GLOBAL EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR SINGULAR DOUBLE PHASE EIGENVALUE PROBLEMS

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Abstract. We study a nonlinear Dirichlet problem eigenvalue driven by a differential operator with unbalanced growth (double phase problem) and a reaction that has the competing effects of a singular term and of a superlinear perturbation. We prove an existence and multiplicity theorem which is global in the parameter $\lambda > 0$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear eigenvalue problem

$$(\mathcal{P}_\lambda) \qquad \begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = \lambda[\beta(z)u(z)^{-\eta} + f(z,u(z))] \ \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0, \lambda > 0, 0 < \eta < 1 < q < p. \end{cases}$$

Let $C^{0,1}(\overline{\Omega})$ be the space of all Lipschitz continuous functions defined on $\overline{\Omega}$. If $a \in C^{0,1}(\overline{\Omega}) \setminus \{0\}$ with $a(z) \geq 0$ for all $z \in \overline{\Omega}$ and $1 < p < \infty$, then by Δ_p^a we denote the weighted p-Laplace differential operator defined by

$$\Delta_p^a u = \operatorname{div}(a(z)|Du|^{p-2}Du).$$

If $a \equiv 1$, then we recover the standard p-Laplace differential operator denoted by Δ_p. Problem $(\mathcal{P}_\lambda)$ is driven by the sum of two such operators with different exponents $q < p$. So, the differential operator in $(\mathcal{P}_\lambda)$ is not homogeneous and this excludes the use of scaling arguments. Another special feature of the differential operator, is that the weight $a(\cdot)$ is not bounded away from zero, that is, we do not assume that $0 < \min_{\overline{\Omega}} a$.

This fact has profound implications on the structure of the problem. Let $\theta(z,t)$ denote the density function associated with the differential operator of $(\mathcal{P}_\lambda)$ and defined by

$$\theta(z,t) = a(z)t^p + t^q \ \text{ for all } \ z \in \overline{\Omega}, \ \text{all } t \geq 0.$$

Since $\min_{\overline{\Omega}} a = 0$, this function exhibits an unbalanced (asymmetric) growth in the variable $t \geq 0$, namely we have

$$t^q \leq \theta(z,t) \leq \hat{c}(1 + t^p) \ \text{ for all } \ z \in \Omega, \ \text{all } t \geq 0, \ \text{some } \hat{c} > 0.$$

We see that now $\theta(z,\cdot)$ is trapped between two different powers of t. This has significant consequences on the structure of $(\mathcal{P}_\lambda)$. A first consequence is that the standard Lebesgue and Sobolev spaces, do not provide on adequate functional framework for the analysis of $(\mathcal{P}_\lambda)$. We need to pass to the broader class of generalized Orlicz spaces. A second and arguably most important consequence of the unbalanced growth of $\theta(z,\cdot)$, is that the global (that is, up to the boundary) regularity theory of Lieberman [18], in this case does not hold. The global regularity theory requires balanced growth for the density function $\theta(z,\cdot)$. This absence of a global regularity theory, implies that many analytical tools which proved

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to be very effective in the case of balanced growth problems, are no longer available. We mention a few of these tools: (a) the equivalence of Sobolev and Hölder local minimizers (see Brezis-Nirenberg [5] and García Azorero-Peral Alonso-Manfredi [10]); (b) nonlinear versions of the Hopf maximum principle (see Pucci-Serrin [31]); (c) comparison principles (see Papageorgiou-Rădulescu-Repovs [28]).

In the reaction of $(\mathcal{P}_\lambda)$, we have the combined effects of a singular term $\beta(z)x^{-\eta}$ ($0 < \eta < 1$) and of a superlinear perturbation. The presence of the singular term adds to the difficulties we face, since the corresponding energy functional of the problem, is not C^1 and so we can not use the minimax theorems of the critical point theory. We have to find techniques to bypass the singularity and deal with C^1-functionals. The perturbation $f(z,\cdot)$ is $(p-1)$-superlinear. However, the superlinearity of $f(z,\cdot)$ is not expressed using the well-known Ambrosetti-Rabinowitz condition or any of its generalizations (see Li-Yang [19]).

Problems driven by such operators, are known as double phase problems or unbalanced growth problems or problems with (p,q)-growth. Such problems and the corresponding integral functionals, were first considered by Marcellini [25, 26] and Zhikov [34, 35, 36], in the context of problems of the calculus of variations (including the Lavrentiev gap phenomenon) and of the nonlinear elasticity theory. Double phase operators arise in the analysis of diffusion-type processes in a space where certain subdomains are distinguished from others. Using double phase operators, we can describe composite materials which have a density of q-growth on $\{a = 0\}$ and of p-growth on $\{a > 0\}$. In mathematical terms, this means that the ellipticity of the differential operator changes as we move in the domain Ω.

Due to these difficulties, singular double phase problems have been studied very little. In the literature, we find very few relevant works. We mention the papers of Arora-Fiscella-Mukherjee-Winkert [2], Liu-Winkert [24], and Papageorgiou *et al.* [4, 8, 21, 22, 33]. None of the aforementioned works examines eigenvalue problems, their approach is based on the Nehari method, the perturbation $f(z,x)$ is either of power type or satisfies more restrictive conditions and their existence and multiplicity results are not global in the parameter $\lambda > 0$. Finally, we mention the works of Ambrosio-Rădulescu [1] and Autuori-Isernia [3], who study nonlocal double phase equations in all of $\mathbb{R}^N$. In [1] the authors examine double phase problems driven by a fractional (p,q)-operator plus a parametric potential term. Using a combination of variational and topological tools, they establish the existence of multiple solutions when the parameter is small and they also prove concentration properties of these solutions as the parameter varies. The reaction is continuous, subcritical, satisfying the Ambrosetti-Rabinowitz condition and a monotonicity hypothesis for the quotient function $(0,\infty) \ni x \mapsto f(x)/x^{q-1}$, where $q > p$. In [3], the authors examine a Kirchhoff double phase problem with a reaction containing a parametric continuous term plus a critical perturbation. Using critical point theory, the Nehari method and the concentration-compactness principle (to accomodate the critical term), they prove the existence of a ground state solution, when the parameter is large.

In this work, following Liu-Papageorgiou [23], we overcome the difficulties resulting from the lack of a global regularity theory, using critical groups. Using them, together with truncations, auxiliary problems and variational methods, we prove an existence and multiplicity theorem, which is global in the parameter $\lambda > 0$ (a bifurcation-type result).

2. Mathematical Background and Hypotheses

We already mentioned in the introduction, that for the analysis of double phase equations, we have to go outside the standard Lebesgue and Sobolev spaces and consider generalized Orlicz spaces. For the theory of these spaces, we refer to the book of Harjulehto-Hästö [13].

Let $L^0(\Omega)$ denote the space of all measurable functions $u : \Omega \to \mathbb{R}$. We identify two such functions which differ only on a Lebesgue-null set. Given $\beta \in L^0(\Omega)$, we write $0 \prec \beta$ if and only if for all $K \subseteq \Omega$ compact we have

$$0 < c_K \le \beta(z) \quad \text{for a.a. } z \in K.$$

Also, let $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \overline{\Omega}$. Our hypotheses on the coefficients $a(\cdot), \beta(\cdot)$ and the exponents p, q, η are the following:

(H_0) $a \in C^{0,1}(\overline{\Omega}) \setminus \{0\}, a(z) \ge 0$ for all $z \in \overline{\Omega}, \beta \in W_0^{1,\infty}(\Omega) \setminus \{0\}, 0 \prec \beta, c_0\hat{d}(z)^\eta \le \beta(z)$ for all $z \in \overline{\Omega}_{\varepsilon_0} = \{z \in \overline{\Omega} : \hat{d}(z) \le \varepsilon_0\}$ for some $\varepsilon_0 > 0$ and $c_0 > 0, 0 < \eta < 1, 2 \le q < p < N$ and $\frac{p}{q} < 1 + \frac{1}{N}$.

Remark 1. We know that the elements of $W_0^{1,\infty}(\Omega)$ admit a representative which belongs in $C^{0,1}(\overline{\Omega})$. Let $C_\eta^0(\overline{\Omega}_{\varepsilon_0}) = \left\{y \in C(\overline{\Omega}) : \frac{y}{\hat{d}^\eta} \in C(\overline{\Omega}_{\varepsilon_0})\right\}$. Then this is an ordered Banach space, with positive (order) cone $C_\eta^0(\overline{\Omega}_{\varepsilon_0})_+ = \left\{y \in C_\eta^0(\overline{\Omega}_{\varepsilon_0}) : y(z) \ge 0 \text{ for all } z \in \overline{\Omega}_{\varepsilon_0}\right\}$. This cone has a nonempty interior $\operatorname{int} C_\eta^0(\overline{\Omega}_{\varepsilon_0})_+$ and if $y \in \operatorname{int} C_\eta^0(\overline{\Omega}_{\varepsilon_0})_+$ then there exists $c_0 > 0$ such that $c_0\hat{d}^\eta \le y$ in $\overline{\Omega}_{\varepsilon_0}$. We mention that the space $C_\eta^0(\overline{\Omega}_{\varepsilon_0})$ arises in the regularity theory of the fractional problems (see, for example, Frassu-Iannizzotto [9]). The last inequality in the above hypotheses, implies that $p < q^* = \frac{Nq}{N-q}$ and this leads to compact embeddings of the relevant spaces (see Proposition 1, below).

Recall that $\theta(z,t) = a(z)t^p + t^q$ for all $z \in \Omega$, all $t \ge 0$. Using this density, we introduce the generalized Lebesgue-Orlicz space $L^\theta(\Omega)$ by

$$L^\theta(\Omega) = \left\{u \in L^0(\Omega) : \ \rho_\theta(u) = \int_\Omega \theta(z, |u|)dz < \infty\right\}.$$

We call $\rho_\theta(\cdot)$ the modular function corresponding to the density $\theta(z,t)$. We equip $L^\theta(\Omega)$ with the so-called "Luxemburg norm" $\|\cdot\|_\theta$ defined by

$$\|u\|_\theta = \inf\left\{t > 0 : \rho_\theta\left(\frac{u}{t}\right) \le 1\right\}.$$

With this norm $L^\theta(\Omega)$ becomes a Banach space which is separable and reflexive (in fact uniformly convex, since $\theta(z,\cdot)$ is an uniformly convex function). Using $L^\theta(\Omega)$ we can define the corresponding generalized Sobolev-Orlicz space $W^{1,\theta}(\Omega)$ by

$$W^{1,\theta}(\Omega) = \left\{u \in L^\theta(\Omega) : |Du| \in L^\theta(\Omega)\right\},$$

with Du being the weak gradient of u. We equip $W^{1,\theta}(\Omega)$ with the norm $\|\cdot\|_{1,\theta}$ defined by

$$\|u\|_{1,\theta} = \|u\|_\theta + \|Du\|_\theta \quad \text{for all } u \in W^{1,\theta}(\Omega),$$

where $\|Du\|_\theta = \|\ |Du|\ \|_\theta$. We also define

$$W_0^{1,\theta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\theta}},$$

with $C_c^\infty(\Omega)$ being the space of all C^∞-functions with compact support. Both spaces $W^{1,\theta}(\Omega)$ and $W_0^{1,\theta}(\Omega)$ are separable and reflexive (in fact uniformly convex) Banach spaces. Moreover, on $W_0^{1,\theta}(\Omega)$ the Poincaré inequality holds, that is, there exists $\hat{c} > 0$ such that

$$\|u\|_\theta \le \hat{c}\|Du\|_\theta, \text{ for all } u \in W_0^{1,\theta}(\Omega) \quad \text{(see [7])}.$$

So, on $W_0^{1,\theta}(\Omega)$, we consider the equivalent norm $\|\cdot\|$ defined by

$$\|u\| = \|Du\|_\theta, \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

We have the following useful embeddings between these spaces. In what follows by $\hookrightarrow$ we denote continuous and dense embedding.

Proposition 1. *(a)* $L^\theta(\Omega) \hookrightarrow L^r(\Omega),\ W_0^{1,\theta}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ *for all* $1 \le r \le q$;
(b) $W_0^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ *for all* $1 \le r \le q^*$ *and the embedding is compact if* $1 \le r < q^*$;
(c) $L^p(\Omega) \hookrightarrow L^\theta(\Omega)$.

