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Multi-bump solutions for critical Schrödinger equations with electromagnetic fields and logarithmic nonlinearity

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In this paper, we are interested in the existence and multiplicity of multi-bump solutions for critical Schrödinger equations with electromagnetic fields and logarithmic nonlinearity of the following type:

$$-(\nabla + \mathrm{i}A(x))^2 u + (\lambda Z(x) + \mathcal{V}(x))u = \vartheta u \log |u|^2 + |u|^{2^*-2}u, \ \ u \in H^1(\mathbb{R}^N, \mathbb{C}),$$

where $N \geq 3$, the magnetic potential $A \in L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N), \vartheta \in (1, +\infty)$, the parameter $\lambda \geq 1$ and $Z(x), \mathcal{V}(x) : \mathbb{R}^N \to \mathbb{R}$ are the non-negative continuous functions. Applying variational methods, we obtain that the above equations have at least $2^k - 1$ multi-bump solutions as $\lambda \geq 1$ is sufficiently large. To some extent, we extend and complement the results of [C. O. Alves and C. Ji, Multi-bump positive solutions for a logarithmic Schrödinger equation with deepening potential well, *Sci. China Math.* **65** (2022) 1577–1598; J. Wang and Z. Yin, Multi-bump solutions for the nonlinear magnetic Schrödinger equation with logarithmic nonlinearity, *Math. Nachr.* **298** (2025) 328–355] from subcritical case to critical case.

Keywords: Logarithmic Schödinger equations; multi-bump solutions; variational methods; deepening potential well.

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1. Introduction and Main Results

In this paper, we consider the following critical Schrödinger equations with electromagnetic fields and logarithmic nonlinearity in $\mathbb{R}^N$:

$$-(\nabla + \mathrm{i}A(x))^2 u + (\lambda Z(x) + \mathcal{V}(x))u = \vartheta u \log |u|^2 + |u|^{2^*-2}u, \quad u \in H^1(\mathbb{R}^N, \mathbb{C}), \tag{1.1}$$

where $2^* = \frac{2N}{N-2}$ is the critical exponent, i is the imaginary unit, $\lambda \geq 1$ is a parameter, $\mathcal{V}(x) \geq 0$, $Z : \mathbb{R}^N \to \mathbb{R}$ is the non-negative continuous function with a potential well $\Omega := \mathrm{int} Z^{-1}(0)$ which has k disjoint bounded components $\Omega = \bigcup_{j=1}^k \Omega_j$.

Recently, logarithmic Schrödinger equations attracted much attention. This class of equations plays a more essential role in physical applications, such as quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum system, effective quantum gravity. For more background on this topic, please see [7, 8, 15, 28, 45, 46]. Such equations originated from the following form:

$$\begin{cases} \partial_t v(t,x) = \mathrm{i}\Delta v(t,x) + \mathrm{i}\lambda v(t,x) \log(|v(t,x)|^2) \\ \qquad\qquad + \mathrm{i}W(t,x,|v|^2)v(t,x), \quad x \in \mathbb{R}^N, \ t > 0, \\ v(0,x) = v_0(x), \quad x \in \mathbb{R}^N, \end{cases}$$

where Δ is the Laplacian operator on $\mathbb{R}^N$, t is time, x is spatial coordinate, $\lambda \in \mathbb{R}\backslash\{0\}$ denotes the force of nonlinear interaction, and W is a real-valued function. In [8], Mycielski and Bialynicki–Birula made the first contribution to the study of logarithmic Schrödinger equations. They obtained the separability of noninteracting systems, i.e. for noninteracting subsystems, the nonlinearity does not introduce correlation. After that, there are many scholars focusing on the research of logarithmic Schrödinger equations.

