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CONCENTRATION PHENOMENA AND COMPETITION EFFECTS FOR FRACTIONAL KIRCHHOFF-CHOQUARD EQUATIONS WITH EXPONENTIAL GROWTH

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Abstract In this paper, we study the fractional Kirchhoff-Choquard equation

$$M\left([u]_{s,p}^p+\varepsilon^{-N}\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\right)(\varepsilon^N(-\Delta)_p^s u+V(x)|u|^{p-2}u)$$
$$=\varepsilon^{\mu-N}\left(\int_{\mathbb{R}^N}\frac{Q(y)F(u(y))}{|x-y|^\mu}\mathrm{d}y\right)Q(x)f(u(x))\quad \text{in } \mathbb{R}^N,$$

where ε is a positive parameter, $N=ps, p\geq 2, s\in(0,1), 0<\mu<N$. The Kirchhoff function $M(t)=a+bt, a>0, b>0$, nonlinear function f has the exponential growth,

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potential functions V and Q are continuous functions satisfying some suitable conditions. Using Ljusternik-Schnirelmann category theory and variational methods, we establish the multiplicity and concentration of positive solutions for small values of the parameter.

Keywords critical exponential; fractional p-Laplace; Ljusternik-Schnirelmann category; Mountain Pass Theorem; Trudinger-Moser inequality; variational techniques

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1 Introduction and Main Results

In this paper, we study the existence, multiplicity and concentration of solutions to Kirchhoff-Choquard equation invloving fractional p-Laplacian with competing potentials as follows:

$$\begin{aligned}&M\left([u]^p_{s,p}+\varepsilon^{-N}\int\limits_{\mathbb{R}^N}V(x)|u|^p\mathrm{d}x\right)(\varepsilon^N(-\Delta)^s_pu+V(x)|u|^{p-2}u)\\&=\varepsilon^{\mu-N}\left(\int\limits_{\mathbb{R}^N}\frac{Q(y)F(u(y))}{|x-y|^\mu}\mathrm{d}y\right)Q(x)f(u(x))\quad\text{in }\mathbb{R}^N,\end{aligned}\tag{1.1}$$

where ε is small positive parameter, $0<\mu<N$, $0<s<1$, $N=ps, p\geq 2$, $M(t)=a+bt, a>0, b>0$, the potential functions V and Q are bounded from below by $V_{\min}>0$ and $Q_{\min}>0$, respectively. The reaction f has critical exponential growth, and $(-\Delta)^s_p$ represents the fractional p-Laplacian, which defines (up to a normalization constant) as

$$(-\Delta)^s_p\varphi(x)=2\lim_{\varepsilon\to 0^+}\int\limits_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\mathrm{d}y$$

for $x\in\mathbb{R}^N$, where $\varphi\in C^\infty_0(\mathbb{R}^N)$ and $B_\varepsilon(x)$ is a ball with center x and radius ε. The absorption potential V and reaction potential Q are bounded continuous functions satisfying some suitable conditions given by:

(V) $V:\mathbb{R}^N\to\mathbb{R}$ is a continuous bounded function on $\mathbb{R}^N$, satisfying

$$0<V_{\min}:=\inf_{x\in\mathbb{R}^N}V(x)<V_\infty=\liminf_{|x|\to\infty}V(x)<+\infty.$$

This kind of hypothesis was introduced by Rabinowitz in [37].

(Q) $Q\in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, we have $Q_{\min}:=\inf_{x\in\mathbb{R}^N}Q(x)>0$ and $Q_{\max}:=\max_{x\in\mathbb{R}^N}Q(x)>Q_\infty$, where $Q_\infty=\limsup_{|x|\to\infty}Q(x)$.

(VQ) $V(0)=V_{\min}$ and $Q(0)=Q_{\max}$, $\mathcal{V}\cap\mathcal{Q}\neq\emptyset$, where

$$\mathcal{V}=\{x\in\mathbb{R}^N:V(x)=V_{\min}\},\ \mathcal{Q}=\{x\in\mathbb{R}^N:Q(x)=Q_{\max}\}.$$

Note that our conditions (V) and (Q) are simpler than the conditions due to Ding and Liu [14] and they are easy to check. Now, we give some assumptions of the nonlinear function f as follows:

(f_1) f is a continuously differentiable function with the property that $f(s)=0$ for all $s\leq 0$, and for each

$$q_1\geq\frac{N}{s},\quad q_2>\frac{N}{s},$$

there exist two positive constants $a_1>0, a_2>0$ such that

$$f'(t)\leq a_1|t|^{q_1-2}+a_2\Phi_{N,s}(\alpha_0|t|^{N/(N-s)})|t|^{q_2-2},$$

where $\Phi_{N,s}(y)=\mathrm{e}^y-\sum_{j=0}^{j_p-2}\dfrac{y^j}{j!}$, $j_p=\min\{j\in\mathbb{N}: j\geq p\}$, $\alpha_0\in(0,\alpha_*)$, and α_* is given in the Lemma 2.1.

(f_2) $\lim_{t\to 0^+}\frac{f'(t)}{t^{p-2}}=0$.

(f_3) There exists $\theta>2p$ such that $2f(t)t\geq\theta F(t)>0$ for all $t>0$, where $F(t)=\int_0^t f(\tau)\mathrm{d}\tau$.

(f_4) There exists $\gamma_1>0$ large enough such that $F(t)\geq\gamma_1|t|^\theta$ for all $t\geq 0$, where $\theta>2p$ is a constant in the condition (f_3).

(f_5) The map $\dfrac{f(t)}{t^{p-1}}$ is strictly increasing on $(0,+\infty)$.

Remark 1.1 We denote $\widetilde{M}(t)=\int\limits_0^t M(\tau)\mathrm{d}\tau$ for all $t>0$. Then

$$\widetilde{M}(t)=at+b\frac{t^2}{2}\leq\gamma(t+t^2)$$

for all $t\geq 0$, where $\gamma=\max\left\{a,\dfrac{b}{2}\right\}$. From the condition ($f_5$), we see that $\frac{f(t)t}{p}-F(t)$ is an increasing function on $(0,+\infty)$. Clearly, M verifies the following conditions:

(M_1) The function $M\in C(\mathbb{R}_0^+,\mathbb{R}^+)$ satisfies

$$\inf_{t\in\mathbb{R}^+}M(t)\geq a>0.$$

(M_2) The function $t\to M(t)$ is increasing on $[0,+\infty)$.

(M_3) For all $t_1\geq t_2>0$, then

$$\frac{M(t_1)}{t_1}-\frac{M(t_2)}{t_2}\leq a\left(\frac{1}{t_1}-\frac{1}{t_2}\right).$$

(M_4) The function $\widetilde{M}(t)-\dfrac{1}{2}M(t)t$ is increasing on $[0,+\infty)$.

When $s\to 1^-$ and $a=1, b=0$, our problem (1.1) becomes the following equation

$$-\varepsilon^N\Delta_N u+V(x)|u|^{N-2}u=\varepsilon^{\mu-N}\left(\int\limits_{\mathbb{R}^N}\frac{Q(y)F(u(y))}{|x-y|^\mu}\mathrm{d}y\right)Q(x)f(u(x))\quad\text{in }\mathbb{R}^N. \tag{1.2}$$

In 2014, Alves-Yang [2] studied the subcritical case of equation (1.2). Namely, they considered the existence of semiclassical ground state solution of the Choquard equation as follows:

$$-\varepsilon^p\Delta_p u+V(x)|u|^{p-2}u=\varepsilon^{\mu-N}\left(\int\limits_{\mathbb{R}^N}\frac{Q(y)F(u(y))}{|x-y|^\mu}\mathrm{d}y\right)Q(x)f(u(x))\quad\text{in }\mathbb{R}^N, \tag{1.3}$$

where $\varepsilon>0$, Δ_p is the p-Laplace operator, $1<p<N$, V and Q are two potential functions satisfying the the conditions due to Ding-Liu [14]. A solution of the equation (1.3) as $\varepsilon\to 0$ is said to be semi-classical. In the physical meaning, the semi-calassical as $\varepsilon\to 0$ should be corresponded to solutions of equation (1.3) and the critical points of potential V and Q which controls the classical dynamics. We see that if u_ε is a solution of equation (1.3) and $x_0\in\mathbb{R}^N$, then the function $v_\varepsilon(x)=u_\varepsilon(x_0+\varepsilon x)$ is a solution of the equation

$$-\Delta_p v_\varepsilon+V(x_0+\varepsilon x)|v_\varepsilon|^{p-2}v_\varepsilon=\left(\int\limits_{\mathbb{R}^N}\frac{Q(x_0+\varepsilon y)F(v_\varepsilon(y))}{|x-y|^\mu}\mathrm{d}y\right)Q(x_0+\varepsilon x)f(v_\varepsilon)\quad\text{in }\mathbb{R}^N. \tag{1.4}$$

If $x_0 \in \mathbb{R}^N$ is a critical point of V and Q with $V(x_0) > 0$ and $Q(x_0) > 0$, then we expect that v_ε should converges to a solution v of the equation

$$-\varepsilon^p \Delta_p u + V(x_0)|u|^{p-2}u = (Q(x_0))^2 \bigg(\int\limits_{\mathbb{R}^N} \frac{F(u(y))}{|x-y|^\mu} \mathrm{d}y \bigg) f(u(x)) \quad \text{in } \mathbb{R}^N.$$

The initial contribution to semi-classical solutions was presented by Floer-Weinstein [16]. Specifically, when $N = 3$ and $p = 2$, equation (1.3) arises in the context of Bose-Einstein condensation, helping to illustrate finite-range many-body interactions among particles.

When $p = 2, Q = 1, \mu = N - \alpha$, and $F(u) = |u|^{\mathfrak{p}}$, the equation (1.3) transforms into the following Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * |u|^{\mathfrak{p}})|u|^{\mathfrak{p}-2}u \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where I_α is the Riesz potential defined for each point $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad \text{where } A_\alpha = \frac{\Gamma\left(\dfrac{N-\alpha}{2}\right)}{\Gamma\left(\dfrac{\alpha}{2}\right)\pi^{N/2}2^\alpha}, \alpha \in (0, N),$$

Γ is the Gamma function and V is a potential function. When $\mathfrak{p} = \alpha = 2, N = 3$, and $V(x) = \nu$, the equation (1.5) reduces by the Choquard-Pekar type equation

$$-\Delta u + \nu u = (I_2 * u^2)u, \quad x \in \mathbb{R}^3, \tag{1.6}$$

which arises from various physical problems. This equation was introduced in 1976 by Choquard [26] in order to describe an electron trapped in its own hole. It also appears in the theory of the polaron at rest [19, 20, 34] and in models of interaction between non relativistic quantum mechanics and gravitation [23, 31, 35] which was proposed by Penrose. It is now called by Schrödinger-Newton equation. The variational method is used to study the Choquard equation from the work of Lieb [26] and Lions [27]. After that, many authors use this method to investigate the existence of weak solutions to Choquard-type equations. In 2016, Alves-Cassani-Tarsi-Yang [3] studied the problem (1.2) with $Q = 1$ in the case $N = 2$. Namely, they considered the following equation:

$$-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{\mu-N} \left[\frac{1}{|x|^\mu} * F(u) \right] f(u) \ \text{ in } \mathbb{R}^2 \ (0 < \mu < 2), \tag{1.7}$$

when f has exponential growth and V satisfies some following conditions:

$(\mathcal{F}_1)$ (i) f is a continuous function sastisfying $f(s) = 0$ for all $s \leq 0$ and $0 \leq f(s) \leq C\mathrm{e}^{4\pi s^2}, s \geq 0$, where $C > 0$ is a constant;

(ii) there are $s_0 > 0, M_0 > 0$ and $q \in (0, 1]$ so that $0 < s^q F(s) \leq M_0 f(s)$ for all $|s| \geq s_0$.

$(\mathcal{F}_2)$ There are $p > \frac{2-\mu}{2}$ and $C_p > 0$ verifying $f(s) \sim C_p s^p$ as $s \to 0$.

$(\mathcal{F}_3)$ There is a positive real number $K > 1$ such that $f(s)s > KF(s)$ for all $s > 0$, where $F(t) = \int\limits_0^t F(s)\mathrm{d}s$.

$(\mathcal{F}_4)$ $\lim_{s\to+\infty} \frac{sf(s)F(s)}{\mathrm{e}^{8\pi s^2}} \geq \beta$, with $\beta > \inf_{\rho>0} \frac{\mathrm{e}^{\frac{4-\mu}{4}V_0\rho^2}}{16\pi^2\rho^{4-\mu}} \frac{(4-\mu)^2}{(2-\mu)(3-\mu)}$.

$(\mathcal{F}_5)$ The map $f(s)$ is strictly increasing on $(0, +\infty)$.

(V_1) $V(x) \geq V_0 > 0$ in $\mathbb{R}^2$ for some $V_0 > 0$;

(V_2) $0 < V_0 = \inf_{x\in\mathbb{R}^2} V(x) < V_\infty = \liminf_{|x|\to\infty} V(x) < \infty$.

Then they obtain the following result:

Theorem 1.2 Assume that the reaction f satisfies the conditions $(\mathcal{F}_1)$–$(\mathcal{F}_5)$ and the potential function V satisfying the assumptions (V_1)–(V_2). Then for any $\varepsilon > 0$ small enough, problem (1.7) admits one positive ground state solution. Moreover, if u_ε is one of these solutions and η_ε is the global maximum of u_ε, then $\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0$.

In the fractional p-Laplacian case, Ambrosio [8] studied the multiplicity and concentration of solution to the following equation

$$(-\Delta)_p^s u + V(\varepsilon x)|u|^{p-2}u = (\frac{1}{|x|^\mu} * F(u))f(u) \quad \text{in } \mathbb{R}^N, \tag{1.8}$$

where f has subcritical growth and V satisfies the condition (V). He uses variational method, Ljusternik-Schnirelmann theory and follow the method due to Szulkin-Weth [41] to prove the main result. In 2023, Sun-Liang-Radulescu-Nguyen [25] studied the multiplicity and concentration of solutions to problem (1.8) when f has exponential growth, and V satisfies the condition (V). In 2021, Clemente-Albuquerque-Barboza [11] studied the following problem in one dimensional related with problem (1.7) as follows:

$$(-\Delta)^{1/2}u + u = [I_\mu * F(u)]f(u) \text{ in } \mathbb{R}, \tag{1.9}$$

where $0 < \mu < 1$, I_μ is the Riesz potential and f has exponential growth. They used the variational methods and minimax estimate to study the existence of weak solution for problem (1.9). We suggest readers to the work of Moroz-Schaftingen [30] for guidance on the Choquard equation. In 2022, Yuan-Tang-Zhang-Zhang [46] have been studied the existence and concentration of weak solution to the problem as follows

$$\varepsilon(-\Delta)^{1/2}u + V(x)u = \varepsilon^{\mu-1}\left(\frac{1}{|x|^\mu}) * F(u)\right)f(u), \ x \in \mathbb{R}.$$

Here, they assume that V satisfies conditions (V_1) and (V_2) in $\mathbb{R}$, f has the Trudinger-Moser growth and satisfies some technique assumptions so that the Mountain Pass Level can be bounded above by a suitable constant. Note that they do not study the multiplicity of weak solutions by using Ljusternik-Schnirelmann category theory. Zhang-Zhang [50] studied the muliplicity and concentration solution for Choquard equation with critical growth in $\mathbb{R}^3$ as follows:

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\mu-3}\int\limits_{\mathbb{R}^3} \frac{|u(y)|^{6-\mu} + Q(y)F(u(y))}{|x-y|^\mu}\mathrm{d}y\left(|u|^{4-\mu}u + \frac{Q(x)f(u)}{6-\mu}\right),$$

where f has subcritical growth and two potential functions V and Q satisfy some suitable assumptions. In 2023, Su-Liu [40] are concerned with the following Choquard equation:

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{-\alpha}(I_\alpha * F(u))F'(u), \ x \in \mathbb{R}^N,$$

where $N \geq 4$, $\alpha \in (0, N)$, I_α is the Riesz potential and $\varepsilon > 0$ is a small parameter. Here, they assume that $F(u) = \frac{1}{q}|u|^q + \frac{1}{2_\alpha^*}|u|^{2_\alpha^*}$, where $2_\alpha^\# < q < 2_\alpha^*$, $2_\alpha^\# = \frac{N+\alpha}{N}$ and $2_\alpha^* = \frac{N+\alpha}{N-2}$ are lower and upper critical exponents respectively, in the sense of the Hardy-Littlewood-Sobolev inequality. In this work, they construct a bound-state concentrating at an isolated component of the positive local minimum points of V as $\varepsilon \to 0$ for each $q \in (2_\alpha^\#, 2_\alpha^*)$ via to variational methods, a truncation technique and a new regularity result. For more results, we refer the readers to [4, 5, 12, 13, 38, 39, 45] and the references therein for more detail information.

When $s=1, p=2$ and $\varepsilon=1$, our problem reduces to

$$M\bigg(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x+\int_{\mathbb{R}^N}V(x)|u|^2\mathrm{d}x\bigg)[-\Delta u+V(x)u]=f(u) \ \text{ in } \ \mathbb{R}^N \tag{1.10}$$

which has analogue of the well-known Kirchhoff equation

$$\rho u_{tt}-\bigg(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|u_x|^2\mathrm{d}x\bigg)u_{xx}=0. \tag{1.11}$$

It was first proposed by Kirchhoff which is an extension of the classical D'Alembert's wave equation to describe the transversal oscillations of a stretched string, where ρ, P_0, E, and L are constant variables with physical meanings. Fiscella and Valdinoci [18] proposed an interesting physical explanation of the fractional Kirchhoff equation.

To the best of our knowledge, there is not any result concerning problems (1.1) with exponential growth. Our aim is to give the first result for these type equations with Trudinger-Moser nonlinearity.

Now let us recall some notations that will be used later. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by

$$W^{s,p}(\mathbb{R}^N):=\{u\in L^p(\mathbb{R}^N):[u]_{s,p}<\infty\},$$

where $[u]_{s,p}$ is the Gagliardo seminorm given by

$$[u]_{s,p}=\bigg(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y\bigg)^{1/p}.$$

We also know that the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is a uniformly convex Banach space (see [36]) equipped with the norm

$$||u||:=\|u\|_{W^{s,p}(\mathbb{R}^N)}=\Big(||u||^p_{L^p(\mathbb{R}^N)}+[u]^p_{s,p}\Big)^{1/p}.$$

For $\eta>0$, we use another norm on $W^{s,p}(\mathbb{R}^N)$ which is given by

$$\|u\|_\eta=\Big(\eta\|u\|^p_{L^p(\mathbb{R}^N)}+[u]^p_{s,p}\Big)^{1/p}.$$

Then two norms $\|.\|$ and $\|.\|_\eta$ are equivalent on $W^{s,p}(\mathbb{R}^N)$. For each $\varepsilon>0$, we denote the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{W_\varepsilon}=\Big([u]^p_{s,p}+||u||^p_{p,V,\varepsilon}\Big)^{1/p}, \quad \|u\|^p_{p,V,\varepsilon}=\int\limits_{\mathbb{R}^N}V(\varepsilon x)|u(x)|^p\mathrm{d}x$$

by space W_ε. It well known that W_ε is uniformly convex Banach space (see [36, Lemma 10] for the proofs). Furthermore, W_ε is also a reflexive space, and it is compact with weakly topology. From the condition (V) and Theorem 6.9 in [32], we have the continuous embedding $W_\varepsilon\hookrightarrow L^\nu(\mathbb{R}^N)$ for all $\nu\in[\frac{N}{s},+\infty)$, then there is a best constant $S_{\nu,\varepsilon}>0$ such that

$$S_{\nu,\varepsilon}=\inf_{u\neq0,u\in W_\varepsilon}\frac{\|u\|_{W_\varepsilon}}{\|u\|_{L^\nu(\mathbb{R}^N)}}$$

for all $\nu\in[\frac{N}{s},+\infty)$. Then it holds that

$$\|u\|_{L^\nu(\mathbb{R}^N)}\leq S^{-1}_{\nu,\varepsilon}\|u\|_{W_\varepsilon} \text{ for all } u\in W_\varepsilon. \tag{1.12}$$

Using again Theorem 6.9 in [32], we also have the continuous embedding $W^{s,N/s}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ for all $\nu \in [\frac{N}{s}, +\infty)$, then there is a best constant $A_{\nu,\eta} > 0$ which is given by

$$A_{\nu,\eta} = \inf_{u \neq 0, u \in W^{s,N/s}(\mathbb{R}^N)} \frac{\|u\|_\eta}{\|u\|_{L^\nu(\mathbb{R}^N)}}$$

for all $\nu \in [\frac{N}{s}, +\infty)$. Hence, we have

$$\|u\|_{L^\nu(\mathbb{R}^N)} \leq A_{\nu,\eta}^{-1} \|u\|_\eta \quad \text{for all } u \in W^{s,N/s}(\mathbb{R}^N). \tag{1.13}$$

By the change of variable $x \mapsto \varepsilon x$, the equation (1.1) is equivalent to the following equation of the form:

$$\begin{aligned} M\bigg([u]_{s,p}^p + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^p\bigg)((-\Delta)_{N/s}^s u + V(\varepsilon x)|u|^{p-2}u) \\ = \bigg[\frac{1}{|x|^\mu} * (Q(\varepsilon y)F(u(y)))\bigg] Q(\varepsilon x) f(u(x)) \ \text{ in } \ \mathbb{R}^N. \end{aligned} \tag{$\mathcal{P}_\varepsilon$}$$

Definition 1.3 The function $u \in W_\varepsilon$ is a weak solution of equation $(\mathcal{P}_\varepsilon)$ if

$$\begin{aligned} &M(||u||_{W_\varepsilon}^p)\bigg(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{\frac{N}{s}-2}u\varphi \mathrm{d}x\bigg) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon y)F(u(y))}{|x-y|^\mu} Q(\varepsilon x) f(u(x))\varphi(x) \mathrm{d}x\mathrm{d}y \end{aligned}$$

for all $\varphi \in W_\varepsilon$.

We denote $\mathrm{cat}_B(A)$ by the category of A with respect to B, namely the least integer k such that $A \subset A_1 \cup \cdots \cup A_k$, where A_i $(i = 1, \cdots, k)$ is closed and contractible in B. We set $\mathrm{cat}_B(\emptyset) = 0$ and $\mathrm{cat}_B(A) = +\infty$ if there is no integer with above property. We refer the reader to [44] for more details on Ljusternik-Schnirelmann theory. Denote

$$(\mathcal{V} \cap \mathcal{Q})_\delta = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \mathcal{V} \cap \mathcal{Q}) \leq \delta\} \quad \text{for } \delta > 0.$$

Now, we state the main result in this paper as follows:

Theorem 1.4 Let $(V), (Q), (VQ)$ and (f_1)–(f_5) hold. Then for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that problem (1.1) has at least $\mathrm{cat}_{(\mathcal{V}\cap\mathcal{Q})_\delta}(\mathcal{V} \cap \mathcal{Q})$ nontrivial nonnegative weak solutions for any $0 < \varepsilon < \varepsilon_\delta$. Moreover, if w_ε denotes one of these solutions and η_ε is its global maximum such that, up to a subsequence, $\eta_\varepsilon \to y \in \mathcal{V} \cap \mathcal{Q}$ and $v_\varepsilon(x) := w_\varepsilon(\varepsilon x + \eta_\varepsilon)$ converges strongly in $W^{s,p}(\mathbb{R}^N)$ to a ground state solution of

$$M(||u||_{V_{\min}}^p)((-\Delta)_p^s u + V_{\min}|u|^{p-2}u) = Q_{\max}^2 \bigg[\frac{1}{|x|^\mu} * F(u)\bigg] f(u) \quad \text{in } \mathbb{R}^N.$$

Furthermore, we have

$$\lim_{\varepsilon \to 0^+} V(\eta_\varepsilon) = V_{\min} \quad \text{and} \quad \lim_{\varepsilon \to 0^+} Q(\eta_\varepsilon) = Q_{\max}.$$

Next, we consider the conditions $(f_1)'$–$(f_5)'$ instead of (f_1)–(f_5) respectively as follows:

$(f_1)'$ f is a continuous differentiable function such that $f(s) = 0$ for all $s \leq 0$.

For each

$$q_1 \geq N, q_2 > N,$$

and there exist two positive constants $a_1 > 0, a_2 > 0$ such that

$$f'(t) \le a_1|t|^{q_1-2} + a_2\Phi_N(\alpha_0|t|^{N/(N-1)})|t|^{q_2-2},$$

where $\Phi_N(y) = \mathrm{e}^y - \sum_{j=0}^{N-2} \dfrac{y^j}{j!}$, and $0 < \alpha_0 \le \alpha_N^*$, α_N^* is defined as in Lemma 2.2.

$(f_2)'$ $\lim_{t\to 0^+} \frac{f'(t)}{t^{N-2}} = 0$.

$(f_3)'$ There exists $\theta > 2N$ such that $2f(t)t \ge \theta F(t) > 0$ for all $t > 0$, where $F(t) = \int_0^t f(\tau)\mathrm{d}\tau$.

$(f_4)'$ There exists $\gamma_1 > 0$ large enough such that $F(t) \ge \gamma_1|t|^\theta$ for all $t \ge 0$, where $\theta > 2N$ is a constant in the condition (f_3).

$(f_5)'$ The map $\dfrac{f(t)}{t^{N-1}}$ is strictly increasing on $(0, +\infty)$.

By arguments as Theorem 1.4, we get the following result for problem (1.2):

Corollary 1 Let $(V), (Q), (VQ)$ and $(f_1)'$–$(f_5)'$ hold. Then for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that problem

$$M\bigg(\int\limits_{\mathbb{R}^N} |\nabla u|^N + \varepsilon^{-N}V(x)|u|^N\mathrm{d}x\bigg)(-\varepsilon^N\Delta_N u + V(x)|u|^{N-2}u)$$
$$= Q_{\max}^2\bigg[\frac{1}{|x|^\mu} * F(u)\bigg]f(u) \quad \text{in } \mathbb{R}^N$$

has at least $\mathrm{cat}_{(\mathcal{V}\cap\mathcal{Q})_\delta}(\mathcal{V}\cap\mathcal{Q})$ nontrivial nonnegative weak solutions for any $0 < \varepsilon < \varepsilon_\delta$. Moreover, if w_ε denotes one of these solutions and η_ε is its global maximum such that, up to a subsequence, $\eta_\varepsilon \to y \in \mathcal{V}\cap\mathcal{Q}$ and $v_\varepsilon(x) := w_\varepsilon(\varepsilon x + \eta_\varepsilon)$ converges strongly in $W^{s,p}(\mathbb{R}^N)$ to a ground state solution of

$$M\bigg(\int\limits_{\mathbb{R}^N} |\nabla u|^N + V_{\min}|u|^N\mathrm{d}x\bigg)(-\Delta_N u + V_{\min}|u|^{N-2}u) = Q_{\max}^2\bigg[\frac{1}{|x|^\mu} * F(u)\bigg]f(u) \quad \text{in } \mathbb{R}^N.$$

Furthermore, we have

$$\lim_{\varepsilon\to 0^+} V(\eta_\varepsilon) = V_{\min} \quad \text{and} \quad \lim_{\varepsilon\to 0^+} Q(\eta_\varepsilon) = Q_{\max},$$

and there exists $C > 0, c > 0$ such that $|w_\varepsilon(x)| \le C\mathrm{e}^{-\frac{c}{\varepsilon}|x-\eta_\varepsilon|}$ for all $x \in \mathbb{R}^N$.

