Existence Results for Inequality Problems with Lack of Convexity*

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Abstract. We establish several existence results of Hartman-Stampacchia type for hemivariational inequalities on bounded and convex sets in a real reflexive Banach space. We also study the cases of coercive and noncoercive variational-hemivariational inequalities.

1 Introduction

The study of variational inequality problems began around 1965 with the pioneering works of G. Fichera, J.-L. Lions and G. Stampacchia (see [4], [7]). The connection of the theory of variational inequalities with the notion of subdifferentiability of convex analysis was achieved by J.J. Moreau (see [8]) who introduced the notion of convex superpotential which permitted the formulation and the solving of a wide ranging class of complicated problems in mechanics and engineering which could not until then be treated correctly by the methods of classical bilateral mechanics. All the inequality problems treated to the middle of the ninth decade were related to convex energy functions and therefore were firmly bound with monotonicity; for instance, only monotone, possibly multivalued boundary conditions and stress-strain laws could be studied. In order to overcome this limitation, P.D. Panagiotopoulos introduced in [14], [15] the notion of nonconvex superpotential by using the generalized gradient of F.H. Clarke. Due to the lack of convexity new types of variational expressions were obtained. These are the so-called *hemivariational inequalities* and they are no longer connected with monotonicity. Generally speaking, mechanical problems involving nonmonotone, possibly multivalued stress-strain laws or boundary conditions derived by nonconvex superpotentials lead to hemivariational inequali-

*This paper is dedicated to the memory of Professor P.D. Panagiotopoulos ties. Moreover, while in the convex case the static variational inequalities generally give rise to minimization problems for the potential or the complementary energy, in the nonconvex case the problem of substationarity of the potential or the complementary energy at an equilibrium position emerges.

Throughout this paper X will denote a real reflexive Banach space, (T,μ) will be a measure space of positive and finite measure and $A : X \to X^\star$ will stand for a nonlinear operator. We also assume that there are given $m \in \mathbf{N}$, $p \geq 1$ and a compact mapping $\gamma : X \to L^p(T, \mathbf{R}^m)$. We shall denote by p' the conjugated exponent of p. If $\varphi : X \to \mathbf{R}$ is a locally Lipschitz functional then $\varphi^0(u; v)$ will stand for the Clarke derivative of φ at $u \in X$ with respect to the direction $v \in X$, that is

$$\varphi^0(u;v) = \limsup_{\substack{w \to u \\ \lambda \downarrow 0}} \frac{\varphi(w+\lambda v) - \varphi(w)}{\lambda} \,.$$

Accordingly, Clarke's generalized gradient $\partial\varphi(u)$ of φ at u is defined by

$$\partial\varphi(u) = \{\xi \in X^\star ;\ \langle \xi, v\rangle \leq \varphi^0(u;v),\ \forall v \in X\}.$$

Let $j : T \times \mathbf{R}^m \to \mathbf{R}$ be a function such that the mapping

$$j(\cdot, y) : T \to \mathbf{R} \qquad \text{is measurable, for every } y \in \mathbf{R}^m \,. \tag{1}$$

We assume that at least one of the following conditions hold: either there exists $k \in L^{p'}(T, \mathbf{R})$ such that

$$|j(x,y_1) - j(x,y_2)| \leq k(x)\, |y_1 - y_2| \,, \qquad \forall x \in T\,, \forall y_1, y_2 \in \mathbf{R}^m \,, \tag{2}$$

or

$$\text{the mapping } j(x,\cdot) \text{ is locally Lipschitz, } \forall x \in T, \tag{3}$$

and there exists $C > 0$ such that

$$|z| \leq C(1 + |y|^{p-1}) \,, \qquad \forall x \in T\,, \forall y_1, y_2 \in \mathbf{R}^m\,, \forall z \in \partial_y j(x,y) \,. \tag{4}$$

Let K be a nonempty closed, convex subset of X, $f \in X^\star$ and $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ a convex, lower semicontinuous functional such that

$$D(\Phi) \cap K \neq \emptyset \,. \tag{5}$$

Throughout this paper $\langle \cdot, \cdot \rangle$ will denote the duality pairing between $X^\star$ and X.

2 The generalized Hartman-Stampacchia theorem for variational-hemivariational inequality problems

Consider the following inequality problem:

Find $u \in K$ such that

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + \int_T j^0(x, \gamma(u(x)); \gamma(v(x) - u(x)))d\mu \geq 0\,, \qquad \forall v \in K\,, \quad (6)$$

where γ denotes the prescribed canonical mapping from X into $L^p(T, \mathbf{R}^m)$.