For $A(x) \equiv 0$, Eq. (1.1) can be reduced to the following equation:

$$-\Delta u + (\lambda Z(x) + \mathcal{V}(x))u = u \log |u|^2, \quad u \in H^1(\mathbb{R}^N). \tag{1.2}$$

In [2], Alves and Ji used penalization method [32] to obtain an auxiliary equation corresponding to Eq. (1.2) with $\mathcal{V}(x) = 0$, together with some useful estimates, they successfully verified that the solutions of auxiliary equation are in fact solutions of Eq. (1.2) when the parameter λ is sufficiently large. Finally, they applied variational methods to obtain the existence and multiplicity of multi-bump positive solutions for Eq. (1.2). Alves and Ji [4] also considered the following logarithmic Schrödinger equations:

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = \lambda u + u \log u^2 \quad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = a^2 \varepsilon^N, \end{cases} \tag{1.3}$$

where a, $\varepsilon > 0$, $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier and the potential $V(x) : \mathbb{R}^N \to [-1, +\infty)$ is a continuous function. With the aid of minimization techniques and Lusternik–Schnirelmann category, they obtained the existence of multiple normalized solutions for Eq. (1.3). In [6], Alves and Ambrosio were interested in the logarithmic Schrödinger equations involving fractional p-Laplacian:

$$\begin{cases} \varepsilon^{sp}(-\Delta)_p^s u + V(x)|u|^{p-2}u = |u|^{p-2}u\log|u|^p & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $(-\Delta)_p^s$ is the fractional Laplacian operator with $p \in [2, +\infty)$, the continuous potential $V(x): \mathbb{R}^N \to \mathbb{R}$ satisfies a local condition [32]. Employing variational arguments, they obtained the existence and concentration of solutions for Eq. (1.4). Shen and Squassina [35] focused on the existence and concentration of solutions for Eq. (1.4) involving the nonlinearity $\lambda|u|^{p-2}u$ in L^p-subcritical and L^p-supcritical cases. Liu and Pucci [25] explored the existence of solutions for a double-phase variable exponent equation without the Ambrosetti–Rabinowitz condition. Note that they obtained the existence of solutions by using the Cerami condition instead of the classical Palais–Smale condition, so that the nonlinearity $f(u)$ does not need to satisfy the Ambrosetti–Rabinowitz condition. Lin *et al.* [26] considered the existence of Mountain-pass-type solutions for Schrödinger equations involving critical exponential growth nonlinearities by the variational methods and Trudinger–Moser inequality. In addition, there are some results on this topic, please refer to [1, 13, 14, 18, 34, 36, 40].

For $A(x) \neq 0$, Xiang *et al.* [44] obtained the existence of multiple solutions for fractional Schrödinger–Kirchhoff equation involving an external magnetic potential. Liang *et al.* [24] explored the existence of multiple solutions for fractional Schrödinger–Kirchhoff equations with electromagnetic fields and critical nonlinearity via concentration compactness principle [31] and variational methods. Song and Shi [37] extended the results of [24] from the classical Laplacian to p-Laplacian. Li *et al.* [23] considered the existence of a nontrivial solution for fractional Schrödinger equations with electromagnetic fields and critical or supercritical nonlinearity by truncation method. Ji and Rădulescu [21] considered the existence and multiplicity of multi-bump solutions for the nonlinear magnetic Choquard equation via variational methods. For more results on the existence and concentration of solutions for nonlinear Schrödinger equations with electromagnetic fields, please see [10, 12, 16, 19, 20, 22, 42]. However, there are few results on the multi-bump solutions of logarithmic Schrödinger equations with electromagnetic fields, even the critical results on this topic. In [41], Wang and Yin considered the following nonlinear magnetic Schrödinger equation with logarithmic nonlinearity:

$$-(\nabla + \mathrm{i}A(x))^2 u + \lambda Z(x)u = u\log|u|^2 + |u|^{q-2}u, \quad u \in H^1(\mathbb{R}^N, \mathbb{C}), \tag{1.5}$$

where $q \in (2, 2^*)$. Using variational methods, they got that Eq. (1.5) possesses at least $2^k - 1$ multi-bump solutions when $\lambda > 0$ is large enough.