Also, there is a close relation between the norm $\|\cdot\|$ and the modular function $\rho_\theta(\cdot)$.

Proposition 2. *(a)* $\|u\| = t > 0 \ \Leftrightarrow\ \rho_\theta\left(\frac{Du}{t}\right) = 1$;
(b) $\|u\| < 1$ $(resp. = 1, > 1)$ $\Leftrightarrow$ $\rho_\theta(Du) < 1$ $(resp. = 1, > 1)$;
(c) $\|u\| < 1 \ \Rightarrow\ \|u\|^p \le \rho_\theta(Du) \le \|u\|^q$;
(d) $\|u\| > 1 \ \Rightarrow\ \|u\|^q \le \rho_\theta(Du) \le \|u\|^p$;
(e) $\|u\| \to 0$ $(resp. \to +\infty)$ $\Leftrightarrow$ $\rho_\theta(Du) \to 0$ $(resp. \to +\infty)$.

In the sequel, we will use also the modular function $\rho_a : W_0^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$\rho_a(u) = \int_\Omega a(z)|Du|^p dz \text{ for all } \ u \in W_0^{1,\theta}(\Omega).$$

This is a continuous, convex function, thus it is weakly lower semicontinuous.

Let $V : W_0^{1,\theta}(\Omega) \to W_0^{1,\theta}(\Omega)$ be the nonlinear operator defined by

$$\langle V(u), h\rangle = \int_\Omega [a(z)|Du|^{p-2} + |Du|^{q-2}](Du, Dh)_{\mathbb{R}^N} dz \ \text{ for all } u, h \in W_0^{1,\theta}(\Omega).$$

This operator has the following properties (see Liu-Dai [20], Liu-Papageorgiou [23] and Papageorgiou-Winkert [30, p.683]).

Proposition 3. *The operator $V(\cdot)$ is bounded (that is, maps bounded sets to bounded ones), continuous, strictly monotone (thus maximal monotone too), coercive and of type $(S)_+$ that is*

$$\text{“}if\ u_n \xrightarrow{w} u \ in\ W_0^{1,\theta}(\Omega)\ and\ \limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0,$$
$$then\ u_n \to u\ in\ W_0^{1,\theta}(\Omega).\text{”}$$

In the study of singular problems, an useful tool is the “Hardy inequality ” (see Papageorgiou-Winkert [30, p.682]).

Proposition 4. $\left\|\frac{u}{\hat{d}}\right\|_r < c^*\|Du\|_r$ *for some* $c^* > 0$, *all* $u \in W_0^{1,r}(\Omega), 1 < r < \infty$. *If* $u \in L^0(\Omega)$, *then we set*

$$u^+ = \max\{u, 0\}, \quad u^- = \max\{-u, 0\}.$$

We have $u = u^+ - u^-, |u| = u^+ + u^-$ *and if* $u \in W_0^{1,\theta}(\Omega)$, *then* $u^\pm \in W_0^{1,\theta}(\Omega)$. *Also by* $|\cdot|_N$ *we denote the Lebesgue measure on* $\mathbb{R}^N$

Let $\mathbb{R}_+^0 = (0, \infty)$ and let $\hat{f} : \Omega \times \mathbb{R}_+^0 \to \mathbb{R}$ be a Carathéodory function (that is, $z \to \hat{f}(z, x)$ is measurable, while $x \to \hat{f}(z, x)$ is continuous), which satisfies

$$\left|\hat{f}(z, x)\right| \le \hat{a}(z)\left(1 + x^{-\eta} + x^{q^*-1}\right)$$

for a.a. $z \in \Omega$, all $x > 0$, with $\hat{a} \in L^\infty(\Omega), 0 < \eta < 1$. We consider the following singular problem

$$(1) \qquad -\Delta_p^a u(z) - \Delta_q u(z) = f(z, u(z)) \ \text{ in } \ \Omega, \quad u|_{\partial\Omega} = 0, \quad u > 0.$$

We say that $u \in W_0^{1,\theta}(\Omega)$ ia a (weak) solution of problem (1) if the following hold

$$u(z) > 0 \ \text{ for a.a. } z \in \Omega, \quad f(\cdot, u(\cdot))h(\cdot) \in L^1(\Omega) \ \text{ for all } \ h \in W_0^{1,\theta}(\Omega),$$

$$\langle V(u), h\rangle = \int_\Omega f(z, u(z))h(z)dz \ \text{ for all } h \in W_0^{1,\theta}(\Omega).$$

We will show that the solutions of (1) are bounded.

For $u \in L^0(\Omega)$, we introduce the sets

$$L_k = \{z \in \Omega : |u| > k\}.$$

The next lemma is an outgrowth of the results of Ladyzhenskaya-Ural'tseva [17, pp. 66, 73-74].

Lemma 1. *If $\mu, \tau > 1, u \in L^\mu(\Omega)$ and*

$$\left[\int_{L_k} (|u|-k)^\mu dz\right]^{\frac{1}{\tau}} \leq c\left[\int_{L_k} (|u|-k)^\mu dz + k^\mu |L_k|_N\right], \quad \textit{for } k \geq k^*, \ \textit{with } c > 0, \ k^* > 0,$$

then $u \in L^\infty(\Omega)$.

Proof. Let $\gamma > 2k^*$. We introduce the sequence $\{k_n\}_{n\in\mathbb{N}_0}$ defined by

$$k_n = \gamma\left(1 - \frac{1}{2^{n+1}}\right) \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{2}$$

Clearly $\{k_n\}_{n\in\mathbb{N}_0}$ is a strictly increasing sequence in $\mathbb{R}^0_+$ and $k_n < \gamma$ for all $n \in \mathbb{N}_0$. Since $k_n < k_{n+1}$ and so $L_{k_{n+1}} \subseteq L_{k_n}$, we have

$$\left(\int_{L_{k_{n+1}}} (|u|-k_{n+1})^\mu dz\right)^{\frac{1}{\tau}} \leq c\left[\int_{L_{k_n}} (|u|-k_n)^\mu dz + k^\mu_{n+1}|L_k|_N\right]. \tag{3}$$

We observe that

$$\begin{aligned}
(k_{n+1}-k_n)^\mu\, |L_{k_{n+1}}|_N &\leq \int_{L_{k_n}} (|u|-k_n)^\mu dz, \\
\Rightarrow \qquad k^\gamma_{n+1}|L_{k_{n+1}}|_N &\leq \left(\frac{k_{n+1}}{k_{n+1}-k_n}\right)^\mu \int_{L_{k_n}} (|u|-k_n)^\mu dz \\
&\leq \left(2^{n+2} - \frac{1}{2}\right)^\mu \int_{L_{k_n}} (|u|-k_n)^\mu dz \quad \text{(see (2))}.
\end{aligned} \tag{4}$$

We return to (3) and use (4), then

$$\begin{aligned}
\int_{L_{k_{n+1}}} (|u|-k_{n+1})^\mu dz &\leq \left[c\left(1+\left(2^{n+2}-\frac{1}{2}\right)^\mu\right)\int_{L_{k_n}} (|u|-k_n)^\mu dz\right]^\tau \\
&\leq c^*\xi^n\left[\int_{L_{k_n}} (|u|-k_n)^\mu dz\right]^{1+\delta}
\end{aligned}$$

with $\xi = 2^{\mu\tau} > 1$, $\delta = \tau - 1$ and $c^* = c^*(c,\mu,\tau) > 0$. We consider the sequence $\left\{y_n = \int_{L_{k_n}} (|u|-k_n)^\mu dz\right\}_{n\in\mathbb{N}_0}$. From Lemma 4.7 of Ladyzhenskaya-Ural'tseva [17, p.66], we know that

$$\text{"if } y_0 \leq (c^*)^{-\frac{1}{\delta}}\xi^{-\frac{1}{\delta^2}}, \text{ then } y_n \to 0 \text{ as } n \to \infty." \tag{5}$$

We have

$$\begin{aligned}
y_0 = \int_{L_{k_0}} (|u|-k_0)^\mu dz &= \int_{L_{k_0}} \left(|u| - \frac{\gamma}{2}\right)^\mu dz \quad \text{(see (2))}, \\
\Rightarrow \ y_0 \to 0^+ \ &\text{ as } \gamma \to +\infty.
\end{aligned}$$

Therefore for $\gamma > 2k^*$ large, we will have

$$\begin{aligned}
&y_0 \leq (c^*)^{-\frac{1}{\delta}}\xi^{-\frac{1}{\delta^2}}, \\
\Rightarrow \ &y_n \to 0 \text{ as } n \to \infty \quad \text{(see (5))}.
\end{aligned} \tag{6}$$

We have

$$\begin{aligned}\int_\Omega (|u|-\gamma)^+ dz &\leq \int_\Omega (|u|-k_n)^+ dz \quad (\text{since } k_n < \gamma \text{ for all } n \in \mathbb{N}_0)\\ &= \int_{L_{k_n}} (|u|-k_n)^+ dz,\\ \Rightarrow \ \int_\Omega (|u|-\gamma)^+ dz &= 0 \ \ (\text{see (6) and recall } L^\mu(\Omega) \hookrightarrow L^1(\Omega)),\\ \Rightarrow \ |u(z)| &\leq \gamma \text{ for a.a. } z \in \Omega \text{ and so } u \in L^\infty(\Omega).\end{aligned}$$

The proof is now complete. □

We will use this lemma, to establish the boundedness of the solutions of (1).

Proposition 5. *Every solution $u \in W_0^{1,\theta}(\Omega)$ of (1) is bounded.*

Proof. By definition, we have

$$u(z) > 0 \text{ for a.a. } z \in \Omega \text{ and } \langle V(u), h\rangle = \int_\Omega f(z,u)hz \ \text{ for all } h \in W_0^{1,\theta}(\Omega). \tag{7}$$

Let $k \in \mathbb{N}, k > 1$ be large so that

$$\|(u-k)^+\| \leq 1. \tag{8}$$

In (7) we choose the test function $h = (u-k)^+ \in W_0^{1,\theta}(\Omega)$. We have

$$\begin{aligned}\|(u-k)^+\|^p &\leq \rho_\theta(D(u-k)^+) \quad (\text{see (8) and Proposition 2})\\ &\leq c_1 \int_\Omega (1 + u^{-\eta} + u^{q^*-1})(u-k)^+ dz \ \ (\text{for some } c_1 > 0).\end{aligned} \tag{9}$$

Since $k > 1$, on L_k we have

$$u^{1-\eta} \leq u \leq u^{q^*} = (u-k+k)^{q^*} \leq 2^{q^*}\left[(u-k)^{q^*} + k^{q^*}\right]. \tag{10}$$

Using (10) in (9) and since $W_0^{1,\theta}(\Omega) \hookrightarrow L^{q^*}(\Omega)$ (see Proposition 1), we obtain

$$\begin{aligned}\left[\int_{L_k} (u-k)^{q^*} dz\right]^{\frac{p}{q^*}} &\leq c_2 \|(u-k)^+\|^p \ \text{ for some } c_2 > 0\\ &\leq c_3 \left[\int_{L_k} (u-k)^{q^*} dz + k^{q^*}|L_k|_N\right] \ \text{ for some } c_3 > 0.\end{aligned}$$

We apply Lemma 1 with $\mu = q^*$ and $\tau = \frac{q^*}{p} > 1$. We infer that $u \in L^\infty(\Omega)$. □

Recall that a function $g : \mathbb{R} \to \mathbb{R}$ is said to be locally Lipschitz, for every $K \subseteq \mathbb{R}$ compact, there exists $\hat{c}_K > 0$ such that

$$|g(x) - g(y)| \leq \hat{c}_K |x-y| \ \text{ for all } x, y \in K.$$

We can easily see that this definition is equivalent to the usual one which says that $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, if for $x \in \mathbb{R}$ we can find a neighborhood U of x such that $g|_U$ is $\hat{c}_U$-Lipschitz (see, for example, Papageorgiou-Winkert [30, p.536]). A function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be an $L^\infty(\Omega)$-locally Lipschitz function, if it is measurable in $z \in \Omega$ and $g(z,\cdot)$ is locally Lipschitz with Lipschitz constant $\hat{c}_K \in L^\infty(\Omega)$ for every $K \subseteq \mathbb{R}$ compact. Such a function is jointly measurable.