The following two situations are of particular interest in applications:
(i) $T = \Omega$, $\mu = dx$, $X = W^{1,q}(\Omega, \mathbf{R}^m)$ and $\gamma : X \to L^p(\Omega, \mathbf{R}^m)$, with $p < q^*$, is the Sobolev embedding operator;
(ii) $T = \partial\Omega$, $\mu = d\sigma$, $X = W^{1,p}(\Omega, \mathbf{R}^m)$ and $\gamma = i \circ \eta$, where $\eta : X \to W^{1-\frac{1}{p},p}(\partial\Omega, \mathbf{R}^m)$ is the trace operator and $i : W^{1-\frac{1}{p},p}(\partial\Omega, \mathbf{R}^m) \to L^p(\partial\Omega, \mathbf{R}^m)$ is the embedding operator.

A direct application of the Knaster-Kuratowski-Mazurkiewicz (KKM, in short) principle (see [6] or [3]) leads to the following basic auxiliary result:

Lemma 1 *Let K be a nonempty, bounded, closed, convex subset of X, $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ a convex, lower semicontinuous functional such that (5) holds. Consider a Banach space Y such that there exists a linear and compact mapping $L : X \to Y$ and let $J : Y \to \mathbf{R}$ be an arbitrary locally Lipschitz function. Suppose in addition that the mapping $K \ni v \longmapsto \langle Av, v - u\rangle$ is weakly lower semicontinuous, for every $u \in K$.*

Then, for every $f \in X^\star$, there exists $u \in K$ such that

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0\left(L(u), L(v - u)\right) \geq 0\,, \forall v \in K\,. \quad (7)$$

Proof. Let us define the set-valued mapping $G : K \cap D(\Phi) \to 2^X$ by

$$G(x) = \{v \in K \cap D(\Phi); \langle Av - f, v - x\rangle - J^0(L(v); L(x) - L(v)) + \Phi(v) - \Phi(x) \leq 0\}\,.$$

We claim that the set $G(x)$ is weakly closed. Indeed, if $G(x) \ni v_n \rightharpoonup v$ then, by our hypotheses,

$$\langle Av, v - x\rangle \leq \liminf_{n\to\infty} \langle Av_n, v_n - x\rangle$$

and

$$\Phi(v) \leq \liminf_{n\to\infty} \Phi(v_n)\,.$$

Moreover, $L(v_n) \to L(v)$ and thus, by the upper semi-continuity of J^0 (see [2]), we also obtain

$$\limsup_{n\to\infty} J^0\left(L(v_n); L(x - v_n)\right) \leq J^0\left(L(v); L(x - v)\right)\,.$$

Therefore

$$-J^0\left(L(v);L(x-v)\right)\leq \liminf_{n\to\infty}\Big(-J^0\left(L(v_n);L(x-v_n)\right)\Big)\,.$$

So, if $v_n\in G(x)$ and $v_n\rightharpoonup v$ then

$$\begin{aligned}&\langle Av-f,v-x\rangle-J^0\left(L(v);L(x-v)\right)+\Phi(v)-\Phi(x)\leq\\&\liminf\Big\{\langle Av_n-f,v_n-x\rangle-J^0\left(L(v_n);L(v-v_n)\right)+\Phi(v_n)-\Phi(x)\Big\}\leq 0\,,\end{aligned}$$

which shows that $v\in G(x)$. Since K is bounded, it follows that $G(x)$ is weakly compact. This implies that

$$\bigcap_{x\in K\cap D(\Phi)}G(x)\neq\emptyset\,,$$

provided that the family $\{G(x);x\in K\cap D(\Phi)\}$ has the finite intersection property. We may conclude by using the KKM principle after showing that G is a KKM-mapping. Suppose by contradiction that there exist $x_1,\cdots,x_n\in K\cap D(\Phi)$ and $y_0\in\operatorname{Conv}\{x_1,\cdots,x_n\}$ such that $y_0\not\in\bigcup_{i=1}^n G(x_i)$. Then

$$\langle Ay_0-f,y_0-x_i\rangle+\Phi(y_0)-\Phi(x_i)-J^0\left(L(y_0);L(x_i-y_0)\right)>0\,,\quad\forall i=1,\cdots,n\,.$$

Therefore

$$x_i\in\Lambda:=\{x\in X;\langle Ay_0-f,y_0-x\rangle+\Phi(y_0)-\Phi(x)-J^0\left(L(y_0);L(x-y_0)\right)>0\}\,,$$

for all $i\in\{1,\cdots,n\}$. The set Λ is convex and thus $y_0\in\Lambda$, leading to an obvious contradiction. So,

$$\bigcap_{x\in K\cap D(\Phi)}G(x)\neq\emptyset\,.$$

This yields an element $u\in K\cap D(\Phi)$ such that, for any $v\in K\cap D(\Phi)$,

$$\langle Au-f,v-u\rangle+\Phi(v)-\Phi(u)+J^0\left(L(u);L(v-u)\right)\geq 0\,.$$

This inequality is trivially satisfied if $v\not\in D(\Phi)$ and the conclusion follows. $\square$