Inspired by [2, 41], our aims are to obtain the existence and multiplicity of multi-bump solutions for Eq. (1.1). Since the appearance of critical and logarithmic nonlinearities, the energy functional corresponding to Eq. (3.1) loses some other good properties. Therefore, we have to apply the useful arguments to verify boundedness of (PS) sequence and recover the compactness via concentration compactness principle [11, 27]. Indeed, different from Eq. (1.1), Eq. in [2] merely contains the logarithmic nonlinearity, so it is more complicated to obtain boundedness of (PS) sequence in this paper. Furthermore, the nonlinearity $t\log|t|^2 + |u|^{2^*} \neq 0$ as $t \to 0$, so del Pino and Felmer's method in [32] cannot be applied directly. In order to get the compactness of (PS) sequence in the whole space, we have to modify penalization methods in [2]. To our best of knowledge, this paper extends and complements the main results obtained in [2, 41] from subcritical case to critical case.

Throughout this paper, we make the following assumptions on $Z(x)$:

- (Z_1) $Z \in C(\mathbb{R}^N, \mathbb{R})$ and $Z(x) \geq 0$.
- (Z_2) $\Omega := \operatorname{int} Z^{-1}(0)$ is a non-empty bounded open subset with smooth boundary and $\overline{\Omega} = Z^{-1}(0)$, where int $Z^{-1}(0)$ denotes the set of the interior points of $Z^{-1}(0)$.
- (Z_3) There exist two positive constants b_0 and M_0 such that the functions $Z(x)$ and $\mathcal{V}(x)$ satisfy

$$0 < b_0 < Z(x) + \mathcal{V}(x)$$

for all $x \in \mathbb{R}^N$ and

$$|\mathcal{V}(x)| \leq M_0 \quad \text{for all } x \in \mathbb{R}^N.$$

- (Z_4) Ω consists of k components:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$$

with $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for all $i \neq j$.

Now we are ready to state the main results of this paper.

Theorem 1.1. *Let $N \geq 3$ and (Z_1)–(Z_4) be satisfied. Then for any non-empty subset Γ of $\{1, 2, \ldots, k\}$, there exist constants $\vartheta^* > 0$ and $\lambda^* = \lambda^*(\vartheta^*) > 0$ such that, for all $\vartheta \geq \vartheta^*$ and $\lambda \geq \lambda^*$, Eq.* (1.1) *has a nontrivial solution u_λ. Moreover, the family $\{u_\lambda\}_{\lambda \geq \lambda^*}$ has the following properties: for any sequence $\lambda_n \to \infty$, we can extract a subsequence λ_{n_i} such that $u_{\lambda_{n_i}}$ converges strongly in $H^1_A(\mathbb{R}^N, \mathbb{C})$ to a function u which satisfies $u(x) = 0$ for $x \notin \Omega_\Gamma$ and the restriction $u|_{\Omega_j}$ is a least* *energy solution of*

$$\begin{cases} -(\nabla + \mathrm{i}A(x))^2 u + \mathcal{V}(x) u = \vartheta u \log |u|^2 + |u|^{2^*-2} u & \text{in } \Omega_\Gamma, \\ u > 0, & x \in \Omega_\Gamma, \\ u = 0 & \text{on } \partial\Omega_\Gamma, \end{cases}$$

where $\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j$.

Theorem 1.2. *Assume that* $N \geq 3$ *and* (Z_1)–(Z_4) *hold, there exist positive constants* ϑ^* *and* $\lambda_* = \lambda_*(\vartheta^*) > 0$ *such that, for all* $\vartheta \geq \vartheta_*$ *and* $\lambda \geq \lambda_*$, *Eq.* (1.1) *has at least* $2^k - 1$ *solutions.*

Remark 1.1. Since the characteristics of Eq. (1.1), there is no doubt that we shall face some difficulties.