Now we can introduce our hypotheses on the perturbation $f(z,x)$.

(H_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^∞-locally Lipschitz function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $f(z,x) \leq \hat{a}(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geq 0$, with $\hat{a} \in L^\infty(\Omega), p < r < q^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)\mathrm{d}s$, then

$$\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty \text{ uniformly for a.a. } z \in \Omega$$

and there exist $\hat{c}_0^*$ and $1 < \tau < q$ such that

$$-\hat{c}_0^*(1 + x^\tau) \le f(z,x)x - pF(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \ge 0;$$

(iii) for every $s > 0$, there exists $l_s^* > 0$ such that

$$0 < l_s^* \le f(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \ge s > 0;$$

(iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function $x \to f(z,x) + \hat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 2. Since we look for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, we may assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \le 0$, Hypothesis H_1(ii) implies that

$$\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \text{ uniformly for a.a. } z \in \Omega.$$

So, the perturbation $f(z,\cdot)$ is $(p-1)$-superlinear. However note that we do not employ the Ambrosetti-Rabinowitz condition (see [16, p.409]) or any of its generalizations (see Li-Yang [19]).

Let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We define

$$K_\varphi = \{u \in X : \ \varphi'(u) = 0\} \text{ (the critical set of } \varphi),$$
$$\varphi^c = \{u \in X : \ \varphi(u) \le c\}.$$

We say that $\varphi(\cdot)$ satisfies the "C-condition", if

"every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

$$\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R} \text{ is bounded}, \quad (1 + \|u_n\|_X)\varphi'(u_n) \to 0 \ \text{ in } \ X^*,$$

admits a strongly convergent subsequence."

For $k \in \mathbb{N}_0$ and $Y_2 \subseteq Y_1 \subseteq X$ by $H_k(Y_1,Y_2)$ we denote the k^{th}-singular homology group with real coefficients. We choose $\mathbb{R}$ as the space of coefficients to avoid torsion phenomena (see [16, p.179]). Then the groups are actually linear spaces. If $u \in K_\varphi$ is isolated, then the critical groups of φ at u are defined by

$$C_k(\varphi,u) = H_k(\varphi^c \cap U, \varphi^c \cap U \backslash \{u\}) \text{ for all } k \in \mathbb{N}_0, \text{ with } c = \varphi(u).$$

Here, U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology, implies that this definition is independent of the isolating neighborhood U (see [16]).

We introduce the following sets

$$\mathcal{L} = \{\lambda > 0 : \text{problem } (\mathcal{P}_\lambda) \text{ has a solution}\} \text{ (set of admissible parameters ("eigenvalues"))},$$

and

$$S_\lambda = \text{solution set of } (\mathcal{P}_\lambda).$$

3. Auxiliary Problems

In this section, we examine two auxiliary problems the solutions of which will help us neutralize the singularity and establish the existence of eigenvalues (that is, that $\mathcal{L} \neq \emptyset$).

First, we consider the purely singular version of $(\mathcal{P}_\lambda)$ (that is, $f = 0$). So, the problem under consideration is

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = \lambda\beta(z)u(z)^{-\eta} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0, \lambda > 0, 0 < \eta < 1. \end{cases} \tag{11}$$

For this problem, we have the following result.

Proposition 6. *If hypotheses H_0 hold and $\lambda > 0$, then problem* (11) *admits a unique solution $\underline{u}_\lambda \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), 0 \prec \underline{u}_\lambda, \{\underline{u}_\lambda\}_{\lambda>0}$ is nondecreasing.*

Proof. From Proposition 3.1 of Bai-Papageorgiou-Zeng [4], we know that problem (11) admits a unique solution

$$\underline{u}_\lambda \in W_0^{1,\theta}(\Omega), 0 \prec \underline{u}_\lambda.$$

From Proposition 5, we deduce that

$$\underline{u}_\lambda \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), 0 \prec \underline{u}_\lambda.$$

Let $0 < \mu < \lambda$ and introduce the Carathéodory function $l_\mu : \Omega \times \mathbb{R}_+^0 \to \mathbb{R}_+^0$ defined by

$$l_\mu(z,x) = \begin{cases} \mu\beta(z)x^{-\eta}, & \text{if } 0 < x \leq \underline{u}_\lambda(z), \\ \mu\beta(z)\underline{u}_\lambda(z)^{-\eta}, & \text{if } \underline{u}_\lambda(z) < x. \end{cases} \tag{12}$$

We consider the following singular problem

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = l_\mu(z,u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0. \end{cases} \tag{13}$$

Reasoning as in the proof of Proposition 3.1 of [4], using regularizations of the equation and eventually a fixed point argument, we show that problem (13) admits a solution $\tilde{u}_\mu \in W_0^{1,\theta}(\Omega) \setminus \{0\}, \tilde{u}_\mu \geq 0, \tilde{u}_\mu \not\equiv 0$. We have

$$\langle V(\tilde{u}_\mu), h\rangle = \int_\Omega l_\mu(z,\tilde{u}_\mu)h\,dz \quad \text{for all } h \in W_0^{1,\theta}(\Omega). \tag{14}$$

In (14) we choose the test function $h = (\tilde{u}_\mu - \underline{u}_\lambda)^+ \in W_0^{1,\theta}(\Omega)$. Then

$$\begin{aligned} \langle V(\tilde{u}_\mu), (\tilde{u}_\mu - \underline{u}_\lambda)^+\rangle &= \int_\Omega \mu\beta(z)\underline{u}_\lambda^{-\eta}\,(\tilde{u}_\mu - \underline{u}_\lambda)^+\,dz \quad \text{(see (12))} \\ &\leq \int_\Omega \lambda\beta(z)\underline{u}_\lambda^{-\eta}\,(\tilde{u}_\mu - \underline{u}_\lambda)^+\,dz \quad \text{(since } \mu < \lambda) \\ &= \langle V(\underline{u}_\lambda), (\tilde{u}_\mu - \underline{u}_\lambda)^+\rangle \end{aligned}$$
$$\Rightarrow \ \tilde{u}_\mu \leq \underline{u}_\lambda \quad \text{(see Proposition 3)}, \quad \tilde{u}_\mu \geq 0, \ \tilde{u}_\mu \not\equiv 0.$$

Then from (12) and (14), we infer that

$$\begin{aligned} &\tilde{u}_\mu = \underline{u}_\mu \\ \Rightarrow \ &\underline{u}_\mu \leq \underline{u}_\lambda \ \text{ and so } \ \{\underline{u}_\lambda\}_{\lambda>0} \text{ is nondecreasing.} \end{aligned}$$

∎

Let $C \subseteq \mathbb{R}^N$ be a measurable set. We consider the positive cone of the Banach lattice $L^\infty(C)$, defined by

$$L^\infty(C)_+ = \{u \in L^\infty(C) : 0 \le u(z) \text{ for a.a. } z \in \Omega\}.$$

This cone has a nonempty interior given by

$$\mathrm{int} L^\infty(C)_+ = \left\{ u \in L^\infty(C)_+ : 0 < \operatorname*{ess\,inf}_C u \right\}.$$

Note that this is in contrast to the positive cones of the Banach lattices $L^p(C)$ $(1 \le p < \infty)$, where $\mathrm{int} L^p(\Omega)_+ = \emptyset$. This fact reflects the different ways the norms of these spaces are defined.

Proposition 7. *If hypotheses H_0 hold and $\lambda > 0$, then there exists $\hat{c}_\lambda > 0$ such that $\hat{c}_\lambda \hat{d} \le \underline{u}_\lambda$ in Ω.*

Proof. From Lemma 14.16, p.355 of Gilbarg-Trudinger [11], we can find $\hat{\varepsilon} > 0$ small such that

$$\hat{d} \in C^2(\overline{\Omega}_{\hat{\varepsilon}}) \ \text{ and } \ |\nabla \hat{d}(z)| = 1 \ \text{ for all } \ z \in \overline{\Omega}_{\hat{\varepsilon}} \tag{15}$$

(recall that $\Omega_{\hat{\varepsilon}} = \{z \in \overline{\Omega} : \hat{d}(z) < \hat{\varepsilon}\}$). For $0 < \delta < \frac{1}{2}\hat{\varepsilon}$, we introduce the function $\hat{\tau} : \overline{\Omega}_{\hat{\varepsilon}} \to \mathbb{R}_+$ defined by

$$\hat{\tau}(z) = \begin{cases} \hat{d}(z), & \text{if } \hat{d}(z) < \delta, \\ \delta + \displaystyle\int_\delta^{\hat{d}(z)} \left(\frac{2\delta - s}{\delta} \right)^{\frac{1}{q-1}} ds, & \text{if } \delta \le \hat{d}(z) \le 2\delta, z \in \overline{\Omega}_{\hat{\varepsilon}}, \\ \delta + \displaystyle\int_\delta^{2\delta} \left(\frac{2\delta - s}{\delta} \right)^{\frac{1}{q-1}} ds, & \text{if } 2\delta < \hat{d}(z). \end{cases} \tag{16}$$

We see that $\hat{\tau} \in C^1(\overline{\Omega}_{\hat{\varepsilon}}), \hat{\tau} \ge 0$. Then from (15) and (16), we obtain that

$$-\Delta_p^a \hat{\tau} - \Delta_q \hat{\tau} = \begin{cases} -\Delta^{a+1} \hat{d}, & \text{if } \hat{d}(z) < \delta, \\ -\Delta^{a_1} \hat{d}, & \text{if } \delta \le \hat{d}(z) \le 2\delta, z \in \overline{\Omega}_{\hat{\varepsilon}}, \\ 0, & \text{if } 2\delta < \hat{d}(z), \end{cases}$$

where $a_1(z) = a(z) \left(\frac{2\delta - \hat{d}(z)}{\delta} \right)^{\frac{p-1}{q-1}} + \frac{2\delta - \hat{d}(z)}{\delta}$. Clearly $a_1 \in C^{0,1}(\overline{\Omega}_{\hat{\varepsilon}})$ and so it follows that

$$-\Delta_p^a \hat{\tau} - \Delta_q \hat{\tau} \in L^\infty(\Omega_{\hat{\varepsilon}}).$$

Moreover, from Lemma 2.3 of Guo-Webb [12] and by choosing $\hat{\varepsilon} > 0$ even smaller, we can have

$$c_4 \hat{d} \le \hat{\tau} \le c_5 \hat{d} \ \text{ in } \ \overline{\Omega}_{\hat{\varepsilon}} \ \text{ with } \ 0 < c_4 < c_5. \tag{17}$$

Let $\varepsilon^* = \min\{\varepsilon_0, \hat{\varepsilon}\}$ (see hypotheses H_0) and choose $\gamma_\lambda \in (0,1)$ small such that

$$\begin{aligned} -\Delta_p^a(\gamma_\lambda \hat{\tau}) - \Delta_q(\gamma_\lambda \hat{\tau}) &\le \frac{\lambda c_0}{c_4^\eta} \quad \text{(see hypotheses } H_0) \\ &\le \lambda \frac{\beta(z)}{(c_4 \hat{d})^\eta} \\ &\le \lambda \beta(z) \hat{\tau}^{-\eta} \ \text{ in } \ \overline{\Omega}_{\varepsilon^*} \quad \text{(see (17))}, \end{aligned}$$

$$\Rightarrow \ \gamma_\lambda \hat{\tau} \ \text{ is a lower solution of (11) on } \ \overline{\Omega}_{\varepsilon^*}.$$

Then on account of the uniqueness of the solution $\underline{u}_\lambda$ of (11), we have

$$\begin{aligned} &\gamma_\lambda \hat{\tau} \le \underline{u}_\lambda \ \text{ in } \ \overline{\Omega}_{\varepsilon^*}, \\ \Rightarrow \ & \hat{c}^0_\lambda \hat{d} \le \underline{u}_\lambda \ \text{ in } \ \overline{\Omega}_{\varepsilon^*}, \text{ for some } \hat{c}^0_\lambda > 0 \ \text{ (see (17))}. \end{aligned} \tag{18}$$

We know that $0 \prec \underline{u}_\lambda$. So, it follows that

$$\underline{u}_\lambda \in L^\infty(\Omega \setminus \overline{\Omega}_{\varepsilon^*})_+.$$