The exponential decay estimate in Corollary 1 is similarly proved as in [2]. Corollary 1 is new up to now. In our present work, there exists the competition between the absorption potential V and the reaction potential Q. The absorption V would like to attract the global maximum point of solutions to its minimum points, and the reaction potential Q want to attract the lobal maximum point of solutions to its maximum points. Therefore, the concentration phenomena of semiclassical states to problem (1.1) is more interesting and delicate than the case the equation contains only on absorption potential V as in [3] and [46]. See again the work of Yuan-Tang-Zhang-Zhang [46], we need overcome some difficulties than their work in proving the concentration of solutions. In their work, the space solution is a Hilbert space and in our work, the solution space is not Hilbert space. Then, some nice properties in Hilbert space is not applied in our work. The proofs about concentration of weak solution in our work is not the same as [46] due to the properties of fractional p-Laplace operator. The fact that, we can not use the s-harmonic extension by Caffareli-Silvestre [10] as in [46]. Our more difficulty is to analyse the compactness of Palais-Smale sequence for energy function due to potential competion and the non-Hilbert structure of space solution. In [47], Yuan-Radulescu-Tang-Zhang studied the concentrating solutions for singularly perturbed fractional $\frac{N}{s}$-Laplacian equations with nonlocal Choquard reaction which comes from (1.1) as $M = 1$ and $Q = 1$. Here, the multiple solutions can not get by using their method. Note that our problem is the lack compactness due to the condition (V). In order to apply the Ljusternik-Schnirelmann category theory, we need establish some tools for this aim. We recommend the readers to read Section 2 to Section 5 for that comments. In our work, we assume that $N = ps$, then it does not have embedding from $W^{s,p}(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$. To overcome this difficulty, we use fractional Trudinger-Moser inequality in every step. It is the main different point in comparing with the work of Ambrosio [8]. Final, we will consider the subcritical case for the equation (1.1) in a forcoming work. The final difficulty is that our problem contains the Kirchhoff function M. Then we need overcome the obstacle problem in estimating the Moutain pass level, the estimation techniques to prove the existence, as well as the multiple solutions of problem (1.1).

The plan of the paper is the following: in Section 2, our focus lies in examining the associated autonomous problem. In Section 3, we study the auxiliary problem. In this section, we prove some technique results about the compactness of (PS) sequence. In Section 4, we prove the existence of ground state solution and concentration of solutions to auxiliary problem and some tools to explore the multiple solutions of auxiliary problem. We also study the limit of sequence of ground states solutions. Finally, the Section 5 is devoted to completing the proof of Theorem 1.4.

2 Autonomous Problem

In this section, we study the autonomous problem connected with $(\mathcal{P}_\varepsilon)$ as follows

$$M(||u||_\eta^p)((-\Delta)^s_{N/s}u + \eta|u|^{\frac{N}{s}-2}u) = \nu^2\left[\frac{1}{|x|^\mu} * F(u)\right]f(u) \ \text{ in } \mathbb{R}^N, \tag{$\mathcal{P}_{\eta\nu}$}$$

where $\eta > 0, \nu > 0$ are constants.

The corresponding energy functional $J_{\eta\nu} : W^{s,N/s}(\mathbb{R}^N) \to \mathbb{R}$ for problem $(\mathcal{P}_{\eta\nu})$ is given by

$$\begin{aligned} J_{\eta\nu}(u) &= \frac{1}{p}\widetilde{M}(\|u\|_\eta^p) - \frac{\nu^2}{2}\int\limits_{\mathbb{R}^N} K(u)(x)F(u(x))\mathrm{d}x \\ &= \frac{1}{p}\widetilde{M}(\|u\|_\eta^p) - \frac{\nu^2}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(u(y))F(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y, \end{aligned}$$

where $K(u)(x) = \int\limits_{\mathbb{R}^N} \frac{F(u(y))}{|x-y|^\mu}\mathrm{d}y$.

We present the following lemmas to prove the results in this section.

Lemma 2.1 ([49]) Let $s \in (0,1)$ and $sp = N$. Then for every $0 \le \alpha < \alpha_* \le \alpha^*_{s,N}$, the following inequality holds:

$$\sup_{u\in W^{s,p}(\mathbb{R}^N), ||u||_{W^{s,p}(\mathbb{R}^N)}\le 1} \int\limits_{\mathbb{R}^N} \Phi_{N,s}(\alpha|u|^{N/(N-s)})\mathrm{d}x < +\infty,$$

where $\Phi_{N,s}(t)=\mathrm{e}^t-\sum_{j=0}^{j_p-2}\dfrac{t^j}{j!}$, $j_p=\min\{j\in\mathbb{N}:j\geq p\}$. Moreover, for $\alpha>\alpha^*_{s,N}$,

$$\sup_{u\in W^{s,p}(\mathbb{R}^N),||u||_{W^{s,p}(\mathbb{R}^N)}\leq 1}\int\limits_{\mathbb{R}^N}\Phi_{N,s}(\alpha|u|^{N/(N-s)})\mathrm{d}x=+\infty,$$

where

$$\alpha^*_{s,N}=N\left(\frac{2(N\omega_N)^2\Gamma(p+1)}{N!}\sum_{k=0}^{+\infty}\frac{(N+k-1)!}{k!}\frac{1}{(N+2k)^p}\right)^{s/(N-s)}=N(\gamma_{s,N})^{s/(N-s)}.$$

The optimal of α_* in Lemma 2.1 is still a open problem, and in the local case, we have the following sharp result:

Lemma 2.2 (see [1, Theorem 1.1]) For every $\alpha>0$ and $\upsilon\in W^{1,N}(\mathbb{R}^N)$, the following inequality holds:

$$\int_{\mathbb{R}^N}\Phi_N(\alpha|\upsilon|^{\frac{N}{N-1}})\mathrm{d}x<+\infty.$$

Moreover, for $\alpha\leq\alpha^*_N$, there holds

$$\sup_{\upsilon\in W^{1,N}(\mathbb{R}^N),\|\upsilon\|_{W^{1,N}(\mathbb{R}^N)}\leq 1}\int_{\mathbb{R}^N}\Phi_N(\alpha|\upsilon|^{\frac{N}{N-1}})\mathrm{d}x<+\infty,$$

where

$$\alpha^*_N=N\omega_{N-1}^{\frac{1}{N-1}}.$$

Meanwhile, the inequality is sharp: for $\alpha>\alpha^*_N$, the supremum is infinity.

Lemma 2.3 ([48, Corollary 2.1]) For any $\alpha>0$ and all $u\in W^{s,N/s}(\mathbb{R}^N)$, it holds

$$\int\limits_{\mathbb{R}^N}\Phi_{N,s}(\alpha|u|^{N/(N-s)})\mathrm{d}x<+\infty.$$

Lemma 2.4 ([28]) Let $r,t>1$ and $0<\mu<N$ such that $\dfrac{1}{r}+\dfrac{\mu}{N}+\dfrac{1}{t}=2$, $f\in L^r(\mathbb{R}^N)$ and $h\in L^t(\mathbb{R}^N)$. Then there exists a sharp constant $C(r,N,\mu,t)>0$ independent of f and h such that

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{f(x)h(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\leq C(r,N,\mu,t)||f||_{L^r(\mathbb{R}^N)}\|h\|_{L^t(\mathbb{R}^N)}.$$

In the application, we usually use $r=t$, then we have $\frac{2}{t}+\frac{\mu}{N}=2$, or $t=\frac{2N}{2N-\mu}$. Then for $F(u)=|u|^q$, we see that $\int\limits_{\mathbb{R}^N}\left[\dfrac{1}{|x|^\mu}*F(u)\right]F(u)\mathrm{d}x$ is well-defined on $L^t(\mathbb{R}^N)$ with $t=\frac{2N}{2N-\mu}$. In order to use the continuous embedding from $W^{s,N/s}(\mathbb{R}^N)$ into $L^r(\mathbb{R}^N)$, $r\in[\frac{N}{s},+\infty)$, we require that $qt\geq\frac{N}{s}$, then $q\geq\frac{N}{st}=\frac{2N-\mu}{2s}$.

Lemma 2.5 Suppose that f satisfies the conditions $(f_1),(f_2)$. Set

$$\psi(u)=\frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(u)\right)F(u)\mathrm{d}x,\quad u\in W^{s,N/s}(\mathbb{R}^N),$$

then $\psi\in C^1(W^{s,N/s}(\mathbb{R}^N),\mathbb{R})$. Furthermore, we have

$$<\psi'(u),v>=\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(u)\right)f(u)v\mathrm{d}x$$

for all $v\in W^{s,N/s}(\mathbb{R}^N)$. Furthermore, we also have $J_{\eta\nu}\in C^1(W^{s,N/s}(\mathbb{R}^N),\mathbb{R})$.

Proof By arguments Lemma 2.6 and using Lemma 2.3, we see that ψ is well defined on $W^{s,N/s}(\mathbb{R}^N)$. We have

$$\begin{aligned}\langle \psi'(u), v\rangle &= \lim_{t\to 0}\frac{\psi(u+tv)-\psi(u)}{t}\\ &= \frac{1}{2t}\lim_{t\to 0}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*(F(u(y)+tv(y)))\right)F(u+tv)-\left(\frac{1}{|x|^\mu}*F(u(y))\right)F(u)\mathrm{d}x.\end{aligned}$$

By Mean value theorem, there are two functions $\theta_1,\theta_2\in(0,1)$ so that

$$\begin{aligned}F(u(y)+tv(y)) &= F(u(y))+f(u(y)+t\theta_1 v(y))tv(y),\\ F(u(x)+tv(x)) &= F(u(x))+f(u(x)+t\theta_2 v(x))tv(x).\end{aligned}$$

It implies that

$$\begin{aligned}&\frac{1}{t}\left[\left(\frac{1}{|x|^\mu}*(F(u(y)+tv(y)))\right)F(u+tv)-\left(\frac{1}{|x|^\mu}*F(u(y))\right)F(u)\right]\\ =&\left(\frac{1}{|x|^\mu}*(F(u(y))+f(u(y)+t\theta_1 v(y))tv(y))\right)(F(u(x))+f(u(x)+t\theta_2 v(x))tv(x))\\ &-\left(\frac{1}{|x|^\mu}*F(u(y))\right)F(u)\\ =&\left(\frac{1}{|x|^\mu}*F(u(y))\right)f(u(x)+t\theta_2 v(x))tv(x))+\left(\frac{1}{|x|^\mu}*(f(u(y)+t\theta_1 v(y))tv(y))\right)F(u(x))\\ &+\left(\frac{1}{|x|^\mu}*(f(u(y)+t\theta_1 v(y))tv(y))\right)f(u(x)+t\theta_2 v(x))tv(x)).\end{aligned}$$

Hence

$$\begin{aligned}&\frac{1}{2t}\int\limits_{\mathbb{R}^N}\left[\left(\frac{1}{|x|^\mu}*(F(u(y)+tv(y)))\right)F(u+tv)-\left(\frac{1}{|x|^\mu}*F(u(y))\right)F(u)\right]\mathrm{d}x\\ =&\frac{1}{2}\int\limits_{\mathbb{R}^N}\left[\left(\frac{1}{|x|^\mu}*F(u(y))\right)f(u(x)+t\theta_2 v(x))v(x))\right.\\ &\left.+\left(\frac{1}{|x|^\mu}*(f(u(y)+t\theta_1 v(y))v(y))\right)F(u(x))\right]\mathrm{d}x\\ &+\frac{t}{2}\int\limits_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*(f(u(y)+t\theta_1 v(y))v(y))\right)f(u(x)+t\theta_2 v(x))v(x))\mathrm{d}x.\end{aligned}$$

By the condition (f_1), we have

$$|f(t)|\le a_1|t|^{q_1-1}+a_2\Phi_{N,s}(\alpha_0|t|^{N/(N-s)})|t|^{q_2-1}.$$

Using Hardy-Littlewood-Sobolev inequality, we deduce

$$\begin{aligned}&\int\limits_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*(f(u(y)+t\theta_1 v(y))v(y))\right)f(u(x)+t\theta_2 v(x))v(x))\mathrm{d}x\\ &\le ||f(u(y)+t\theta_1 v(y))v(y)||_{L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)}||f(u(x)+t\theta_2 v(x))v(x))||_{L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)}.\end{aligned}\tag{2.1}$$

When t small enough, we see that

$$\begin{aligned}&|f(u(y)+t\theta_1 v(y))v(y)|\\ &\le a_1 2^{q_1-1}(|u|^{q_1-1}+|v|^{q_1-1})|v|\end{aligned}$$

$$+ 2^{q_2-1}a_2|v|\Phi_{N,s}(\alpha_0 2^{N/(N-s)}(|u|^{N/(N-s)}+|v|^{N/(N-s)})(|u|^{q_2-1}+|v|^{q_2-1}). \tag{2.2}$$

Since $\Phi_{N,s}(t)$ is a convex function, then

$$\begin{aligned}&\Phi_{N,s}(\alpha_0 2^{N/(N-s)}(|u|^{N/(N-s)}+|v|^{N/(N-s)})\\ \le\ &\frac{1}{2}\left(\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|u|^{N/(N-s)})+\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|v|^{N/(N-s)})\right).\end{aligned} \tag{2.3}$$

Combine (3.5) and (3.6), there exists a suitable constant

$$\begin{aligned}|f(u(y)+t\theta_1 v(y))v(y)| \le\ & C(|u|^{q_1-1}|v|+|v|^{q_1}+(\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|u|^{N/(N-s)})\\ &+\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|v|^{N/(N-s)}))(|u|^{q_2-1}|v|+|v|^{q_2})\\ =\ & C(|u|^{q_1-1}|v|+|v|^{q_1}+\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|u|^{N/(N-s)})|u|^{q_2-1}|v|\\ &+\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|u|^{N/(N-s)})|v|^{q_2}\\ &+\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|v|^{N/(N-s)})|u|^{q_2-1}|v|\\ &+\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|v|^{N/(N-s)})|v|^{q_2}).\end{aligned}$$

Then

$$\begin{aligned}|f(u(y)+t\theta_1 v(y))v(y)|^{\frac{2N}{2N-\mu}} \le\ & (6C)^{\frac{2N}{2N-\mu}}\Big((|u|^{q_1-1}|v|)^{\frac{2N}{2N-\mu}}+(|v|^{q_1})^{\frac{2N}{2N-\mu}}\\ &+(\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|u|^{N/(N-s)})|u|^{q_2-1}|v|)^{\frac{2N}{2N-\mu}}\\ &+(\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|u|^{N/(N-s)})|v|^{q_2})^{\frac{2N}{2N-\mu}}\\ &+(\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|v|^{N/(N-s)})|u|^{q_2-1}|v|)^{\frac{2N}{2N-\mu}}\\ &+(\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|v|^{N/(N-s)})|v|^{q_2})^{\frac{2N}{2N-\mu}}\Big).\end{aligned} \tag{2.4}$$

Using Hölder inequality,

$$\int_{\mathbb{R}^N}(|u|^{q_1-1}|v|)^{\frac{2N}{2N-\mu}}\mathrm{d}x \le \Big(\int_{\mathbb{R}^N}|u|^{\frac{2Nq_1}{2N-\mu}}\mathrm{d}x\Big)^{\frac{q_1-1}{q_1}}\Big(\int_{\mathbb{R}^N}|v|^{\frac{2Nq_1}{2N-\mu}}\mathrm{d}x\Big)^{\frac{1}{q_1}}.$$

Using the Hölder inequality for $q_*>q_2$ with $\frac{q_2-1}{q_*}+\frac{q_*-q_2}{q_*}+\frac{1}{q_*}=1$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|u|^{N/(N-s)})|u|^{q_2-1}|v|)^{\frac{2N}{2N-\mu}}\mathrm{d}x\\ \le\ &\Big(\int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|u|^{N/(N-s)}))^{\frac{q_*}{q_*-q_2}}\Big)^{\frac{q_*}{q_*-q_2}}\Big(\int_{\mathbb{R}^N}|u|^{\frac{2Nq_2}{2N-\mu}}\mathrm{d}x\Big)^{\frac{q_*}{q_2-1}}\\ &\times\Big(\int_{\mathbb{R}^N}|v|^{\frac{2Nq_*}{2N-\mu}}\mathrm{d}x\Big)^{\frac{1}{q_*}} \le C_{1,u}||v||^{\frac{2N}{2N-\mu}}_{L^{\frac{2Nq_*}{2N-\mu}}(\mathbb{R}^N)}.\end{aligned} \tag{2.5}$$

By [24, Lemma 2.3], there is $c>\frac{q_*}{q_*-q_2}$ and near $\frac{q_*}{q_*-q_2}$ so that

$$(\Phi_{N,s}(\alpha_0 2.2^{N/(N-s)}|u|^{N/(N-s)}))^{\frac{q_*}{q_*-q_2}}\Big)^{\frac{q_*}{q_*-q_2}} \le \Phi_{N,s}(\alpha_0 c 2.2^{N/(N-s)}|u|^{N/(N-s)}). \tag{2.6}$$

Combining (3.7), (2.6) and Lemma 2.3, also computing similarly for remain factor in (2.4), we get that

$$||f(u(y)+t\theta_1 v(y))v(y)||_{L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)} < +\infty. \tag{2.7}$$

By arguments as before, we also have

$$||f(u(x)+t\theta_2 v(x))v(x))||_{L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)} < +\infty. \tag{2.8}$$

Combining (2.1), (2.7) and (2.8), we get

$$\left(\frac{1}{|x|^\mu} * (f(u(y)+t\theta_1 v(y))v(y))\right) f(u(x)+t\theta_2 v(x))v(x) \in L^1(\mathbb{R}^N).$$

Hence,

$$\lim_{t\to 0} \frac{t}{2} \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * (f(u(y)+t\theta_1 v(y))v(y))\right) f(u(x)+t\theta_2 v(x))v(x)\mathrm{d}x = 0. \tag{2.9}$$

Similarly, we also have $\left(\frac{1}{|x|^\mu} * F(u(y))\right) f(u(x)+t\theta_2 v(x))v(x) \in L^1(\mathbb{R}^N)$. Then there is a constant $A>0$ so that $\left|\left(\frac{1}{|x|^\mu} * F(u(y))\right) f(u(x)+t\theta_2 v(x))v(x)\right| \le A$ almost everywhere. For any $\varepsilon>0$, there exists $\delta=\varepsilon/A$, and any measurable set $E\subset\mathbb{R}^N$ satisfying $|E|<\delta$, we obtain

$$\int_E \left|\left(\frac{1}{|x|^\mu} * F(u(y))\right) f(u(x)+t\theta_2 v(x))v(x)\right|\mathrm{d}x \le A|E| = \delta A = \varepsilon \tag{2.10}$$

for all $|\eta|\in[0,1]$. Note that $\left(\frac{1}{|x|^\mu} * F(u(y))\right) f(u(x)+t\theta_2 v(x))v(x) \in L^1(\mathbb{R}^N)$, then there is a positive real number $R>0$ so that

$$\int_{\mathbb{R}^N\setminus B_R(0)} \left|\left(\frac{1}{|x|^\mu} * F(u(y))\right) f(u(x)+t\theta_2 v(x))v(x)\right|\mathrm{d}x < \varepsilon. \tag{2.11}$$

By (2.10) and (2.11), we deduce $\left|\left(\frac{1}{|x|^\mu} * F(u(y))\right) f(u(x)+t\theta_2 v(x))v(x)\right|$ is equi-integrable. Furthermore,

$$\left(\frac{1}{|x|^\mu} * F(u(y))\right) f(u(x)+t\theta_2 v(x))v(x) \to \left(\frac{1}{|x|^\mu} * F(u(y))\right) f(u(x))v(x)$$

for all $x\in\mathbb{R}^N$ and $t\to 0$. Therefore, by Vitali's theorem, we get

$$\lim_{t\to 0} \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u(y))\right) f(u(x)+t\theta_2 v(x))v(x)\mathrm{d}x = \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u(y))\right) f(u(x))v(x)\mathrm{d}x. \tag{2.12}$$

Similarly, we also have

$$\lim_{t\to 0} \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * (f(u(y)+t\theta_1 v(y))v(y))\right) F(u(x))\mathrm{d}x = \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * f(u(y))v(y)\right) F(u(x))\mathrm{d}x. \tag{2.13}$$

By chnaging variables between x and y, using Fubini's theorem, it holds

$$\int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * f(u(y))v(y)\right) F(u(x))\mathrm{d}x = \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u(y))\right) f(u(x))v(x)\mathrm{d}x.$$

Combine (2.8), (2.12) and (2.13), we get

$$\langle \psi'(u), v\rangle = \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)v\mathrm{d}x = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(u(y))f(u(x))v(x)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y. \tag{2.14}$$

Then we have that ψ is Gâteaux differentiable. From the condition (f_1), we have

$$\frac{1}{|x|^{\mu}} * F(u) = \int_{\mathbb{R}^N} \frac{F(u(y))}{|x-y|^{\mu}} \mathrm{d}y \in L^{\infty}(\mathbb{R}^N). \tag{2.15}$$

Indeed, by the condition (f_1), we have

$$\left|\frac{1}{|x|^{\mu}} * F(u)\right| \le \int_{\mathbb{R}^N} \frac{a_1|u|^{q_1} + a_2\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})|u|^{q_2}}{|x-y|^{\mu}} \mathrm{d}y.$$

Choosing $t > \dfrac{N}{N-\mu}$ or $N > \dfrac{\mu t}{t-1}$, then we get

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|u|^{q_1}}{|x-y|^{\mu}} \mathrm{d}y &= \int_{\{y:|x-y|\ge 1\}} \frac{|u|^{q_1}}{|x-y|^{\mu}} \mathrm{d}y + \int_{\{y:|x-y|\le 1\}} \frac{|u|^{q_1}}{|x-y|^{\mu}} \mathrm{d}y \\ &\le \int_{\mathbb{R}^N} |u|^{q_1} \mathrm{d}x + \left(\int_{\{y:|x-y|\le 1\}} \frac{1}{|x-y|^{\mu t/(t-1)}} \mathrm{d}y\right)^{\frac{t-1}{t}} \left(\int_{\mathbb{R}^N} |u|^{q_1 t} \mathrm{d}x\right)^{\frac{1}{t}}.\end{aligned}$$

Also, we get

$$\int_{\mathbb{R}^N} \frac{\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})|u|^{q_2}}{|x-y|^{\mu}} \mathrm{d}y \le \left(\int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0|u|^{N/(N-s)}))^{t/(t-1)}\right)^{\frac{t-1}{t}} \left(\int_{\mathbb{R}^N} \frac{|u|^{q_2 t}}{|x-y|^{\mu t}} \mathrm{d}y\right)^{\frac{1}{t}}, \tag{2.16}$$

where we choose $t > 1$ so that $0 < t\mu < N$. By arguments before, we $\left|\int_{\mathbb{R}^N} \frac{|u|^{q_2 t}}{|x-y|^{\mu t}} \mathrm{d}y\right| < \mathcal{C}_u$. Then there exists a constant $C_u > 0$ such that

$$\left|\frac{1}{|x|^{\mu}} * F(u)\right| \le C_u \tag{2.17}$$

for all $x \in \mathbb{R}^N$. Combine (2.14) and (2.17), we deduce

$$|\langle \Psi'(u), v\rangle| \le C_u \int_{\mathbb{R}^N} |f(u)v| \mathrm{d}x. \tag{2.18}$$

By the condition (f_1), using Hölder inequality, for $q_* > q_2$ with $\frac{q_2-1}{q_*} + \frac{q_*-q_2}{q_*} + \frac{1}{q_*} = 1$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N} \Phi_{N,s}(\alpha_0|u|^{N/(N-s)})|u|^{q_2-1}|v| \mathrm{d}x \qquad (2.19)\\ &\le \left(\int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0|u|^{N/(N-s)}))^{\frac{q_*}{q_*-q_2}}\right)^{\frac{q_*}{q_*-q_2}} \left(\int_{\mathbb{R}^N} |u|^{q_2 *} \mathrm{d}x\right)^{\frac{q_2-1}{q_*}} \left(\int_{\mathbb{R}^N} |v|^{q_*} \mathrm{d}x\right)^{\frac{1}{q_*}} \le C_{2,u}||v||_{L^{q_*}(\mathbb{R}^N)}.\end{aligned}$$

Hence, it holds

$$\begin{aligned}\int_{\mathbb{R}^N} |f(u)v| \mathrm{d}x &\le a_1 \int_{\mathbb{R}^N} |u|^{q_1-1}|v| \mathrm{d}x + a_2 \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha_0|u|^{N/(N-s)})|u|^{q_2-1}|v| \mathrm{d}x \\ &\le a_1||u||^{q_1-1}_{L^{q_1}(\mathbb{R}^N)}||v||_{L^{q_1}(\mathbb{R}^N)} + a_2C_{2,u}||v||_{L^{q_*}(\mathbb{R}^N)} \le C_{3,u}||v||_{W^{s,N/s}(\mathbb{R}^N)},\end{aligned}$$

$$|\langle \psi'(u), v\rangle| \le C_{3,u}||v||_{W^{s,N/s}(\mathbb{R}^N)}. \tag{2.20}$$

This implies that $\psi \in (W^{s,N/s}(\mathbb{R}^N))'$. We now prove that ψ' is continuous on $(W^{s,N/s}(\mathbb{R}^N))'$. Then it is Fréchet differentiable. We claim that

$$\begin{aligned}||\psi'(u_n)-\psi'(u)|| &= \sup_{||v||_{W^{s,N/s}(\mathbb{R}^N)}=1} |\langle \psi'(u_n)-\psi'(u),v\rangle| \\ &= \sup_{||v||_{W^{s,N/s}(\mathbb{R}^N)}=1} \left| \int\limits_{\mathbb{R}^N} \left[\left(\frac{1}{|x|^\mu}*F(u_n)\right)f(u_n)-\left(\frac{1}{|x|^\mu}*F(u)\right)f(u)\right] v\mathrm{d}x\right| \to 0\end{aligned}$$

as $n\to\infty$. Conversely, if for some $\varepsilon_0>0$, there exist n large enough so that $||\psi'(u_n)-\psi'(u)||>\varepsilon_0>0$. Then there exists $v\in W^{s,N/s}(\mathbb{R}^N)$ such that

$$|\langle \psi'(u_n)-\psi'(u),v\rangle| \geq ||\psi'(u_n)-\psi'(u)|| - \frac{\varepsilon_0}{2} > \frac{\varepsilon_0}{2}. \tag{2.21}$$

Now we show that $\lim_{n\to\infty} < \psi'(u_n)-\psi'(u), v >= 0$. Then it is contradiction with (2.21) as n large enough. By (2.14), we deduce

$$\langle \psi'(u_n)-\psi'(u),v\rangle = \int\limits_{\mathbb{R}^N} \left[\left(\frac{1}{|x|^\mu}*F(u_n)\right)f(u_n)-\left(\frac{1}{|x|^\mu}*F(u)\right)f(u)\right] v\mathrm{d}x \tag{2.22}$$

By arguments before, we have $\left(\frac{1}{|x|^\mu}*F(u)\right)f(u)v\in L^1(\mathbb{R}^N)$. Now, we show that

$$\left(\frac{1}{|x|^\mu}*F(u_n)\right)f(u_n)v\in L^1(\mathbb{R}^N). \tag{2.23}$$

Using Hardy-Littlewood-Sobolev inequality, we get

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|F(u_n(y))f(u_n(x))v(x)|}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \leq ||F(u_n)||_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)} ||f(u_n)v||_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}.$$

By the condition (f_1), one has

$$\begin{aligned}|F(u_n)|^{\frac{2N}{N-\mu}} &\leq (a_1|u_n|^{q_1}+a_2|u_n|^{q_2}\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{\frac{2N}{2N-\mu}} \\ &\leq C(|u_n|^{\frac{2Nq_1}{2N-\mu}}+|u_n|^{\frac{2Nq_2}{2N-\mu}}(\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{\frac{2N}{2N-\mu}}.\end{aligned} \tag{2.24}$$