(i) Compared with equations of [2, 41], Eq. (1.1) contains critical nonlinearity, and the lack of compactness shall occur. We apply the concentration compactness principle [11, 27] to overcome this difficulty. In contrast to equations in [2], Eq. (1.1) also contains electromagnetic nonlinearity, so we shall apply diamagnetic inequality (see (2.1)) to make some more detailed estimates.

(ii) If we try to use variational method to study the existence of solutions for Eq. (1.1), the energy functional $\mathcal{J}_\lambda$ of Eq. (1.1) cannot be well defined in $H^1_A(\mathbb{R}^N, \mathbb{C})$. In fact, since there exists a function $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$ such that $\int_{\mathbb{R}^N} |u|^2 \log |u|^2 dx = -\infty$, which makes the possibility that $\mathcal{J}_\lambda(u) = +\infty$. Furthermore, it is impossible to directly apply the critical points theory of C^1 functional. In order to overcome this obstacle, inspired by [17, 38], we consider a decomposition on Eq. (1.1) (see Sec. 2). In addition, based on the fact that the energy $\mathcal{J}_\lambda$ is of class C^1 in $H^1_A(\Lambda, \mathbb{C})$ with a bounded domain $\Lambda \subset \mathbb{R}^N$, we consider to search for a solution $u_{\lambda,R} \in H^1_A(B_R(0))$ for each $R > 0$ and $\lambda \geq 1$ large enough. After that, passing the limit as $R \to +\infty$, we obtain the existence of a solution for the original equation.

(iii) Different from the nonlinearity t^p of equations in [32], it is possible to obtain the facts that $\lim_{t\to 0} \frac{t^p}{|t|} = 0$ and the function $\frac{t^p}{t}$ is increasing for all $t \in (0, +\infty)$. The above two facts play important roles to apply the powerful arguments in [32]. However, the nonlinearity $u \log |u|^2 + |u|^{2^*} \neq 0$ as $t \to 0$. Hence, it is impossible to employ directly del Pino and Felmer's method in [32] which brings some difficulties to deal with Eq. (1.1).

The framework of this paper is as follows. In Sec. 2, we provide some useful facts which will be used later. In the following three sections, we shall verify that some important lemmas are true in preparation for proving Theorem 1.1. Finally, we obtain Theorem 1.1 and Corollary 1.2.

2. Preliminary Results

In this section, we show the variational framework for Eq. (1.1) and give some powerful results. For any $v : \mathbb{R}^N \to \mathbb{C}$, we define

$$\nabla_A v := (\nabla + \mathrm{i}A)v$$

and

$$H^1_A(\mathbb{R}^N, \mathbb{C}) := \{v \in L^2(\mathbb{R}^N, \mathbb{C}) : |\nabla_A v| \in L^2(\mathbb{R}^N, \mathbb{R})\}.$$

The space $H^1_A(\mathbb{R}^N, \mathbb{C})$ is a Hilbert space equipped with the scalar product

$$\langle u, v\rangle := \mathrm{Re}\int_{\mathbb{R}^N} ((\nabla_A u + \mathrm{i}A(x)u)\overline{(\nabla v + \mathrm{i}A(x)v)} + u\overline{v})dx$$

for any $u, v \in H^1_A(\mathbb{R}^N, \mathbb{C})$, where Re denotes the real part of a complex number, and the bar is the complex conjugation. Moreover, we use notion $\|u\|_A$ to denote the norm induced by this inner product.