So, we can find $\hat{c}^1_\lambda > 0$ such that

$$\hat{c}^1_\lambda \hat{d} \le \underline{u}_\lambda \ \text{ in } \ \Omega \setminus \overline{\Omega}_{\varepsilon^*}. \tag{19}$$

Let $\hat{c}_\lambda = \min\{\hat{c}^0_\lambda, \hat{c}^1_\lambda\} > 0$. We conclude that

$$\hat{c}_\lambda \hat{d} \le \underline{u}_\lambda \ \text{ in } \ \Omega \quad \text{(see (18), (19))}.$$

This completes the proof. ☐

Proposition 8. *If hypotheses H_0 hold and $\lambda > 0$, then $\beta(\cdot)\underline{u}_\lambda(\cdot)^{-\eta} \in L^\infty(\Omega)$.*

Proof. Let $s > 1$. We have

$$\begin{aligned} \int_\Omega \left(\frac{\beta(z)}{\underline{u}_\lambda^\eta}\right)^s dz = & \int_\Omega \left(\underline{u}_\lambda^{1-\eta} \frac{\beta(z)}{\underline{u}_\lambda}\right)^s dz \\ \le & \|\underline{u}_\lambda\|_\infty^{(1-\eta)s} \int_\Omega \left(\frac{\beta(z)}{\underline{u}_\lambda}\right)^s dz \\ \le & \|\underline{u}_\lambda\|_\infty^{(1-\eta)s} \frac{1}{\hat{c}^s_\lambda} \int_\Omega \left(\frac{\beta(z)}{\hat{d}}\right)^s dz \quad \text{(see Proposition 7)} \\ \le & \left(\frac{\|\underline{u}_\lambda\|_\infty^{1-\eta}}{\hat{c}_\lambda}\right)^s c_5 \|D\beta\|_s^s \quad \text{for some } c_5 > 0 \quad \text{(see Proposition 4)}, \end{aligned}$$

$$\Rightarrow \ \|\beta \underline{u}_\lambda^{-\eta}\|_s \le c_6(\lambda)\|D\beta\|_s \quad \text{for some } c_6(\lambda) > 0. \tag{20}$$

Since in (12) $s > 1$ is arbitrary, we let $s \to +\infty$ and obtain

$$\begin{aligned} &\|\beta \underline{u}_\lambda^{-\eta}\|_\infty \le c_6(\lambda)\|D\beta\|_\infty, \\ \Rightarrow \ & \beta \underline{u}_\lambda^{-\eta} \in L^\infty(\Omega), \end{aligned}$$

which completes the proof. ☐

Now we introduce the second auxiliary problem that we will examine.

$$\begin{cases} -\Delta^a_p u(z) - \Delta_q u(z) = \beta(z)\underline{u}_\lambda(z)^{-\eta} + 1 \ \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0. \end{cases} \tag{21}$$

For this problem, we have the following result.

Proposition 9. *If hypotheses H_0 hold, then problem* (21) *admits a unique solution $\overline{u}_\lambda \in W^{1,\theta}_0(\Omega) \cap L^\infty(\Omega), 0 \prec \overline{u}_\lambda$ and $\underline{u}_\lambda \le \overline{u}_\lambda$ for all $0 < \lambda \le 1$.*

Proof. From Proposition 3, we know that $V(\cdot)$ is continuous, strictly monotone, coercive. Hence $V(\cdot)$ is surjective (see [30, p.536]). Therefore there exists $\overline{u}_\lambda \in W^{1,\theta}_0(\Omega), \overline{u}_\lambda \not\equiv 0, \overline{u}_\lambda \ge 0$ such that

$$V(\overline{u}_\lambda) = \beta(\cdot)\underline{u}_\lambda^{-\eta} + 1 \ \text{ in } \ W^{1,\theta}_0(\Omega)^*.$$

In fact the strict monotonicity of $V(\cdot)$ implies that this solution is unique. If $0 < \lambda \leq 1$, then

$$V(\underline{u}_\lambda) = \lambda\beta(\cdot)\underline{u}_\lambda^{-\eta} \leq \beta(\cdot)\underline{u}_\lambda^{-\eta} + 1 = V(\overline{u}_\lambda) \text{ in } W_0^{1,\theta}(\Omega),$$
$$\Rightarrow \underline{u}_\lambda \leq \overline{u}_\lambda \text{ for all } \lambda \in (0,1] \quad \text{(see [31, p.61])}. \tag{22}$$

From (22) and Proposition 5, it follows that

$$0 \prec \overline{u}_\lambda \text{ and } \overline{u}_\lambda \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega).$$

The proof is now complete. □

4. Solutions of $(\mathcal{P}_\lambda)$

In this section we prove the global in $\lambda > 0$ existence and multiplicity theorem for problem $(\mathcal{P}_\lambda)$.

We start by showing the existence of eigenvalues and also determine the properties of the corresponding eigenfunctions.

Proposition 10. *If hypotheses H_0 and H_1 hold, then $\mathcal{L} \neq \emptyset$ and for $\lambda \in \mathcal{L}, \emptyset \neq S_\lambda \subseteq W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), 0 \prec u$ for all $u \in S_\lambda$.*

Proof. Let $0 < \lambda \leq 1$ and let $\overline{u}_\lambda$ be the unique solution of (21) (see Proposition 9). We know that

$$\overline{u}_\lambda \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), \quad \underline{u}_\lambda \leq \overline{u}_\lambda \quad (0 < \lambda \leq 1). \tag{23}$$

Then (23) and hypothesis H_1(i) imply that we can find $\hat{\lambda}^* \in (0,1]$ such that

$$0 \leq \lambda f(z, \overline{u}_\lambda(z)) \leq 1 \text{ for a.a. } z \in \Omega, \text{ all } 0 < \lambda \leq \hat{\lambda}^*. \tag{24}$$

Then for $0 < \lambda \leq \hat{\lambda}^*$, we introduce the Carathéodory function $\hat{k}_\lambda(z, x)$ defined by

$$\hat{k}_\lambda(z,x) = \begin{cases} \lambda\left[\beta(z)\underline{u}_\lambda(z)^{-\eta} + f(z, \underline{u}_\lambda(z))\right], & \text{if } x < \underline{u}_\lambda(z), \\ \lambda\left[\beta(z)x^{-\eta} + f(z,x)\right], & \text{if } \underline{u}_\lambda(z) \leq x \leq \overline{u}_\lambda(z), \\ \lambda\left[\beta(z)\overline{u}_\lambda(z)^{-\eta} + f(z, \overline{u}_\lambda(z))\right], & \text{if } \overline{u}_\lambda(z) < x. \end{cases} \quad \text{(see(23))}. \tag{25}$$

We set $\hat{K}_\lambda(z,x) = \int_0^x \hat{k}_\lambda(z,s)ds$ and consider the C^1-functional $\hat{\psi}_\lambda : W_0^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\lambda(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \hat{K}_\lambda(z,u)dz \text{ for all } u \in W_0^{1,\theta}(\Omega).$$

It is clear from (25) and Proposition 2, that $\hat{\psi}_\lambda(\cdot)$ is coercive. Also using Proposition 1, we see that $\hat{\psi}_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_\lambda \in W_0^{1,\theta}(\Omega)$ such that

$$\hat{\psi}_\lambda(u_\lambda) = \inf\left\{\hat{\psi}_\lambda(u) : u \in W_0^{1,\theta}(\Omega)\right\},$$
$$\Rightarrow \langle \hat{\psi}'_\lambda(u_\lambda), h\rangle = 0 \text{ for all } h \in W_0^{1,\theta}(\Omega),$$
$$\Rightarrow \langle V(u_\lambda), h\rangle = \int_\Omega \hat{k}_\lambda(z, u_\lambda)h dz \text{ for all } h \in W_0^{1,\theta}(\Omega). \tag{26}$$

In (26) first we choose the test function $h = (\underline{u}_\lambda - u_\lambda)^+ \in W_0^{1,\theta}(\Omega)$. Then

$$\begin{aligned}\langle V(u_\lambda), (\underline{u}_\lambda - u_\lambda)^+\rangle &= \int_\Omega \lambda \left[\beta(z)\underline{u}_\lambda^{-\eta} + f(z, \underline{u}_\lambda)\right](\underline{u}_\lambda - u_\lambda)^+ dz \quad \text{(see (25))}\\ &\geq \int_\Omega \lambda\beta(z)\underline{u}_\lambda^{-\eta}(\underline{u}_\lambda - u_\lambda)^+ dz \quad \text{(since } f \geq 0)\\ &= \langle V(\underline{u}_\lambda), (\underline{u}_\lambda - u_\lambda)^+\rangle \quad \text{(see Proposition 6)},\\ \Rightarrow\ \underline{u}_\lambda \leq u_\lambda.\end{aligned}$$

Next in (26) we choose the test function $h = (u_\lambda - \overline{u}_\lambda)^+ \in W_0^{1,\theta}(\Omega)$. We have

$$\begin{aligned}\langle V(u_\lambda), (u_\lambda - \overline{u}_\lambda)^+\rangle &= \int_\Omega \lambda \left[\beta(z)\overline{u}_\lambda^{-\eta} + f(z, \overline{u}_\lambda)\right](u_\lambda - \overline{u}_\lambda)^+ dz\\ &\leq \int_\Omega \lambda \left[\beta(z)\overline{u}_\lambda^{-\eta} + 1\right](u_\lambda - \overline{u}_\lambda)^+ dz \quad \text{(see (24) and recall } 0 < \lambda \leq \hat{\lambda}^* \leq 1)\\ &\leq \int_\Omega \left[\beta(z)\underline{u}_\lambda^{-\eta} + 1\right](u_\lambda - \overline{u}_\lambda)^+ dz \quad \text{(see Proposition 9)}\\ &= \langle V(\overline{u}_\lambda), (u_\lambda - \overline{u}_\lambda)^+\rangle,\\ \Rightarrow\ u_\lambda \leq \overline{u}_\lambda.\end{aligned}$$

We have proved that

$$\underline{u}_\lambda \leq u_\lambda \leq \overline{u}_\lambda \quad (0 < \lambda \leq \hat{\lambda}^* \leq 1). \tag{27}$$

Then (27), (25) and (26) imply that

$$u_\lambda \in S_\lambda \ \text{ and so } \ (0, \hat{\lambda}^*] \subseteq \mathcal{L} \neq \emptyset.$$

Using Proposition 5, we see that for all $\lambda \in \mathcal{L}$, we have

$$\emptyset \neq S_\lambda \subseteq W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega).$$

Finally if $u \in S_\lambda$, then

$$\begin{aligned}&-\Delta_p^a u - \Delta_q u \geq 0 \ \text{ in } \ \Omega,\\ \Rightarrow\ &0 \prec u \quad \text{(see Papageorgiou-Vetro-Vetro [29, Proposition 2.4])}.\end{aligned}$$

This completes the proof. ☐

In the next result we produce a lower bounded for the elements of S_λ ($\lambda \in \mathcal{L}$).

Proposition 11. *If hypotheses H_0 and H_1 hold and $\lambda \in \mathcal{L}$, then $\underline{u}_\lambda \leq u$ for all $u \in S_\lambda$.*

Proof. Let $\hat{u} \in S_\lambda$ and introduce the the Carathéodory function $\hat{\gamma}_\lambda : \Omega \times \mathbb{R}_+^0 \to \mathbb{R}_+^0$ defined by

$$\hat{\gamma}_\lambda(z, x) = \begin{cases}\lambda\beta(z)x^{-\eta}, & \text{if } 0 < x \leq \hat{u}(z),\\ \lambda\beta(z)\hat{u}(z)^{-\eta}, & \text{if } \hat{u}(z) < x,\end{cases} \tag{28}$$

(see also the proof of Proposition 6). We consider the following singular problem

$$\begin{cases}-\Delta_p^a u(z) - \Delta_q u(z) = \hat{\gamma}_\lambda(z, u(z)) \ \text{ in } \Omega,\\ u|_{\partial\Omega} = 0, \ u > 0.\end{cases} \tag{29}$$

As in [4] (Proposition 3.1), we show that (29) has a solution $\widetilde{u}_\lambda \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), 0 \prec \widetilde{u}_\lambda$. We have

$$\langle V(\widetilde{u}_\lambda), h\rangle = \int_\Omega \hat{\gamma}_\lambda(z, \widetilde{u}_\lambda) h dz \ \text{ for all } h \in W_0^{1,\theta}(\Omega). \tag{30}$$

In (30), we choose the test function $h = (\widetilde{u}_\lambda - \hat{u})^+ \in W_0^{1,\theta}(\Omega)$. Then

$$\begin{aligned}\langle V(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - \hat{u})^+\rangle &= \int_\Omega \lambda\beta(z)\hat{u}^{-\eta}(\widetilde{u}_\lambda - \hat{u})^+ dz \quad \text{(see (28))}\\ &\leq \int_\Omega \lambda\left[\beta(z)\hat{u}^{-\eta} + f(z,\hat{u})\right](\widetilde{u}_\lambda - \hat{u})^+ dz \quad (\text{since } f \geq 0)\\ &= \langle V(\hat{u}), (\widetilde{u}_\lambda - \hat{u})^+\rangle \quad (\text{since } \hat{u} \in S_\lambda),\\ \Rightarrow\ & \widetilde{u}_\lambda \leq \hat{u}, 0 \prec \widetilde{u}_\lambda. \end{aligned} \tag{31}$$

From (31), (28), (30) and Proposition 6, it follows that

$$\begin{aligned} &\widetilde{u}_\lambda = \underline{u}_\lambda,\\ \Rightarrow\ &\underline{u}_\lambda \leq u \quad \text{for all } u \in S_\lambda, \end{aligned}$$

which completes the proof. □

The set $\mathcal{L}$ of eigenvalues is connected (an interval).