We may now derive a result applicable to the inequality problem (6). Indeed, suppose that the above hypotheses are satisfied and set $Y=L^p(T,\mathbf{R}^m)$. Let $J:Y\to\mathbf{R}$ be the function defined by

$$J(u)=\int_T j(x,u(x))d\mu\,. \tag{8}$$

The conditions (2) or (3)-(4) on j ensure that J is locally Lipschitz on Y and

$$\int_T j^0(x,u(x);v(x))d\mu\geq J^0(u;v)\,,\quad\forall u,v\in X\,.$$

It follows that

$$\int_T j^0(x,\gamma(u(x));\gamma(v(x)))d\mu \geq J^0(\gamma(u);\gamma(v))\,, \quad \forall u,v \in X\,. \tag{9}$$

It results that if $u \in K$ is a solution of (7) then u solves the inequality problem (6), too. The following result follows.

Theorem 1 *Assume that the hypotheses of Lemma 1 are fulfilled for $Y = L^p(T, \mathbf{R}^m)$ and $L = \gamma$. Then the problem (6) has at least a solution.*

In order to establish a variant of Lemma 1 for monotone and hemicontinuous operators we need the following result which is due to Mosco (see [9]):

Mosco's Theorem. *Let K be a nonempty convex and compact subset of a topological vector space X. Let $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function such that $D(\Phi) \cap K \neq \emptyset$. Let $f, g : X \times X \to \mathbf{R}$ be two functions such that*
(i) $g(x,y) \leq f(x,y)$, for every $x, y \in X$;
(ii) the mapping $f(\cdot, y)$ is concave, for any $y \in X$;
(iii) the mapping $g(x, \cdot)$ is lower semicontinuous, for every $x \in X$.

Let λ be an arbitrary real number. Then the following alternative holds: either
- there exists $y_0 \in D(\Phi) \cap K$ such that $g(x, y_0) + \Phi(y_0) - \Phi(x) \leq \lambda$, for any $x \in X$,
or
- there exists $x_0 \in X$ such that $f(x_0, x_0) > \lambda$.

We notice that two particular cases of interest for the above result are if $\lambda = 0$ or $f(x,x) \leq 0$, for every $x \in X$.

Lemma 2 *Let K be a nonempty, bounded, closed subset of the real reflexive Banach space X, and $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ a convex and lower semicontinuous function such that (5) holds. Consider a linear subspace Y of $X^\star$ such that there exists a linear and compact mapping $L : X \to Y$. Let $J : Y \to \mathbf{R}$ be a locally Lipschitz function. Suppose in addition that the operator $A : X \to X^\star$ is monotone and hemicontinuous.*

Then for each $f \in X^\star$, the inequality problem (7) has at least a solution .

Proof. Set

$$g(x,y) = \langle Ax - f, y - x\rangle - J^0(L(y); L(x) - L(y))$$

and

$$f(x,y) = \langle Ay - f, y - x\rangle - J^0(L(y); L(x) - L(y))\,.$$

The monotonicity of A implies that

$$g(x,y) \leq f(x,y)\,, \qquad \forall x,y \in X\,.$$

The mapping $x \longmapsto f(x,y)$ is concave while the mapping $y \longmapsto g(x,y)$ is weakly lower semicontinuous. Applying Mosco's Theorem with $\lambda = 0$, we obtain the existence of $u \in K \cap D(\Phi)$ satisfying

$$g(w,u) + \Phi(u) - \Phi(w) \leq 0\,, \qquad \forall w \in K\,,$$

that is

$$\langle Aw - f, w-u\rangle + \Phi(w) - \Phi(u) + J^0(L(u); L(w-u)) \geq 0\,, \qquad \forall w \in K\,. \tag{10}$$

We use in what follows an argument which is in the same spirit as that used in the proof of Minty's Lemma (see [5, Lemma III.1.5]). Fix $v \in K$ and set $w = u + \lambda(v-u) \in K$, for $\lambda \in [0,1)$. So, by (10),

$$\lambda\,\langle A(u+\lambda(v-u)) - f, v-u\rangle + \Phi(\lambda v + (1-\lambda)u)) - \Phi(u) + J^0(L(u); \lambda L(v-u)) \geq 0\,.$$

