^n (-1)^k/k!$. (This is the *derangement number* of n, that is, the number of permutations of $\{1, \dots, n\}$ that fix no element.) Prove that

$$\sum_{n=1}^\infty H_{n,2}\frac{(-1)^n}{n!} = \frac{\pi^2}{6e} - \sum_{n=0}^\infty \frac{D_n}{n!(n+1)^2}.$$

SOLUTIONS

An Uncountable Linearly Independent Set of Binary Sequences

11658 [2012, 608]. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.* Let V be the vector space over $\mathbb{R}$ of all (countably infinite) sequences $(x_1, x_2, \dots)$ of real numbers, equipped with the usual addition and scalar multiplication. For $v \in V$, say that v is *binary* if $v_k \in \{0, 1\}$ for $k \geq 1$, and let B be the set of all binary members of V. Prove that there exists a subset I of B with cardinality $2^{\aleph_0}$ that is linearly independent over $\mathbb{R}$. (An infinite subset of a vector space is linearly independent if all of its finite subsets are linearly independent.)

Solution by Bruce S. Burdick, Roger Williams University, Bristol, RI. Given a bijection $\phi\colon \mathbb{N} \to \mathbb{Q}$, for each $r \in \mathbb{R}$, define $v(r) \in B$ by

$$v(r)_k = \begin{cases} 1 & \text{if } \phi(k) \leq r, \\ 0 & \text{if } \phi(k) > r. \end{cases}$$

Let $I = \{v(r)\colon\ r \in \mathbb{R}\}$. We claim first that v is injective. Given $r, r' \in \mathbb{R}$ with $r < r'$, let q be a rational number between r and r'. Let $k = \phi^{-1}(q)$. Since $v(r)_k = 0$ and $v(r')_k = 1$, we have $v(r) \neq v(r')$. Thus I, $\mathbb{R}$, and B have the same cardinality, $2^{\aleph_0}$.

We show also that I is a linearly independent subset of B. Given $\sum_{i=1}^n a_i v(r_i) = \vec{0}$ for distinct real numbers $r_1, \dots, r_n$, we may assign indices so that $r_1 < \cdots < r_n$. Let