Proposition 12. *If hypotheses H_0 and H_1 hold, $\lambda \in \mathcal{L}$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$.*

Proof. From Propositions 6 and 11, we have

$$\underline{u}_\mu \leq \underline{u}_\lambda \leq \hat{u} \quad \text{for all } \hat{u} \in S_\lambda.$$

Fix $\hat{u} \in S_\lambda$ and introduce the Carathéodory function $\hat{e}_\mu(z,x)$ defined by

$$\hat{e}_\mu(z,x) = \begin{cases} \mu\left[\beta(z)\underline{u}_\mu(z)^{-\eta} + f(z,\underline{u}_\mu(z))\right], & \text{if } x < \underline{u}_\mu(z),\\ \mu\left[\beta(z)x^{-\eta} + f(z,x)\right], & \text{if } \underline{u}_\mu(z) \leq x \leq \hat{u}(z),\\ \mu\left[\beta(z)\hat{u}(z)^{-\eta} + f(z,\hat{u}(z))\right], & \text{if } \hat{u}(z) < x. \end{cases} \tag{32}$$

We set $\hat{E}_\mu(z,x) = \int_0^x \hat{e}_\mu(z,s)ds$ and consider the C^1-functional $\hat{\sigma}_\mu : W_0^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\sigma}_\mu(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \hat{E}_\mu(z,u)dz \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

Evidently $\hat{\sigma}_\mu(\cdot)$ is coercive (see (32)) and sequentially weakly lower semicontinuous (use Proposition 1). So, there exists $u_\mu \in W_0^{1,\theta}(\Omega)$ such that

$$\begin{aligned} &\hat{\sigma}_\mu(u_\mu) = \inf\left\{\hat{\sigma}_\mu(u) :\ u \in W_0^{1,\theta}(\Omega)\right\},\\ \Rightarrow\ &\langle V(u_\mu), h\rangle = \int_\Omega \hat{e}_\mu(z,u_\mu)h dz \quad \text{for all } h \in W_0^{1,\theta}(\Omega). \end{aligned} \tag{33}$$

In (33) first we choose the test function $h = (\underline{u}_\mu - u_\mu)^+ \in W_0^{1,\theta}(\Omega)$ and using (32) and that $f \geq 0$, we obtain

$$\underline{u}_\mu \leq u_\mu.$$

If in (33) we choose the test function $h = (u_\mu - \hat{u})^+ \in W_0^{1,\theta}(\Omega)$, then using (32) and since $\mu < \lambda$, we obtain

$$u_\mu \leq \hat{u}.$$

So, we have proved that

$$\underline{u}_\mu \leq u_\mu \leq \hat{u}. \tag{34}$$

From (34), (32) and (33) it follows that

$$u_\mu \in S_\mu, \quad \text{hence} \quad \mu \in \mathcal{L}.$$

The proof is now complete. □

Included in the above proof, is the following weak monotonicity property of the solution multifunction $\lambda \to S_\lambda$.

Corollary 1. *If hypotheses H_0 and H_1 hold, $\lambda \in \mathcal{L}, \hat{u} \in S_\lambda$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in S_\mu$ such that $u_\mu \leq \hat{u}$.*

We can show that if $\lambda \in \mathcal{L}$, then S_λ admits a smallest element (minimal positive solution).

Proposition 13. *If hypotheses H_0 and H_1 hold and $\lambda \in \mathcal{L}$, then there exists $u_\lambda^* \in S_\lambda$ such that*

$$u_\lambda^* \leq u \quad \text{for all} \quad u \in S_\lambda.$$

Proof. We know that S_λ is downward directed, that is, if $u_1, u_2 \in S_\lambda$, then we can find $u \in S_\lambda$ such that $u \leq u_1, u \leq u_2$ (see [28], Proposition 19). Then according to Theorem 5.109, p.308, of Hu-Papageorgiou [15], we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq S_\lambda$ decreasing such that

$$\inf S_\lambda = \inf_{n\in\mathbb{N}} u_n.$$

We have

$$\langle V(u_n), h\rangle = \int_\Omega \lambda \left[\beta(z) u_n^{-\eta} + f(z, u_n)\right] h dz \quad \text{for all } h \in W_0^{1,\theta}(\Omega), \text{ all } n \in \mathbb{N}, \tag{35}$$

$$\underline{u}_\lambda \leq u_n \leq u_1 \quad \text{for all } n \in \mathbb{N} \qquad \text{(see Proposition 11)}. \tag{36}$$

In (35) we choose the test function $h = u_n \in W_0^{1,\theta}(\Omega)$. Then

$$\rho_\theta(Du_n) = \int_\Omega \lambda \left[\beta(z) u_n^{1-\eta} + f(z, u_n) u_n\right] dz \quad \text{for all } n \in \mathbb{N}.$$

Then from (36), hypothesis H_1(i) (recall $u_1 \in L^\infty(\Omega)$) and Proposition 2, we obtain that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega) \quad \text{is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W_0^{1,\theta}(\Omega), \quad u_n \to u_\lambda^* \text{ in } L^r(\Omega) \quad \text{(see Proposition 1)}. \tag{37}$$

In (35), we choose the test function $h = u_n - u_\lambda^* \in W_0^{1,\theta}(\Omega)$. Then

$$\int_\Omega f(z, u_n)(u_n - u_\lambda^*) dz \to 0 \text{ as } n \to \infty \qquad \text{(see (37) and hypothesis } H_1\text{(i))}. \tag{38}$$

Also for every $n \in \mathbb{N}$, we have

$$\begin{aligned} \left|\int_\Omega \beta(z) u_n^{-\eta}(u_n - u_\lambda^*) dz\right| &\leq \int_\Omega \beta(z) \underline{u}_\lambda^{-\eta} |u_n - u_\lambda^*| dz \qquad \text{(see (36))} \\ &\leq \|\beta \underline{u}_\lambda^{-\eta}\|_\infty \|u_n - u_\lambda^*\|_1 \qquad \text{(see Proposition 8)}, \end{aligned}$$

$$\Rightarrow \int_\Omega \beta(z) u_n^{-\eta}(u_n - u_\lambda^*) dz \to 0 \text{ as } n \to \infty. \tag{39}$$

From (38), (39) and (35) (with $h = u_n - u_\lambda^*$), we infer that

$$\lim_{n\to\infty} \langle V(u_n), u_n - u_\lambda^*\rangle = 0,$$

$$\Rightarrow u_n \to u_\lambda^* \text{ in } W_0^{1,\theta}(\Omega) \text{ (see Proposition 3)}, \quad \underline{u}_\lambda \leq u_\lambda^*. \tag{40}$$

In (35) we pass to the limit as $n \to \infty$ and use (40), then

$$\begin{aligned} &\langle V(u_\lambda^*), h\rangle = \int_\Omega \lambda \left[\beta(z)(u_\lambda^*)^{-\eta} + f(z, u_\lambda^*)\right] h dz \quad \text{for all } h \in W_0^{1,\theta}(\Omega), \\ &\Rightarrow u_\lambda^* \in S_\lambda \text{ and } u_\lambda^* = \inf S_\lambda. \end{aligned}$$

The proof is now complete. □

Let $\lambda^* = \sup \mathcal{L}$.

Proposition 14. *If hypotheses H_0 and H_1 hold, then $\lambda^* < \infty$.*

Proof. Hypotheses H_1 imply that there exists $\tilde{\lambda} > 0$ such that

$$\tilde{\lambda}\left[\beta(z)x^{-\eta} + f(z,x)\right] \geq \beta(z)x^{p-1} \quad \text{for a.a. } x \in \Omega, \text{ all } x \geq 0. \tag{41}$$

Let $\lambda > \tilde{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. We can find $\hat{u} \in S_\lambda \subseteq W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), 0 \prec \hat{u}$. Let $\Omega_0 \subseteq \Omega$ open with $\overline{\Omega_0} \subseteq \Omega$. Then

$$0 < m_0 = \operatorname*{ess\,inf}_{\Omega_0} \hat{u}.$$

For $\delta > 0$, let $m_0^\delta = m_0 + \delta$. We have

$$m_0^{-\eta} - (m_0^\delta)^{-\eta} \leq \frac{(m_0+\delta)^\eta - m_0^\eta}{m_0^{2\eta}} \leq \left(\frac{\delta}{m_0^2}\right)^\eta \quad (\text{since } 0 < \eta < 1),$$

$$\Rightarrow \; -(m_0^\delta)^{-\eta} \leq \left(\frac{\delta}{m_0^2}\right)^\eta - m_0^{-\eta}. \tag{42}$$

Let $\rho = \|\hat{u}\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by the hypothesis H_1(iv). We have

$$\begin{aligned}
&-\Delta_p^a m_0^\delta - \Delta_q m_0^\delta + \lambda\hat{\xi}_\rho (m_0^\delta)^{p-1} - \lambda\beta(z)(m_0^\delta)^{-\eta} \\
\leq& \lambda\hat{\xi}_\rho m_0^{p-1} + \chi_\lambda(\delta) - \lambda\beta(z)m_0^{-\eta} \quad \text{with} \;\; \chi_\lambda(\delta) \to 0^+ \;\; \text{as} \;\; \delta \to 0^+ \\
\leq& \left[\lambda\hat{\xi}_\rho + \beta(z)\right] m_0^{p-1} + \chi_\lambda(\delta) - \lambda\beta(z)m_0^{-\eta} \\
\leq& \lambda\hat{\xi}_\rho m_0^{p-1} + \tilde{\lambda}\beta(z)m_0^{-\eta} + f(z,m_0) + \chi_\lambda(\delta) - \lambda\beta(z)m_0^{-\eta} \\
=& \lambda\left[f(z,m_0) + \hat{\xi}_\rho m_0^{p-1}\right] + \chi_\lambda(\delta) - \left(\lambda - \tilde{\lambda}\right)\beta(z)m_0^{-\eta} \;\; \text{in} \;\; \Omega_0.
\end{aligned} \tag{43}$$

By hypothesis H_1(iii), we have

$$0 < l_{m_0}^* \leq f(z,m_0) \;\; \text{for a.a.} \;\; z \in \Omega. \tag{44}$$

Also recall that $0 \prec \beta$ (see hypotheses H_0). Therefore

$$0 < \hat{\mu}_0 \leq \beta(z) \;\; \text{for a.a.} \;\; z \in \Omega_0. \tag{45}$$

We return to (43) and use (44) and (45), then

$$\begin{aligned}
&-\Delta_p^a m_0^\delta - \Delta_q m_0^\delta + \lambda\hat{\xi}_\rho (m_0^\delta)^{p-1} - \lambda\beta(z)(m_0^\delta)^{-\eta} \\
\leq& \lambda\left[f(z,m_0) + \hat{\xi}_\rho m_0^{p-1}\right] - \left(\lambda - \tilde{\lambda}\right)\left(l_{m_0}^* + \hat{\mu}_0 m_0^{-\eta}\right) + \chi_\lambda(\delta) \quad (\text{since } \tilde{\lambda} < \lambda) \\
\leq& \lambda\left[f(z,\hat{u}) + \hat{\xi}_\rho \hat{u}^{p-1}\right] \quad \text{for } \delta \in (0,1) \text{ small and using hypothesis } H_1(iv) \\
=& -\Delta_p^a \hat{u} - \Delta_q \hat{u} + \lambda\hat{\xi}_\rho \hat{u}^{p-1} - \lambda\beta(z)\hat{u}^{-\eta} \;\; \text{in } \Omega_0, \\
\Rightarrow& \;\; m_0^\delta \leq \hat{u} \;\; \text{for a.a. } z \in \Omega_0 \text{ and for } \delta \in (0,1) \text{ small},
\end{aligned}$$

a contradiction to the definition of m_0. We conclude that

$$\lambda \notin \mathcal{L} \;\; \text{and so} \;\; \lambda^* \leq \tilde{\lambda} < \infty.$$

The proof is now complete. ∎

We have

$$(0, \lambda^*) \subseteq \mathcal{L} \subseteq (0, \lambda^*]. \tag{46}$$

We show that for $\lambda \in (0, \lambda^*)$, we have multiplicity of solutions.