By using Hölder inequality, for $t>1$ and $t'>1$ so that $\frac{1}{t}+\frac{1}{t'}=1$, we deduce

$$\begin{aligned}&\int\limits_{\mathbb{R}^N} |u_n|^{\frac{2Nq_2}{2N-\mu}}(\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{\frac{2N}{2N-\mu}}\mathrm{d}x \\ &\leq \left(\int\limits_{\mathbb{R}^N}|u_n|^{\frac{2Nq_2t}{2N-\mu}}\mathrm{d}x\right)^{1/t}\left((\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{\frac{2Nt'}{2N-\mu}}\mathrm{d}x\right)^{1/t'}.\end{aligned} \tag{2.25}$$

By [24, Lemma 2.3], for $\mathfrak{c}>\frac{2Nt'}{2N-\mu}$ and it is choosen near $\frac{2Nt'}{2N-\mu}$, we have

$$(\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{\frac{2Nt'}{2N-\mu}} \leq \Phi_{N,s}(\alpha_0\mathfrak{c}|u_n|^{N/(N-s)}) \quad \text{for all } n.$$

Note that

$$\begin{aligned}&\Phi_{N,s}(\alpha_0\mathfrak{c}|u_n|^{N/(N-s)}) \leq \Phi_{N,s}(\alpha_0\mathfrak{c}(|u_n-u|+|u|)^{N/(N-s)}) \\ &\leq \Phi_{N,s}(\alpha_0 2^{N/(N-s)}\mathfrak{c}(|u_n-u|^{N/(N-s)}+|u|^{N/(N-s)})) \\ &\leq \frac{1}{2}\Big(\Phi_{N,s}(2\alpha_0 2^{N/(N-s)}\mathfrak{c}|u_n-u|^{N/(N-s)})+\Phi_{N,s}(2\alpha_0 2^{N/(N-s)}\mathfrak{c}|u|^{N/(N-s)})\Big).\end{aligned} \tag{2.26}$$

Since $u_n \to u$ in W, then $||u_n - u||_{W^{s,N/s}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Hence $\lim_{n\to\infty} 2\alpha_0 2^{N/(N-s)}||u_n - u||^{N/(N-s)}_{W^{s,N/s}(\mathbb{R}^N)} = 0$ and it is small enough as n large enough. Hence, we can apply fractional Trudinger-Moser inequality to get

$$\int_{\mathbb{R}^N} \Phi_{N,s}(2\alpha_0 2^{N/(N-s)}\mathfrak{c}|u_n - u|^{N/(N-s)})\mathrm{d}x \tag{2.27}$$
$$= \int_{\mathbb{R}^N} \Phi_{N,s}\bigg(2\alpha_0 2^{N/(N-s)}\mathfrak{c}||u_n - u||^{N/(N-s)}_{W^{s,N/s}(\mathbb{R}^N)}\bigg(\frac{|u_n - u|}{||u_n - u||_{W^{s,N/s}(\mathbb{R}^N)}}\bigg)^{N/(N-s)}\bigg)\mathrm{d}x \le D < +\infty$$

for a suitable consant $D > 0$ as n large enough. From (2.24), (2.25) and (2.27), there exists a positive constant $C_0 > 0$ so that

$$\int_{\mathbb{R}^N} |F(u_n)|^{\frac{2N}{N-\mu}}\mathrm{d}x \le C_0 \tag{2.28}$$

for all n large enough. Similarly, we also have

$$\int_{\mathbb{R}^N} |f(u_n)v|^{\frac{2N}{2N-\mu}}\mathrm{d}x$$
$$\le D_*\bigg(\int_{\mathbb{R}^N}(|u_n|^{q_1-1}|v|)^{\frac{2N}{2N-\mu}}\mathrm{d}x + \int_{\mathbb{R}^N}(|u_n|^{q_2-1}|v|\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{\frac{2N}{2N-\mu}}\mathrm{d}x\bigg), \tag{2.29}$$

where $D_* > 0$ is a suitable constant. Using Hölder inequality,

$$\int_{\mathbb{R}^N}(|u_n|^{q_1-1}|v|)^{\frac{2N}{2N-\mu}}\mathrm{d}x \le \bigg(\int_{\mathbb{R}^N}|u_n|^{\frac{2Nq_1}{2N-\mu}}\mathrm{d}x\bigg)^{\frac{q_1-1}{q_1}}\bigg(\int_{\mathbb{R}^N}|v|^{\frac{2Nq_1}{2N-\mu}}\mathrm{d}x\bigg)^{\frac{1}{q_1}}. \tag{2.30}$$

Using the Hölder inequality for $q_* > q_2$ with $\frac{q_2-1}{q_*} + \frac{q_*-q_2}{q_*} + \frac{1}{q_*} = 1$, we get

$$\int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)})|u_n|^{q_2-1}|v|)^{\frac{2N}{2N-\mu}}\mathrm{d}x \tag{2.31}$$
$$\le \bigg(\int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{\frac{q_*}{q_*-q_2}}\bigg)^{\frac{q_*}{q_*-q_2}}\bigg(\int_{\mathbb{R}^N}|u_n|^{\frac{2Nq_2}{2N-\mu}}\mathrm{d}x\bigg)^{\frac{q_*}{q_2-1}}\bigg(\int_{\mathbb{R}^N}|v|^{\frac{2Nq_*}{2N-\mu}}\mathrm{d}x\bigg)^{\frac{1}{q_*}}.$$

By Lemma 2.3, there is $c > \frac{q_*}{q_*-q_2}$ and near $\frac{q_*}{q_*-q_2}$ so that

$$\int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{\frac{q_*}{q_*-q_2}}\mathrm{d}x$$
$$\le \int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0 c|u_n|^{N/(N-s)}))\mathrm{d}x$$
$$\le \int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0 c2^{N/(N-s)}(|u_n - u|^{N/(N-s)})) + \Phi_{N,s}(\alpha_0 c2^{N/(N-s)}|u|^{N/(N-s)})\mathrm{d}x \tag{2.32}$$

By arguments (2.27) and continuous embedding from $W^{s,N/s}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N), q \ge \dfrac{N}{s}$, from (3.7) and (2.32), there exists $\mathcal{C}_v > 0$ so that

$$\int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)})|u_n|^{q_2-1}|v|)^{\frac{2N}{2N-\mu}}\mathrm{d}x \le C_v \tag{2.33}$$

for all n large enough. Combining (2.29), (2.30) and (2.33), we get

$$\int_{\mathbb{R}^N} |f(u_n)v|^{\frac{2N}{2N-\mu}} \le C_1 \tag{2.34}$$

for all n large enough and $C_1 > 0$ is a suitable constant. From (2.28) and (2.34), we get that (2.68). Similarly, we also have

$$\left(\frac{1}{|x|^\mu} * F(u)\right) f(u)v \in L^1(\mathbb{R}^N). \tag{2.35}$$

Then, by (2.68) and (2.35), we have

$$\left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n) - \left(\frac{1}{|x|^\mu} * F(u)\right) f(u) \in L^1(\mathbb{R}^N).$$

Hence there is a suitable constant κ so that

$$\left|\left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n) - \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)\right| \le \kappa$$

on $\mathbb{R}^N$ almost everywhere in $\mathbb{R}^N$. For any $\varepsilon > 0$, there exists $\delta = \varepsilon/\kappa$ and for all measurable set $E \subset \mathbb{R}^N$ such that $|E| < \delta$, we have

$$\int_E \left|\left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n) - \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)\right| \mathrm{d}x \le \kappa |E| < \delta\kappa = \varepsilon, \tag{2.36}$$

and then $\left\{\left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n) - \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)\right\}_n$ is equi-integrable. Since $u_n \to u$ and $v \in W^{s,N/s}(\mathbb{R}^N)$, and $W^{s,N/s}(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N), q \ge \frac{N}{s}$, there is $R > 0$ large enough so that

$$\left(\int\limits_{\mathbb{R}^N \setminus B_R(0)} |v|^{\frac{2Nq_1}{2N-\mu}} \mathrm{d}x\right)^{\frac{1}{q_1}} \le \varepsilon \ \text{ and } \ \left(\int\limits_{\mathbb{R}^N \setminus B_R(0)} |v|^{\frac{2Nq_*}{2N-\mu}} \mathrm{d}x\right)^{\frac{1}{q_*}} < \varepsilon \tag{2.37}$$

By arguments (2.68) and (2.35), and we only take integral on $\mathbb{R}^N \setminus B_R(0)$ to get

$$\int\limits_{\mathbb{R}^N \setminus B_R(0)} \left|\left[\left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n) - \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)\right] v\right| \mathrm{d}x < C_* \varepsilon,$$

where $C_* > 0$ is a suitable constant. Since

$$\left[\left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n) - \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)\right] v \to 0$$

almost every where in $\mathbb{R}^N$. Therefore, apply Vitali's theorem, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left|\left[\left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n) - \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)\right] v\right| \mathrm{d}x = 0.$$

It implies that

$$\lim_{n\to\infty} \left|\int_{\mathbb{R}^N} \left[\left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n) - \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)\right] v \mathrm{d}x\right| = 0,$$

which is a contradiction with (2.21). Hence $\lim_{n\to\infty} ||\psi'(u_n) - \psi'(u)|| = 0$ and ψ' is continuous in $(W^{s,N/s}(\mathbb{R}^N))'$. By arguments as above, using Hölder inequality and Vitali's theorem, we see that for any sequence $u_n \to u$ in $W^{s,N/s}(\mathbb{R}^N)$, we get $(||u_n||_\eta^p)' \to (||u||_\eta^p)'$ in $(W^{s,N/s}(\mathbb{R}^N))'$. Hence, $||u||_\eta^p$ belongs $C^1(W^{s,N/s}(\mathbb{R}^N), \mathbb{R})$ and we have $\varphi(u) = \frac{1}{p}\widetilde{M}(||u||_\eta^p) \in$

$C^1(W^{s,N/s}(\mathbb{R}^N),\mathbb{R})$. Indeed, we see that $\varphi'(u)=M(||u||_\eta^p)(||u||_\eta^p)'$, and for any sequence $u_n\to u$ in $W^{s,N/s}(\mathbb{R}^N)$, it holds $\varphi'(u_n)=M(||u_n||_\eta^p)(||u_n||_\eta^p)'\to M(||u||_\eta^p)(||u||_\eta^p)'$ in $(W^{s,N/s}(\mathbb{R}^N))'$. In conclusion, we get that $J_{\eta\nu}\in C^1(W^{s,N/s}(\mathbb{R}^N),\mathbb{R})$. □

Furthermore, we have

$$\langle J'_{\eta\nu}(u),\varphi\rangle = M(||u||_\eta^p)\Big(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y$$
$$+\eta\int_{\mathbb{R}^N}|u|^{\frac{N}{s}-2}u\varphi\mathrm{d}x\Big)-\nu^2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(y))f(u(x))\varphi(x)}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y.$$

Lemma 2.6 Suppose that (f_1) and (f_5) hold. Then there are two positive constants t_0,ρ_0 satisfying $J_{\eta\nu}(u)\ge\rho_0$ for all $u\in W^{s,N/s}(\mathbb{R}^N):\|u\|_{W^{s,N/s}(\mathbb{R}^N)}=t_0$.

Proof From the condition (f_1), for each $q_1\ge\dfrac{N}{s}>\dfrac{2N-\mu}{2s}, q_2>\dfrac{N}{s}$, there exist $a_1>0, a_2>0$ such that

$$f(t)\le a_1|t|^{q_1-1}+a_2\Phi_{N,s}(\alpha_0|t|^{N/(N-s)})|t|^{q_2-1}.$$

Therefore, we get

$$|F(t)|\le a_1|t|^{q_1}+a_2|t|^{q_2}\Phi_{N,s}(\alpha_0|t|^{N/(N-s)})$$

for all $t\in\mathbb{R}$. Apply Lemma 2.4, we obtain that

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(y))F(u(x))}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y\le C(r,N,\mu)||F(u)||^2_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}. \tag{2.38}$$

Note that

$$\|F(u)\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}\le a_1\|u^{q_1}\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}+a_2\||u|^{q_2}\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}. \tag{2.39}$$

Using the Hölder's inequality for $t>1, t'>1:\dfrac{1}{t}+\dfrac{1}{t'}=1$, and t' is choosen near 1, for any $\mathfrak{b}>\frac{2Nt'}{2N-\mu}$, together with [24, Lemma 2.3], show that there is a constant $C(\mathfrak{b})>0$ satisfying

$$\Big(\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\Big)^{\frac{2Nt'}{2N-\mu}}\le C(\mathfrak{b})\Phi_{N,s}(\mathfrak{b}\alpha_0|u|^{N/(N-s)}) \tag{2.40}$$

on $\mathbb{R}^N$, and then we deduce

$$\begin{aligned}&\||u|^{q_2}\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\|_{\frac{2N}{2N-\mu}(\mathbb{R}^N)}\\ &=\Big(\int_{\mathbb{R}^N}(|u|^{q_2}\Phi_{N,s}(\alpha_0|u|^{N/(N-s)}))^{\frac{2N}{2N-\mu}}\mathrm{d}x\Big)^{\frac{2N-\mu}{2N}}\\ &\le\|u\|^{q_2}_{L^{\frac{2Ntq_2}{2N-\mu}}}\Big(\int_{\mathbb{R}^N}C(\mathfrak{b})\Phi_{N,s}(\mathfrak{b}\alpha_0|u|^{N/(N-s)})\mathrm{d}x\Big)^{\frac{2N-\mu}{2N}}.\end{aligned} \tag{2.41}$$

By Lemma 2.1, for $\|u\|_\eta$ small enough such that

$$\mathfrak{b}\alpha_0\|u\|_\eta^{N/(N-s)}<\alpha_*, \tag{2.42}$$

we obtain

$$\int_{\mathbb{R}^N}\Phi_{N,s}(\mathfrak{b}\alpha_0|u|^{N/(N-s)})\mathrm{d}x=\int_{\mathbb{R}^N}\Phi_{N,s}\Big(\mathfrak{b}\alpha_0\|u\|_\eta^{N/(N-s)}\Big(\frac{|u|}{\|u\|_\eta}\Big)^{N/(N-s)}\Big)\mathrm{d}x<+\infty. \tag{2.43}$$

Together with (2.42)–(2.43), for $\|u\|_\eta$ small enough, there exists suitable constants $\mathfrak{a}_1 > 0$ and $D > 0$ such that

$$\frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(u(y))F(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \le \mathfrak{a}_1\|u\|^{2q_1}_{L^{\frac{2Nq_1}{2N-\mu}}(\mathbb{R}^N)} + D\|u\|^{q_2}_{L^{\frac{2Ntq_2}{2N-\mu}}}.$$

Thus, we have

$$J_{\eta\nu}(u) \ge \frac{as}{N}\|u\|_\eta^{N/s} + \frac{bs}{2N}\|u\|_\eta^{2N/s} - \mathfrak{a}_1\nu^2\|u\|^{2q_1}_{L^{\frac{2Nq_1}{2N-\mu}}(\mathbb{R}^N)} - D\nu^2\|u\|^{2q_2}_{L^{\frac{2Ntq_2}{2N-\mu}}}. \tag{2.44}$$

Since the continuous embedding $W^{s,N/s}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for all $q \ge \dfrac{N}{s}$, then (2.44) implies that

$$J_{\eta\nu}(u) \ge \frac{as}{N}\|u\|_\eta^{N/s} + \frac{bs}{2N}\|u\|_\eta^{2N/s} - \mathfrak{a}_1\nu^2 A^{-2q_1}_{\frac{2Nq_1}{2N-\mu},\eta}\|u\|_\eta^{2q_1} - D\nu^2 A^{-2q_2}_{\frac{2Ntq_2}{2N-\mu},\eta}\|u\|_\eta^{2q_2}. \tag{2.45}$$

Let

$$h(t) = \frac{as}{N} + \frac{bs}{2N}t^{N/s} - \mathfrak{a}_1\nu^2 A^{-2q_1}_{\frac{2Nq_1}{2N-\mu},\eta} t^{2q_1-\frac{N}{s}} - D\nu^2 A^{-2q_2}_{\frac{2Ntq_2}{2N-\mu},\eta} t^{2q_2-\frac{N}{s}}, \quad t \ge 0.$$

We now verify there exists $t_0 > 0$ small satisfying $h(t_0) \ge \frac{as}{2N}$. We see that h is continuous function on $[0, +\infty)$ and $\lim_{t\to 0^+} h(t) = \frac{as}{N}$, then there exists t_0 such that $h(t) \ge \frac{as}{N} - \varepsilon_1$ for all $0 \le t \le t_0$, t_0 is small enough such that $||u||_\eta = t_0$ satisfying (2.42). If we choose $\varepsilon_1 = \frac{as}{2N}$, we have $h(t) \ge \frac{as}{2N}$ for all $0 \le t \le t_0$. Especially, $h(t_0) \ge \frac{as}{2N}$ and we obtain $J_{\eta\nu}(u) \ge \frac{as}{2N}\cdot t_0^{N/s} = \rho_0$ for $||u||_\eta = t_0$. □

Lemma 2.7 Suppose that (f_4) holds. Then there exists a function $v \in C_0^\infty(\mathbb{R}^N)$ with $||v||_\eta > t_0$, such that $J_{\eta\nu}(v) < 0$, where $t_0 > 0$ is the number given in Lemma 2.6.

Proof We denote $\mathcal{K}(u) = \dfrac{1}{2}\int\limits_{\mathbb{R}^N} K(u)(x)F(u(x))\mathrm{d}x$. Fix $u_0 \in W^{s,N/s}(\mathbb{R}^N)\setminus\{0\}$ such that $u_0 \ge 0$. We set $h(t) = \mathcal{K}\left(\dfrac{tu_0}{\|u_0\|_\eta}\right)$ for $t > 0$. By the condition (f_4), we have

$$h'(t) = h'\left(\frac{tu_0}{\|u\|_\eta}\right)\frac{u_0}{\|u_0\|_\eta} = \int\limits_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu}\right] * F\left(\frac{tu_0}{\|u_0\|_\eta}\right) f\left(\frac{tu_0}{\|u_0\|_\eta}\right)\frac{u_0}{||u_0||_\eta}\mathrm{d}x > \frac{\theta}{t}h(t).$$

Then integrating above inequality on $[1, t\|u_0\|_\eta]$ with $t > \frac{1}{\|u_0\|_\eta}$, we get $h(t\|u_0\|_\eta) \ge h(1)(t\|u_0\|_\eta)^\theta$. Hence, we deduce

$$\mathcal{K}(tu_0) \ge \mathcal{K}\left(\frac{u_0}{\|u_0\|_\eta}\right)\|u_0\|_\eta^\theta t^\theta.$$

Consequently, we have

$$\begin{aligned} J_{\eta\nu}(tu_0) &= \frac{ast^{N/s}}{N}\|u_0\|_\eta^{N/s} + \frac{bst^{2N/s}}{2N}\|u_0\|_\eta^{2N/s} - \nu^2\int\limits_{\mathbb{R}^N} K(tu_0)(x)F(tu_0)\mathrm{d}x \\ &\le \frac{st^{N/s}}{N}||u_0||_\eta^{N/s} + \frac{bst^{2N/s}}{2N}\|u_0\|_\eta^{2N/s} - \nu^2\mathcal{K}\left(\frac{u_0}{\|u_0\|_\eta}\right)\|u_0\|_\eta^\theta t^\theta \end{aligned}$$

for all $t > \frac{1}{\|u_0\|_\eta}$. Since $\theta > 2p$, set $e = tu_0$ and t large enough, we get the conclusion of Lemma 2.7. □

Using Lemma 2.6, Lemma 2.7, and an alternative form of the Mountain Pass Theorem that doesn't require the Palais-Smale condition, we obtain a sequence $u_n \subset W^{s,N/s}(\mathbb{R}^N)$ such that

$$J_{\eta\nu}(u_n) \to c_{\eta\nu} \ \text{ and } \ J'_{\eta\nu}(u_n) \to 0 \ \text{ as } \ n \to \infty,$$

where the level $c_{\eta\nu}$ is characterized by

$$c_{\eta\nu} = \inf_{\zeta \in \Sigma} \max_{t \in [0,1]} J_{\eta\nu}(\zeta(t))$$

and $\Sigma = \{\zeta \in C([0,1], W^{s,N/s}(\mathbb{R}^N)) : \zeta(0) = 0, J_{\eta\nu}(\zeta(1)) < 0\}$.

Lemma 2.8 ([25]) Assume that f satisfies the condition (f_1) and $\{u_n\}$ is a sequence verifying $\limsup_{n\to\infty} \|u_n\|_\eta^{N/(N-s)} < \frac{\alpha_*}{\mathfrak{b}\alpha_0}\mathfrak{d}^{s/(N-s)}$ for some $\mathfrak{b} > 1$, where $\mathfrak{d} = \min\{1, \eta\}$. Then there exists $C_0 > 0$ such that

$$\left| \frac{1}{|x|^\mu} * F(u_n) \right| \le C_0 \ \text{ for all } \ n.$$

Lemma 2.9 Assume that $\{u_n\}$ is a $(PS)_{c_{\eta\nu}}$ sequence of $J_{\eta\nu}$. Then, there is a constant C_{γ_1} such that $\rho_0 \le c_{\eta\nu} \le C_{\gamma_1}$.

Proof We consider the function $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ which is satisfied the conditions $\varphi(x) = 1$ if $|x| \le 1$, $\varphi(x) = 0$ if $|x| \ge 2$ and $\nabla\varphi(x) \le 1$. Note that $B_{1-|x|}(0) \subset B_1(x)$ since $|x| \le 1$, hence we get

$$\begin{aligned}
\int_{B_1(0)} \int_{B_1(0)} \frac{\varphi^\theta(x)\varphi^\theta(y)}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y &= \int_{B_1(0)} \int_{B_1(0)} \frac{1}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \\
&= \int_{B_1(0)} \int_{B_1(x)} \frac{1}{|z|^\mu} \mathrm{d}z\mathrm{d}x \ge \int_{B_1(0)} \int_{B_{1-|x|}(0)} \frac{1}{|z|^\mu} \mathrm{d}z\mathrm{d}x \\
&= N|B_1(0)| \int_{B_1(0)} \mathrm{d}x \int_0^{1-|x|} r^{N-\mu-1} \mathrm{d}r = \frac{N|B_1(0)|}{N-\mu} \int_{B_{1(0)}} (1-|x|)^{N-\mu} \mathrm{d}x \\
&= \frac{|NB_1(0)|^2}{N-\mu} \int_0^1 (1-r)^{N-\mu} r^{N-1} \mathrm{d}r = \frac{|NB_1(0)|^2 B(N, N-\mu+1)}{N-\mu},
\end{aligned}$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$ $(x>0, y>0)$ is Beta function. From the assumption (M_3), we can induce that there is a positive constant γ such that

$$\widetilde{M}(t) \le \gamma(t + t^2) \quad \text{for all } \ t \ge 0. \tag{2.46}$$

By assumption (f_4), we have

$$\begin{aligned}
J_{\eta\nu}(t\varphi) &= \frac{1}{p}\widetilde{M}(\|t\varphi\|_\eta^p) - \frac{\nu^2}{2} \int_{\mathbb{R}^N} F(t\varphi)\mathrm{d}x \int_{\mathbb{R}^N} \frac{1}{|x-y|^\mu} F(t\varphi)\mathrm{d}y \\
&\le \frac{\gamma t^p}{p}\|\varphi\|_\eta^p + \frac{\gamma t^{2p}}{2p}\|\varphi\|_\eta^{2p} - \frac{\gamma_1^2\nu^2 t^{2\theta}}{2} \int_{\mathbb{R}^N} |\varphi|^\theta \mathrm{d}x \int_{\mathbb{R}^N} \frac{|\varphi|^\theta}{|x-y|^\mu} \mathrm{d}y \\
&\le \frac{\gamma}{p} t^p \|\varphi\|_\eta^p + \frac{\gamma}{2p} t^{2p} \|\varphi\|_\eta^{2p} - \frac{\gamma_1^2\nu^2 t^{2\theta}}{2} \int_{B_1(0)} \int_{B_1(0)} \frac{1}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{2.47}$$

Hence, we obtain that

$$J_{\eta\nu}(t\varphi) \le \frac{\gamma}{p} t^p \|\varphi\|_\eta^p + \frac{\gamma}{2p} t^{2p} \|\varphi\|_\eta^{2p} - \frac{\gamma_1^2\nu^2 t^{2\theta}}{2} \cdot \frac{|NB_1(0)|^2 B(N, N-\mu+1)}{N-\mu}$$

and

$$\begin{aligned} c_{\eta\nu} &\le \max_{t\ge 0} J_{\eta\nu}(t\varphi) \\ &\le \max_{t\ge 0}\left\{\frac{\gamma}{p}t^p\|\varphi\|_\eta^p + \frac{\gamma}{2p}t^{2p}\|\varphi\|_\eta^{2p} - \frac{\gamma_1^2\nu^2 t^{2\theta}}{2}\cdot\frac{|NB_1(0)|^2B(N,N-\mu+1)}{N-\mu}\right\}. \end{aligned} \tag{2.48}$$

Set $a_1 := \frac{\gamma}{p}\|\varphi\|_\eta^p, a_2 = \frac{\gamma}{2p}\|\varphi\|_\eta^{2p}$ and $b := \frac{\gamma_1^2\nu^2|NB_1(0)|^2B(N,N-\mu+1)}{2(N-\mu)}$. We denote

$$g(t) = a_1t^p + a_2t^{2p} - bt^{2\theta}$$

on $[0,+\infty)$. We have

$$c \le \max_{t\in[0,1]} g(t) + \max_{t\ge 1} g(t). \tag{2.49}$$

As $t\in[0,1]$, we get $g(t)\le h(t) = (a_1+a_2)t^p - bt^{2\theta}$. Compute directly and obtain

$$\max_{t\in[0,1]} g(t) \le h(\theta_{\gamma_1}) = C_{\gamma_1}, \tag{2.50}$$

where

$$\theta_{\gamma_1} = \left(\frac{(a_1+a_2)p}{2\theta b}\right)^{1/(2\theta-p)} \le 1$$

as $b \ge \dfrac{(a_1+a_2)p}{2\theta}$. It implies that

$$\gamma_1 \ge \frac{1}{\nu N|B_1(0)|}\left(\frac{(a_1+a_2)p(N-\mu)}{\theta B(N,N-\mu+1)}\right)^{1/2} := \gamma^*. \tag{2.51}$$

Hence, we have

$$C_{\gamma_1} = h(\theta_{\gamma_1}) = (a_1+a_2)(1-\frac{p}{2\theta})\left(\frac{(a_1+a_2)p}{2\theta b}\right)^{1/(2\theta-p)}. \tag{2.52}$$

We see that $\lim_{\gamma_1\to+\infty}\theta_{\gamma_1} = 0$, then $\lim_{\gamma_1\to+\infty} h(\theta_{\gamma_1}) = 0$. By arguments as above, for all $t\ge 1$, we get

$$g(t) \le h_*(t) = (a_1+a_2)t^{2p} - bt^{2\theta}$$

and h_* has uniqueness local maximum point at $\beta_{\gamma_1} = \left(\frac{(a_1+a_2)p}{\theta b}\right)^{1/(2\theta-2p)}$ on $(0,+\infty)$. Note that if we choose $\gamma_1\ge\gamma_*$, where γ_* satisfies $b\ge\dfrac{(a_1+a_2)p}{\theta}$, and we have $\beta_{\gamma_1}\le 1$. Then we need

$$\gamma_1 \ge \frac{1}{N\nu|B_1(0)|}\left(\frac{2(a_1+a_2)p(N-\mu)}{\theta B(N,N-\mu+1)}\right)^{1/2} := \gamma_*. \tag{2.53}$$

Hence, we deduce

$$\max_{t\ge 1} g(t) \le h_*(1) = a_1 + a_2 - b.$$

Denoting $\gamma_{**} = \frac{1}{N\nu|B_1(0)|}\left(\frac{2(N-\mu)(a_1+a_2)}{B(N,N-\mu+1)}\right)^{1/2}$, we have

$$\max_{t\ge 1} g(t) \le 0 \text{ for all } \gamma_1 \ge \max\{\gamma_*,\gamma_{**}\}. \tag{2.54}$$

Combining (2.49), (2.50), (2.52) and (2.54), we get

$$c_{\eta\nu} \le C_{\gamma_1} = (a_1+a_2)(1-\frac{p}{2\theta})\left(\frac{(a_1+a_2)p}{2\theta b}\right)^{1/(2\theta-p)} \tag{2.55}$$

for $\gamma_1 \geq \max\{\gamma^*, \gamma_*, \gamma_{**}\}$. Therefore, the Mountain Pass level c is small enough when γ_1 is large enough, which will be used later. Combining the result and Lemma 2.6, we get $\rho_0 \leq c_{\eta\nu} \leq C_{\gamma_1}$. □

The following result is a version of Lions's result:

Lemma 2.10 ([9]) If $\{u_n\}$ is a bounded sequence in $W^{s,N/s}(\mathbb{R}^N)$ and

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{B_R(y)} |u_n(x)|^{N/s}\mathrm{d}x = 0$$

for some $R > 0$, then $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (\frac{N}{s}, +\infty)$.