Since $A \in L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$, there exists diamagnetic inequality on $H^1_A(\mathbb{R}^N, \mathbb{C})$ (see [29, Theorem 7.21]):

$$|\nabla_A u(x)| \geq |\nabla|u(x)||. \tag{2.1}$$

Let

$$E_\lambda := \left\{u \in H^1_A(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} \lambda Z(x)|u|^2 dx < \infty\right\}$$

with the following norm:

$$\|u\|^2_\lambda = \int_{\mathbb{R}^N} (|\nabla_A u|^2 + (\lambda Z(x) + \mathcal{V}(x))|u|^2)dx.$$

It is clear to see that $(E_\lambda, \|\cdot\|_\lambda)$ is also a Hilbert space and $E_\lambda \subset H^1_A(\mathbb{R}^N, \mathbb{C})$ for any $\lambda \geq 1$. For an open set $B_R(0) \subset \mathbb{R}^N$, we consider

$$H^1_A(B_R(0)) := \{u \in L^2(B_R(0), \mathbb{C}) : |\nabla_A u| \in L^2(B_R(0), \mathbb{R})\},$$

$$\|u\|_{H^1_{A,R}} = \left(\int_{B_R(0)} (|\nabla_A u|^2 + |u|^2)dx\right)^{\frac{1}{2}}$$

and

$$E_{\lambda,R}(B_R(0), \mathbb{C}) := \left\{u \in H^1_A(B_R(0), \mathbb{C}) : \int_{B_R(0)} \lambda Z(x)|u|^2 dx < \infty\right\},$$

$$\|u\|^2_{\lambda,R} = \int_{B_R(0)} (|\nabla_A u|^2 + (\lambda Z(x) + \mathcal{V}(x))|u|^2)dx.$$

Let $H^{0,1}_A(B_R(0), \mathbb{C})$ be the Hilbert space endowed with norm $\|u\|_{H^1_{A,R}}$, as the closure of $C_0^\infty(B_R(0), \mathbb{C})$. From (2.1), we see that if $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$. Therefore, there exist the continuous embedding $E_\lambda \hookrightarrow L^s(\mathbb{R}^N, \mathbb{C})$ for all $s \in [2, 2^*]$ and the compact embedding $E_\lambda \hookrightarrow\hookrightarrow L^s_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C})$ for all $s \in [1, 2^*)$.

Since the appearance of logarithmic nonlinearity in Eq. (1.1), we shall encounter some interesting difficulties. The energy functional $\mathcal{J}_\lambda : E_\lambda \to \mathbb{R}$ corresponding to Eq. (1.1) is defined by

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla_A v|^2 + (\lambda Z(x) + \mathcal{V}(x))|v|^2)dx - \vartheta \int_{\mathbb{R}^N} F(v)dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |v|^{2^*} dx$$

with

$$F(v) = \int_0^v t\log|t|^2 dt = \frac{1}{2}|v|^2\log|v|^2 - \frac{|v|^2}{2}$$

for $v \in E_\lambda$. Furthermore, the Fréchet derivative of $\mathcal{J}_\lambda$ is given by

$$\langle \mathcal{J}_\lambda'(v), \phi\rangle = \mathrm{Re}\bigg(\int_{\mathbb{R}^N} (\nabla_A v \overline{\nabla_A \phi} + (\lambda Z(x) + \mathcal{V}(x))v\overline{\phi})dx - \vartheta\int_{\mathbb{R}^N} F'(v)\overline{\phi}dx - \int_{\mathbb{R}^N} |v|^{2^*-2} v\overline{\phi}dx\bigg)$$

for v, $\phi \in E_\lambda$. Then there exist functions $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$ such that $\int_{\mathbb{R}^N} |u|^2 \log |u|^2 dx = -\infty$, which implies that $\mathcal{J}_\lambda(u) = +\infty$. Therefore, the energy functional $\mathcal{J}_\lambda$ cannot well be defined on $H^1_A(\mathbb{R}^N, \mathbb{C})$. In order to overcome this obstacle, inspired by [3, 5, 38], we consider a decomposition of the following type:

$$F_2(t) - F_1(t) = \frac{1}{2}|t|^2\log|t|^2, \quad \forall\, t \in \mathbb{C},$$