Using the convexity of Φ, the fact that $J^0(u;\cdot)$ is positive homogeneous (see [1], p. 103) and dividing then by $\lambda > 0$ we find

$$\langle A(\lambda v + (1-\lambda)u) - f, v-u\rangle + \Phi(v) - \Phi(u) + J^0(L(u); L(v-u)) \geq 0\,.$$

Now, taking $\lambda \to 0$ and using the hemicontinuity of A we find that u solves (7). $\square$

The analogue of Theorem 1 for monotone and hemicontinuous operators can now be stated as follows:

Theorem 2 *Assume that the hypotheses of Lemma 2 are fulfilled for $Y = L^p(T, \mathbf{R}^m)$ and $L = \gamma$. Then the inequality problem (6) admits at least a solution.*

3 Coercive variational-hemivariational inequalities

We observe that if j satisfies conditions (1) and (2) then, by the Cauchy-Schwarz Inequality,

$$|\int_T j^0(x, \gamma(u(x)); \gamma(v(x)))d\mu| \leq \int_T k(x)|\gamma(v(x))|d\mu \leq |k|_{p'} \cdot |\gamma(v)|_p \leq C\,|k|_{p'}\,\|v\|\,, \tag{11}$$

where $|\cdot|_p$ denotes the norm in the space $L^p(T, \mathbf{R}^m)$ and $\|\cdot\|$ stands for the norm in X. On the other hand, if j satisfies conditions (1), (3) and (4) then

$$|j^0(x, \gamma(u(x)); \gamma(v(x)))| \leq C\,(1 + |\gamma(u(x))|^{p-1})\,|\gamma(v(x))|$$

and thus

$$|\int_T j^0(x,\gamma(u(x));\gamma(v(x)))d\mu| \le C\left(|\gamma(v)|_1 + |\gamma(u)|_{p-1}^p|\gamma(v)|_p\right) \le C_1\,\|v\| + C_2\,\|u\|^{p-1}\|v\|\,, \quad (12)$$

for some suitable constants $C_1, C_2 > 0$. We discuss in this framework the solvability of coercive variational-hemivariational inequalities.

Theorem 3 *Let K be a nonempty closed convex subset of X, $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous function such that $K \cap D(\Phi) \neq \emptyset$ an $A : X \to X^\star$ and operator such that the mapping $v \longmapsto \langle Av, v-x\rangle$ is weakly lower semicontinuous, for all $x \in K$. The following hold*

(i) If j satisfies conditions (1) and (2), and if there exists $x_0 \in K \cap D(\Phi)$ such that

$$\frac{\langle Aw, w-x_0\rangle + \Phi(w)}{\|w\|} \to +\infty\,, \qquad \text{as } \|w\| \to +\infty \quad (13)$$

then for each $f \in X^\star$, there exists $u \in K$ such that

$$\langle Au - f, v-u\rangle + \Phi(v) - \Phi(u) + \int_T j^0(x,\gamma(u(x));\gamma(v(x)) - \gamma(u(x)))d\mu \ge 0\,, \quad \forall v \in K\,. \quad (14)$$

(ii) If j satisfies conditions (1), (3) and (4) and if there exist $x_0 \in K \cap D(\Phi)$ and $\theta \ge p$ such that

$$\frac{\langle Aw, w-x_0\rangle}{\|w\|^\theta} \to +\infty\,, \qquad \text{as } \|w\| \to +\infty \quad (15)$$

then for each $f \in X^\star$, there exists $u \in K$ satisfying (14).

Proof. There exists a positive integer n_0 such that

$$x_0 \in K_n := \{x \in K; \|x\| \le n\}\,, \qquad \forall n \ge n_0\,.$$

Applying Lemma 1 with J as defined in (8) we find some $u_n \in K_n$ such that, for every $n \ge n_0$ and any $v \in K_n$,

$$\langle Au_n - f, v-u_n\rangle + \Phi(v) - \Phi(u_n) + J^0(\gamma(u_n);\gamma(v) - \gamma(u_n)) \ge 0\,. \quad (16)$$

We claim that the sequence (u_n) is bounded. Suppose by contradiction that $\|u_n\| \to +\infty$. Then, passing eventually to a subsequence, we may assume that

$$v_n := \frac{u_n}{\|u_n\|} \rightharpoonup v\,.$$

Setting $v = x_0$ in (16) and using (9), we obtain

$$\begin{aligned}\langle Au_n, u_n - x_0\rangle + \Phi(u_n) \leq\ & \Phi(x_0) + \langle f, u_n - x_0\rangle + J^0(\gamma(u_n); \gamma(x_0) - u_n)) \leq \\ & \Phi(x_0) + \langle f, u_n - x_0\rangle + \left|\int_T j^0(x, \gamma(u_n); \gamma(x_0 - u_n))d\mu\right|.\end{aligned} \tag{17}$$