By arguments as in [9], we get the following result:

Lemma 2.11 Suppose that $\{u_n\}$ is a sequence in $W^{s,N/s}(\mathbb{R}^N)$ which converges weakly to 0 and $\limsup_{n\to\infty} ||u_n||_\eta^{N/(N-s)} < \dfrac{\alpha_*}{\mathfrak{b}\alpha_0}\mathfrak{d}^{s/(N-s)}$, where $\mathfrak{b} > 1$ is a constant and near 1. Assume that (f_1) holds and $\lim_{t\to 0^+} \frac{f(t)}{t^{\frac{N}{s}-1}} = 0$, and there exists $R > 0$ such that $\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{B_R(y)} |u_n|^{N/s}\mathrm{d}x = 0$. Then

$$\int\limits_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)u_n \to 0 \quad \text{and} \quad \int\limits_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(u_n)\right] F(u_n) \to 0.$$

Proposition 1 Suppose the conditions (f_1)–(f_5) are fulfilled. Then, problem $(\mathcal{P}_{\eta\nu})$ possesses a nontrivial nonnegative weak solution.

Proof Using Lemma 2.6 and Lemma 2.7, it is straightforward to verify that the energy function $J_{\eta\nu}$ satisfies the geometric conditions required by the Mountain Pass Theorem. Consequently, a (PS) sequence exists $\{u_n\} \subset W^{s,N/s}(\mathbb{R}^N)$ such that

$$J_{\eta\nu}(u_n) \to c_{\eta\nu} \quad \text{and} \quad J'_{\eta\nu}(u_n) \to 0 \quad \text{as } n \to \infty,$$

where the level c is characterized by

$$c_{\eta\nu} = \inf_{\zeta\in\Sigma} \max_{t\in[0,1]} J_{\eta\nu}(\zeta(t))$$

and $\Sigma = \{\zeta \in C([0,1], W^{s,N/s}(\mathbb{R}^N)) : \zeta(0) = 0, J_{\eta\nu}(\zeta(1)) < 0\}$. This implies that

$$\sup_{||\varphi||_\eta=1} |\langle J'_{\eta\nu}(u_n), \varphi\rangle| \to 0 \tag{2.56}$$

as $n \to \infty$, and it holds

$$J_{\eta\nu}(u_n) - \frac{1}{\theta}\langle J'_{\eta\nu}(u_n), u_n\rangle = c_{\eta\nu} + o_n(1) + o_n(1)||u_n||_\eta. \tag{2.57}$$

From the condition (M_3), we get

$$\widetilde{M}(t) \geq \frac{M(t) + a}{2}t \quad \text{for all } t \geq 0. \tag{2.58}$$

Then, we obtain

$$\begin{aligned} & J_{\eta\nu}(u_n) - \frac{1}{\theta}\langle J'_{\eta\nu}(u_n), u_n\rangle \\ = & \frac{s}{N}\widetilde{M}(||u_n||_\eta^{N/s}) - \frac{\nu^2}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(u_n(y))F(u_n(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \end{aligned}$$

$$
-\frac{1}{\theta}\bigg[M(||u_n||_\eta^{N/s})||u_n||_\eta^{N/s}-\nu^2\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{F(u_n(y))f(u_n(x))u_n(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\bigg]
$$

$$
\geq a\bigg(\frac{s}{N}-\frac{1}{\theta}\bigg)||u_n||_\eta^{N/s}s+\nu^2\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{F(u_n(y))}{|x-y|^\mu}[\frac{1}{\theta}f(u_n(x))u_n(x)-\frac{1}{2}F(u_n(x))]\mathrm{d}y\mathrm{d}x.
$$

Therefore, we have

$$
J_{\eta\nu}(u_n)-\frac{1}{\theta}\langle J'_{\eta\nu}(u_n),u_n\rangle\geq a\bigg(\frac{s}{N}-\frac{1}{\theta}\bigg)||u_n||_\eta^{N/s}. \tag{2.59}
$$

By (2.57) and (2.59), we get

$$
a\bigg(\frac{s}{N}-\frac{1}{\theta}\bigg)||u_n||_\eta^{N/s}\leq c_{\eta\nu}+o_n(1)+o_n(1)||u_n||_\eta. \tag{2.60}
$$

Combine (2.52) and (2.60), with

$$
J_{\eta\nu}(u_n)-\frac{1}{\theta}\langle J'_{\eta\nu}(u_n),u_n\rangle\to c_{\eta\nu}
$$

as $n\to\infty$, to get

$$
\limsup_{n\to\infty}\|u_n\|_\eta^{N/s}\leq(a_1+a_2)(1-\frac{p}{2\theta})\left(\frac{(a_1+a_2)p}{2\theta b}\right)^{1/(2\theta-p)}\cdot\frac{a^{-1}}{\frac{s}{N}-\frac{1}{\theta}}. \tag{2.61}
$$

From that result, we have

$$
\mathfrak{c}\alpha_0\mathfrak{d}^{-s/(N-s)}\sup_n\|u_n\|_\eta^{N/(N-s)}\leq\beta_*<\alpha_*, \tag{2.62}
$$

when $\gamma_1\geq\gamma_0$, where γ_0 satisfies

$$
\mathfrak{c}\alpha_0\mathfrak{d}^{-s/(N-s)}\left(\frac{(a_1+a_2)p}{2\theta b}\right)^{\frac{Ns}{(2\theta s-N)(N-s)}}\left\{(a_1+a_2)(1-\frac{p}{2\theta})\cdot\frac{m_0^{-1}}{\frac{s}{N}-\frac{1}{\theta}}\right\}^{\frac{s}{N-s}}\leq\beta_*, \tag{2.63}
$$

with $b=\frac{\nu^2\gamma_0^2|NB_1(0)|^2B(N,N-\mu+1)}{2(N-\mu)}$. It means that

$$
\gamma_1\geq\frac{1}{\nu|NB_1(0)|}\sqrt{\frac{(N-\mu)(a_1+a_2)p}{B(N,N-\mu+1)\theta}}
$$

$$
\times\left[\left(\frac{\beta_*\mathfrak{d}^{s/(N-s)}}{\mathfrak{c}\alpha_0}\right)^{\frac{(2\theta s-N)(N-s)}{2Ns}}\left(\frac{a^{-1}(a_1+a_2)(1-\frac{p}{2\theta})}{\frac{s}{N}-\frac{1}{\theta}}\right)^{\frac{2\theta s-N}{2N}}\right]:=\gamma_0. \tag{2.64}
$$

Therefore, apply to Lemma 2.1, we deduce

$$
\sup_n\int\limits_{\mathbb{R}^N}\Phi_{N,s}(\mathfrak{c}\alpha_0|u_n|^{N/(N-s)})\mathrm{d}x
$$

$$
=\sup_n\int\limits_{\mathbb{R}^N}\Phi_{N,s}(\mathfrak{c}\alpha_0\mathfrak{d}^{-s/(N-s)}||u_n||_\eta^{N/(N-s)}(\mathfrak{d}^{s/N}u/||u_n||_\eta)^{N/(N-s)})\mathrm{d}x<+\infty. \tag{2.65}
$$

Choose a subsequence if necessary, for any $q\geq\dfrac{N}{s}$, we may assume that

$$
u_n\rightharpoonup u\text{ weak in }W^{s,N/s}(\mathbb{R}^N),
$$

$$
u_n\to u\text{ strong in }L^q_{\mathrm{loc}}(\mathbb{R}^N),
$$

$$u_n(x) \to u(x) \text{ almost everywhere in } \mathbb{R}^N.$$

If $u \equiv 0$, we will get a nontrivial solution as follows. We claim that there exists a sequence $\{y_n\} \subset \mathbb{R}^N$, the positive number $R > 0$ and $\delta > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{B_R(y)} |u_n(x)|^p \mathrm{d}x \geq \delta > 0. \tag{2.66}$$

By a contradiction, we assume that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{B_R(y)} |u_n(x)|^p \mathrm{d}x = 0,$$

then by Lemma 2.10, we get $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $q > p = \frac{N}{s}$. Using the condition (2.66) and Lemma 2.11 together with Trudinger-Moser inequality (2.65), we get

$$\int\limits_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n) u_n \mathrm{d}x \to 0 \text{ as } n \to \infty.$$

Then

$$\begin{aligned} o_n(1) = \langle J'_{\eta\nu}(u_n), u_n\rangle &= M(\|u_n\|_\eta^p)\|u_n\|_\eta^p - \nu^2 \int\limits_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n) u_n \mathrm{d}x \\ &= M(\|u_n\|_\eta^p)\|u_n\|_\eta^p + o(1) \end{aligned}$$

as $n \to \infty$. Hence $u_n \to 0$ strongly in $W^{s,N/s}(\mathbb{R}^N)$. It implies that

$$J_{\eta\nu}(u_n) = \frac{1}{p}\widetilde{M}(\|u_n\|_\eta^p) - \frac{\nu^2}{2} \int\limits_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(u_n)\right] F(u_n) \mathrm{d}x \to 0$$

as $n \to \infty$. It contradicts with $c_{\eta\nu} > 0$. Therefore (2.66) holds. We denote $v_n(x) = u_n(x + y_n)$, then from (2.66) we get

$$\int\limits_{B_R(0)} |v_n|^p \mathrm{d}x \geq \delta/2 \text{ for } n \text{ large enough.} \tag{2.67}$$

Because $J_{\eta\nu}$ and $J'_{\eta\nu}$ are both invariant by the translation, it implies that

$$J_{\eta\nu}(v_n) \to c_{\eta\nu} \text{ and } J'_{\eta\nu}(v_n) \to 0 \text{ in } W^{s,N/s}(\mathbb{R}^N)^*.$$

Since $||v_n||_\eta = ||u_n||_\eta$, then $\{v_n\}$ is also bounded in $W^{s,N/s}(\mathbb{R}^N)$, then exists $v \in W^{s,N/s}(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $W^{s,N/s}(\mathbb{R}^N)$. Up to a subsequence, we may assume that $\lim_{n\to\infty} ||v_n||_\eta^{N/s} = r_0 > 0$. From (2.67), we get $\int_{B_R(0)} |v|^p \mathrm{d}x \geq \delta/2 > 0$, then $v \not\equiv 0$. Now, we prove that $J'_{\eta\nu}(v) = 0$. By arguments as [51, Lemma 12], we get the following results:

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(v_n(y))f(v_n(x))\varphi(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \to \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(v(y))f(v(x))\varphi(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x,$$

$$\int\limits_{\mathbb{R}^N} |v_n|^{p-2} v_n \varphi \mathrm{d}x \to \int\limits_{\mathbb{R}^N} |v|^{p-2} v \varphi \mathrm{d}x,$$

$$\int\limits_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y$$

$$\to \int\limits_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y \tag{2.68}$$

for all $\varphi \in W^{s,p}(\mathbb{R}^N)$. We now show that $M(||v||_\eta^p) = M(r_0^p)$. By the Fatou's lemma, we obtain

$$||v||_\eta^p \leq \liminf_{n\to\infty} ||v_n||_\eta^p = r_0^p.$$

By the condition (M_2), we get $M(||v||_\eta^p) \leq M(r_0^p)$. Assume that $M(||v||_\eta^p) < M(r_0^p)$, it follows that

$$M(||v||_\eta^p)||v||_\eta^p < M(r_0^p)||v||_\eta^p = \int_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{F(v(y))f(v(x))v(x)}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y,$$

which yields $< J_{\eta\nu}^{'}(v), v >< 0$. Then there is $\tilde{r} \in (0,1)$ such that $\tilde{r}v \in \mathcal{N}_{\eta\nu}$ which is the Nehari manifold associated with problem $(\mathcal{P}_{\eta\nu})$:

$$\mathcal{N}_{\eta\nu} = \{u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\} :< J_{\eta\nu}^{'}(u), u >= 0\}.$$

Together with the characterization $c_{\eta\nu}$, M satisfies the condition (M_4), (f_5) and the Fatou's lemma, we have

$$\begin{aligned}
c_{\eta\nu} &\leq J_{\eta\nu}(\tilde{r}v) = J_{\eta\nu}(\tilde{r}v) - \frac{1}{2p}\langle J_{\eta\nu}'(\tilde{r}v), \tilde{r}v\rangle \\
&= \frac{1}{p}\widetilde{M}(||\tilde{r}v||_\eta^p) - \frac{1}{2p}M(||\tilde{r}v||_\eta^p)||\tilde{r}v||_\eta^p \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(\tilde{r}v(y))}{|x-y|^\mu}\left[\frac{f(\tilde{r}v(x))\tilde{r}v(x)}{2p} - \frac{1}{2}F(\tilde{r}v(x))\right]\mathrm{d}x\mathrm{d}y \\
&< \frac{1}{p}\widetilde{M}(||v||_\eta^p) - \frac{1}{2p}M(||v||_\eta^p)||v||_\eta^p \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(v(y))}{|x-y|^\mu}\left[\frac{f(v(x))v(x)}{2p} - \frac{1}{2}F(v(x))\right]\mathrm{d}x\mathrm{d}y \\
&\leq \liminf_{n\to\infty}\left[J_{\eta\nu}(v_n) - \frac{1}{2p}\langle J_{\eta\nu}'(v_n), v_n\rangle\right] = c_{\eta\nu},
\end{aligned} \tag{2.69}$$

which is a contradiction. Hence $M(||v||_\eta^p) = M(r_0^p)$. Therefore, we deduce that

$$\langle J_{\eta\nu}'(v), v\rangle = 0.$$

If $u \not\equiv 0$, by arguments as before, then we can show that u is a solution of $(\mathcal{P}_{\eta\nu})$.

Now, we show that v is a ground state solution to problem $(\mathcal{P}_{\eta\nu})$. We note that

$$\begin{aligned}
c_{\eta\nu} &\leq J_{\eta\nu}(v) = J_{\eta\nu}(v) - \frac{1}{2p}\langle J_{\eta\nu}^{'}(v), v\rangle \\
&= \frac{1}{p}\widetilde{M}(||v||_\eta^p) - \frac{1}{2p}M(||v||_\eta^p)||v||_\eta^p \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(v(y))}{|x-y|^\mu}\left[\frac{f(v(x))v(x)}{2p} - \frac{1}{2}F(v(x))\right]\mathrm{d}x\mathrm{d}y \\
&\leq \liminf_{n\to\infty}\left[J_{\eta\nu}(v_n) - \frac{1}{2p}\langle J_{\eta\nu}'(v_n), v_n\rangle\right] = c_{\eta\nu},
\end{aligned}$$

Hence $J_{\eta\nu}(v) = c_{\eta\nu}$, and v is a ground state solution of equation $(\mathcal{P}_{\eta\nu})$. □

Lemma 2.12 Suppose that $\eta_i > 0$ and $\nu_i > 0$ for all $i = 1, 2$ with $\min\{\eta_2-\eta_1, \nu_1-\nu_2\} \geq 0$. Then $c_{\eta_1\nu_1} \leq c_{\eta_2\nu_2}$. Furthermore, if $\max\{\eta_2 - \eta_1, \nu_1 - \nu_2\} > 0$, then $c_{\eta_1\nu_1} < c_{\eta_2\nu_2}$.

Proof Let $u \in \mathcal{N}_{\eta_2\nu_2}$ with $J_{\eta_2\nu_2}(u) = c_{\eta_2\nu_2}$, then we have

$$c_{\eta_2\nu_2} = J_{\eta_2\nu_2}(u) = \max_{t\geq 0} J_{\eta_2\nu_2}(tu).$$

Furthermore, there exists uniquely $t_0 > 0$ such that $u_0 = t_0 u \in \mathcal{N}_{\eta_1\nu_1}$ satisfying

$$J_{\eta_1\nu_1}(u_0) = \max_{t\geq 0} J_{\eta_1\nu_1}(tu_0).$$

Since M is an increasing function, it implies that $\widetilde{M}$ is also an increasing function. Then $\widetilde{M}(||u_0||^p_{\eta_2}) \geq \widetilde{M}(||u_0||^p_{\eta_1})$ since $||u_0||_{\eta_2} \geq ||u_0||_{\eta_1}$. Clearly, we have

$$\begin{aligned} c_{\eta_2\nu_2} &= J_{\eta_2\nu_2}(u) \geq J_{\eta_2\nu_2}(u_0) \\ &= J_{\eta_1\nu_1}(u_0) + \frac{1}{p}(\widetilde{M}(||u_0||^p_{\eta_2}) - \widetilde{M}(||u_0||^p_{\eta_1})) \\ &\quad + \frac{\nu_1^2 - \nu_2^2}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_0(x))F(u_0(y))}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \geq c_{\eta_1\nu_1}. \end{aligned}$$

The proof is now complete. □

3 The Auxiliary Problem

Using the change variable $x \mapsto \varepsilon x$, the problem (1.1) is equivalent to the problem of the form:

$$M(||u||^p_{W_\varepsilon})((-\Delta)^s_p u + V(\varepsilon x)|u|^{p-2}u) = \Big[\frac{1}{|x|^\mu} * (Q(\varepsilon y)F(u(y)))\Big]Q(\varepsilon x)f(u). \qquad (\mathcal{P}_\varepsilon)$$

Definition 3.1 We say that $u \in W_\varepsilon$ is a weak solution of problem $(\mathcal{P}_\varepsilon)$ if

$$\begin{aligned} &M(||u||^p_{W_\varepsilon})\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y \\ &+ \int_{\mathbb{R}^N} V(\varepsilon x)|u(x)|^{\frac{N}{s}-2}u(x)\varphi(x)\mathrm{d}x\Big) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon y)F(u(y))Q(\varepsilon x)f(u(x))\varphi(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \end{aligned}$$

for any $\varphi \in W_\varepsilon$.

In the studying of problem $(\mathcal{P}_\varepsilon)$, we use the energy functional $I_\varepsilon : W_\varepsilon \to \mathbb{R}$ which is given by

$$I_\varepsilon(u) = \frac{1}{p}\widetilde{M}(\|u\|^p_{W_\varepsilon}) - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon y)F(u(y))Q(\varepsilon x)F(u(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

By the condition (f_1), we see that I_ε is well defined on W_ε, and $I_\varepsilon \in C^2(W_\varepsilon, \mathbb{R})$. We denote the Nehari manifold $\mathcal{N}_\varepsilon$ associated to I_ε by

$$\mathcal{N}_\varepsilon = \{u \in W_\varepsilon \setminus \{0\} : \langle I'_\varepsilon(u), u\rangle = 0\},$$

where

$$\begin{aligned} \langle I'_\varepsilon(u), \varphi\rangle = M(||u||^p_{W_\varepsilon})\Big(&\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\ &+ \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{p-2}u\varphi\mathrm{d}x\Big) - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon y)F(u(y))Q(\varepsilon x)f(u(x))\varphi(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x, \end{aligned}$$

for any $u, \varphi \in W_\varepsilon$.

Proposition 2 There exists $r_* > 0$ such that

$$||u||_{W_\varepsilon} \geq r_* > 0 \text{ for all } u \in \mathcal{N}_\varepsilon.$$

Proof Clearly, we have the following inequality

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} \leq \min\{1, V_{\min}\}^{-1/p} \|u\|_{W_\varepsilon}. \tag{3.1}$$

Thus, it follows from Lemma 2.1 and (3.1) that

$$\begin{aligned} &\sup_{u\in W_\varepsilon, ||u||_{W_\varepsilon} \leq (\min\{1,V_{\min}\})^{s/N}} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha|u|^{N/(N-s)})\mathrm{d}x \\ \leq &\sup_{u\in W^{s,N/s}(\mathbb{R}^N), ||u||_{W^{s,N/s}(\mathbb{R}^N)}\leq 1} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha|u|^{N/(N-s)})\mathrm{d}x < +\infty \end{aligned} \tag{3.2}$$

for all $0 \leq \alpha < \alpha_* \leq \alpha^*_{s,N}$. From the Hardy-Littlewood-Sobolev inequality and (f_3), it follows that

$$\begin{aligned} \int_{\mathbb{R}^{2N}} [\frac{1}{|x|^\mu} * (Q(\varepsilon y)F(u(y))]Q(\varepsilon x)f(u)u\mathrm{d}x &\leq CQ^2_{\max}\|F(u)\|_{L^{\frac{2N}{2N-\mu}}} \|f(u)u\|_{L^{\frac{2N}{2N-\mu}}} \\ &\leq CQ^2_{\max}\|f(u)u\|^2_{L^{\frac{2N}{2N-\mu}}}, \end{aligned}$$

where $C > 0$ is a suitable constant. By the conditions (f_1) and (f_2), for any $\varepsilon_* > 0$ and $q \geq \dfrac{N}{s}$, there is a constant $C_{q,\varepsilon_*} > 0$ satisfying

$$|f(u)u|_{\frac{2N}{2N-\mu}} \leq \varepsilon_* \||u|^{N/s}\|_{L^{\frac{2N}{2N-\mu}}} + C_{q,\varepsilon_*}\||u|^q\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\|_{L^{\frac{2N}{2N-\mu}}} \tag{3.3}$$

for all $u \in W_\varepsilon$. Using inequality (3.2) and by arguments as Lemma 2.6, then for all $u \in \mathcal{N}_\varepsilon$ which $\|u\|_{W_\varepsilon}$ is small enough, there exists a suitable constant $C(\varepsilon_*)$ such that the following inequality holds

$$|f(u)u|^2_{\frac{2N}{2N-\mu}} \leq \varepsilon_* S^{-\frac{2N}{s}}_{\frac{2N^2}{s(2N-\mu)},\varepsilon} \|u\|^{\frac{2N}{s}}_{W_\varepsilon} + C(\varepsilon_*)\|u\|^{2q}_{W_\varepsilon} \tag{3.4}$$

for some $q > \dfrac{N}{s}$. Assume that there is $\{u_n\} \subset \mathcal{N}_\varepsilon$ satisfying $\|u_n\|_{W_\varepsilon} \to 0$ as $n \to \infty$. Since (3.4) is true when we substitute $u = u_n$ as n sufficiently large. Then we have

$$\begin{aligned} &a\|u_n\|^{N/s}_{W_\varepsilon} \leq M(\|u_n\|^{N/s}_{W_\varepsilon})\|u_n\|^{N/s}_{W_\varepsilon} \\ =& \int_{\mathbb{R}^{2N}} [\frac{1}{|x|^\mu} * (Q(\varepsilon y)F(u_n(y)))]Q(\varepsilon x)f(u_n)u_n\mathrm{d}x \\ \leq& \varepsilon_* CQ^2_{\max}S^{-\frac{2N}{s}}_{\frac{2N^2}{s(2N-\mu)},\varepsilon}\|u_n\|^{\frac{2N}{s}}_{W_\varepsilon} + C(\varepsilon_*)CQ^2_{\max}\|u_n\|^{2q}_{W_\varepsilon}. \end{aligned}$$

Dividing both sides of above inequality to $\|u_n\|^{N/s}_{W_\varepsilon}$ and taking $n \to \infty$, from $q > \dfrac{N}{s}$, we get a contradiction when ε_* is sufficiently enough. Hence, we finish the proof. □

Lemma 3.2 It holds that

(i) There exist two positive constants $\alpha > 0, \rho > 0$ satisfying $I_\varepsilon(u) \geq \alpha$ with $u \in W_\varepsilon$ so that $||u||_{W_\varepsilon} = \rho$;

(ii) There is a function e in W_ε so that $||e||_{W_\varepsilon} > \rho$ and $I_\varepsilon(e) < 0$.

Proof The proof of Lemma 3.2 is standard. We omit the details here. □

In view of Lemma 3.2, there exists a $(PS)_{c_\varepsilon}$ sequence $\{u_n\} \subset W_\varepsilon$ satisfying

$$I_\varepsilon(u_n) \to c_\varepsilon \quad \text{and} \quad I'_\varepsilon(u_n) \to 0,$$

where

$$c_\varepsilon = \inf_{\xi \in \Xi} \max_{t\in[0,1]} I_\varepsilon(\xi(t))$$

and $\Xi = \{\xi \in C([0,1], W_\varepsilon) : \xi(0) = 0, I_\varepsilon(\xi(1)) < 0\}$.

Inspired by [37, Proposition 3.11], we have the following result:

Proposition 3 It holds that $c_\varepsilon = \inf_{u\in W_\varepsilon\setminus\{0\}} \sup_{t\geq 0} I_\varepsilon(tu) = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u)$.