Proposition 15. *If hypotheses H_0 and H_1 hold and $0 < \lambda < \lambda^*$, then problem $(\mathcal{P}_\lambda)$ has at least two solutions*

$$u_0, \hat{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega),\ 0 \prec u_0, 0 \prec \hat{u}.$$

Proof. Let $\tau \in (\lambda, \lambda^*)$. Then $\tau \in \mathcal{L}$ (see (46)) and if $u_\tau \in S_\tau$, then

$$\underline{u}_\lambda \le u_\tau \quad \text{(see Propositions 6 and 11).}$$

We can introduce the Carathéodory function $\hat{d}_\lambda(z,x)$ defined by

$$\hat{d}_\lambda(z,x) = \begin{cases} \lambda\left[\beta(z)\underline{u}_\lambda(z)^{-\eta} + f(z,\underline{u}_\lambda(z))\right], & \text{if } x < \underline{u}_\lambda(z), \\ \lambda\left[\beta(z)x^{-\eta} + f(z,x)\right], & \text{if } \underline{u}_\lambda(z) \le x \le u_\tau(z), \\ \lambda\left[\beta(z)u_\tau(z)^{-\eta} + f(z,u_\tau(z))\right], & \text{if } u_\tau(z) < x. \end{cases} \tag{47}$$

We set $\hat{D}_\lambda(z,x) = \int_0^x \hat{d}_\lambda(z,s)ds$ and consider the C^1-functional $\hat{i}_\lambda : W_0^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$\hat{i}_\lambda(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \hat{D}_\lambda(z,u)dz \ \text{ for all } u \in W_0^{1,\theta}(\Omega).$$

From (47) we see that $\hat{i}_\lambda(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W_0^{1,\theta}(\Omega)$ such that

$$\hat{i}_\lambda(u_0) = \inf\left\{\hat{i}_\lambda(u) :\ u \in W_0^{1,\theta}(\Omega)\right\},$$

$$\Rightarrow\ \langle \hat{i}'_\lambda(u_0), h\rangle = 0 \ \text{ for all } h \in W_0^{1,\theta}(\Omega),$$

$$\Rightarrow\ \langle V(u_0), h\rangle = \int_\Omega \hat{d}_\lambda(z,u_0)hdz \ \text{ for all } h \in W_0^{1,\theta}(\Omega). \tag{48}$$

As before (see the proof of Proposition 12), choosing the test functions $h = (\underline{u}_\lambda - u_0)^+ \in W_0^{1,\theta}(\Omega)$ and $h = (u_0 - u_\tau)^+ \in W_0^{1,\theta}(\Omega)$, we obtain

$$\underline{u}_\lambda \le u_0 \le u_\tau,$$

$$\Rightarrow\ u_0 \in S_\lambda \subseteq W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), 0 \prec u_0 \quad \text{(see (47)).}$$

We introduce the Carathéodory function $d_\lambda(z,x)$ defined by

$$d_\lambda(z,x) = \begin{cases} \lambda\left[\beta(z)\underline{u}_\lambda(z)^{-\eta} + f(z,\underline{u}_\lambda(z))\right], & \text{if } x < \underline{u}_\lambda(z), \\ \lambda\left[\beta(z)x^{-\eta} + f(z,x)\right], & \text{if } \underline{u}_\lambda(z) \le x. \end{cases} \tag{49}$$

We set $D_\lambda(z,x) = \int_0^x d_\lambda(z,s)ds$ and consider the C^1-functional $i_\lambda : W_0^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$i_\lambda(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega D_\lambda(z,u)dz \ \text{ for all } u \in W_0^{1,\theta}(\Omega).$$

If

$$[\underline{u}_\lambda, u_\tau] = \left\{h \in W_0^{1,\theta}(\Omega) : \underline{u}_\lambda(z) \le h(z) \le u_\tau(z) \ \text{ for a.a } z \in \Omega\right\}$$

and

$$[\underline{u}_\lambda) = \left\{h \in W_0^{1,\theta}(\Omega) : \underline{u}_\lambda(z) \le h(z) \ \text{ for a.a } z \in \Omega\right\},$$

then using (47) and (49), we show that

$$K_{\hat{i}_\lambda} \subseteq [\underline{u}_\lambda, u_\tau], \quad K_{i_\lambda} \subseteq [\underline{u}_\lambda) \cap L^\infty(\Omega) \quad \text{(see Proposition 5).} \tag{50}$$

From (50) we see that we may assume that

$$K_{i_\lambda} \ \text{ is finite} \tag{51}$$

or otherwise we already have an infinity of solutions for $(\mathcal{P}_\lambda)$ which are bounded and so we are done.

We compare the critical groups of $\hat{i}_\lambda(\cdot)$ and $i_\lambda(\cdot)$ at u_0.

Claim 1: $C_k(\hat{i}_\lambda, u_0) = C_k(i_\lambda, u_0)$ for all $k \in \mathbb{N}_0$.

Let $Y = W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega)$ and let $u \in Y$. We have

$$\begin{aligned} \left|i_\lambda(u) - \hat{i}_\lambda(u)\right| &\leq \int_\Omega \left|D_\lambda(z,u) - \hat{D}_\lambda(z,u)\right| dz \\ &\leq \int_\Omega |D_\lambda(z,u) - D_\lambda(z,u_0)|\, dz + \int_\Omega \left|\hat{D}_\lambda(z,u_0) - \hat{D}_\lambda(z,u)\right| dz \\ &\qquad \text{(since } \underline{u}_\lambda \leq u_0 \leq u_\tau, \text{ see (48) and (50)).} \end{aligned} \tag{52}$$

We estimate the two integrals in the right hand side of (52). We have

$$\begin{aligned} &\int_\Omega |D_\lambda(z,u) - D_\lambda(z,u_0)|\, dz \\ \leq &\int_{\{u<\underline{u}_\lambda\}} \Big|\lambda\beta(z)\underline{u}_\lambda^{-\eta} u + \lambda f(z,\underline{u}_\lambda)u - \lambda\beta(z)\underline{u}_\lambda^{1-\eta} - \frac{\lambda\beta(z)}{1-\eta}\left(u_0^{1-\eta} - \underline{u}_\lambda^{1-\eta}\right) \\ &- \lambda f(z,\underline{u}_\lambda)\underline{u}_\lambda - (F(z,u_0) - F(z,\underline{u}_\lambda))\Big| dz \\ &+ \int_{\{\underline{u}_\lambda \leq u\}} \Big|\lambda\beta(z)\underline{u}_\lambda^{1-\eta} + \frac{\lambda\beta(z)}{1-\eta}\left(u^{1-\eta} - \underline{u}_\lambda^{1-\eta}\right) + f(z,\underline{u}_\lambda)\underline{u}_\lambda + (F(z,u) - F(z,\underline{u}_\lambda)) \\ &- \lambda\beta(z)\underline{u}_\lambda^{1-\eta} - \frac{\lambda\beta(z)}{1-\eta}\left(u_0^{1-\eta} - \underline{u}_\lambda^{1-\eta}\right) - f(z,\underline{u}_\lambda)\underline{u}_\lambda - (F(z,u_0) - F(z,\underline{u}_\lambda))\Big| dz \\ &\qquad\qquad \text{(see (49) and recall } \underline{u}_\lambda \leq u_0 \leq u_\tau) \\ \leq &\int_{\{u<\underline{u}_\lambda\}} \frac{\lambda\beta(z)}{1-\eta}(u_0 - \underline{u}_\lambda)^{1-\eta} dz + \int_{\{\underline{u}_\lambda \leq u\}} (F(z,u_0) - F(z,\underline{u}_\lambda))\, dz \\ &\qquad\qquad \text{(see Rudin [32, p.78])} \\ \leq &c_7\left(\|u-u_0\|^{1-\eta} + \|u-u_0\|\right) \quad \text{for some } c_7 > 0 \end{aligned}$$

(see Hewitt-Stromberg [14, p.196] and note that $F(z,\cdot)$ is L^∞-locally Lipschitz). So, if $\|u-u_0\| \leq 1$, then

$$\int_\Omega |D_\lambda(z,u) - D_\lambda(z,u_0)|\, dz \leq 2c_7\|u-u_0\|^{1-\eta}. \tag{53}$$

Similarly, using (47), we show that

$$\int_\Omega \left|\hat{D}_\lambda(z,u) - \hat{D}_\lambda(z,u_0)\right| dz \leq c_8\|u-u_0\|^{1-\eta}. \tag{54}$$

Returning to (52) and using(53) and (54), we obtain

$$\left|i_\lambda(u) - \hat{i}_\lambda(u_0)\right| \leq c_9\|u-u_0\|^{1-\eta} \text{ for some } c_9 > 0 \quad (\text{for } \|u-u_0\| \leq 1). \tag{55}$$

Next let $u, h \in Y$. We have

$$\begin{aligned} \left|\langle i_\lambda'(u) - \hat{i}_\lambda'(u), h\rangle\right| &\leq \int_\Omega \left|d_\lambda(z,u) - \hat{d}_\lambda(z,u)\right| |h| dz \\ &\leq \int_\Omega |d_\lambda(z,u) - d_\lambda(z,u_0)|\, |h| dz + \int_\Omega \left|\hat{d}_\lambda(z,u_0) - \hat{d}_\lambda(z,u)\right| |h| dz. \end{aligned} \tag{56}$$

We estimate the two integrals in the right hand side of (55). We have

$$\begin{aligned}&\int_\Omega |d_\lambda(z,u)-d_\lambda(z,u_0)|\,|h|dz\\ &\leq \int_{\{u<\underline{u}_\lambda\}}\left|\lambda\beta(z)\left(\underline{u}_\lambda^{-\eta}-u_0^{-\eta}\right)+\lambda\left(f(z,\underline{u}_\lambda)-f(z,u_0)\right)\right||h|dz\\ &\quad+\int_{\{\underline{u}_\lambda\leq u\}}\left|\lambda\beta(z)\left(u^{-\eta}-u_0^{-\eta}\right)+\lambda\left(f(z,u)-f(z,u_0)\right)\right||h|dz \quad \text{(see (49))}.\end{aligned}$$

On the set $\{u<\underline{u}_\lambda\}$, we have

$$0\leq\left(\underline{u}_\lambda^{-\eta}-u_0^{-\eta}\right)|h|\leq\frac{u_0^\eta-\underline{u}_\lambda^\eta}{\underline{u}_\lambda^{2\eta}}|h|\leq c_{10}\frac{u_0-\underline{u}_\lambda}{\hat{d}^{2\eta}}|h| \ \text{ for some } c_{10}>0.$$

In the last inequality we have used that $x\to x^\eta$ is continuous concave $(0<\eta<1)$, thus locally Lipschitz and Proposition 7. We have

$$\begin{aligned}&\frac{u_0-\underline{u}_\lambda}{\hat{d}^{2\eta}}\leq\frac{u_0-u}{\hat{d}^{2\eta}} \quad \text{on } \ \{u<\underline{u}_\lambda\},\\ \Rightarrow\ &\frac{u_0-u}{\hat{d}^{2\eta}}|h|\leq\frac{|u_0-u|}{\hat{d}^\eta}\frac{|h|}{\hat{d}} \quad \text{on } \ \{u<\underline{u}_\lambda\}.\end{aligned}$$