Case (i). Using (11) we obtain

$$\langle Au_n, u_n - x_0\rangle + \Phi(u_n) \leq \Phi(x_0) + \langle f, u_n - x_0\rangle + c|k|_{p'}\, \|u_n - x_0\|$$

and thus

$$\frac{\langle Au_n, u_n - x_0\rangle + \Phi(u_n)}{\|u_n\|} \leq \frac{\Phi(x_0)}{\|u_n\|} + \langle f, v_n - x_0\|u_n\|^{-1}\rangle + c|k|_{p'}\, \|v_n - x_0\|u_n\|^{-1}\|\,. \tag{18}$$

Passing to the limit as $n \to \infty$ we observe that the left-hand term in (18) tends to $+\infty$ while the right-hand term remains bounded which yields a contradiction.

Case (ii). The function Φ being convex and lower semicontinuous, we may apply the Hahn-Banach separation theorem to find that

$$\Phi(x) \geq \langle \alpha, x\rangle + \beta\,, \qquad \forall x \in X\,,$$

for some $\alpha \in X^\star$ and $\beta \in \mathbf{R}$. This means that

$$\Phi(x) \geq -\|\alpha\|_\star\, \|x\| + \beta\,, \qquad \forall x \in X\,.$$

From (17) and (12) we deduce that

$$\langle Au_n, u_n - x_0\rangle \leq \Phi(x_0) + \|\alpha\|_\star\, \|u_n\| - \beta + \langle f, u_n - x_0\rangle + C_1\, \|u_n - x_0\| + C_2\, \|u_n\|^{p-1}\|u_n - x_0\|\,.$$

Thus

$$\begin{aligned}\frac{\langle Au_n, u_n - x_0\rangle}{\|u_n\|^\theta} \leq\ & \|\alpha\|_\star\, \|u_n\|^{1-\theta} + (\Phi(x_0) - \beta)\|u_n\|^{-\theta} + \langle f, v_n\|u_n\|^{1-\theta} - x_0\|u_n\|^{-\theta}\rangle + \\ & C_1\|v_n\|u_n\|^{1-\theta} - x_0\|u_n\|^{-\theta}\| + C_2\|v_n - x_0\|u_n\|^{-1}\| \cdot \|u_n\|^{p-\theta}\end{aligned}$$

and taking the limit as $n \to \infty$ we obtain a contradiction, since $\theta \geq p \geq 1$.

Thus in both cases (i) and (ii), the sequence $\{u_n\}$ is bounded. This implies that, up to a subsequence, $u_n \rightharpoonup u \in K$. Let $v \in K$ be given. For all n large enough we have $v \in K_n$ and thus by (16),

$$\langle Au_n - f, u_n - v\rangle + \Phi(u_n) - \Phi(v) - J^0(\gamma(u_n); \gamma(v) - \gamma(u_n)) \leq 0. \tag{19}$$

Passing to the limit as $n \to \infty$ we obtain

$$\langle Au - f, u - v\rangle \leq \liminf_{n\to\infty} \langle Au_n - f, u_n - v\rangle$$

$$\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n)$$

$$\gamma(u) = \lim_{n\to\infty} \gamma(u_n)$$

and

$$-J^0(\gamma(u); \gamma(v) - \gamma(u)) \leq \liminf_{n\to\infty} \left(-J^0(\gamma(u_n); \gamma(v) - \gamma(u_n))\right).$$

Taking the inferior limit in (19) we obtain

$$\langle Au - f, u - v\rangle + \Phi(u) - \Phi(v) - J^0(\gamma(u); \gamma(v) - \gamma(u)) \leq 0.$$

Since v has been chosen arbitrarily we obtain

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0(\gamma(u); \gamma(v) - \gamma(u)) \geq 0, \qquad \forall v \in K.$$

Using now again (9) we conclude that u solves (14). $\square$

The following result gives a corresponding variant for monotone hemicontinuous operators.