Proof First, we show that for each $u \in W_\varepsilon \setminus \{0\}$, there exists uniquely $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$. Set $h(t) = I_\varepsilon(tu)$. By Lemma 3.2, we have $h(t) > 0$ for all $t > 0$ small enough and $h(t) < 0$ for t large enough. Therefore, $\max_{t\geq 0} h(t)$ is attained at some $t = t_u > 0$ and by Fermat's theorem, we get $h'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. Note that $tu \in \mathcal{N}_\varepsilon$ iff

$$\begin{aligned}\frac{M(||tu||^p_{W_\varepsilon})}{||tu||^p_{W_\varepsilon}} &= \frac{1}{||u||^{2p}_{W_\varepsilon}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon y)F(tu(y))}{t^p|x-y|^\mu}\frac{Q(\varepsilon x)f(tu(x))}{t^{p-1}}u(x)\mathrm{d}x\mathrm{d}y\\ &= \int_{\{x:u(x)>0\}}\int_{\{y:u(y)>0\}} \frac{F(tu(y))}{(tu(y))^p}\frac{f(tu(x))}{(tu(x))^{p-1}}\frac{Q(\varepsilon y)Q(\varepsilon x)u(y)^p u(x)^p}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

From the condition (f_5), we see that $\frac{F(t)}{t^p}$ and $\frac{f(t)}{t^{p-1}}$ are increasing for $t > 0$. Suppose that for each u, there exists $t_1 > t_2$ such that $t_1u, t_2u \in \mathcal{N}_\varepsilon$, then we have

$$\begin{aligned}0 &> a||u||^{-p}_{W_\varepsilon}\left(\frac{1}{t_1}-\frac{1}{t_2}\right) \geq \frac{M(||t_1u||^p_{W_\varepsilon})}{||t_1u||^p_{W_\varepsilon}} - \frac{M(||t_2u||^p_{W_\varepsilon})}{||t_2u||^p_{W_\varepsilon}}\\ &= ||u||^{-2p}_{W_\varepsilon}\int_{\{x:u(x)>0\}}\int_{\{y:u(y)>0\}}\left(\frac{F(t_1u(y))}{(t_1u(y))^p}\frac{f(t_1u(x))}{(t_1u(x))^{p-1}} - \frac{F(t_2u(y))}{(t_2u(y))^p}\frac{f(t_2u(x))}{(t_2u(x))^{p-1}}\right)\\ &\quad\times\frac{Q(\varepsilon y)Q(\varepsilon x)u(y)^p u(x)^p}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y > 0.\end{aligned}$$

It is a contradiction. Hence t_u is unique. We denote $c^*_\varepsilon = \inf_{u\in W_\varepsilon\setminus\{0\}} \sup_{t\geq 0} I_\varepsilon(tu)$ and $c^{**}_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u)$. Then it holds

$$\sup_{t\geq 0} I_\varepsilon(tu) = I_\varepsilon(t(u)u)$$

and $t(u)u \in \mathcal{N}_\varepsilon$, and we get

$$c^*_\varepsilon = c^{**}_\varepsilon. \tag{3.5}$$

We now fix $u \in W_\varepsilon \setminus \{0\}$, and observe that $I_\varepsilon(tu) < 0$ as t sufficiently large. It implies that there is $t_0 >> 0$ such that for all $t \geq t_0$, we have $I_\varepsilon(tu) < 0$. Let $g_u : [0,1] \to W_\varepsilon$ define by $g_u(t) = tt_0u$ for all $t \in [0,1]$, and then $g_u \in \Gamma$, and $\max_{t\geq 0} I_\varepsilon(tu) = \max_{t\in[0,1]} I_\varepsilon(g_u(t))$. Thus, we deduce

$$c^*_\varepsilon = \inf_{u\in W_\varepsilon\setminus\{0\}}\max_{t\geq 0} I_\varepsilon(tu) = \inf_{u\in W_\varepsilon\setminus\{0\}}\max_{t\in[0,1]} I_\varepsilon(g_u(t)) \geq \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\varepsilon(\gamma(t)) = c_\varepsilon. \tag{3.6}$$

By arguments [33, Proposition 3], we have

$$c_\varepsilon \geq c^{**}_\varepsilon. \tag{3.7}$$

Combining (3.5)–(3.7), we finish the proof. □

Lemma 3.3 Suppose that $\{u_n\}$ is a bounded sequence in W_ε such that

$$\limsup_{n\to\infty}\|u_n\|_{W_\varepsilon}^{N/(N-s)}<\frac{\beta_*\mathfrak{d}_*^{s/(N-s)}}{2^{N/(N-s)}\mathfrak{c}\alpha_0},$$

where $\mathfrak{d}_*=\min\{1,V_{\min}\}$, $\mathfrak{c}>1$ is a constant and it is near 1. Furthermore, we assume that (f_1) and (f_5) hold, $u_n\rightharpoonup u$ in W_ε. Then we have the following statements:

(i) $\lim_{n\to\infty}|\Theta(v_n+u)-\Theta(v_n)-\Theta(u)|=0$.

(ii) For any $\varphi\in W_\varepsilon$ such that $\|\varphi\|_\varepsilon\le 1$, we have

$$\lim_{n\to\infty}\langle\Theta'(v_n+u)-\Theta'(v_n)-\Theta'(u),\varphi\rangle=0,$$

where $v_n=u_n-u$ and $\Theta:=\frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{Q(\varepsilon y)F(u(y))Q(\varepsilon x)F(u(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x$.

Proof By arguments as [43, Lemma 8], we get

$$\int\limits_{\mathbb{R}^N}|F(v_n+u)-F(v_n)-F(u)|^{\frac{2N}{2N-\mu}}\mathrm{d}x\to 0. \tag{3.8}$$

We see that

$$\begin{aligned}
&\Theta(u_n)-\Theta(v_n)-\Theta(u)\\
&=\frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{Q(\varepsilon x)Q(\varepsilon y)(F(u_n(x))F(u_n(y))-F(v_n(x))F(v_n(y))-F(u(x))F(u(y)))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&=\frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{Q(\varepsilon x)Q(\varepsilon y)F(u_n(x))[F(u_n(y))-F(v_n(y))-F(u(y))]}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&\quad+\frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{Q(\varepsilon x)Q(\varepsilon y)F(v_n(x))[F(u_n(y))-F(v_n(y))-F(u(y))]}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&\quad+\frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{Q(\varepsilon x)Q(\varepsilon y)F(u(x))[F(u_n(y))-F(v_n(y))-F(u(y))]}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&\quad+\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{Q(\varepsilon x)Q(\varepsilon y)F(u(x))F(v_n(y))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&:=I_1+I_2+I_3+I_4.
\end{aligned}$$

Using (3.8) and Hardy-Sobolev-inequality inequality and note that $Q\in L^\infty(\mathbb{R}^N)$, we are easy to get $I_i\to 0$ as $n\to\infty$ for all $i=1,2,3$. Finally, since $F(v_n)\to 0$ weak in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ and $\frac{1}{|x|^\mu}*F(u)\in L^{\frac{2N}{\mu}}(\mathbb{R}^N)$, we get $I_4\to 0$. Hence (i) is proved. The proof of statement (ii) is the same as (i). We omit the details. □

Lemma 3.4 Suppose that $\{u_n\}\subset W_\varepsilon$ is a $(PS)_d$ sequence of I_ε satisfying $u_n\rightharpoonup 0$ in W_ε and

$$\limsup_{n\to\infty}\|u_n\|_{W_\varepsilon}^{N/(N-s)}<\frac{\beta_*\mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

where $\mathfrak{d}_*=\min\{1,V_0\}$, $\mathfrak{c}>1$ is a constant and it is choosen near 1. Then we have either:

(i) $u_n\to 0$ in W_ε or

(ii) there is a sequence $\{y_n\} \subset \mathbb{R}^N$, and positive constants $R > 0, \beta > 0$ with

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |u_n|^{N/s} \mathrm{d}x \geq \beta > 0.$$

Proof Assume that (ii) is not true. From Lemma 2.10, it holds $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in (\frac{N}{s}, +\infty)$. By arguments as Lemma 2.11, together with conditions (f_1) and (f_2), we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(\varepsilon y)F(u_n(y))Q(\varepsilon x)f(u_n(x))u_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x = 0.$$

Since $\langle I'_\varepsilon(u_n), u_n\rangle \to 0$ as $n \to \infty$, so we have $u_n \to 0$ in W_ε. □

Lemma 3.5 Suppose that $\{v_n\} \subset W_\varepsilon$ is a $(PS)_d$ sequence which converges weakly to 0 and verifying $\limsup_{n\to\infty} \|v_n\|_{W_\varepsilon}^{N/(N-s)} \leq \frac{\beta_* \mathfrak{d}_*{}^{s/(N-s)}}{\mathfrak{c}\alpha_0}$, where $\mathfrak{c} > 1$ is constant and it is choosen near 1. If $v_n \not\to 0$ in W_ε, then $d \geq c_{V_\infty Q_\infty}$, where

$$c_{V_\infty Q_\infty} = \inf_{\zeta\in\Sigma} \max_{t\in[0,1]} J_{V_\infty Q_\infty}(\zeta(t))$$

and $\Sigma = \{\zeta \in C([0,1], W^{s,N/s}(\mathbb{R}^N)) : \zeta(0) = 0, J_{V_\infty Q_\infty}(\zeta(1)) < 0\}$.

Proof We denote by $\{t_n\} \subset (0, +\infty)$ satisfying $\{t_n v_n\} \subset \mathcal{N}_{V_\infty Q_\infty}$.

Claim 1 We have $\limsup_{n\to\infty} t_n \leq 1$.

In fact, if that claim is not true, then there exists $\delta > 0$ and a subsequence still denoted by $\{t_n\}$ such that

$$t_n \geq 1 + \delta \text{ for all } n \in \mathbb{N}. \tag{3.9}$$

We see that $\{v_n\}$ is bounded sequence in W_ε, and we have $\langle I'_\varepsilon(v_n), v_n\rangle = o_n(1)$ as $n \to \infty$. It means that

$$a||v_n||_{W_\varepsilon}^p + b||v_n||_{W_\varepsilon}^{2p} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(\varepsilon y)F(v_n(y))Q(\varepsilon x)f(v_n(x))v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x + o_n(1).$$

Moreover, reminder that $\{t_n v_n\} \subset \mathcal{N}_{V_\infty Q_\infty}$, we get

$$at_n^p||v_n||_{V_\infty}^p + bt_n^{2p}||v_n||_{V_\infty}^{2p} = Q_\infty^2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(t_n v_n(y))f(t_n v_n(x))t_n v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x.$$

Two above equalities give that

$$\begin{aligned}
&Q_\infty^2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(t_n v_n(y))f(t_n v_n(x))v_n(x)}{t_n^{2p-1}|x-y|^\mu} \mathrm{d}y\mathrm{d}x - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\
&= Q_\infty^2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{F(t_n v_n(y))f(t_n v_n(x))v_n(x)}{t_n^{2p-1}|x-y|^\mu} - \frac{F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \right) \mathrm{d}y\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(Q_\infty^2 - Q(\varepsilon y)Q(\varepsilon x))F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\
&\leq a \int_{\mathbb{R}^N} [V_\infty - V(\varepsilon x)|v_n|^p \mathrm{d}x + b(||v_n||_{V_\infty}^{2p} - ||v_n||_{W_\varepsilon}^{2p}) + o_n(1).
\end{aligned} \tag{3.10}$$

For any $\xi > 0$, there is $R = R(\xi) > 0$ verifying

$$V(\varepsilon x) \geq V_\infty - \xi \text{ and } Q(\varepsilon x) \leq V_\infty + \xi \text{ for any } |x| \geq R \tag{3.11}$$

via the condition (V). Hence, we have

$$
\begin{aligned}
|Q_\infty^2 - Q(\varepsilon y)Q(\varepsilon x)| &= |(Q_\infty - Q(\varepsilon y))Q_\infty + Q(\varepsilon y)(Q_\infty - Q(\varepsilon x))| \\
&\le |(Q_\infty - Q(\varepsilon y))|Q_\infty + Q_{\max}|Q_\infty - Q(\varepsilon x))| \le 2\varepsilon Q_{\max} \qquad (3.12)
\end{aligned}
$$

for all $|x| \ge R$ and $|y| \ge R$. Then there exists a suitable constant $C_* > 0$ such that

$$
\begin{aligned}
a\int_{\mathbb{R}^N} [V_\infty - V(\varepsilon x)]|v_n|^p \mathrm{d}x &= a\int_{B_R(0)} [V_\infty - V(\varepsilon x)]|v_n|^p \mathrm{d}x + a\int_{\mathbb{R}^N\setminus B_R(0)} [V_\infty - V(\varepsilon x)]|v_n|^p \mathrm{d}x \\
&\le C_*\xi. \qquad (3.13)
\end{aligned}
$$

Similarly, we also have

$$
\begin{aligned}
||v_n||_{V_\infty}^{2p} - ||v_n||_{W_\varepsilon}^{2p} &= ([v_n]_{s,p}^p + \int_{\mathbb{R}^N} V_\infty |v_n|^p \mathrm{d}x)^2 - ([v_n]_{s,p}^p + \int_{\mathbb{R}^N} V(\varepsilon x)|v_n|^p \mathrm{d}x)^2 \\
&= 2[v_n]_{s,p}^p \int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x))|v_n|^p \mathrm{d}x \\
&\quad + \Big(\int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x))|v_n|^p \mathrm{d}x\Big)\Big(\int_{\mathbb{R}^N} (V_\infty + V(\varepsilon x))|v_n|^p \mathrm{d}x\Big). \qquad (3.14)
\end{aligned}
$$

Since $V \in L^\infty(\mathbb{R}^N)$ and $\{v_n\}$ is a bounded sequence, then from (3.13) and (3.14), there exists a constant $C_{**} > 0$ such that

$$
b(||v_n||_{V_\infty}^{2p} - ||v_n||_{W_\varepsilon}^{2p}) \le C_{**}\xi.
$$

We see that

$$
\begin{aligned}
& Q_\infty^2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{F(t_n v_n(y)) f(t_n v_n(x)) v_n(x)}{t_n^{2p-1}|x-y|^\mu} - \frac{F(v_n(y)) f(v_n(x)) v_n(x)}{|x-y|^\mu} \right) \mathrm{d}y\mathrm{d}x \\
\le & \int_{B_R(0)}\int_{B_R(0)} \frac{(Q_\infty^2 - Q(\varepsilon y)Q(\varepsilon x))F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\
& + \int_{B_R^c(0)}\int_{B_R^c(0)} \frac{(Q_\infty^2 - Q(\varepsilon y)Q(\varepsilon x))F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\
& + 2\int_{B_R^c(0)}\int_{B_R(0)} \frac{(Q_\infty^2 - Q(\varepsilon y)Q(\varepsilon x))F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\
& + a\int_{\mathbb{R}^N} [V_\infty - V(\varepsilon x)|v_n|^p \mathrm{d}x + b(||v_n||_{V_\infty}^{2p} - ||v_n||_{W_\varepsilon}^{2p}) + o_n(1).
\end{aligned}
$$

Combine (3.10)–(3.13), we deduce

$$
\begin{aligned}
& Q_\infty^2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{F(t_n v_n(y)) f(t_n v_n(x)) v_n(x)}{t_n^{2p-1}|x-y|^\mu} - \frac{F(v_n(y)) f(v_n(x)) v_n(x)}{|x-y|^\mu} \right) \mathrm{d}y\mathrm{d}x \\
\le & 2\varepsilon Q_{\max} \int_{B_R^c(0)}\int_{B_R^c(0)} \frac{F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\
& + 2Q_{\max}^2 \int_{B_R(0)}\int_{B_R(0)} \frac{F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x
\end{aligned}
$$

$$+ 4Q_{\max}^2 \int\limits_{B_R^c(0)} \int\limits_{B_R(0)} \frac{F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x + (C_* + C_{**})\xi + o_n(1). \tag{3.15}$$

Note that $v_n \to 0$ weak in W_ε, then $v_n \to 0$ strong in $L^q(B_R(0))$ for all $q \geq 1$. From the assumption, we have

$$\limsup_{n\to\infty} \|v_n\|_{V_{\min}}^{N/(N-s)} \leq \limsup_{n\to\infty} \|v_n\|_{W_\varepsilon}^{N/(N-s)} \leq \frac{\beta_* \mathfrak{d}_*{}^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

then apply Lemma 2.8, there exists $C_0 > 0$ such that $\left|\frac{1}{|x|^\mu} * F(v_n)\right| \leq C_0$ as n large enough. Using Trudinger-Moser inequality, there exits $D_* > 0$ such that

$$\begin{aligned} &2Q_{\max}^2 \int\limits_{B_R(0)} \int\limits_{B_R(0)} \frac{F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\ &\quad + 4Q_{\max}^2 \int\limits_{B_R^c(0)} \int\limits_{B_R(0)} \frac{F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\ &\leq 6Q_{\max}^2 \int\limits_{B_R(0)} \Big[\frac{1}{|x|^\mu} * F(v_n(y))\Big] f(v_n(x))v_n(x)\mathrm{d}x \\ &\leq 6Q_{\max}^2 C_0 \int\limits_{B_R(0)} f(v_n(x))v_n(x)\mathrm{d}x \leq D_*\xi \end{aligned} \tag{3.16}$$

for all n large enough. We still use the Trudinger-Moser inequality and the bound property of the sequence $\{v_n\}$, there exists $E_* > 0$ such that

$$2\xi Q_{\max} \int\limits_{B_R^c(0)} \int\limits_{B_R^c(0)} \frac{F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \leq E_*\xi \tag{3.17}$$

for all n large enough. From (3.15)–(3.17), we get

$$\begin{aligned} &Q_\infty^2 \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \left(\frac{F(t_n v_n(y))f(t_n v_n(x))v_n(x)}{t_n^{2p-1}|x-y|^\mu} - \frac{F(v_n(y))f(v_n(x))v_n(x)}{|x-y|^\mu} \right) \mathrm{d}y\mathrm{d}x \\ &\leq (C_* + C_{**} + D_* + E_*)\xi + o_n(1). \end{aligned} \tag{3.18}$$

From $v_n \not\to 0$ in W_ε, then there is a sequence $\{y_n\} \subset \mathbb{R}^N$ and two positive real numbers $R_* > 0$ and $\beta > 0$ satisfying

$$\int\limits_{B_{R_*}(y_n)} |v_n|^{N/s} \mathrm{d}x \geq \beta > 0 \tag{3.19}$$

via Lemma 3.4. We use the symbols $v_n^-(x) = \min\{v_n(x), 0\}$ and $v_n^+(x) = \max\{v_n(x), 0\}$. By argumemts as [43] and $f(t) = 0$, for $t \in (-\infty, 0]$, we get $\|v_n^-\|_{W_\varepsilon} \to 0$ as $n \to \infty$/ Set $\overline{v}_n(x) = v_n(x + y_n)$, then we have

$$\|\overline{v}_n\|_{V_{\min}}^p = \|v_n\|_{V_{\min}}^p \leq [v_n]_{s,p}^p + \int\limits_{\mathbb{R}^N} V(\varepsilon x)|v_n|^p = \|v_n\|_{W_\varepsilon}^p.$$

Since $\{\overline{v}_n\}$ is bounded sequence in $W^{s,N/s}(\mathbb{R}^N)$, then up to a subsequence, we may assume that there is $\overline{v} \in W^{s,N/s}(\mathbb{R}^N)$ with $\overline{v}_n \rightharpoonup \overline{v}$ in $W^{s,N/s}(\mathbb{R}^N)$. By Fatou's lemma, we get

$$||\overline{v}^-||_{V_{\min}} \leq \liminf_{n\to\infty} ||\overline{v}_n^-||_{V_{\min}} \leq \liminf_{n\to\infty} ||v_n^-||_{W_\varepsilon} = 0.$$

Then $\overline{v}=\overline{v}^+$ and we can assume $\overline{v}_n(x)\to\overline{v}(x)$ on $\Omega\subset B_{R_*}(0)$, where Ω has positive measure. Then $v(x)>a_*, a_*>0$ is a constant and $\overline{v}_n(x)>\dfrac{a_*}{2}>0$ for all $x\in\Omega$ and n large enough. Combine (3.10) and (3.18), we deduce

$$\begin{aligned}
&Q_\infty^2\int\limits_{\mathrm{supp}(\overline{v}_n^+)}\int\limits_{\mathrm{supp}(\overline{v}_n^+)}[\frac{F(t_n\overline{v}_n(y))f(t_n\overline{v}_n(x))\overline{v}_n(x)}{t_n^{2p-1}|x-y|^\mu}-\frac{F(t_n\overline{v}_n(y))f(t_n\overline{v}_n(x))\overline{v}_n(x)}{|x-y|^\mu}]\mathrm{d}y\mathrm{d}x\\
&=Q_\infty^2\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\left[\frac{F(t_n\overline{v}_n(y))f(t_n\overline{v}_n(x))\overline{v}_n(x)}{t_n^{2p-1}|x-y|^\mu}-\frac{F(t_n\overline{v}_n(y))f(t_n\overline{v}_n(x))\overline{v}_n(x)}{|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x\\
&=Q_\infty^2\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\left[\frac{F(t_nv_n(y))f(t_nv_n(x))v_n(x)}{t_n^{2p-1}|x-y|^\mu}-\frac{F(t_nv_n(y))f(t_nv_n(x))v_n(x)}{|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x\\
&\le(C_*+C_{**}+D_*+E_*)\xi+o_n(1)
\end{aligned}\tag{3.20}$$

for any $\xi>0$. From (f_5), we have $\dfrac{F(t)}{t^p}$ and $\frac{f(t)}{t^{p-1}}$ are increasing function for all $t>0$. By Fatou's lemma, (3.9), (3.20) and $f(t)=0$ for all $t\in(-\infty,0]$, we have

$$\begin{aligned}
0&<Q_\infty^2\int\limits_\Omega\int\limits_\Omega\left[\frac{F((1+\delta)\overline{v}(y))f((1+\delta)\overline{v}(x))\overline{v}(x)}{(1+\delta)^{2p-1}|x-y|^\mu}-\frac{F(\overline{v}(y))f(\overline{v}(x))\overline{v}(x)}{|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x\\
&=Q_\infty^2\int\limits_\Omega\int\limits_\Omega|\overline{v}(y)|^p|\overline{v}(x)|^p\left[\frac{F((1+\delta)\overline{v}(y))f((1+\delta)\overline{v}(x))}{((1+\delta)\overline{v}(y))^p((1+\delta)\overline{v}(x))^{p-1}|x-y|^\mu}\right.\\
&\quad\left.-\frac{F(\overline{v}(y))f(\overline{v}(x))}{|\overline{v}(y)|^p|\overline{v}(x)|^{p-1}|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x\\
&\le Q_\infty^2\liminf_{n\to\infty}\int\limits_\Omega\int\limits_\Omega|\overline{v}_n(y)|^p|\overline{v}_n(x)|^p\left[\frac{F((1+\delta)\overline{v}_n(y))f((1+\delta)\overline{v}_n(x))}{((1+\delta)\overline{v}_n(y))^p((1+\delta)\overline{v}_n(x))^{p-1}|x-y|^\mu}\right.\\
&\quad\left.-\frac{F(\overline{v}_n(y))f(\overline{v}_n(x))}{|\overline{v}_n(y)|^p|\overline{v}_n(x)|^{p-1}|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x\\
&\le Q_\infty^2\liminf_{n\to\infty}\int\limits_\Omega\int\limits_\Omega\left[\frac{F(t_n\overline{v}_n(y))f(t_n\overline{v}_n(x))\overline{v}_n(x)}{{t_n}^{2p-1}|x-y|^\mu}-\frac{F(\overline{v}_n(y))f(\overline{v}_n(x))\overline{v}_n(x)}{|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x\\
&\le Q_\infty^2\liminf_{n\to\infty}\int\limits_{\mathrm{supp}(\overline{v}_n^+)}\int\limits_{\mathrm{supp}(\overline{v}_n^+)}\left[\frac{F(t_n\overline{v}_n(y))f(t_n\overline{v}_n(x))\overline{v}_n(x)}{{t_n}^{2p-1}|x-y|^\mu}-\frac{F(\overline{v}_n(y))f(\overline{v}_n(x))\overline{v}_n(x)}{|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x\\
&\le(C_*+C_{**}+D_*+E_*)\varepsilon+o_n(1)
\end{aligned}$$

for n sufficiently large. This is impossible if we take $\xi>0$ small enough. Now, we investigate the cases as follows:

Case 1 $\limsup_{n\to\infty}t_n=1$. Then up to a subsequence, we may suppose that $t_n\to1$. Remind that $I_\varepsilon(v_n)\to d$ as $n\to\infty$, and $J_{V_\infty Q_\infty}(t_nv_n)\ge c_{V_\infty Q_\infty}$, then we get

$$d+o_n(1)\ge I_\varepsilon(v_n)-J_{V_\infty}(t_nv_n)+c_{V_\infty Q_\infty}.\tag{3.21}$$

We now evaluate the following quantity

$$\begin{aligned}
&I_\varepsilon(v_n)-J_{V_\infty Q_\infty}(t_nv_n)\\
&=\frac{a(1-t_n^p)}{p}[v_n]_{s,p}^p+\frac{b(1-t_n^{2p})}{2p}[v_n]_{s,p}^{2p}
\end{aligned}$$

$$+\frac{a}{p}\int\limits_{\mathbb{R}^N}(V(\varepsilon x)-t_n^pV_\infty)|v_n|^p\mathrm{d}x+\frac{b}{p}[v_n]_{s,p}^p\int\limits_{\mathbb{R}^N}(V(\varepsilon x)|-V_\infty t_n^{2p})|v_n|^p$$
$$+\frac{b}{2p}\bigg(\int\limits_{\mathbb{R}^N}(V(\varepsilon x)-t_n^pV_\infty)|v_n|^p\mathrm{d}x\bigg)\bigg(\int\limits_{\mathbb{R}^N}(V(\varepsilon x)+t_n^pV_\infty)|v_n|^p\mathrm{d}x\bigg)$$
$$+\frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\bigg[\frac{Q_\infty^2F(t_nv_n(y))F(t_nv_n(x))}{|x-y|^\mu}-\frac{Q(\varepsilon y)F(v_n(y))Q(\varepsilon y)F(v_n(x))}{|x-y|^\mu}\bigg]\mathrm{d}y\mathrm{d}x. \tag{3.22}$$

From the assumption (V), (3.11) and $v_n\to 0$ in $L^{N/s}(B_R(0))$, together with $t_n\to 1$ and

$$V(\varepsilon x)-t_n^pV_\infty=(V(\varepsilon x)-V_\infty)+(1-t_n^p)V_\infty\geq-\xi+(1-t_n^p)V_\infty\quad\text{for all } |x|\geq R,$$

we deduce

$$\int\limits_{\mathbb{R}^N}(V(\varepsilon x)-t_n^pV_\infty)|v_n|^p\mathrm{d}x\geq o_n(1)-\xi C^*, \tag{3.23}$$

and

$$\int\limits_{\mathbb{R}^N}(V(\varepsilon x)-t_n^{2p}V_\infty)|v_n|^p\mathrm{d}x\geq o_n(1)-\xi C^*, \tag{3.24}$$

for a suitable constant $C^*>0$. We have

$$\lim_{n\to\infty}\frac{(1-t_n^l)}{p}[v_n]_{s,p}^l=0,\quad l\in\{p,2p\}. \tag{3.25}$$

due to the bounded of $\{v_n\}$ in W_ε. Then from the assumption

$$\limsup_{n\to\infty}||v_n||_{W_\varepsilon}^{N/(N-s)}\leq\frac{\beta_*\mathfrak{d}_*{}^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

using Vitali convergence theorem, we get

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\bigg[\frac{F(t_nv_n(y))F(t_nv_n(x))}{|x-y|^\mu}-\frac{F(v_n(y))F(v_n(x))}{|x-y|^\mu}\bigg]\mathrm{d}y\mathrm{d}x=0. \tag{3.26}$$

We see that

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\bigg[\frac{Q_\infty^2F(t_nv_n(y))F(t_nv_n(x))}{|x-y|^\mu}-\frac{Q(\varepsilon y)F(v_n(y))Q(\varepsilon y)F(v_n(x))}{|x-y|^\mu}\bigg]\mathrm{d}y\mathrm{d}x$$
$$=\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}Q_\infty^2\bigg[\frac{F(t_nv_n(y))F(t_nv_n(x))}{|x-y|^\mu}-\frac{F(v_n(y))F(v_n(x))}{|x-y|^\mu}\bigg]\mathrm{d}y\mathrm{d}x$$
$$+\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}(Q_\infty^2-Q(\varepsilon y)Q(\varepsilon x))\frac{F(v_n(y))F(v_n(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

By arguments above, we get

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}(Q_\infty^2-Q(\varepsilon y)Q(\varepsilon x))\frac{F(v_n(y))F(v_n(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x=0. \tag{3.27}$$

From (3.21)–(3.27), we obtain

$$d+o_n(1)\geq c_{V_\infty Q_\infty}-D^*\xi+o_n(1)$$

for a suitable constant $D^* > 0$ due to the bounded of $[v_n]_{s,p}^p$ and

$$\left\{\int\limits_{\mathbb{R}^N}(V(\varepsilon x)+t_n^p V_\infty)|v_n|^p\mathrm{d}x\right\}_n.$$

In above inequality, we get $d \geq c_{V_\infty Q_\infty}$ by letting $n \to \infty$.