Also on $\{\underline{u}_\lambda\leq u\}$, we have

$$\left|u^{-\eta}-u_0^{-\eta}\right||h|\leq\frac{|u_0^\eta-u^\eta|}{\underline{u}_\lambda^{2\eta}}|h|\leq c_{11}\frac{|u_0-u|^\eta}{\hat{d}^\eta}\frac{|h|}{\hat{d}^\eta} \ \text{ for some } c_{11}>0.$$

Via Hardy's inequality (see Proposition 4), we have

$$\left(\frac{u_0-u}{\hat{d}}\right)^\eta\in L^{\frac{q}{\eta}}(\Omega), \quad \frac{|h|}{\hat{d}^\eta}\in L^{\frac{q}{q-\eta}}(\Omega) \quad (\text{since } 1+\eta<2\leq q).$$

Then from (56), Hölder's inequality and the fact that $f(z,\cdot)$ is L^∞-locally Lipschitz, we obtain

$$\int_\Omega|d_\lambda(z,u)-d_\lambda(z,u_0)|\,|h|dz\leq c_{12}\|u-u_0\|\|h\| \ \text{ for some } c_{12}>0.$$

Similarly, using (47), we show that

$$\int_\Omega\left|\hat{d}_\lambda(z,u)-\hat{d}_\lambda(z,u_0)\right||h|dz\leq c_{13}\|u-u_0\|\|h\| \ \text{ for some } c_{13}>0.$$

Then from (56) and since $h\in W_0^{1,\theta}(\Omega)$ is arbitrary, it follows that

$$\|i'_\lambda(u)-\hat{i}'_\lambda(u)\|\leq c_{14}\|u-u_0\| \ \text{ for some } c_{14}>0. \tag{57}$$

From (55), (57) and the C^1-continuity property of critical groups (see Hu-Papageorgiou [16, p.179]), we have

$$C_k(i_\lambda|_Y,u_0)=C_k(\hat{i}_\lambda|_Y,u_0) \ \text{ for all } k\in\mathbb{N}_0.$$

Since $Y\hookrightarrow W_0^{1,\theta}(\Omega)\hookrightarrow H_0^1(\Omega)$ (recall that $2\leq q$), it follows that

$$C_k(i_\lambda,u_0)=C_k(\hat{i}_\lambda,u_0) \ \text{ for all } k\in\mathbb{N}_0 \quad \text{(see Palais [27])}.$$

This proves Claim 1.

From (48) we have that

$$\begin{aligned}&C_k(\hat{i}_\lambda,u_0)=\delta_{k,0}\mathbb{R} \ \text{ for all } k\in\mathbb{N}_0,\\ \Rightarrow\ &C_k(i_\lambda,u_0)=\delta_{k,0}\mathbb{R} \ \text{ for all } k\in\mathbb{N}_0 \quad \text{(see Claim 1)}.\end{aligned}$$

Then Theorem 4.6 of Chang [6, p.43] implies that

$$u_0 \text{ is a local minimizer of } i_\lambda(\cdot). \tag{58}$$

Claim 2: The functional $i_\lambda(\cdot)$ satisfies the C-condition.

We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega)$ such that

$$|i_\lambda(u_n)| \leq c_{15} \text{ for some } c_{15} > 0, \text{ all } n \in \mathbb{N}, \tag{59}$$

$$(1 + \|u_n\|)i'_\lambda(u_n) \to 0 \text{ in } W_0^{1,\theta}(\Omega)^*, \text{ all } n \in \mathbb{N}, \text{ as } n \to \infty. \tag{60}$$

From (60) we have

$$\left|\langle V(u_n), h\rangle - \int_\Omega d_\lambda(z, u_n)hdz\right| \leq \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \tag{61}$$
$$\text{for all } h \in W_0^{1,\theta}(\Omega), \text{ all } n \in \mathbb{N}, \text{ with } \varepsilon_n \to 0^+.$$

In (61) we choose the test function $h = -u_n^- \in W_0^{1,\theta}(\Omega)$. Then

$$\rho_\theta(Du_n^-) \leq \varepsilon_n \text{ for all } n \in \mathbb{N},$$
$$\Rightarrow \quad u_n^- \to 0 \text{ in } W_0^{1,\theta}(\Omega) \text{ (see Proposition 2).} \tag{62}$$

We will show that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega)$ is bounded. Arguing indirectly, we assume that at least for a subsequence, we have

$$\|u_n^+\| \to \infty. \tag{63}$$

We set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,\theta}(\Omega), \quad y_n \to y \text{ in } L^r(\Omega), \ y \geq 0. \tag{64}$$

First assume that $y \neq 0$ and let $\Omega_+ = \{z \in \Omega : y(z) > 0\}$. Then $|\Omega_+|_N > 0$ (see (64)) and we have

$$u_n^+(z) \to +\infty \text{ for a.a. } z \in \Omega_+ \text{ (see (63), (64)).}$$

On account of Hypothesis H_1(ii), we have

$$\lim_{n\to\infty} \frac{F(z, u_n^+(z))}{u_n^+(z)^p} = +\infty \text{ for a.a. } z \in \Omega_+,$$
$$\Rightarrow \lim_{n\to\infty} \int_{\Omega_+} \frac{F(z, u_n^+)}{\|u_n^+\|^p}dz = \lim_{n\to\infty} \int_{\Omega_+} \frac{F(z, u_n^+)}{(u_n^+)^p}y_n^p dz = +\infty \text{ (by Fatou's Lemma).} \tag{65}$$

We have

$$\int_\Omega \frac{F(z, u_n^+)}{\|u_n^+\|^p}dz = \int_{\Omega_+} \frac{F(z, u_n^+)}{\|u_n^+\|^p}dz + \int_{\Omega\setminus\Omega_+} \frac{F(z, u_n^+)}{\|u_n^+\|^p}dz,$$
$$\geq \int_{\Omega_+} \frac{F(z, u_n^+)}{\|u_n^+\|^p}dz \quad \text{(since } F \geq 0\text{)},$$
$$\Rightarrow \int_\Omega \frac{F(z, u_n^+)}{\|u_n^+\|^p}dz \to +\infty \text{ as } n \to \infty \quad \text{(see (65)).} \tag{66}$$

From (59) and (62), we obtain

$$\int_\Omega \frac{F(z, u_n^+)}{\|u_n^+\|^p}dz \leq c_{16} + \frac{1}{p}\rho_a(Dy_n) \leq c_{16} + \frac{1}{p} \tag{67}$$
$$\text{for some } c_{16} > 0, \text{ all } n \in \mathbb{N} \text{ (see Proposition 2).}$$

Comparing (66) and (67), we have a contradiction.

Next assume that $y = 0$. From (61) and (62), we have

$$\left|\int_\Omega \frac{d_\lambda(z,u_n^+)}{\|u_n^+\|^{p-1}} h dz\right| \le \varepsilon_n' \|h\| + \frac{1}{\|u_n^+\|^{p-1}} |\langle V(u_n), h\rangle|$$
$$\text{for all } h \in W_0^{1,\theta}(\Omega), \text{ all } n \in \mathbb{N}, \text{ with } \varepsilon_n' \to 0^+,$$
$$\Rightarrow \left|\int_\Omega \frac{d_\lambda(z,u_n^+)}{\|u_n^+\|^{p-1}} h dz\right| \le c_{17}\|h\| \text{ for some } c_{17} > 0, \text{ all } n \in \mathbb{N}. \tag{68}$$

Let $\hat{w}_\lambda : W_0^{1,\theta}(\Omega) \to \mathbb{R}$ be the linear functional defined by

$$\hat{w}_\lambda(h) = \int_\Omega \frac{d_\lambda(z,u_n^+)}{\|u_n^+\|^{p-1}} h dz, \quad \text{for all } h \in W_0^{1,\theta}(\Omega).$$

From (68), we see that

$$|\hat{w}_\lambda(h)| \le c_{17}\|h\| \text{ for all } h \in W_0^{1,\theta}(\Omega),$$
$$\Rightarrow \hat{w}_\lambda \in W_0^{1,\theta}(\Omega)^*, \quad \|\hat{w}_\lambda\|_* \le c_{17}. \tag{69}$$

From Proposition 1 we know that $W_0^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$. So, by the Hahn-Banach theorem, there exists $\hat{w}_\lambda^* \in L^r(\Omega)^*$ such that

$$\hat{w}_\lambda^*\big|_{W_0^{1,\theta}(\Omega)} = \hat{w}_\lambda \text{ and } \|\hat{w}_\lambda^*\|_{L^r(\Omega)^*} = \|\hat{w}_\lambda\|_* \le c_{17} \text{ (see (69))}. \tag{70}$$

The Riesz Representation theorem for the Lebesgue spaces says that $L^r(\Omega)^* = L^{r'}(\Omega)$ with $r' = \frac{r}{r-1}$ and there exists unique $g_\lambda \in L^{r'}(\Omega)$ such that

$$\hat{w}_\lambda^*(h) = \int_\Omega g_\lambda h dz \text{ for all } h \in L^r(\Omega), \quad \|\hat{w}_\lambda^*\|_{L^r(\Omega)^*} = \|g_\lambda\|_{r'} \le c_{17} \text{ (see (70))}. \tag{71}$$

From (70) and (71), we have

$$\int_\Omega \frac{d_\lambda(z,u_n^+)}{\|u_n^+\|^{p-1}} h dz = \int_\Omega g_\lambda h dz \text{ for all } h \in W_0^{1,\theta}(\Omega). \tag{72}$$

We choose the test function $h = y_n \in W_0^{1,\theta}(\Omega)$. Since $y = 0$ from (64) we have

$$\int_\Omega g_\lambda y_n dz \to 0,$$
$$\Rightarrow \int_\Omega \frac{d_\lambda(z,u_n^+)}{\|u_n^+\|^{p-1}} y_n dz \to 0 \text{ as } n \to \infty \quad \text{(see (72))}. \tag{73}$$

We return to (61), multiply with $\frac{1}{\|u_n^+\|^{p-1}}$ and choose the test function $h = y_n \in W_0^{1,\theta}(\Omega)$. Then

$$\rho_a(Dy_n) \to 0 \text{ as } n \to \infty \quad \text{(see (73), (63) and recall that } q < p). \tag{74}$$

We know that

$$\|y_n\| = 1 \text{ for all } n \in \mathbb{N},$$
$$\Rightarrow \rho_\theta(Dy_n) = 1 \text{ for all } n \in \mathbb{N} \text{ (see Proposition 2)},$$
$$\Rightarrow 1 = \rho_a(Dy_n) + \|Dy_n\|_q^q \text{ for all } n \in \mathbb{N},$$
$$\Rightarrow \|Dy_n\|_q \to 1 \text{ (see (74))},$$
$$\Rightarrow \|Du_n^+\|_q \to +\infty \text{ as } n \to \infty \quad \left(\text{recall } y_n = \frac{u_n^+}{\|u_n^+\|} \text{ and see (63)}\right). \tag{75}$$

From (59) and (62), we have

$$\rho_a(Du_n^+) + \frac{p}{q}\|Du_n^+\|_q^q - \int_\Omega pD_\lambda(z,u_n^+)dz \le c_{18} \ \text{ for some } \ c_{18}>0, \ \text{all } n\in\mathbb{N}. \tag{76}$$

Also from (61) with $h = u_n^+ \in W_0^{1,\theta}(\Omega)$, we have

$$-\rho_a(Du_n^+) - \|Du_n^+\|_q^q + \int_\Omega d_\lambda(z,u_n^+)u_n^+ dz \le \varepsilon_n \ \text{ for all } n\in\mathbb{N}. \tag{77}$$