where $F_1 \in C^1(\mathbb{R}, \mathbb{R})$ and F_1 is a non-negative convex function, $F_2 \in C^1(\mathbb{R}, \mathbb{R})$ satisfies Sobolev subcritical growth. Indeed, fixed $\delta > 0$ small enough, we define the following functions:

$$F_1(t) := \begin{cases} 0, & t = 0, \\ -\frac{1}{2}|t|^2\log|t|^2, & 0 < |t| < \delta, \\ -\frac{1}{2}|t|^2(\log\delta^2 + 3) + 2\delta|t| - \frac{\delta^2}{2}, & |t| \geq \delta \end{cases}$$

and

$$F_2(t) := \begin{cases} 0, & |t| < \delta, \\ \frac{1}{2}|t|^2\log\left(\frac{|t|^2}{\delta^2}\right) + 2\delta|t| - \frac{3}{2}|t|^2 - \frac{\delta^2}{2}, & |t| \geq \delta \end{cases}$$

for every $t \in \mathbb{C}$. Therefore,

$$F_2(t) - F_1(t) = \frac{1}{2}|t|^2\log|t|^2, \quad \forall\, t \in \mathbb{R}. \tag{2.2}$$

From [17, 38], we see that F_1 and F_2 satisfy the following properties:

(f_1) For $\delta \approx 0^+$, F_1 is an even and convex function with $F_1'(t)t \geq 0$ and $F_1 \geq 0$.

(f_2) $F_2 \in C^1(\mathbb{R},\mathbb{R}) \cap C^2((\delta,+\infty),\mathbb{R})$ and there exists $C = C_p > 0$ such that

$$|F_2'(t)| \le C|t|^{p-1}, \quad \forall\, t \in \mathbb{R},\ \ p \in (2, 2^*).$$

(f_3) The function $t \mapsto \frac{F_2'(t)}{t}$ is nondecreasing for $t > 0$, and is also a strictly increasing function for $t > \delta$.

(f_4) $\lim_{t\to\infty} \frac{F_2'(t)}{t} = \infty$.

According to the above facts, the energy functional $\mathcal{J}_\lambda$ can be rewritten as

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla_A u|^2 + (\lambda Z(x) + \mathcal{V}(x) + 1)|u|^2)dx$$
$$+ \vartheta \int_{\mathbb{R}^N} F_1(u)dx - \vartheta \int_{\mathbb{R}^N} F_2(u)dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx.$$

From this method, we see that $\mathcal{J}_\lambda$ can be decomposed as a sum of a C^1-functional with a convex and lower semi-continuous functional. Therefore, we can apply the critical point theory of functionals in [39] to obtain the existence of solutions for Eq. (1.1).

For any open set $K \subset \mathbb{R}^N$, it follows from (Z_3) that

$$b_0 |||u|||_{2,K}^2 \le \int_K (|\nabla_A u|^2 + (\lambda Z(x) + \mathcal{V}(x))|u|^2)dx$$

for all $u \in E_\lambda(K,\mathbb{C})$ and $\lambda \ge 1$, where $|||u|||_{2,K}^2 = \int_K |u|^2 dx$.

The following result is a consequence of the above considerations.

Lemma 2.1 ([13, Corollary 1.4]). *There exist $a_0, b_0 > 0$ such that for any open set $K \subset \mathbb{R}^N$,*

$$a_0\|u\|_{\lambda,K}^2 \le \|u\|_{\lambda,K}^2 - b_0|||u|||_{2,K}^2$$

for all $u \in E_\lambda(K,\mathbb{C})$ and $\lambda \ge 1$.

We recall the second concentration-compactness principle in [11, 27] which plays an essential role to recover the compactness in the whole space.