Case 2 $\limsup_{n\to\infty} t_n = t_0 < 1$. Then choose a subsequence if necessary, we may assume that $t_n \to t_0$ (< 1) and $t_n < 1$ for all $n \in \mathbb{N}$. We see that

$$\begin{aligned}
&d + o_n(1)\\
&= I_\varepsilon(v_n) - \frac{1}{2p}\langle I'_\varepsilon(v_n), v_n\rangle \qquad (3.28)\\
&= \frac{a}{2p}||v_n||_{W_\varepsilon}^p + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{Q(\varepsilon y)Q(\varepsilon x)F(t_nv_n(y))}{|x-y|^\mu}\left[\frac{1}{2p}f(t_nv_n(x))t_nv_n(x) - \frac{1}{2}F(t_nv_n(x))\right]\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Recalling that $t_nv_n \in \mathcal{N}_{V_\infty Q_\infty}$, using the condition (f_5) and (3.28) which leads to that

$$\begin{aligned}
c_{V_\infty Q_\infty} &\leq J_{V_\infty Q_\infty}(t_nv_n) = J_{V_\infty Q_\infty}(t_nv_n) - \frac{1}{2p}\langle J'_{V_\infty Q_\infty}(t_nv_n), t_nv_n\rangle\\
&= \frac{a}{2p}||v_n||_{V_\infty}^p + Q_\infty^2\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{F(t_nv_n(y))}{|x-y|^\mu}[\frac{1}{2p}f(t_nv_n(x))t_nv_n(x) - \frac{1}{2}F(t_nv_n(x))]\mathrm{d}y\mathrm{d}x\\
&\leq \frac{a}{2p}||v_n||_{W_\varepsilon}^p + Q_\infty^2\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{F(v_n(y))}{|x-y|^\mu}[\frac{1}{2p}f(v_n(x))v_n(x) - \frac{1}{2}F(v_n(x))]\mathrm{d}y\mathrm{d}x + o_n(1)\\
&= d + o_n(1).
\end{aligned}$$

From above inequality, we get $d \geq c_{V_\infty Q_\infty}$ by letting $n \to \infty$. □

Lemma 3.6 Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ε satisfying

$$\limsup_{n\to\infty}||u_n||_{W_\varepsilon}^{N/(N-s)} \leq \frac{\beta_*\mathfrak{d}_*{}^{s/(N-s)}}{2^{N/(N-s)}\mathfrak{c}\alpha_0},$$

$\mathfrak{c} > 1$ is constant and it is choosen near 1. Assume that $c < c_{V_\infty Q_\infty}$. Then $\{u_n\}$ has a convergent subsequence in W_ε.

Proof This lemma is similarly proved as [43, Lemma 11]. We recall some main steps. By the condition (f_3), we get that $\{u_n\}$ is a bounded sequence in W_ε. Up to a subsequence, we may assume that $u_n \rightharpoonup u$ weak in W_ε. Similar to Proposition 1, we obtain $I'_\varepsilon(u) = 0$. Denote $v_n = u_n - u$, then we deduce

$$||u_n - u||_{W_\varepsilon}^p = ||u_n||_{W_\varepsilon}^p - ||u||_{W_\varepsilon}^p + o_n(1) \leq ||u_n||_{W_\varepsilon}^p + o_n(1)$$

as $n \to \infty$ via Brezis-Lieb's lemma. Thus,

$$\limsup_{n\to\infty}||u_n - u||_{W_\varepsilon}^p \leq \sup_{n\in\mathbb{N}}||u_n||_{W_\varepsilon}^p < \left(\frac{\beta_*}{2^{N/(N-s)}\mathfrak{c}\alpha_0}\right)^{(N-s)/s}\mathfrak{d}_*.$$

Therefore, there is a natural number n_0 such that

$$\sup_{n\geq n_0}||u_n - u||_{W_\varepsilon}^p < \left(\frac{\beta_*}{2^{N/(N-s)}\mathfrak{c}\alpha_0}\right)^{(N-s)/s}\mathfrak{d}_*. \qquad (3.29)$$

By Fatou's lemma, we have

$$||u||_{W_\varepsilon}^p \le \liminf_{n\to\infty} ||u_n||_{W_\varepsilon}^p < \left(\frac{\beta_*}{2^{N/(N-s)}\mathfrak{c}\alpha_0}\right)^{(N-s)/s} \mathfrak{d}_*. \tag{3.30}$$

By arguments as Thin [43, Lemma 4], we get

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} f(v_n)\varphi \mathrm{d}x \to 0, \tag{3.31}$$

and

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y = 0 \tag{3.32}$$

as $n\to\infty$, for all $\varphi \in W^{s,N/s}(\mathbb{R}^N)$. Combining (3.31) and (3.32), we obtain

$$\langle I_\varepsilon'(v_n), \varphi\rangle \to 0 \tag{3.33}$$

as $n\to\infty$, for all $\varphi\in W_\varepsilon(\mathbb{R}^N)$. By the condition (f_2), we have

$$I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{2p}\langle I_\varepsilon'(u), u\rangle = \frac{a}{2p}\|u\|_{W_\varepsilon}^p + \int\limits_{\mathbb{R}^N}\left[\frac{1}{2p}f(u)u - F(u)\right]\mathrm{d}x \ge 0. \tag{3.34}$$

Choose a subsequence if necessary, we have $\lim_{n\to\infty}||u_n||_{W_\varepsilon} = r_2 \ge 0$. From Brezis-Lieb's and Lemma 3.3, we deduce lemma it holds

$$\begin{aligned} I_\varepsilon(v_n) &= \frac{a}{p}||u_n||_{W_\varepsilon}^p + \frac{b}{2p}||u_n||_{W_\varepsilon}^{2p} - \left(\frac{a}{p}||u||_{W_\varepsilon}^p + \frac{b}{2p}||u||_{W_\varepsilon}^{2p}\right) - \frac{b}{p}||u_n||_{W_\varepsilon}^p||u||_{W_\varepsilon}^p \\ &\quad - \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u_n(y))F(u_n(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\ &\quad + \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u(y))F(u(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x + o_n(1) \\ &= I_\varepsilon(u_n) - I_\varepsilon(u) - \frac{b}{p}||u_n||_{W_\varepsilon}^p||u||_{W_\varepsilon}^p + o_n(1) \\ &= c - I_\varepsilon(u) - \frac{br_2^p}{p}||u||_{W_\varepsilon}^p + o_n(1) = d + o_n(1), \end{aligned} \tag{3.35}$$

where $d = c - I_\varepsilon(u) - \frac{br_2^p}{p}||u||_{W_\varepsilon}^p$. Hence $\{v_n\}$ is a $(PS)_d$ sequence of I_ε with $d \le c < c_{V_\infty Q_\infty}$. Lemma 3.5 gives that $v_n \to 0$ in $W_\varepsilon(\mathbb{R}^N)$. It implies that $u_n\to u$ in $W_\varepsilon(\mathbb{R}^N)$. □

Lemma 3.7 Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ε constrained to $\mathcal{N}_\varepsilon$ with

$$\limsup_{n\to\infty} ||u_n||_{W_\varepsilon}^{N/(N-s)} \le \frac{\beta_*\mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}, \tag{3.36}$$

where $\mathfrak{c} > 1$ is a suitable constant. Assume that $c < c_{V_\infty Q_\infty}$. Then $\{u_n\}$ has a convergent subsequence in W_ε.

Proof From Proposition 2, we have

$$||u_n||_{W_\varepsilon} \ge r_* > 0 \text{ for all } n. \tag{3.37}$$

Then there is $u \in W^{s,N/s}(\mathbb{R}^N)$ which $u_n \to u$ weak in W_ε, $u_n \to u$ strong in $L^q_{\rm loc}(\mathbb{R}^N)$, $q \in [\frac{N}{s}, +\infty)$ and $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^N$. Futhermore,

$$\lim_{n\to\infty} ||u_n||_{W_\varepsilon} = l,\ r_* \le l \le \sup_n ||u_n||_{W_\varepsilon} < +\infty \tag{3.38}$$

Thanks to $\{u_n\} \subset \mathcal{N}_\varepsilon$, we get

$$M(||u_n||^p_{W_\varepsilon})||u_n||^p_{W_\varepsilon} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u_n(y))f(u_n(x))u_n(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

We consider the case $u \not\equiv 0$. Then there exists uniquely $t \in (0, +\infty)$ such that $tu \in \mathcal{N}_{V_{\min}Q_{\max}}$, and

$$M(\|tu\|^p_{V_{\min}})\|tu\|^p_{V_{\min}} = Q^2_{\max}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(tu(y))f(tu(x))(tu(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x. \tag{3.39}$$

We know that there exists $r > 0$ such that $\|u\|_{V_{\min}} \ge r > 0$, for all $u \in \mathcal{N}_{V_{\min}Q_{\max}}$, then (3.39) implies $u^+ \not\equiv 0$ and there is a postive real number ζ_0 so that $u(x) \ge \zeta_0 > 0$ on a measure set $\Omega \subset \mathbb{R}^N$ with $|\Omega| > 0$. By the method of Lagrange multipliers, there exists a real sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$I'_\varepsilon(u_n) = \lambda_n K'_\varepsilon(u_n) + o_n(1), \tag{3.40}$$

inwhich $K_\varepsilon : W_\varepsilon \to \mathbb{R}$ is defined by

$$K_\varepsilon(u) = \langle I'_\varepsilon(u), u\rangle = a||u||^p_{W_\varepsilon} + b||u||^{2p}_{W_\varepsilon} - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u(y))f(u(x))u(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

Consequently, we have

$$\begin{aligned}
\langle K'_\varepsilon(u_n), u_n\rangle &= pa||u_n||^p_{W_\varepsilon} + 2pb||u_n||^{2p}_{W_\varepsilon}\\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)f(u_n(y))f(u_n(x))u_n(x)u_n(y)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u_n(y))f'(u_n(x))u_n(x)^2}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u_n(y))f(u_n(x))u_n(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \qquad (3.41)\\
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u_n(y))}{|x-y|^\mu}\left[\left(2p-1-\frac{\theta}{2}\right)f(u_n(x))u_n(x) - f'(u_n(x))u_n^2(x)\right]\mathrm{d}y\mathrm{d}x\\
&= \int_{\Omega_n}\int_{\Omega_n} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u_n(y))}{|x-y|^\mu}\left[\left(2p-1-\frac{\theta}{2}\right)f(u_n^+)u_n^+ - f'(u_n^+){u_n^+}^2\right]\mathrm{d}y\mathrm{d}x,
\end{aligned}$$

where $\Omega_n = \{x \in \mathbb{R}^N : u_n(x) > 0\}$. By the condition (f_5), we have $(p-1)f(t) - tf'(t) < 0$ for all $t > 0$. For all n, we have

$$(2p-1-\frac{\theta}{2})f(u_n^+)u_n^+ - f'(u_n^+)(u_n^+)^2 < 0$$

via (f_5) and $\theta > 2p$. The equality (3.41) show that $\sup_{n\in\mathbb{N}}\langle K'_\varepsilon(u_n), u_n\rangle \le 0$. Indeed, if

$$\sup_{n\in\mathbb{N}}\langle K'_\varepsilon(u_n), u_n\rangle = 0.$$

Then by choosing a subsequence if necessary, we have $\lim_{n\to\infty}\langle K'_\varepsilon(u_n),u_n\rangle = 0$. From (3.41), it holds

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u_n(y))}{|x-y|^\mu}\left[\left(2p-1-\frac{\theta}{2}\right)f(u_n^+)u_n^+ - f'(u_n^+){u_n^+}^2\right]\mathrm{d}y\mathrm{d}x \geq 0.$$

Thus, we get

$$\int_{\Omega_n}\int_{\Omega_n} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u_n(y))}{|x-y|^\mu}\left[\left(2p-1-\frac{\theta}{2}\right)f(u_n^+)u_n^+ - f'(u_n^+){u_n^+}^2\right]\mathrm{d}y\mathrm{d}x \geq 0. \tag{3.42}$$

Applying (f_5) again, we deduce that

$$\left(2p-1-\frac{\theta}{2}\right)f(u^+)u^+ - f'(u^+){u^+}^2 < 0 \tag{3.43}$$

on $\Omega \subset \Omega_n$, for n large enough. Hence, combine (3.42) and (3.43), it contradicts. Final, we consider the case $u \equiv 0$, then (3.37) implies that $u_n \not\to 0$ in W_ε. Similar to Lemma 3.4, there is a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R > 0, \beta > 0$ verifying

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^{N/s}\mathrm{d}x \geq \beta > 0. \tag{3.44}$$

Set $v_n(x) = u_n(x+y_n)$, we have $\|v_n\|_{V_{\min}} = ||u_n||_{V_{\min}} \leq \|u_n\|_{W_\varepsilon}$. Then sequence $\{v_n\}$ is bounded in $W^{s,N/s}(\mathbb{R}^N)$, and up to a subsequence, we can suppose that there is a function $v \in W^{s,N/s}(\mathbb{R}^N)$ with $v_n \to v$ weakly in W_ε, $v_n \to v$ strong in $L^q_{\text{loc}}(\mathbb{R}^N)$, $q \in [\frac{N}{s}, +\infty)$, and $v_n(x) \to v(x)$ almost everywhere in $\mathbb{R}^N$. From (3.44), we obtain $v \not\equiv 0$. We see that

$$\begin{aligned} & -\langle K'_\varepsilon(u_n),u_n\rangle \\ = & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon x)Q(\varepsilon y)F(u_n(y))}{|x-y|^\mu}[f'(u_n(x))u_n(x)^2 - \left(2p-1-\frac{\theta}{2}\right)f(u_n(x))u_n(x)]\mathrm{d}y\mathrm{d}x \\ \geq & Q^2_{\min}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(v_n(y))}{|x-y|^\mu}[f'(v_n(x))v_n(x)^2 - \left(2p-1-\frac{\theta}{2}\right)f(v_n(x))v_n(x)]\mathrm{d}y\mathrm{d}x \geq 0. \end{aligned} \tag{3.45}$$

Now, by arguments as $u \not\equiv 0$, and we get a contradiction. Hence, we must have $\sup_{n\in\mathbb{N}}\langle K'_\varepsilon(u_n), u_n\rangle < 0$, and from (3.40), we deduce $\lim_{n\to\infty}\lambda_n = 0$. Therefore, $\{u_n\}$ is a $(PS)_c$ sequence of I_ε and Lemma 3.7 is proved by applying Lemma 3.6. □

Corollary 2 The critical points of $I_\varepsilon|_{\mathcal{N}_\varepsilon}$ are also critical points of I_ε in W_ε.

Proof This result is similarly proved as Proposition 2.1 [17] and we omit the details. □

4 Existence of a Ground State Solution

In this section, we denote the energy function of problem $(\mathcal{P}_{V_{\min}Q_{\max}})$ by

$$J_{V_{\min}Q_{\max}}(u) = \frac{1}{p}\widetilde{M}(||u||^p_{V_0}) - \frac{Q^2_{\max}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(y))F(u(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

We remind that $c_{V_{\min}Q_{\max}}$ is given as follows

$$c_{V_{\min}Q_{\max}} = \inf_{\zeta\in\Sigma}\max_{t\in[0,1]} J_{V_{\min}Q_{\max}}(\zeta(t))$$

and $\Sigma = \{\zeta \in C([0,1], W^{s,N/s}(\mathbb{R}^N)) : \zeta(0) = 0, J_{V_{\min}Q_{\max}}(\zeta(1)) < 0\}$, and $\mathcal{N}_{V_{\min}Q_{\max}}$ is the Nehari manifold associated with $J_{V_{\min}Q_{\max}}$ which is defined as

$$\mathcal{N}_{V_{\min}Q_{\max}} = \Big\{u \in W^{s,N/s}(\mathbb{R}^N) \setminus \{0\} : M(||u||_{V_0}^p)||u||_{V_0}^p = Q_{\max}^2 \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(u(y))f(u(x))u(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\Big\}.$$

Now we are ready to state the main result of this section:

Theorem 4.1 Assume that (f_1)–(f_5) and (V), (Q) and (VQ) hold. Then there exists $\overline{\varepsilon} > 0$ such that for all $0 < \varepsilon < \overline{\varepsilon}$, problem $(\mathcal{P}_\varepsilon)$ has a ground state solution.

Proof We claim that there is $\overline{\varepsilon} > 0$ satisfying $c_\varepsilon < c_{V_{\min}Q_{\max}}$ for all $\varepsilon \in (0, \overline{\varepsilon})$. Since $c_{V_{\min}Q_{\max}} < c_{V_\infty Q_\infty}$, then by Lemma 3.6, we get that I_ε satisfies the $(PS)_{c_\varepsilon}$ condition. Furthermore, together that result and Lemma 3.2, I_ε admits a critical point with level c_ε. From the condition (VQ), we have $V(0) = V_{\min}$ and $Q(0) = Q_{\max}$.

Choose the smooth function $\Phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ satisfying that

$$\Phi(x) = \begin{cases} 1 & \text{if } x \in B_1(0), \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_2(0). \end{cases}$$

We denote $v_r(x) = \Phi(\dfrac{x}{r})w(x)$ for each $r > 0$, then there is $t_{\varepsilon,r} > 0$ so that $t_{\varepsilon,r}v_r \in \mathcal{N}_\varepsilon$, and we get

$$c_\varepsilon \le I_\varepsilon(t_{\varepsilon,r}v_r) = \frac{1}{p}\widetilde{M}(||t_{\varepsilon,r}v_r||^p) - \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(t_{\varepsilon,r}v_r(y))F(t_{\varepsilon,r}v_r(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

For any $u \in \mathcal{N}_\varepsilon$, we get

$$a\|u\|_{W_\varepsilon}^p + b\|u\|_{W_\varepsilon}^{2p} = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u(y))f(u(x))u(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

Thus, we deduce

$$\begin{aligned} I_\varepsilon(u)|_{\mathcal{N}_\varepsilon} &= \frac{a}{p}||u||_{W_\varepsilon}^p + \frac{b}{2p}||u||_{W_\varepsilon}^{2p} - \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u(y))F(u(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\ &\ge \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{Q(\varepsilon y)Q(\varepsilon x)F(u(y))}{|x-y|^\mu}[\frac{1}{2p}f(u(x))u(x) - \frac{1}{2}F(u(x))]\mathrm{d}y\mathrm{d}x \ge 0. \end{aligned} \tag{4.1}$$

From (4.1), for each $r > 0$, sequence $\{t_{\varepsilon,r}\}$ is a bounded sequence when ε small enough. Indeed, by a contradiction that $\lim_{\varepsilon\to 0^+} t_{\varepsilon,r} = +\infty$, for each fixed r, then we get

$$I_\varepsilon(t_{\varepsilon,r}v_r) \ge \frac{at_{\varepsilon,r}^p}{p}||v_r||_{W_\varepsilon}^p - \gamma_1^2 Q_{\min}^2 t_{\varepsilon,r}^{2\theta}\||v_r|^\theta\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}^2 \to -\infty$$

via the condition (f_4). It is a impossible with (4.1). Hence, we can suppose that $t_{\varepsilon,r} \to t_r$ as $\varepsilon \to 0^+$. Note that the support of v_r is a compact set, it holds

$$\limsup_{\varepsilon\to 0^+} c_\varepsilon \le \frac{1}{p}\widetilde{M}(||t_rv_r||_{V_{\min}}^p) - \frac{Q_{\max}^2}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(t_rv_r(y))F(t_rv_r(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x = J_{V_{\min}Q_{\max}}(t_rv_r)$$

via Vitali's theorem. We remind that $t_r v_r, w \in \mathcal{N}_{V_{\min}Q_{\max}}$ and by [6, Lemma 2.2.], $\lim_{r\to+\infty} v_r \to w$ in $W^{s,N/s}(\mathbb{R}^N)$. Hence, we deduce $\lim_{r\to\infty} t_r = 1$ by (f_5), and

$$\limsup_{\varepsilon\to 0^+} c_\varepsilon \le \lim_{r\to+\infty} J_{V_{\min}Q_{\max}}(t_r v_r) = J_{V_{\min}Q_{\max}}(w) = c_{V_{\min}Q_{\max}}.$$

By arguments as Lemma 2.9, we get $c_{V_{\min}Q_{\max}} \le C_{\gamma_1}$. Thus, if we take γ_1 sufficiently large as in (2.64) with $\eta = V_{\min}, \nu = Q_{\max}$. Then for any $(PS)_{c_\varepsilon}$ sequence $\{u_n\}$ for I_ε, which satisfies

$$\limsup_{n\to\infty} ||u_n||_{W_\varepsilon}^{N/(N-s)} \le \frac{\beta_* \mathfrak{d}_*{}^{s/(N-s)}}{2^{N/(N-s)}\mathfrak{c}\alpha_0}, \text{where } \mathfrak{d}_* = \min\{1, V_{\min}\}.$$

Apply Lemma 3.6, we get the result of this lemma. □

Lemma 4.2 Let $\varepsilon_n \to 0^+$ and $\{u_n\} := \{u_{\varepsilon_n}\}$ be ground state solution of (P_{ε_n}). Then there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that up to a subsequence, $\{y_n\} : y_n = \varepsilon_n \tilde{y}_n \to y \in \mathcal{V} \cap \mathcal{Q}$. Furthermore, the sequence $v_n(x) = u_n(x + \tilde{y}_n)$ converges strongly in $W^{s,N/s}(\mathbb{R}^N)$ to ground state solution v of

$$M(||u||_{V_{\min}}^p)((-\Delta)_p^s u + V_{\min}|u|^{p-2}u) = Q_{\max}^2 \left[\frac{1}{|x|^\mu} * F(u(y))\right] f(u) \text{ in } \mathbb{R}^N, \tag{4.2}$$

up to a subsequence.

Proof Let $\{u_n\}$ be a sequence of solutions due to from Theorem 4.1 with $\varepsilon_n \to 0$. Then we have

$$\begin{aligned} I_{\varepsilon_n}(u_n) &= I_{\varepsilon_n}(u_n) - \frac{1}{\theta}\langle I'_{\varepsilon_n}(u_n), u_n\rangle = a\left(\frac{1}{p} - \frac{1}{\theta}\right)||u_n||_{W_{\varepsilon_n}}^p + b\left(\frac{1}{2p} - \frac{1}{\theta}\right)||u_n||_{W_{\varepsilon_n}}^{2p} \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon_n y)Q(\varepsilon_n x)F(u_n(y))}{|x-y|^\mu}[\frac{1}{\theta}f(u_n(x))u_n(x) - \frac{1}{2}F(u(x))]\mathrm{d}x \\ &\ge \frac{\theta - p}{\theta p} a||u_n||_{W_{\varepsilon_n}}^p \end{aligned} \tag{4.3}$$

From (4.3), we get

$$||u_n||_{W_\varepsilon}^{N/s} \le \frac{(\theta s - N)a}{N\theta} c_{\varepsilon_n} \le \frac{(\theta s - N)a c_{V_{\min}Q_{\max}}}{N\theta} \le \frac{(\theta s - N)aC_{\gamma_1}}{N\theta}.$$

First, we prove that there are sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$, $R > 0$ and $\delta > 0$ satisfying

$$\lim_{n\to\infty} \int_{B_R(\tilde{y}_n)} |u_n|^{N/s} \mathrm{d}s \ge \delta > 0. \tag{4.4}$$

Indeed, if for any $R > 0$, we have

$$\lim_{n\to\infty} \sup\nolimits_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^{N/s}\mathrm{d}s = 0.$$

From Lemma 2.10, we see that $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (\dfrac{N}{s}, +\infty)$. Choose γ_1 large enough such that

$$\limsup_{n\to\infty} ||u_n||_{V_{\min}}^{N/(N-s)} \le \limsup_{n\to\infty} ||u_n||_{W_\varepsilon}^{N/(N-s)} \le \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}, 0 < \beta_* < \alpha_*,$$

where $\mathfrak{c} > 1$ and near 1. From the condition (f_1) and compute as Lemma 3.4, we deduce

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon_n x)Q(\varepsilon_n y)F(u_n(y))f(u_n(x))u_n(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x = 0.$$

Since $u_n \in \mathcal{N}_{\varepsilon_n}$, then we see that

$$a||u_n||^p_{W_{\varepsilon_n}} + b||u_n||^{2p}_{W_{\varepsilon_n}} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon_n x)Q(\varepsilon_n y)F(u_n(y))f(u_n(x))u_n(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \to 0$$

as $n\to\infty$. Then $||u_n||_{W_{\varepsilon_n}} \to 0$ as $n\to\infty$. It contradicts with Proposition 2. Hence (4.4) holds. We set $v_n = u_n(x+\tilde{y}_n)$, $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$ and $Q_n(x) = Q(\varepsilon_n x + \varepsilon_n\tilde{y}_n)$. Then v_n is a solution of following equation

$$\begin{aligned} &M\left([v_n]^p_{s,p} + \int_{\mathbb{R}^N} V_n(x)|v_n|^p \mathrm{d}x\right)((-\Delta)^s_p u + V_n(x)|u|^{p-2}u) \\ &= \left[\frac{1}{|x|^\mu} * (Q_n(y)F(u(y)))\right]Q_n(x)f(u),\ x\in\mathbb{R}^N \end{aligned} \tag{4.5}$$

with energy function

$$\begin{aligned} \tilde{I}_{\varepsilon_n}(v_n) &= \frac{1}{p}\widetilde{M}([v_n]^p_{s,p} + \int_{\mathbb{R}^N} V_n(x)|v_n|^p\mathrm{d}x) - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q_n(y)Q_n(x)F(v_n(y))F(v_n(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\ &= I_{\varepsilon_n}(u_n) = c_{\varepsilon_n}. \end{aligned} \tag{4.6}$$

Note that from Theorem 4.1, we have $\limsup_{\varepsilon\to 0} c_\varepsilon \le c_{V_{\min}Q_{\max}}$.