We add (76) and (77), then

$$\begin{aligned}
&\left(\frac{p}{q}-1\right)\|Du_n^+\|_q^q + \int_\Omega \left[d_\lambda(z,u_n^+)u_n^+ - pD_\lambda(z,u_n^+)\right]dz \le c_{19}\\
&\qquad\qquad \text{for some } c_{19}>0, \ \ \text{all } n\in\mathbb{N},\\
\Rightarrow &\left(\frac{p}{q}-1\right)\|Du_n^+\|_q^q + \int_\Omega \lambda\left[f(z,u_n^+)u_n^+ - pF(z,u_n^+)\right]dz\\
&\qquad \le c_{20}\left(1+\|Du_n^+\|_q^{1-\eta}\right) \quad \text{for some } c_{20}>0, \ \text{all } n\in\mathbb{N}\\
&\qquad\qquad \text{(see Hewitt-Stromberg [14, p.196])}\\
&\qquad \le c_{21}\left(1+\|Du_n^+\|_q^{\tau}\right) \quad \text{for some } c_{21}>0, \ \text{all } n\in\mathbb{N},
\end{aligned}$$

$$\Rightarrow \ \{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,q}(\Omega) \ \text{ is bounded } \ (\text{since } \tau<q). \tag{78}$$

We compare (78) and (75) and have a contradiction. It follows that

$$\begin{aligned}
&\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega) \ \text{ is bounded},\\
\Rightarrow \ &\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega) \ \text{ is bounded} \quad \text{(see (62))}.
\end{aligned}$$

We may assume that

$$u_n \xrightarrow{w} u \ \text{ in } W_0^{1,\theta}(\Omega), \ \ u_n \to u \ \text{ in } L^r(\Omega) \text{ as } n\to\infty. \tag{79}$$

We have

$$\begin{aligned}
\int_\Omega d_\lambda(z,u_n)(u_n-u)dz &= \int_{\{u_n<\underline{u}_\lambda\}} \lambda\beta(z)\underline{u}_\lambda^{-\eta}(u_n-u)dz + \int_{\{\underline{u}_\lambda\le u_n\}} \lambda\beta(z)\underline{u}_\lambda^{-\eta}(u_n-u)dz\\
&\quad + \int_{\{u_n<\underline{u}_\lambda\}} \lambda f(z,\underline{u}_\lambda)(u_n-u)dz + \int_{\{\underline{u}_\lambda\le u_n\}} \lambda f(z,u_n)(u_n-u)dz\\
&\le \int_\Omega \lambda\beta(z)\underline{u}_\lambda^{-\eta}(u_n-u)dz + c_{22}\|u_n-u\|_r + \int_\Omega \lambda f(z,u_n)|u_n-u|dz\\
&\qquad \text{for some } c_{22}>0, \ \text{all } n\in\mathbb{N}\\
&\le c_{23}\|u_n-u\|_r + \int_\Omega \lambda f(z,u_n)|u_n-u|dz\\
&\qquad \text{for some } c_{23}>0, \ \text{all } n\in\mathbb{N} \quad \text{(see Proposition 8)}
\end{aligned}$$

$$\Rightarrow \ \limsup_{n\to\infty}\int_\Omega d_\lambda(z,u_n)(u_n-u)dz \le 0 \quad \text{(see (79))}. \tag{80}$$

If in (61), we choose the test function $h = u_n - u \in W_0^{1,\theta}(\Omega)$, pass to the limit as $n\to\infty$ and use (58), then

$$\begin{aligned}
&\limsup_{n\to\infty}\langle V(u_n), u_n-u\rangle \le 0,\\
\Rightarrow \ &u_n \to u \ \text{ in } \ W_0^{1,\theta}(\Omega).
\end{aligned}$$

Therefore $i_\lambda(\cdot)$ satisfies the C-condition and this proves Claim 2.

On account of (51), (58), Claim 2 and Proposition 3.132 of Hu-Papageorgiou [16, p.179], we can find $\rho \in (0,1)$ small such that

$$i_\lambda(u_0) < \inf\{i_\lambda(u) : \|u - u_0\| = \rho\} = m_\lambda. \tag{81}$$

Let $u \in C_0^1(\overline{\Omega})$ with $u(z) > 0$ for all $z \in \Omega$. Then on account of hypothesis H_1(ii), we have

$$i_\lambda(tu) \to -\infty \ \text{ as } \ t \to +\infty. \tag{82}$$

Then Claim 2, (81), (82) and the mountain pass theorem, imply that there exists $\hat{u} \in W_0^{1,\theta}(\Omega)$ such that

$$\hat{u} \in K_{i_\lambda} \subseteq [\underline{u}_\lambda) \cap L^\infty(\Omega) \quad \text{(see (50))}, \qquad i_\lambda(u_0) < m_\lambda \le i_\lambda(\hat{u}),$$
$$\Rightarrow \ \hat{u} \in S_\lambda \ \text{ (see (49))}, \ \ \hat{u} \ne u_0 \ \text{ and } \ 0 \prec \hat{u}.$$

This completes the proof. □

We have to decide about the critical parameter λ^* (whether it is an eigenvalue).

Proposition 16. *If hypotheses H_0 and H_1 hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Let $\lambda_n \uparrow \lambda^*$ and let $u_n^* = u_{\lambda_n}^*$, $n \in \mathbb{N}$. Then $\{u_n^*\}_{n\in\mathbb{N}}$ is increasing and $\underline{u}_{\lambda_1} \le u_n^*$ for all $n \in \mathbb{N}$ (see Propositions 6 and 11). We have

$$\langle V(u_n), h\rangle = \int_\Omega \lambda_n \left[\beta(z) u_n^{-\eta} + f(z,u_n)\right] h dz \ \text{ for all } h \in W_0^{1,\theta}(\Omega), \ \text{ all } n \in \mathbb{N}. \tag{83}$$

We claim that $\{u_n^*\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega)$ is bounded. If this not the case, we may assume that $\|u_n^*\| \to \infty$. We set $y_n^* = \frac{u_n^*}{\|u_n^*\|}, n \in \mathbb{N}$. Then $\|y_n^*\| = 1, y_n^* \ge 0$ for all $n \in \mathbb{N}$. We may assume that

$$y_n^* \xrightarrow{w} y^* \ \text{ in } W_0^{1,\theta}(\Omega), \ \ y_n^* \to y^* \ \text{ in } L^r(\Omega). \tag{84}$$

From (83) with $h = y_n^* \in W_0^{1,\theta}(\Omega)$, we have

$$\rho_a(Dy_n^*) + \frac{1}{\|u_n^*\|^{p-q}}\|Dy_n^*\|_q^q = \int_\Omega \lambda_n \left[\beta(z)\frac{(u_n^*)^{1-\eta}}{\|u_n^*\|^p} + \frac{f(z,u_n^*)}{\|u_n^*\|^{p-1}} y_n^*\right] dz \quad \text{for all } n \in \mathbb{N}.$$

Since $\rho_a(Dy_n^*) \le 1$ for all $n \in \mathbb{N}$ and $\|u_n^*\| \to \infty$, then on account of hypothesis H_1(ii), we must have $y^* = 0$ (see (84)). Then reasoning as in Claim 2 in the proof of Proposition 15 (case $y = 0$), we obtain

$$u_n \to u^* \ \text{ in } \ W_0^{1,\theta}(\Omega), \ \ \underline{u}_{\lambda_1} \le u^*.$$

From (83), it follows that

$$\langle V(u^*), h\rangle = \int_\Omega \lambda^* \left[\beta(z)(u^*)^{-\eta} + f(z,u^*)\right] h dz \ \text{ for all } h \in W_0^{1,\theta}(\Omega),$$
$$\Rightarrow \ u^* \in S_{\lambda^*} \ \text{ and } \ \lambda^* \in \mathcal{L},$$

which completes the proof. □

We can say that $\mathcal{L} = (0, \lambda^*]$.

Next we examine the minimal solution map $\lambda \to u_\lambda^*$ (see Proposition 13).

Proposition 17. *If hypotheses H_0 and H_1 hold, then the minimal solution map $\lambda \to u_\lambda^*$ from $\mathcal{L} = (0, \lambda^*]$ into $W_0^{1,\theta}(\Omega)$ is nondecreasing and left continuous.*

Proof. Let $0 < \mu < \lambda \le \lambda^*$. Corollary 1 says that we can find $u_\mu \in S_\mu$ such that

$$u_\mu \le u_\lambda^*,$$
$$\Rightarrow \ u_\mu^* \le u_\lambda^*.$$

Therefore the minimal solution map is nondecreasing.

Let $\lambda_n \in \mathcal{L},\ n \in \mathbb{N}$ and assume that $\lambda_n \to \lambda^-$. We set $u_n^* = u_{\lambda_n}^*,\ n \in \mathbb{N}$ (see Proposition 13) and have

$$\langle V(u_n^*), h\rangle = \int_\Omega \lambda_n \left[\beta(z)(u_n^*)^{-\eta} + f(z, u_n^*)\right] hdz \ \text{ for all } h \in W_0^{1,\theta}(\Omega), \ \text{ all } n \in \mathbb{N}, \tag{85}$$

$$\underline{u}_{\lambda_1} \le u_n^* \le u_{\lambda^*}^* \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega). \tag{86}$$

If in (85) we choose the test function $h = u_n^* \in W_0^{1,\theta}(\Omega)$ and use (86) and hypothesis H_1(i), we see that

$$\{u_n^*\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega) \ \text{ is bounded.}$$

So, we may assume that

$$u_n^* \xrightarrow{w} \hat{u}^* \ \text{ in } W_0^{1,\theta}(\Omega), \ \ u_n^* \to \hat{u}^* \ \text{ in } L^r(\Omega). \tag{87}$$

Choosing the test function $h = u_n^* - \hat{u}^* \in W_0^{1,\theta}(\Omega)$ in (85), passing to the limit as $n \to \infty$ and using (87), we obtain

$$\limsup_{n\to\infty} \langle V(u_n^*), u_n^* - \hat{u}^*\rangle = 0,$$

$$\Rightarrow \ u_n^* \to \hat{u}^* \ \text{ in } \ W_0^{1,\theta}(\Omega), \ \ \hat{u}^* \in S_\lambda. \tag{88}$$

We claim that $\hat{u}^* = u_\lambda^*$. Let $u \in S_\lambda$. Then u is an upper solution for $(\mathcal{P}_{\lambda_n}), n \in \mathbb{N}$. Therefore truncating from below at $\underline{u}_{\lambda_1}(z)$ and from above at $u(z)$ and using the Weierstrass-Tonelli theorem, we obtain $u_n \in S_{\lambda_n}$ such that $\underline{u}_{\lambda_1} \le u_n \le u$. Evidently $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega)$ is bounded and as above via the $(S)_+$-property of $V(\cdot)$, we have

$$u_n \to \hat{u} \ \text{ in } \ W_0^{1,\theta}(\Omega),$$

$$\Rightarrow \ \underline{u}_{\lambda_1} \le \hat{u} \le u. \tag{89}$$

Note that

$$\underline{u}_{\lambda_1} \le u_n^* \le u_n \le u,$$

$$\Rightarrow \ \underline{u}_{\lambda_1} \le \hat{u}^* \le \hat{u} \le u \quad \text{(see (88), (89))}.$$

Since $u \in S_\lambda$ is arbitrary, we conclude that

$$\hat{u}^* = u_\lambda^*,$$

$$\Rightarrow \ \lambda \to u_\lambda^* \text{ is left continuous.}$$

The proof is now complete. ☐

Summarizing, we have the following existence and multiplicity theorem for problem $(\mathcal{P}_\lambda)$ which is global in $\lambda > 0$ (bifurcation-type theorem).

Theorem 1. *If hypotheses H_0, H_1 hold, then there exists $\lambda^* > 0$ such that*

(a) *for all $\lambda \in (0, \lambda^*)$ problem $(\mathcal{P}_\lambda)$ has at least two solutions*

$$u_0, \ \hat{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), \ 0 \prec u_0, \ 0 \prec \hat{u};$$

(b) *for $\lambda = \lambda^*$ problem $(\mathcal{P}_\lambda)$ has at least one solution*

$$u_* \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), \ 0 \prec u_*;$$

(c) *for all $\lambda > \lambda^*$ problem $(\mathcal{P}_\lambda)$ has no solution;*

(d) *for all $\lambda \in \mathcal{L} = (0, \lambda^*]$ problem $(\mathcal{P}_\lambda)$ has a smallest solution*

$$u_\lambda^* \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), \ 0 \prec u_\lambda^*$$

and the minimal solution map $\lambda \to u_\lambda^*$ from $\mathcal{L} = (0, \lambda^*]$ into $W_0^{1,\theta}(\Omega)$ is nondecreasing and left continuous.

Declarations

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