Theorem 4 *Let K be a nonempty closed convex subset of X, $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ a proper convex and lower semicontinuous function such that $D(\Phi) \cap K \neq \emptyset$. Let $A : X \to X^\star$ be a monotone and hemicontinuous operator. Assume (13) or (15) as in Theorem 3. Then the conclusions of Theorem 3 hold true.*

Proof. Using Lemma 2 we find a sequence $u_n \in K_n$ such that

$$\langle Au_n - f, v - u_n\rangle + \Phi(v) - \Phi(u_n) + J^0(\gamma(u_n); \gamma(v) - \gamma(u_n)) \geq 0, \qquad \forall v \in K_n. \tag{20}$$

As in the proof of Theorem 3 we justify that $\{u_n\}$ is bounded and thus, up to a subsequence, we may assume that $u_n \rightharpoonup u$. By (20) and the monotonicity of A we deduce that

$$\langle Av - f, v - u_n\rangle + \Phi(v) - \Phi(u_n) + J^0(\gamma(u_n); \gamma(v) - \gamma(u_n)) \geq 0.$$

Let $v \in K$ be given. For n large enough we obtain

$$\langle Av - f, u_n - v\rangle + \Phi(u_n) - \Phi(v) - J^0(\gamma(u_n); \gamma(v) - \gamma(u_n)) \leq 0$$

and taking the inferior limit we obtain

$$\langle Av - f, u - v\rangle + \Phi(u) - \Phi(v) - J^0(\gamma(u); \gamma(v) - \gamma(u)) \leq 0.$$

Since v has been chosen arbitrarily it follows that

$$\langle Av - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0(\gamma(u); \gamma(v) - \gamma(u)) \geq 0, \qquad \forall v \in K.$$

Using now the same argument as in the proof of Lemma 2 we obtain that

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0(\gamma(u); \gamma(v) - \gamma(u)) \geq 0, \qquad \forall v \in K$$

and the conclusion follows now by (9). □

4 Noncoercive variational-hemivariational inequalities

In order to treat noncoercive cases we use in this Section a minimax approach for studying the inequality problem (7) (in particular, (6)). To this end we present the necessary background of nonsmooth critical point theory developed in Motreanu-Panagiotopoulos ([10], Chapter III).

Definition 1 (Definition 3.1 in Motreanu-Panagiotopoulos [10]). *Let X be a real Banach space, let $F : X \to \mathbf{R}$ be a locally Lipschitz function and let $G : X \to \mathbf{R} \cup \{+\infty\}$ be a proper (i.e., $\not\equiv +\infty$), convex and lower semicontinuous function. An element $u \in X$ is called a critical point of the functional $I = F + G : X \to \mathbf{R} \cup \{+\infty\}$ if the inequality below holds*

$$F^0(u; v - u) + G(v) - G(u) \geq 0, \ \forall \ v \in X.$$

Definition 2 (Definition 3.2 in Motreanu-Panagiotopoulos [10]). *The functional $I = F + G : X \to \mathbf{R} \cup \{+\infty\}$ as in Definition 1 is said to satisfy the Palais - Smale condition if every sequence $\{u_n\} \subset X$ for which $I(u_n)$ is bounded and*

$$F^0(u_n; v - u_n) + G(v) - G(u_n) \geq -\varepsilon_n \|v - u_n\|, \ \ \forall v \in X,$$

for a sequence $\{\varepsilon_n\} \subset \mathbf{R}^+$ with $\varepsilon_n \to 0$, contains a strongly convergent subsequence in X.

Remark. Definitions 1 and 2 extend and unify the nonsmooth critical point theories due to Chang [1] and Szulkin [19]. Precisely, if $G = 0$ Definitions 1 and 2 reduce to the corresponding definitions of Chang [1], while if $F \in C^1(X, \mathbf{R})$ Definitions 1 and 2 coincide with those in Szulkin [19].

Mountain Pass Theorem. (Corollary 3.2 in Motreanu-Panagiotopoulos [10]) *Let* $I = F + G : X \to \mathbf{R} \cup \{+\infty\}$ *be a functional as in Definition 1 which satisfies the Palais-Smale condition in the sense of Definition 2. Assume that there exist a number* $\rho > 0$ *and a point* $e \in X$ *with* $\|e\|_X > \rho$ *such that*

$$\inf_{\|u\|_X = \rho} I > \max\{I(0), I(e)\}.$$

Then the number

$$c = \inf\{ \sup_{t\in[0,1]} I(f(t)) : \quad f \in C([0,1], X),\ f(0) = 0,\ f(1) = e\} \geq \inf_{\|u\|_X=\rho} I$$

is a critical value of I*, i.e., there exists* $u \in X$ *such that* $I(u) = c$ *and* u *is a critical point of* I *in the sense of Definition 1.*

Let us describe now the abstract functional framework of our variational approach in studying the inequality problem (7) without the assumptions of boundedness for set K or of coerciveness as in Theorem 3. Let X and Y be Banach spaces, with X reflexive, and let $L : X \to Y$ be a linear compact operator. Consider the functionals $E \in C^1(X, \mathbf{R})$ (in (7) we will take $A := E' : X \to X^*$), $\Phi : X \to \mathbf{R}$ convex, lower semicontinuous, Gâteaux differentiable and $J : Y \to \mathbf{R}$ locally Lipschitz. Given a closed convex cone K of X, with $0 \in K$, let I_K denote the indicator function of K. We apply the aforementioned nonsmooth version of Mountain Pass Theorem for the following choices: $F := E + J \circ L$, $G := \Phi + I_K$ and thus $I = F + G$.