Claim 1 The sequence $\{\varepsilon_n\tilde{y}_n\}$ is a bounded sequence. Otherwise if the sequence $\{\varepsilon_n\tilde{y}_n\}$ is not bounded, then up to a subsequence, still denote by $\{\varepsilon_n\tilde{y}_n\}$ such that $\varepsilon_n\tilde{y}_n\to\infty$. By the boundedness of V and Q, up to a subsequence, we may suppose that $V(\varepsilon_n\tilde{y}_n)\to V_0 \ge V_\infty > V_{\min}$ and $Q(\varepsilon_n\tilde{y}_n) \to Q_0 \le Q_\infty < Q_{\max}$ as $n\to\infty$. Since V and Q are uniformly continuous on $B_R(0)$ for any $R>0$, then we have

$$|V_n(x) - V_0| \le |V(\varepsilon_n(x+\tilde{y}_n)) - V(\varepsilon_n\tilde{y}_n)| + |V(\varepsilon_n\tilde{y}_n) - V_0| \to 0$$

as $n\to\infty$ on $B_R(0)$. Similarly, we also have

$$|Q_n(x) - Q_0| \to 0 \ \text{ as } n\to\infty \text{ on } B_R(0). \tag{4.7}$$

Since the norm in $W^{s,N/s}(\mathbb{R}^N)$ is invariant with the change of variable $z = x+\tilde{y}_n$, we have

$$\limsup_{n\to\infty} ||v_n||^{N/(N-s)}_{V_{\min}} \le \frac{\beta_*\mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}, 0<\beta_*<\alpha_* \tag{4.8}$$

where $\mathfrak{c}>1$ and near 1. From [9], for any $\varphi\in W^{s,N/s}(\mathbb{R}^N)$, we have

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{\frac{N}{s}-2}(v_n(x)-v_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y \\ &\to \int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{\frac{N}{s}-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y \end{aligned} \tag{4.9}$$

as $n\to\infty$. Next, we prove that

$$\int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * (Q_n(y)F(v_n(y)))\right]Q_n(x)f(v_n)\varphi\mathrm{d}x \to Q^2_\infty \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(v(y))\right)f(v(x))\varphi(x)\mathrm{d}x. \tag{4.10}$$

by using Vitali's convergence theorem. Note that $\{v_n\}$ is a $W^{s,N/s}(\mathbb{R}^N)$, then choose a subsequence if necessary, we find that $v \in W^{s,N/s}(\mathbb{R}^N)$ with $v_n \to v$ weakly in $W^{s,N/s}(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ on $\mathbb{R}^N$ outside a set with measure zero. From (4.4), we have $v \not\equiv 0$. For any $\tau > 0$ and $q > \frac{N}{s}$, there exists $C = C(q,\tau)$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} |f(v_n)\varphi|\mathrm{d}x &\le \tau \int_{\mathbb{R}^N} |v_n|^{\frac{N}{s}-1}|\varphi|\mathrm{d}x + C\int_{\mathbb{R}^N} |v_n|^{q-1}\Phi_{N,s}(\alpha_0|v_n|^{N/(N-s)})|\varphi|\mathrm{d}x \\ &\le \tau||v_n||^{\frac{N}{s}-1}_{L^{N/s}(\mathbb{R}^N)}||\varphi||_{L^{N/s}(\mathbb{R}^N)} \\ &\quad + C||v_n||^{q-1}_{L^q(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0|v_n|^{N/(N-s)}))^l\mathrm{d}x\Big)^{1/l}\Big(\int_{\mathbb{R}^N}|\varphi|^t\mathrm{d}x\Big)^{1/t}, \end{aligned} \tag{4.11}$$

where $t \ge \frac{N}{s}$, $l > 1$ and near 1 such that $\frac{q-1}{q} + \frac{1}{l} + \frac{1}{t} = 1$. By [24, Lemma 2.3], there exists $\mathfrak{c} > 1$ and near 1 such that

$$(\Phi_{N,s}(\alpha_0|v_n|^{N/(N-s)}))^l \le \Phi_{N,s}(\alpha_0\mathfrak{c}|v_n|^{N/(N-s)})$$

for all n. Then from (4.8), using Trudinger-Moser inequality (Lemma 2.2), we have

$$\begin{aligned}\int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0|v_n|^{N/(N-s)}))^l\mathrm{d}x &\le \int_{\mathbb{R}^N}\Phi_{N,s}(\alpha_0\mathfrak{c}|v_n|^{N/(N-s)})\mathrm{d}x \\ &= \int_{\mathbb{R}^N}\Phi_{N,s}(\alpha_0\mathfrak{c}||v_n||^{N/(N-s)}_{V_{\min}}\Big(\frac{|v_n|}{||v_n||_{V_{\min}}}\Big)^{N/(N-s)})\mathrm{d}x \le D_* < +\infty\end{aligned} \tag{4.12}$$

for all n, where $D_* > 0$ is a suitable constant. Combine (4.11) and (4.12), and apply Lemma 2.8, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N}\Big|\Big[\frac{1}{|x|^\mu} * (Q_n(y)F(v_n(y)))\Big]Q_n(x)f(v_n)\varphi\Big|\mathrm{d}x \le C\int_{\mathbb{R}^N}|f(v_n)\varphi|\mathrm{d}x < +\infty. \tag{4.13}$$

Similarly, we also get $\int_{\mathbb{R}^N} |[\frac{1}{|x|^\mu} * F(v(y))]f(v(x))\varphi(x)|\mathrm{d}x < +\infty$. Therefore,

$$\Big\{\Big(\Big[\frac{1}{|x|^\mu} * (Q_n(y)F(v_n(y)))\Big]Q_n(x)f(v_n) - \frac{Q_0^2}{|x|^\mu} * F(v(y))\Big)f(v(x))\varphi\Big\} \in L^1(\mathbb{R}^N).$$

Then there exists a constant $K > 0$ such that

$$\Big|\Big(\Big(\Big[\frac{1}{|x|^\mu} * (Q_n(y)F(v_n(y)))\Big]Q_n(x)f(v_n) - \frac{Q_0^2}{|x|^\mu} * F(v(y))\Big)f(v(x))\varphi\Big| \le K$$

for all $x \in \mathbb{R}^N$ outside a set with measure zero. Thus for any $\delta > 0$, there exist $\tau = \delta/K$ such that for any measure E with $|E| < \delta/K$, we have

$$\int_E\Big|\Big(\Big[\frac{1}{|x|^\mu} * (Q_n(y)F(v_n(y)))\Big]Q_n(x)f(v_n) - \frac{Q_0^2}{|x|^\mu} * F(v(y))\Big)f(v(x))\varphi\Big|\mathrm{d}x \le |E|K = \tau$$

for all n. Hence

$$\Big\{\Big(\Big[\frac{1}{|x|^\mu} * (Q_n(y)F(v_n(y)))\Big]Q_n(x)f(v_n) - \frac{Q_0^2}{|x|^\mu} * F(v(y))\Big)f(v(x))\varphi\Big\}_n$$

is uniform integrability on $\mathbb{R}^N$. Since $\varphi \in L^{N/s}(\mathbb{R}^N)$ and $\varphi \in L^t(\mathbb{R}^N)$, then for any $\tau_* > 0$, we can choose $R >> 0$ such that

$$||\varphi||_{L^{N/s}(B_R^c(0))} < \tau_* \text{ and } ||\varphi||_{L^t(B_R^c(0))} < \tau_*. \tag{4.14}$$

By arguments as before, we only take integral in $B_R^c(0)$, there exist $C_* > 0$ such that

$$\int\limits_{B_R^c(0)} \left|\left(\left[\frac{1}{|x|^\mu} * (Q_n(y)F(v_n(y)))\right] Q_n(x) f(v_n) - \frac{Q_0^2}{|x|^\mu} * F(v(y))\right)] f(v(x))\varphi\right| \mathrm{d}x < C_*\tau_*. \tag{4.15}$$

Note that

$$\left(\left[\frac{1}{|x|^\mu} * (Q_n(y)F(v_n(y)))\right] Q_n(x) f(v_n) - \frac{Q_0^2}{|x|^\mu} * F(v(y))\right)] f(v(x))\varphi \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}^N$ outside a set with measure zero. Then all conditions of Vitali's covergence theorem are satisfied, we get (4.10).

By arguments as above, we are easy to get

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^B} V_n(x)|v_n|^{p-2} v_n \varphi \mathrm{d}x = V_0 \int\limits_{\mathbb{R}^N} |v|^{p-2} v\varphi \mathrm{d}x. \tag{4.16}$$

Since $[v_n]_{s,p}^p + \int\limits_{\mathbb{R}^N} V_n(x)|v_n|^p \mathrm{d}x = ||u_n||_{W_\varepsilon}^p$, then up to a subsequence, we can assume that

$$\lim_{n\to\infty} \left([v_n]_{s,p}^p + \int\limits_{\mathbb{R}^N} V_n(x)|v_n|^p \mathrm{d}x\right) = \lim_{n\to\infty} ||u_n||_{W_\varepsilon}^p = r_1^p.$$

Therefore, we have

$$\begin{aligned} &M(r_1^p)\Big(\int\limits_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} + \int\limits_{\mathbb{R}^N} V_0|v|^{p-2} v\varphi \mathrm{d}x\Big) \\ &= Q_0^2 \int\limits_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(u)\right] f(u(x))\varphi(x) \mathrm{d}x \quad x \in \mathbb{R}^N. \end{aligned} \tag{4.17}$$

We now show that $M(||v||_{V_0}^p) = M(r_1^p)$. By the Fatou's lemma, we obtain

$$||v||_{V_0}^p \le \liminf_{n\to\infty} \left([v_n]_{s,p}^p + \int\limits_{\mathbb{R}^N} V_n(x)|v_n|^p \mathrm{d}x\right) = r_1^p.$$

Hence, we get $M(||v||_{V_0}^p) \le M(r_1^p)$. We will prove that

$$M(||v||_{V_0}^p) = M(r_1^p).$$

By a contradiction that $M(||v||_{V_0}^p) < M(r_1^p)$, it follows that

$$M(||v||_{V_0}^p)||v||_{V_0}^p < M(r_1^p)||v||_{V_0}^p = Q_0^2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(v(y))f(v(x))v(x)}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y,$$

which yields $\langle J'_{V_0Q_0}(v), v\rangle < 0$. Then there is $\tilde{r} \in (0,1)$ such that $\tilde{r}v \in \mathcal{N}_{V_0Q_0}$ which is Nehari manifold associated with $J_{V_0Q_0}$. Together with the characterization $c_{V_0Q_0}$, M satisfies the condition (M_4), (f_5) and the Fatou's lemma, we have

$$c_{V_0Q_0} \le J_{V_0Q_0}(\tilde{r}v) = J_{V_0Q_0}(\tilde{r}v) - \frac{1}{2p}\langle J'_{V_0Q_0}(\tilde{r}v), \tilde{r}v\rangle$$

$$
\begin{aligned}
&= \frac{1}{p}\widetilde{M}(||\tilde{r}v||^p_{V_0}) - \frac{1}{2p}M(||\tilde{r}v||^p_{V_0})||\tilde{r}v||^p_{V_0} \\
&\quad + Q_0^2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\tilde{r}v(y))}{|x-y|^\mu}[\frac{f(\tilde{r}v(x))\tilde{r}v(x)}{2p} - \frac{1}{2}F(\tilde{r}v(x))]\mathrm{d}x\mathrm{d}y \\
&< \frac{1}{p}\widetilde{M}(||v||^p_{V_0}) - \frac{1}{2p}M(||v||^p_{V_0})||v||^p_{V_0} \\
&\quad + Q_0^2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(v(y))}{|x-y|^\mu}[\frac{f(v(x))v(x)}{2p} - \frac{1}{2}F(v(x))]\mathrm{d}x\mathrm{d}y \\
&\le \liminf_{n\to\infty}\left[\tilde{I}_{\varepsilon_n}(v_n) - \frac{1}{2p}\langle\tilde{I}'_{\varepsilon_n}(v_n), v_n\rangle\right] = \liminf_{n\to\infty}\left[I_{\varepsilon_n}(u_n) - \frac{1}{2p}\langle I'_{\varepsilon_n}(u_n), u_n\rangle\right] \\
&= \liminf_{n\to\infty} c_{\varepsilon_n} \le c_{V_{\min}Q_{\max}} < c_{V_0Q_0},
\end{aligned} \tag{4.18}
$$

which is a contradiction. Hence $M(||v||^p_{V_0}) = M(r_1^p)$. Therefore, from (4.17), we deduce that $\langle J'_{V_0Q_0}(v), \varphi\rangle = 0$ for all $\varphi \in W^{s,N/s}(\mathbb{R}^N)$.

On combining (4.9), (4.16) and (4.17), we get v is a solution of equation

$$
M(||v||^p_{V_0})(-\Delta)^s_p v + V_0|u|^{p-2}u = Q_0^2\left[\frac{1}{|x|^\mu} * F(u)\right]f(u),\ x \in \mathbb{R}^N.
$$

Using Fatou's lemma and (4.6), Lemma 2.12, we get

$$
\begin{aligned}
c_{V_{\min}Q_{\max}} &< c_{V_0,Q_0} \le J_{V_0Q_0}(v) \\
&= J_{V_0Q_0}(v) - \frac{1}{2p}\langle J'_{V_0Q_0}(v), v\rangle \\
&= \frac{a}{2p}||v||^p_{V_0} + Q_0^2\int\limits_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(v(y))\right]\left(\frac{1}{2p}f(v)v - \frac{1}{2}F(v)\right)\mathrm{d}x \\
&\le \liminf_{n\to\infty}(\tilde{I}_{\varepsilon_n}(v_n) - \frac{1}{2p}\langle\tilde{I}'_{\varepsilon_n}(v_n), v_n\rangle \\
&= \liminf_{n\to\infty}\left(I_{\varepsilon_n}(u_n) - \frac{1}{2p}\langle\tilde{I}'_{\varepsilon_n}(u_n), u_n\rangle\right) = \liminf_{n\to\infty} c_{\varepsilon_n} \le c_{V_{\min}Q_{\max}}.
\end{aligned}
$$

It is a contradiction. Hence $\{\varepsilon_n\tilde{y}_n\}$ must be bounded in $\mathbb{R}^N$, then up to a subsequence, we may assume that there exists $y \in \mathbb{R}^N$ such that $\varepsilon_n\tilde{y}_n \to y$ as $n \to \infty$.

Claim 2 $y \in \mathcal{V} \cap \mathcal{Q}$.

If $y \notin \mathcal{V} \cap \mathcal{Q}$, then we have the following cases:

Case 2.1 $y \in \mathcal{V}$ an $y \notin \mathcal{Q}$, then $V(y) = V_{\min}$ and $Q(y) < Q_{\max}$.

Case 2.2 $y \notin \mathcal{V}$, and $y \in \mathcal{Q}$, then $V_{\min} < V(y)$ and $Q(y) = Q_{\max}$.

Case 2.3 $y \notin \mathcal{V}$, and $y \notin \mathcal{Q}$, then $V_{\min} < V(y)$ and $Q(y) < Q_{\max}$.

Using Lemma 2.12, we get

$$
\begin{aligned}
c_{V_{\min}Q_{\max}} &< c_{V(y),Q(y)} \le J_{V(y)Q(y)}(v) \\
&= J_{V(y)Q(y)}(v) - \frac{1}{2p}\langle J'_{V(y)Q(y)}(v), v\rangle \\
&= \frac{a}{2p}||v||_{V(y)^p} + Q(y)^2\int\limits_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(v(y))\right]\left(\frac{1}{2p}f(v)v - \frac{1}{2}F(v)\right)\mathrm{d}x \\
&\le \liminf_{n\to\infty}(\tilde{I}_{\varepsilon_n}(v_n) - \frac{1}{2p}\langle\tilde{I}'_{\varepsilon_n}(v_n), v_n\rangle) = \liminf_{n\to\infty} c_{\varepsilon_n} \le c_{V_{\min}Q_{\max}}
\end{aligned}
$$

It is an impossible and Claim 2 is proved.

Claim 3 $v_n \to v$ strong in $W^{s,N/s}(\mathbb{R}^N)$. Calculate as Claim 1 to get $J'_{V_{\min}Q_{\max}}(v) = 0$. We next prove

$$\lim_{n\to\infty} ||v_n||^p_{V_{\min}} = ||v||^p_{V_{\min}}. \tag{4.19}$$

Then from (4.19), $v_n \to v$ strong in $W^{s,N/s}(\mathbb{R}^N)$ via Brezis-Lieb's lemma. We also have

$$||v||^p_{V_{\min}} \le \liminf_{n\to\infty} ||v_n||^p_{V_{\min}} \tag{4.20}$$

via Fatou's lemma. Assume that by contradiction that

$$||v||^p_{V_{\min}} < \limsup_{n\to\infty} ||v_n||^p_{V_{\min}}.$$

Note that

$$\begin{aligned} c_{V_{\min}Q_{\max}} + o_n(1) &= J_{V_{\min}Q_{\max}}(v_n) - \frac{1}{2p}\langle J'_{V_{\min}Q_{\max}}(v_n), v_n\rangle \\ &= \frac{a}{2p}||v_n||^p_{V_{\min}} + Q^2_{\max}\int\limits_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(v_n(y))\right]\left[\frac{1}{2p}f(v_n)v_n - \frac{F(v_n)}{2}\right]\mathrm{d}x. \end{aligned}$$

Using the condition (f_2), and Fatou's lemma, we get

$$\begin{aligned} c_{V_{\min}Q_{\max}} &\ge \frac{a}{2p}\limsup_{n\to\infty}||v_n||^p_{V_{\min}Q_{\max}} \\ &\quad + Q^2_{\max}\liminf_{n\to\infty}\int\limits_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(v_n(y))\right]\left[\frac{1}{2p}f(v_n)v_n - \frac{F(v_n)}{2}\right]\mathrm{d}x \\ &> \frac{a}{2p}||v||^p_{V_{\min}} + \int\limits_{\mathbb{R}^N} Q^2_{\max}\left[\frac{1}{|x|^\mu} * F(v)\right]\left[\frac{1}{2p}f(v)v - \frac{F(v)}{2}\right]\mathrm{d}x \\ &= J_{V_{\min}Q_{\max}}(v) - \frac{1}{2p}\langle J'_{V_{\min}Q_{\max}}(v), v\rangle = J_{V_{\min}Q_{\max}}(v) \ge c_{V_{\min}Q_{\max}}, \end{aligned}$$

which is a contradiction. Then

$$||v||^p_{V_{\min}} \ge \limsup_{n\to\infty} ||v_n||^p_{V_{\min}}. \tag{4.21}$$

Combining (4.20) and (4.21), we get (4.19).

Claim 4 v is a ground state solution of (4.2). First we see that v is a solution of (4.2). Hence,

$$\begin{aligned} c_{V(y)Q(y)} &\le J_{V(y)Q(y)}(v) = J_{V(y)Q(y)}(v) - \frac{1}{2p}\langle J'_{V(y)Q(y)}(v), v\rangle \\ &= \frac{a}{2p}||v||^p_{V(y)} + Q(y)^2\int\limits_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(v(y))\right]\left(\frac{1}{2p}f(v)v - \frac{F(v)}{2}\right)\mathrm{d}x. \end{aligned}$$

On the other hand, by Fatou's lemma, and $y \in \mathcal{V}\cap\mathcal{Q}$, we get

$$\begin{aligned} c_{V_{\min}Q_{\max}} = c_{V(y)Q(y)} &\le \frac{a}{2p}\liminf_{n\to\infty}\left([v_n]^p_{s,p} + \int_{\mathbb{R}^N} V_n(x)|v_n|^p\mathrm{d}x\right) \\ &\quad + \liminf_{n\to\infty}\int\limits_{\mathbb{R}^N} Q_n(x)Q_n(y)\left[\frac{1}{|x|^\mu} * F(v_n(y))\right]\left(\frac{1}{2p}f(v_n(x))v_n(x) - \frac{F(v_n(x))}{2}\right)\mathrm{d}x \\ &= \liminf_{n\to\infty}(\tilde{I}_{\varepsilon_n}(v_n) - \frac{1}{2p}\langle\tilde{I}'_{\varepsilon_n}(v_n), v_n\rangle) = \liminf_{n\to\infty}\left(I_{\varepsilon_n}(u_n) - \frac{1}{2p}\langle I'_{\varepsilon_n}(u_n), u_n\rangle\right) \end{aligned}$$

$$= \liminf_{n\to\infty} c_{\varepsilon_n} \le c_{V_{\min}Q_{\max}}.$$

Hence $J_{V(y)Q(y)}(v) = J_{V_{\min}Q_{\max}}(v) = c_{V_{\min}Q_{\max}}$, and v is a ground state solution of equation (4.2). □

5 Multiplicity of Solutions to Problem $(\mathcal{P}_\varepsilon)$

The main result in this section is stated as follows:

Theorem 5.1 Let the conditions (f_1)–(f_5) and (V), (Q) and (VQ) hold. Then for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that for any $0 < \varepsilon < \varepsilon_\delta$, problem $(\mathcal{P}_\varepsilon)$ has at least $\mathrm{cat}_{M_\delta}(M)$ nontrival nonnegative solutions, Moreover, assume that u_ε is a solution and z_ε is global maximum of u_ε, then

$$\lim_{\varepsilon\to 0^+} V(\varepsilon z_\varepsilon) = V_{\min} \text{ and } \lim_{\varepsilon\to 0^+} Q(\varepsilon z_\varepsilon) = Q_{\max}.$$

Proof Fix $\delta > 0$ and suppose that w is a ground state solution of problem $(\mathcal{P}_{V_{\min}Q_{\max}})$. It clears that $J_{V_{\min}Q_{\max}}(w) = c_{V_{\min}Q_{\max}}$ and $J'_{V_{\min}Q_{\max}}(w) = 0$. Choose a be a smooth nonincreasing cut-off function $\eta : [0, +\infty) \to [0, 1]$ satisfying $\eta(s) = 1$ if $0 \le s \le \frac{\delta}{2}$ and $\eta(s) = 0$ if $s \ge \delta$. For $y \in \mathcal{V} \cap \mathcal{Q}$, set

$$\psi_{\varepsilon,y} = \eta(|\varepsilon x - y|) w\left(\frac{\varepsilon x - y}{\varepsilon}\right)$$

and we consider the function $\Phi_\varepsilon : \mathcal{V} \cap \mathcal{Q} \to \mathcal{N}_\varepsilon$ given by $\Phi_\varepsilon(y) = t_\varepsilon \psi_{\varepsilon,y}$, where $t_\varepsilon > 0$ satisfies

$$\max_{t\ge 0} I_\varepsilon(t\psi_{\varepsilon,y}) = I_\varepsilon(t_\varepsilon \psi_{\varepsilon,y}).$$

From the construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in \mathcal{V} \cap \mathcal{Q}$.

Lemma 5.2 The function Φ_ε satisfies the following limit

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = c_{V_{\min}Q_{\max}} \text{ uniformly in } y \in \mathcal{V} \cap \mathcal{Q}.$$

Proof Assume that the statement of Lemma 5.2 doesnot occur, then there exists $\delta_0 > 0$, $\{y_n\} \subset \mathcal{V} \cap \mathcal{Q}$ and $\varepsilon_n \to 0$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_{\min}Q_{\max}}| \ge \delta_0. \tag{5.1}$$

By [6, Lemma 2.2], we have

$$\lim_{n\to\infty} \|\psi_{\varepsilon_n,y_n}\|^p_{W_{\varepsilon_n}} = \|w\|^p_{V_{\min}}. \tag{5.2}$$

Since $\langle I'_{\varepsilon_n}(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}), t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}\rangle = 0$, using the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$ and $\bar{z} = \frac{\varepsilon_n y - y_n}{\varepsilon_n}$, then we get

$$\begin{aligned}
&a||t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}||^p_{W_{\varepsilon_n}} + b||t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}||^{2p}_{W_{\varepsilon_n}} \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon_n x)Q(\varepsilon_n y)F(t_\varepsilon \psi_{\varepsilon_n,y_n}(y))f(t_\varepsilon \psi_{\varepsilon_n,y_n}(x))t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \qquad (5.3)\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(\varepsilon_n z + y_n)Q(\varepsilon_n \bar{z} + y_n)F(t_{\varepsilon_n}\eta(|\varepsilon_n \bar{z}|)w(\bar{z}))f(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)}{|z-\bar{z}|^\mu}\mathrm{d}\bar{z}\mathrm{d}z.
\end{aligned}$$

Now we prove that $t_{\varepsilon_n} \to 1$. First we show that $t_{\varepsilon_n} \to t_0 < +\infty$. Conversly if $t_{\varepsilon_n} \to +\infty$, Since $\frac{f(t)}{t^{p-1}}$ and $\frac{F(t)}{t^p}$ are increasing for $t > 0, \eta = 1$ in $B_{\frac{\delta}{2}}(0)$ and $B_{\frac{\delta}{2}}(0) \subset B_{\frac{\delta}{2\varepsilon_n}}(0)$ for all n big enough, we obtain

$$\begin{aligned}&\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{Q(\varepsilon_n x)Q(\varepsilon_n y)F(t_\varepsilon \psi_{\varepsilon_n,y_n}(y))f(t_\varepsilon \psi_{\varepsilon_n,y_n}(x))\psi_{\varepsilon_n,y_n}(x)}{t_\varepsilon^{2p-1}|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\ &\geq Q_{\min}^2 |B_{\frac{\delta}{2}}(0)| \frac{F(t_\varepsilon \varpi(\bar{z}))}{(t_\varepsilon \varpi(\bar{z}))^p} \cdot \frac{f(t_\varepsilon \varpi(\bar{z}))}{(t_\varepsilon \varpi(\bar{z}))^{p-1}} \int\limits_{B_{\frac{\delta}{2}}(0)} |\omega(z)|^p \mathrm{d}z, \end{aligned} \tag{5.4}$$

where $\varpi(\bar{z}) = \min_{z\in \overline{B}_{\frac{\delta}{2}}(0)} \omega(z) > 0$ (we recall that $\omega \in C^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0,1)$ by [21, Corollary 5.5] and $\omega > 0$ in $\mathbb{R}^N$ by maximum principle in [15]). From the conditions (f_3) and (f_4), we have $f(t) \geq \frac{\gamma_1 \theta}{2}|t|^{\theta-1}$ for all $t \geq 0$. Hence, if $t_\varepsilon \to \infty$, we can obtain that

$$\lim_{t\to\infty} \frac{F(t)}{t^p} = \lim_{t\to\infty} \frac{f(t)}{pt^{p-1}} \geq \lim_{t\to\infty} \frac{\gamma_1\theta}{2p} t^{\theta-p} \to \infty,$$

which together with (5.2), (5.3) and (5.4) give a contradiction by dividing both sides of (5.3) to $t_{\varepsilon_n}^{2p}$ and taking the limit as $n \to \infty$.

Hence, choose a subsequence if necessary, we find that $t_{\varepsilon_n} \to t_0 \geq 0$ as $n \to \infty$. If $t_0 = 0$, from $t_{\varepsilon_n}\psi_{\varepsilon_n,y_n} \in \mathcal{N}_{\varepsilon_n}$, by Lemma 2, there is a positive real number $r_* > 0$ such that

$$||t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}||_{W_{\varepsilon_n}} \geq r_* > 0$$

for all n large enough. It is impossible since $t_{\varepsilon_n} \to 0$ and $||\psi_{\varepsilon_n,y_n}||_{W_{\varepsilon_n}} \to ||w||_{V_{\min}} > 0$ as $n \to \infty$. Now we prove that $t_0 = 1$. From (5.3) and uses Lebesgue Dominated convergence theorem to get

$$\frac{M(\|tw\|_{V_{\min}}^p)}{\|tw\|_{V_{\min}}^p} = \frac{Q_{\max}^2}{\|w\|_{V_{\min}}^p} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(t_0\omega(y))f(t_0\omega(x))\omega(x)}{t_0^{2p-1}|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

Note that $\omega \in \mathcal{N}_{V_{\min}Q_{\max}}$, and (f_5) implies that $\dfrac{f(t)}{t^{p-1}}$ and $\dfrac{F(t)}{t^p}$ are increasing functions on $(0,\infty)$, and together with the assumption $\dfrac{M(t)}{t}$ is a decreasing function on $(0,\infty)$, we get $t_0 = 1$. Apply Vitali's theorem, we deduce that

$$\begin{aligned}&\lim_{n\to\infty} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{Q(\varepsilon_n x)Q(\varepsilon_n y)F(t_\varepsilon \psi_{\varepsilon_n,y_n}(y))F(t_\varepsilon \psi_{\varepsilon_n,y_n}(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\\ &= Q_{\max}^2 \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(\omega(y))F(\omega(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\lim_{n\to\infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))\\ &= \lim_{n\to\infty}\left[\frac{1}{p}\widetilde{M}(||t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}||_{W_{\varepsilon_n}}^p) - \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{Q(\varepsilon_n x)Q(\varepsilon_n y)F(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}(y))F(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\right]\\ &= \frac{\widetilde{M}(||w||_{V_{\min}}^p)}{p} - \frac{Q_{\max}^2}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(\omega(y))F(\omega(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x = J_{V_{\min}Q_{\max}}(\omega) = c_{V_{\min}Q_{\max}}\end{aligned}$$

which contradicts with (5.1). □

For any $\delta > 0$, there exists $\rho = \rho(\delta) > 0$ satisfying $(\mathcal{V} \cap \mathcal{Q})_\delta \subset B_\rho(0)$. Let us define the map $\chi : \mathbb{R}^N \to \mathbb{R}^N$ as follows:

$$\mathfrak{T}(x) = \begin{cases} x & \text{if } |x| < \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

Next, we give the definition of the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ by

$$\beta_\varepsilon(u) = \frac{\int\limits_{\mathbb{R}^N} \mathfrak{T}(\varepsilon x)|u(x)|^p \mathrm{d}x}{\int\limits_{\mathbb{R}^N} |u|^p \mathrm{d}x}.$$

Lemma 5.3 ([6, Lemma 3.13]) The following limit holds

$$\lim_{\varepsilon \to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \text{ uniformly in } y \in \mathcal{V} \cap \mathcal{Q}. \tag{5.5}$$

Lemma 5.4 Assume that $\varepsilon_n \to 0^+$ and $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ satisfying $I_{\varepsilon_n}(u_n) \to c_{V_{\min} Q_{\max}}$. Then there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that up to a subsequence, $v_n(x) = u_n(x + \tilde{y}_n)$ converges to v in $W^{s,N/s}(\mathbb{R}^N)$ and $y_n = \varepsilon \tilde{y}_n \to y \in \mathcal{V} \cap \mathcal{Q}$.

Proof Since $\theta > 2p$, $\langle I'_{\varepsilon_n}(u_n), u_n \rangle = 0$ and $I_{\varepsilon_n}(u_n) \to c_{V_{\min} Q_{\max}}$, then we have

$$\begin{aligned} I_{\varepsilon_n}(u_n) &= I_{\varepsilon_n}(u_n) - \frac{1}{\theta}\langle I'_{\varepsilon_n}(u_n), u_n \rangle \\ &\geq a\left(\frac{s}{N} - \frac{1}{\theta}\right)||u_n||^p_{W_{\varepsilon_n}} \\ &\quad + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{Q(\varepsilon x)Q(\varepsilon y)F(u_n(y))}{|x-y|^\mu}\left[\frac{1}{\theta} f(u_n(x))u_n(x) - \frac{1}{2}F(u_n(x))\right]\mathrm{d}y\mathrm{d}x \\ &\geq a\left(\frac{s}{N} - \frac{1}{\theta}\right)||u_n||^p_{W_{\varepsilon_n}}. \end{aligned}$$

Thus, there exists a constant $C = \left(\frac{a^{-1}c_{V_{\min}Q_{|max}}}{\frac{s}{N}-\frac{1}{\theta}}\right)^{s/N} \leq \left(\frac{a^{-1}C_{\gamma_1}}{\frac{s}{N}-\frac{1}{\theta}}\right)^{s/N}$ such that

$$\limsup_{n\to\infty} ||u_n||_{W_{\varepsilon_n}} \leq C.$$

Hence, the sequence $\{u_n\}$ is a bounded in $W^{s,N/s}(\mathbb{R}^N)$. We claim that

$$\limsup_{n\to\infty} \int\limits_{B_r(\tilde{y}_n)} |u_n|^{N/s} \mathrm{d}x \geq \beta > 0 \tag{5.6}$$

for some sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$, and constants $r > 0$ and $\beta > 0$. Conversely, if (5.6) is not true, then for any $r > 0$, we have

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int\limits_{B_r(y)} |u_n|^{N/s} \mathrm{d}x = 0.$$

From Lemma 2.10, we deduce that u_n converges strongly to 0 in $L^q(\mathbb{R}^N)$, $q \in (\frac{N}{s}, +\infty)$. Choosing γ_1 sufficiently large as the method of (2.64), we deduce

$$\limsup_{n\to\infty} ||u_n||^{N/(N-s)}_{V_{\min}} \leq \limsup_{n\to\infty} ||u_n||^{N/(N-s)}_{W_\varepsilon} \leq \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

where $\mathfrak{c} > 1$ is a constant and c near 1. Appling Lemma 2.11, we have

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{Q(\varepsilon y)Q(\varepsilon x)F(u_n(y))f(u_n(x))u_n(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x=0.$$

Since $u_n \in \mathcal{N}_{\varepsilon_n}$, we get $||u_n||_{W_{\varepsilon_n}} \to 0$ as $n \to \infty$. It contradicts with Proposition 2. Therefore, (5.6) must hold. We denote by $v_n := u_n(x + \tilde{y}_n)$. We remark that $\{v_n\}$ is a bounded sequence in $W^{s,N/s}(\mathbb{R}^N)$ due to norm $||.||_{V_{\min}}$ invariant under the translation. Choose a subsequence if necessary, we find $v \in W^{s,N/s}(\mathbb{R}^N)$ so that v_n converges weakly to v in $W^{s,N/s}(\mathbb{R}^N)$ and $v_n \to v$ in $L^q_{\rm loc}(\mathbb{R}^N)$ with $q \in [\frac{N}{s}, +\infty)$. From (5.6), we have $v \not\equiv 0$. Assume that $t_n > 0$ such that $w_n = t_n v_n \in \mathcal{N}_{V_{\min}Q_{\max}}$ and we set $y_n := \varepsilon_n \tilde{y}_n$. Thus, using the transformation $z = x + \tilde{y}_n$, $V_n(x) := V(\varepsilon_n(x + \tilde{y}_n)) \geq V_{\min}$, $Q_n(x) := Q(\varepsilon_n(x + \tilde{y}_n)) \leq Q_{\max}$, and the invariance by translation, we can see that

$$\begin{aligned}
&c_{V_{\min}Q_{\max}} \leq J_{V_{\min}Q_{\max}}(w_n)\\
&\leq \frac{1}{p}\widetilde{M}([w_n]^p_{s,p} + \frac{1}{p}\int\limits_{\mathbb{R}^N} V_n(x)|w_n|^p\mathrm{d}x) - \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{Q_n(x)Q_n(y)F(\omega_n(y))F(\omega_n(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\\
&= I_{\varepsilon_n}(t_n u_n) \leq I_{\varepsilon_n}(u_n) \leq c_{V_{\min}Q_{\max}} + o_n(1).
\end{aligned}$$

Hence, $J_{V_{\min}Q_{\max}}(w_n) \to c_{V_{\min}Q_{\max}}$. Because $\{w_n\} \subset \mathcal{N}_{V_{\min}Q_{\max}}$, use (f_3), there is a suitable constant $K > 0$ satisfying $\|w_n\|_{V_{\min}} \leq K$ for all n. We observe that $v_n \not\to 0$ strongly in $W^{s,N/s}(\mathbb{R}^N)$. Conversely, if $v_n \to 0$ strong in $W^{s,N/s}(\mathbb{R}^N)$, then $v_n \to 0$ weak in $W^{s,N/s}(\mathbb{R}^N)$, it is a contradiction since $v \not\equiv 0$. Hence, $\|v_n\|_{V_{\min}} \geq \alpha > 0$ for all n and some constant $\alpha > 0$. Consequently, we deduce

$$t_n\alpha \leq \|t_n v_n\|_{V_{\min}} = \|w_n\|_{V_{\min}} \leq K,$$

which leads to that $t_n \leq \dfrac{K}{\alpha}$ for all $n \in \mathbb{N}$. Now choosing a subsequence if necessary, we find that $\lim_{n\to\infty} t_n = t_0 \geq 0$. We claim that $t_0 > 0$. If $t_0 = 0$, then $w_n \to 0$ strong in $W^{s,N/s}(\mathbb{R}^N)$, and $\lim_{n\to\infty} J_{V_{\min}Q_{\max}}(w_n) \to 0$. It is impossible since $c_{V_{\min}Q_{\max}} > 0$. Then we may assume that $w_n \to w := t_0 v \not\equiv 0$ weak in $W^{s,N/s}(\mathbb{R}^N)$. By the same arguments as Lemma 2.9, we obtain $J'_{V_{\min}Q_{\max}}(w) = 0$. Next, we show that

$$\lim_{n\to\infty}\|w_n\|^p_{V_{\min}} = \|w\|^p_{V_{\min}}. \tag{5.7}$$

If (5.7) is proved, we can get that $w_n \to w$ strong in $W^{s,N/s}(\mathbb{R}^N)$ via Brézis-Lieb's lemma. Using Fatou's lemma, we get

$$\|w\|^p_{V_{\min}} \leq \liminf_{n\to\infty}\|w_n\|^p_{V_0}. \tag{5.8}$$

Assume that by contradiction that

$$\|w\|^p_{V_{\min}} < \limsup_{n\to\infty}\|w_n\|^p_{V_{\min}}.$$

We have

$$\begin{aligned}
c_{V_{\min}Q_{\max}} + o_n(1) &= J_{V_{\min}Q_{\max}}(\omega_n) - \frac{1}{2p}\langle J'_{V_{\min}Q_{\max}}(\omega_n), \omega_n\rangle\\
&= \frac{a}{2p}\|w_n\|^p_{V_{\min}} + Q^2_{\max}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{F(\omega_n(y))}{|x-y|^\mu}\left[\frac{1}{2p}f(\omega_n(x))\omega_n(x) - \frac{1}{2}F(\omega_n(x))\right]\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Then, it holds

$$\begin{aligned} c_{V_{\min}Q_{\max}} &\geq \frac{a}{2p}\limsup_{n\to\infty}\|\omega_n\|^p_{V_{\min}} \\ &\quad + \liminf_{n\to\infty} Q^2_{\max}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{F(\omega_n(y))}{|x-y|^\mu}\Big[\frac{1}{2p}f(\omega_n(x))\omega_n(x)-\frac{1}{2}F(\omega_n(x))\Big]\mathrm{d}y\mathrm{d}x \\ &> \frac{a}{2p}\|w\|^p_{V_{\min}} + Q^2_{\max}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{F(\omega(y))}{|x-y|^\mu}\Big[\frac{1}{2p}f(\omega(x))\omega(x)-\frac{1}{2}F(\omega(x))\Big]\mathrm{d}y\mathrm{d}x \\ &= J_{V_{\min}Q_{\max}}(\omega) - \frac{1}{2p}\langle J'_{V_{\min}Q_{\max}}(\omega),\omega\rangle = J_{V_{\min}Q_{\max}}(\omega) \geq c_{V_{\min}Q_{\max}} \end{aligned}$$

via condition (f_3) and Fatou's lemma, which is a contradiction. Then

$$\|\omega\|^p_{V_{\min}} \geq \limsup_{n\to\infty}\|\omega_n\|^p_{V_{\min}}. \tag{5.9}$$

Hence (5.7) is proved by combining (5.8) and (5.9). By the result $t_n \to t_0$ as $n \to \infty$, we obtain $v_n \to v$ in $W^{s,N/s}(\mathbb{R}^N)$ as $n\to\infty$. Now we prove $y_n \to y \in \mathcal{V}\cap\mathcal{Q}$ up to a subsequence. Indeed, if $\{y_n\}$ is not bounded, then there exists a subsequence, still denoted by itselft such that $|y_n| \to +\infty$. Because $w_n \to w$ strongly in $W^{s,N/s}(\mathbb{R}^N)$ and the conditions (V) and (Q), use the transformations $z = x+\tilde{y}_n$ and $\bar{z} = y + \tilde{y}_n$, by Lemma 2.12, we have

$$\begin{aligned} & c_{V_{\min}Q_{\max}} \\ &= J_{V_{\min}Q_{\max}}(w) < J_{V_\infty Q_\infty}(w) \\ &\leq \liminf_{n\to\infty}\Big[\frac{1}{p}\widetilde{M}\Big([w_n]^p_{s,p} + \int\limits_{\mathbb{R}^N} V_n(x)|w_n|^p\mathrm{d}x\Big) - \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{Q_n(x)Q_n(y)F(w_n(y))F(w_n(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\Big] \\ &= \liminf_{n\to\infty}\Big[\frac{1}{p}\widetilde{M}(\|t_nu_n\|^p_{W_{\varepsilon_n}}) - \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{Q(\varepsilon_n z)Q(\varepsilon_n\bar{z})F(t_nu_n(\bar{z}))F(t_nu_n(z))}{|z-\bar{z}|^\mu}\mathrm{d}\bar{z}\mathrm{d}z\Big] \\ &= \liminf_{n\to\infty} I_{\varepsilon_n}(t_nu_n) \leq \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = c_{V_{\min}Q_{\max}}, \end{aligned} \tag{5.10}$$

which is a contradiction. Then the sequence $\{y_n\}$ is bounded. By choosing a subsequence if necessary, we may assume that $y_n \to y$. If $y \notin \mathcal{V}\cap\mathcal{Q}$, then we have the following cases:

Case 2.1 $y\in\mathcal{V}$ and $y\notin\mathcal{Q}$, then $V(y) = V_{\min}$ and $Q(y) < Q_{\max}$.

Case 2.2 $y\notin\mathcal{V}$ and $y\in\mathcal{Q}$, then $V_{\min} < V(y)$ and $Q(y) = Q_{\max}$.

Case 2.3 $y\notin\mathcal{V}$ and $y\notin\mathcal{Q}$, then $V_{\min} < V(y)$ and $Q(y) < Q_{\max}$.

By argument as (5.10), it is impossible. Hence $y\in\mathcal{V}\cap\mathcal{Q}$. □

We use the postive function $\mathfrak{h}:\mathbb{R}^+\to\mathbb{R}^+$ verifying $\lim_{\varepsilon\to 0^+}\mathfrak{h}(\varepsilon) = 0$ and denote by

$$\tilde{\mathcal{N}_\varepsilon} = \{u\in\mathcal{N}_\varepsilon : I_\varepsilon(u) \leq c_{V_0} + \mathfrak{h}(\varepsilon)\}.$$

From Lemma 5.3, if we choose $\mathfrak{h}(\varepsilon) = |I_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}|$, then we have $\lim_{\varepsilon\to0^+}\mathfrak{h}(\varepsilon) = 0$. Therefore $\Phi_\varepsilon(y)$ belongs $\mathcal{N}_\varepsilon$ and $\tilde{\mathcal{N}_\varepsilon}$ is non empty set for any $\varepsilon > 0$.

Lemma 5.5 ([6, Lemma 3.14]) For any $\delta > 0$, we have the following limit

$$\lim_{\varepsilon\to0^+}\sup_{u\in\tilde{\mathcal{N}_\varepsilon}}\mathrm{dist}(\beta_\varepsilon(u),(\mathcal{V}\cap\mathcal{Q})_\delta) = 0.$$

Lemma 5.6 Suppose that the conditions (V) and (f_1)–(f_5) hold. Denote v_n by a nontrivial nonnegative solution of equation

$$M\left([v_n]_{s,p}^p + \int\limits_{\mathbb{R}^N} V_n(x)|v_n|^p \mathrm{d}x\right)((-\Delta)_{N/s}^s v_n + V_n(x)|v_n|^{\frac{N}{s}-2}v_n)$$
$$= \left[\frac{1}{|x|^\mu} * (Q_n(y)F(v_n))\right]Q_n(x)f(v_n) \ \text{ in } \ \mathbb{R}^N, \tag{5.11}$$

where $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n), Q_n(x) = Q(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$ and $\varepsilon_n \tilde{y}_n \to y \in \mathcal{V} \cap \mathcal{Q}$. If $v_n \to v$ strong in $W^{s,N/s}(\mathbb{R}^N)$ and the following inequality holds

$$\limsup_{n\to\infty} \|v_n\|_{V_{\min}}^{N/(N-s)} \leq \frac{\beta_* \mathfrak{d}_*{}^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

where $\mathfrak{c} > 1$ is a constant and it is choosen near 1, then $v_n \in L^\infty(\mathbb{R}^N)$ and there is a suitable constant $C > 0$ so that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \leq C$ for all $n \in \mathbb{N}$. Furthermore, we also have

$$\lim_{|x|\to+\infty} v_n(x) = 0 \text{ uniformly in } n.$$

Proof For any positive real number $T > 0$ and $\alpha > 1$, we denote $\zeta(t) = t(\min\{t, T\})^{p(\alpha-1)}$ and

$$\zeta(v_n) = \zeta_{T,\alpha}(v_n) = v_n v_{T,n}^{p(\alpha-1)} \in W_\varepsilon, \quad v_{T,n} = \min\{v_n, T\}.$$

Set

$$\Lambda(t) = \frac{|t|^p}{p} \text{ and } \Theta(t) = \int\limits_0^t (\zeta'(t))^{\frac{1}{p}} \mathrm{d}\tau.$$

By the similar arguments in [6], we have

$$\Lambda'(a-b)(\zeta(a) - \zeta(b)) \geq |\Theta(a) - \Theta(b)|^p \ \text{ for any } \ a, b \in \mathbb{R}. \tag{5.12}$$

From (5.12), we get

$$|\Theta(v_n(x)) - \Theta(v_n(y))|^p$$
$$\leq |v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))((v_n v_{T,n}^{p(\alpha-1)})(x) - (v_n v_{T,n}^{p(\alpha-1)})(y)). \tag{5.13}$$

Therefore, taking $\zeta(v_n) = v_n v_{T,n}^{p(\alpha-1)}$ as a test function in (5.11) and together with (5.13), we have

$$a\left([\Theta(v_n)]_{s,p}^p + \int\limits_{\mathbb{R}^N} V_n(x)|v_n|^p v_{T,n}^{p(\alpha-1)} \mathrm{d}x\right)$$
$$\leq M\left([\Theta(v_n)]_{s,p}^p + \int\limits_{\mathbb{R}^N} V_n(x)|u_n|^p \mathrm{d}x\right)$$
$$\times \left(\int\limits_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))((v_n v_{T,n}^{p(\alpha-1)})(x) - (v_n v_{T,n}^{p(\beta-1)})(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y\right.$$
$$\left. + \int\limits_{\mathbb{R}^N} V_n(x)|v_n|^p v_{T,n}^{p(\alpha-1)} \mathrm{d}x\right)$$
$$= \int\limits_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * (Q_n(y)F(v_n))\right] Q_n(x) f(v_n) v_n v_{T,n}^{p(\beta-1)} \mathrm{d}x. \tag{5.14}$$

Using (5.12), we have $v_n v_{T,n}^{\alpha-1} \geq |\Theta(v_n)|$. Since $\Theta(v_n) \geq \dfrac{1}{\alpha} v_n v_{T,n}^{\alpha-1}$ and the embedding continuous $W^{s,N/s}(\mathbb{R}^N) \hookrightarrow L^{N^*}(\mathbb{R}^N)$ $(N^* > \frac{N}{s})$, then there exists a suitable constant $S_* > 0$ such that

$$\|\Theta(v_n)\|^p_{V_{\min}/2} \geq S_* \|\Theta(v_n)\|^p_{L^{N^*}(\mathbb{R}^N)} \geq \frac{1}{\alpha^p} S_* \|v_n v_{L,n}^{\alpha-1}\|^p_{L^{N^*}(\mathbb{R}^N)}. \tag{5.15}$$

On the other hand, from the boundedness of $\{v_n\}$ and Lemma 2.8, it follows that there exists $C_0 > 0$ such that

$$\Big|\frac{1}{|x|^\mu} * F(v_n)\Big| \leq C_0.$$

By the assumptions (f_1) for $q = \frac{N}{s}$ and (f_2), for any $\xi > 0$, there exists $C(\xi) > 0$ such that

$$|f(t)| \leq \xi |t|^{p-1} + C(\xi)|t|^{p-1}\Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$$

for all $t \in \mathbb{R}$. Consequently, we have

$$\begin{aligned} &\frac{a}{\alpha^p} S_* \|v_n v_{T,n}^{\beta-1}\|^p_{L^{N^*}(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N} V_n(x)|v_n|^p v_{T,n}^{p(\alpha-1)} \mathrm{d}x \\ &\leq C_0 \xi Q^2_{\max} \int_{\mathbb{R}^N} |v_n v_{T,n}^{\alpha-1}|^p \mathrm{d}x + C_0 C(\xi) Q^2_{\max} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha_0 |v_n|^{N/(N-s)}) |v_n v_{T,n}^{\alpha-1}|^p \mathrm{d}x. \end{aligned} \tag{5.16}$$

Choosing $0 < \xi < \dfrac{V_{\min}}{4C_0 Q^2_{\max}}$, then (5.16) implies

$$\frac{a}{\alpha^p} S_* \|v_n v_{T,n}^{\alpha-1}\|^p_{L^{N^*}(\mathbb{R}^N)} \leq C_0 C(\xi) \Big(\int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0 |v_n|^{N/(N-s)}))^{q'} \mathrm{d}x\Big)^{\frac{1}{q'}} \Big(\int_{\mathbb{R}^N} |v_n v_{T,n}^{\alpha-1}|^{qp} \mathrm{d}x\Big)^{\frac{1}{q}}.$$

Apply fractional Trudinger-Moser inequality in $W^{s,N/s}(\mathbb{R}^N)$, then there is a constant $D > 0$ satisfying

$$\|v_n v_{T,n}^{\alpha-1}\|^p_{L^{N^*}(\mathbb{R}^N)} \leq C_0 D \alpha^p \|v_n v_{T,n}^{\alpha-1}\|^p_{L^{qp}(\mathbb{R}^N)},$$

where $q \geq \frac{N}{s}$, $N^{**} = qp < N^*$, $q' > 1$ and near 1. Let $T \to +\infty$ in that inequality to get

$$\|v_n\|_{L^{N^*\alpha}} \leq C_0^{\frac{1}{p\alpha}} D^{\frac{1}{p\alpha}} \alpha^{\frac{1}{\alpha}} \|v_n\|_{L^{N^{**}\alpha}(\mathbb{R}^N)}. \tag{5.17}$$

Set $\alpha = \frac{N^*}{N^{**}} > 1$. It implies that $\alpha^2 N^{**} = \alpha N^*$. Now, in (5.17), we use α^2 instead of α, and get

$$\begin{aligned} \|v_n\|_{L^{N^*\alpha^2}} &\leq C_0^{\frac{1}{p\alpha^2}} D^{\frac{1}{p\alpha^2}} \alpha^{\frac{2}{\alpha^2}} \|v_n\|_{L^{N^{**}\alpha^2}(\mathbb{R}^N)} \\ &= C_0^{\frac{1}{p\alpha^2}} D^{\frac{1}{p\alpha^2}} \alpha^{\frac{2}{\alpha^2}} ||v_n||_{L^{N^*\alpha}(\mathbb{R}^N)} \\ &\leq C_0^{\frac{1}{p}(\frac{1}{\alpha}+\frac{1}{\alpha^2})} D^{\frac{1}{p}(\frac{1}{\alpha}+\frac{1}{\alpha^2})} \alpha^{\frac{1}{\alpha}+\frac{2}{\alpha^2}} \|v_n\|_{L^{N^{**}\alpha}(\mathbb{R}^N)}. \end{aligned}$$

Continue that process, for any m, we get

$$\|v_n\|_{L^{N^*\alpha^m}} \leq (C_0 D)^{\sum_{j=1}^m \frac{1}{p\alpha^j}} \alpha^{\sum_{j=1}^m j\alpha^{-j}} ||v_n||_{L^{N^{**}\alpha}(\mathbb{R}^N)}. \tag{5.18}$$

Taking the limit in (5.18) as $m \to \infty$, we get

$$\|v_n\|_{L^\infty(\mathbb{R}^N)} \leq C$$

for all n, where $C = (C_0 D)^{\sum_{j=1}^\infty \frac{1}{p\alpha^j}} \alpha^{\sum_{j=1}^\infty j\alpha^{-j}} \sup_n ||v_n||_{L^{N^{**}\alpha}(\mathbb{R}^N)} < +\infty$. □

Now, we continue to prove Theorem 5.1. Fix $\varepsilon > 0$ sufficiently small. From Lemma 5.2 and Lemma 5.5, we see that $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopic with the inclusion map id : $\mathcal{V} \cap \mathcal{Q} \to (\mathcal{V} \cap \mathcal{Q})_\delta$. Therefore, we deduce

$$\text{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon) \geq \text{cat}_{(\mathcal{V}\cap\mathcal{Q})_\delta}(\mathcal{V} \cap \mathcal{Q}).$$

Since I_ε satisfies the $(PS)_c$ condition with $c \in (c_{V_{\min}Q_{\max}}, c_{V_{\min}Q_{\max}} + h(\varepsilon))$, then by using the critical points theorem (see Willem [44]), we get that I_ε has at least $\text{cat}_{(\mathcal{V}\cap\mathcal{Q})_\delta}(\mathcal{V} \cap \mathcal{Q})$ critical points on $\mathcal{N}_\varepsilon$. We now apply Corollary 2 to deduce that I_ε has at least $\text{cat}_{(\mathcal{V}\cap\mathcal{Q})_\delta}(\mathcal{V} \cap \mathcal{Q})$ critical points in W_ε.

Assume that u_{ε_n} is a solution of problem $(\mathcal{P}_\varepsilon)$, then $v_n(x) = u_{\varepsilon_n}(x + \tilde{y}_n)$ is a solution of (5.11). Moreover, choose a subsequence if necessary, we find that $v_n \to v$ strong in $W^{s,N/s}(\mathbb{R}^N)$ for some $v \in W^{s,N/s}(\mathbb{R}^N)$ and $y_n = \varepsilon_n \tilde{y}_n \to y \in \mathcal{V} \cap \mathcal{Q}$. In the following, we prove that there exists $\delta > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \geq \delta$ for all n large enough. Indeed, by Lemma 5.4 (see (5.6)), we have

$$0 < \frac{\beta}{2} \leq \int\limits_{B_r(0)} |v_n|^{N/s} \mathrm{d}x \leq |B_r(0)| \|v_n\|^{N/s}_{L^\infty(\mathbb{R}^N)} \tag{5.19}$$

for all n sufficiently large, where $\delta = \left(\frac{\beta}{2|B_r(0)|}\right)^{N/s}$. Note that $v_n \to v$ in $W^{s,N/s}(\mathbb{R}^N)$, which implies that $\lim_{|x|\to\infty} v_n(x) = 0$ uniformly in $n \in \mathbb{N}$. Set p_n is the global maximum of v_n, then from Lemma 5.6 and (5.19), there is a positive real number $R > 0$ so that $|p_n| \leq R$ for all $n \in \mathbb{N}$. Hence, $z_{\varepsilon_n} = p_n + \tilde{y}_n$ is the maximum point of u_{ε_n} and $\varepsilon_n z_{\varepsilon_n} \to y \in \mathcal{V} \cap \mathcal{Q}$. Since V and Q are continuous functions, we get $V(\varepsilon_n z_{\varepsilon_n}) \to V(y) = V_{\min}$ and $Q(\varepsilon_n z_{\varepsilon_n}) \to Q(y) = Q_{\max}$ as $n \to \infty$.

If u_ε is a nontrivial nonnegative solution of problem $(\mathcal{P}_\varepsilon)$, then $w_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$ is a nontrivial nonnegative solution of (1.1). Then $\eta_\varepsilon = \varepsilon z_\varepsilon$ is maximum point of w_ε. Setting $v_{\varepsilon_n}(x) := w_{\varepsilon_n}(\varepsilon_n x + \eta_{\varepsilon_n}) = u_{\varepsilon_n}(x + z_{\varepsilon_n})$. Then by arguments as Lemma 4.2, v_ε converges strongly to v in $W^{s,p}(\mathbb{R}^N)$, which is a ground state solution of equation

$$M(||u||^p_{V_{\min}})((-\Delta)^s_p u + V_{\min}|u|^{p-2}v) = Q^2_{\max}\left[\frac{1}{|x|^\mu} * F(u(y))\right] f(u) \text{ in } \mathbb{R}^N.$$

This completes the proof of Theorem 5.1. □

Authors' contributions

The authors contributed equally both to the design of this paper and to the preparation of the final version of the present work.

Conflict of Interest The authors declare no conflict of interest.

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