The following result follows readily from Definition 1.

Lemma 3 *Every critical point* $u \in X$ *of the functional* I *in the sense of Definition 1 is a solution to problem (7) with* $A = E'$.

Lemma 4 *Assume in addition that the following hypotheses are satisfied:*
(H1) There exist positive constants a_0, a_1, α *with* $\alpha < a_0$ *such that*

$$E(v) + \Phi(v) + J(Lv) - \alpha(\langle E'(v) + \Phi'(v), v\rangle + J^0(Lv; Lv))$$

$$\geq a_0\|v\| - a_1,\ \forall\ u \in K,$$

and
(H2) If $\{u_n\}$ *is a sequence in* K *provided* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty}\langle E'(u_n), u_n - u\rangle \leq 0$ *for some* $u \in X$*, then* $\{u_n\}$ *contains a subsequence denoted again by* $\{u_n\}$ *with* $u_n \to u$ *in* X*.*
Then the functional I *satisfies the Palais-Smale condition in the sense of Definition 2.*

Proof. Let $\{u_n\}$ be a sequence in X with the properties required in Definition 1. In particular, we know that $\{u_n\} \subset K$ and there exist a constant $M > 0$ and a sequence $\{\varepsilon_n\} \subset \mathbf{R}^+$ with $\varepsilon_n \to 0$ such that

$$|I(u_n)| \leq M, \quad \forall n \geq 1,$$

and

$$\langle E'(u_n), v - u_n\rangle + J^0(Lu_n; Lv - Lu_n) + \Phi(v) - \Phi(u_n) \geq -\varepsilon_n\|v - u_n\|, \quad \forall v \in K.$$

Using the convexity and the Gâteaux differentiabiliy of Φ, setting $v = (1+t)u_n$, with $t > 0$, in the inequality above and then letting $t \to 0$ one obtains that

$$\langle E'(u_n) + \Phi'(u_n), u_n\rangle + J^0(Lu_n; Lu_n) \geq -\varepsilon_n\|u_n\|, \quad \forall n \geq 1.$$

The inequalities above ensure that for n sufficiently large (so that $\varepsilon_n \leq 1$) one has

$$M + \alpha\|u_n\|$$

$$\geq E(u_n) + \Phi(u_n) + J(Lu_n) - \alpha[\langle E'(u_n) + \Phi'(u_n), u_n\rangle + J^0(Lu_n; Lu_n)].$$

Here α denotes the positive constant entering assumption $(H1)$. Then on the basis of condition $(H1)$ we deduce that the sequence $\{u_n\}$ is bounded in X.

Consequently, the sequence $\{u_n\}$ contains a subsequence again denoted by $\{u_n\}$ such that $u_n \rightharpoonup u$ in X and $Lu_n \to Lu$ in Y for some $u \in K$. On the other hand if we set $v = u$, we derive that

$$\langle E'(u_n), u - u_n\rangle + J^0(Lu_n; Lu - Lu_n) + \Phi(u) - \Phi(u_n) \geq -\varepsilon_n\|u - u_n\|.$$

Since J^0 is upper semicontinuous and Φ is lower semicontinuous, this yields that

$$\limsup_{n\to\infty}\langle E'(u_n), u_n - u\rangle \leq 0.$$

Assumption $(H2)$ completes the proof. □

The main result of this Section is stated below.

Theorem 5 *Assume* $(H1)$, $(H2)$,
$(H3)$ *There exist an element* $\overline{u} \in K \setminus \{0\}$ *satisfying* $\|\overline{u}\| > a_1/a_0$, *for the constants* a_0, a_1 *in* $(H1)$, *and* $E(\overline{u}) + \Phi(\overline{u}) + J(\overline{u}) \leq 0$,
and
$(H4)$ *There exist a constant* $\rho > 0$ *such that*

$$\inf_{\|v\|=\rho}(E(v) + \Phi(v) + J(v)) > E(0) + \Phi(0) + J(0).$$

Then problem (7) with $A = E'$ *admits at least a solution* $u \in K \setminus \{0\}$.

Proof. Let us apply the nonsmooth version of Mountain Pass Theorem to our functional I. Lemma 4 establishes that I satisfies the Palais-Smale condition in the sense of Definition 2.

The calculus with generalized gradients (see Clarke [2]) shows that

$$\partial_t(t^{-\frac{1}{\alpha}}(E+\Phi)(tu)+t^{-\frac{1}{\alpha}}J(tLu))$$

$$\subset -\frac{1}{\alpha}t^{-\frac{1}{\alpha}-1}(E+\Phi)(tu)+t^{-\frac{1}{\alpha}}\langle(E'+\Phi')(tu),u\rangle$$

$$-\frac{1}{\alpha}t^{-\frac{1}{\alpha}-1}J(tLu)+t^{-\frac{1}{\alpha}}\partial J(tLu)u,\ \forall t>0,\ \forall u\in X,$$

where the notation ∂_t stands for the generalized gradient with respect to t. Lebourg's mean value theorem allows to find some $\tau=\tau(u)\in(1,t)$ such that

$$t^{-\frac{1}{\alpha}}(E(tu)+\Phi(tu)+J(tLu))-(E(u)+\Phi(u)+J(u))$$

$$\in\frac{1}{\alpha}\tau^{-\frac{1}{\alpha}-1}[\alpha(\langle E'(\tau u)+\Phi'(\tau u),\tau u\rangle+\partial J(\tau u)\tau u)$$

$$-(E(\tau u)+\Phi(\tau u)+J(\tau u))](t-1),\quad \forall t>1,\ \forall u\in X.$$

Combining with assumption $(H1)$ it follows that

$$t^{-\frac{1}{\alpha}}(E(tu)+\Phi(tu)+J(tLu))-(E(u)+\Phi(u)+J(u))$$

$$\leq\frac{1}{\alpha}\tau^{-\frac{1}{\alpha}-1}(-a_0\tau\|u\|+a_1)(t-1),\quad \forall t>1,\ \forall u\in K.$$

It is then clear from assumption $(H3)$ that one can write

$$I(t\overline{u})=E(t\overline{u})+\Phi(t\overline{u})+J(t\overline{u})\leq t^{\frac{1}{\alpha}}[E(\overline{u})+\Phi(\overline{u}))+J(\overline{u})],\quad \forall t>1.$$

This fact in conjunction with assumption $(H3)$ leads to the conclusion that

$$\lim_{t\to+\infty}I(t\overline{u})=-\infty.$$

Then assumption $(H4)$ enables us to apply the nonsmooth version of Mountain Pass Theorem for $e=t\overline{u}$, with a sufficiently large positive number t. According to Mountain Pass Theorem the functional I possesses a nontrivial critical point $u\in X$ in the sense of Definition 1. Finally, Lemma 3 shows that u is a (nontrivial) solution of problem (7) with $A=E'$. The proof of Theorem 5 is thus complete. □

We end this Section with an example of application of Theorem 5 in the case of variational-hemivariational inequality (6). For the sake of simplicity we consider a uniformly convex Banach space X, a convex closed cone K in X with $0 \in K$, $f = 0$, $\Phi = 0$ and a self-adjoint linear continuous operator $A : X \to X^*$ satisfying $\langle Av, v\rangle \geq c_0\|v\|^2$, for all $v \in X$, with a constant $c_0 > 0$.

Assume that the function $j : T \times \mathbf{R}^m \to \mathbf{R}$ verifies the conditions (1), (3), (4) with $p > 2$, as well as the following assumptions of Ambrosetti–Rabinowitz type:

(i) there exist constants $0 < \alpha < 1/2$ and $c \in \mathbf{R}$ such that

$$j(x, y) \geq \alpha j_y^0(x, y; y) + c, \quad \text{for a.e. } x \in T, \ \forall y \in \mathbf{R}^m;$$

(ii) $\liminf_{y\to 0} \frac{1}{|y|^2} j(x, y) \geq 0$ uniformly with respect to $x \in T$, and $j(x, 0) = 0$ a.e. $x \in T$;

(iii) there exists an element $u_0 \in K \setminus \{0\}$ such that

$$\liminf_{t\to\infty} \left[\frac{1}{2}\langle Au_0, u_0\rangle t^2 + \int_T j(x, tu_0(x))dx\right] < 0.$$

Let us apply Theorem 5 for the functional J given by (8) and $E(v) = (1/2)\langle Av, v\rangle$, $\forall v \in X$. We see that hypotheses (i) and (ii) imply $(H1)$ and $(H4)$, respectively. Taking $\overline{u} = tu_0$ for $t > 0$ sufficiently large, we get $(H3)$ from (iii). It is straightforward to check that condition $(H2)$ holds true. Therefore Theorem 5 yields a nontrivial solution of variational-hemivariational inequality (6) in our setting.

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