Lemma 2.2 ([27, Lemma 1.2]). *Let $\{u_n\}$ be a sequence weakly convergent to u in $H^1(\mathbb{R}^N)$ such that $|u_n|^{2^*} \rightharpoonup \nu$ and $|\nabla u_n|^2 \rightharpoonup \mu$ in the sense of measures. Then, for some at most countable index set I,*

(i) $\nu = |u|^{2^*} + \sum_{j\in I} \delta_{x_j}\nu_j$, $\nu_j > 0$,
(ii) $\mu \ge |\nabla u|^2 + \sum_{j\in I} \delta_{x_j}\mu_j$, $\mu_j > 0$,
(iii) $\mu_j \ge S\nu_j^{2/2^*}$,

where S is the best Sobolev constant, i.e. $S = \inf\{\int_{\mathbb{R}^N} |\nabla u|^2 dx : \int_{\mathbb{R}^N} |u|^{2^*} dx = 1\}$, $x_j \in \mathbb{R}^N$, δ_{x_j} are Dirac measures at x_j and μ_j, ν_j are constants.

Lemma 2.3 ([11]). *Let $\{u_n\}$ be a sequence weakly convergent to u in $H^1(\mathbb{R}^N)$ and define*

(i) $\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{2^*} dx$,

(ii) $\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^2 dx$.

The quantities ν_∞ *and* μ_∞ *exist and satisfy*

(iii) $\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty$,
(iv) $\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty$,
(v) $\mu_\infty \geq S\nu_\infty^{2/2^*}$.

3. Auxiliary Problem

Fix a bounded open subset Ω'_j with smooth boundary such that

$$\overline{\Omega_j} \subset \Omega'_j$$

for any $j \in \{1, \ldots, k\}$ and

$$\overline{\Omega'_j} \cap \overline{\Omega'_l} = \emptyset \quad \text{for all } j \neq l.$$

In the following, fix a non-empty subset $\Gamma \subset \{1, \ldots, k\}$ and $R > 0$ such that $\Omega'_\Gamma \subset B_R(0)$ and

$$\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j, \quad \Omega'_\Gamma = \bigcup_{j\in\Gamma} \Omega'_j.$$

If we try to employ critical point theory for the energy $\mathcal{J}_\lambda$, we have to use some powerful methods to obtain some compactness property. However, due to the unboundedness of $\mathbb{R}^N$, the usual Sobolev embedding is merely continuous. Therefore, it is impossible to prove that (PS) condition holds. We shall make a minor adjustments of penalization methods [13, 32] to overcome this obstacle.

Next, we consider the auxiliary equation corresponding to Eq. (1.1). Fix constants $b_0 > 0$ and $a_0 > 1 > \delta$, and $\theta > 2$, $\zeta > \frac{\theta}{\theta-2} > 1$ satisfying $\vartheta F'_2(a_0) + a_0^{2^*-1} = \zeta^{-1} b_0$.

Now we set

$$\widetilde{F}'_2(t) := \begin{cases} \vartheta F'_2(t) + |t|^{2^*-2}t, & 0 \leq |t| \leq a_0, \\ \zeta^{-1} b_0 |t|, & |t| \geq a_0 \end{cases}$$

and

$$g_2(x,t) := \chi_\Gamma(x)(\vartheta F'_2(t) + |t|^{2^*-2}t) + (1 - \chi_\Gamma(x))\widetilde{F}'_2(t),$$

where

$$\chi_\Gamma(x) := \begin{cases} 1, & x \in \Omega'_\Gamma, \\ 0, & x \in B_R(0) \backslash \Omega'_\Gamma. \end{cases}$$

Moreover, the energy functional $\mathcal{J}_{\lambda,R}(u) : E_{\lambda,R} \to \mathbb{R}$ is defined by

$$\mathcal{J}_{\lambda,R}(u) = \int_{B_R(0)} (|\nabla_A u|^2 + (\lambda Z(x) + \mathcal{V}(x) + 1)|u|^2) dx + \vartheta \int_{B_R(0)} F_1(u) dx - \int_{B_R(0)} G_2(x,u) dx,$$

where $G_2(x,t) = \int_0^t g_2(x,s)ds$ for all $(x,t) \in B_R(0) \times \mathbb{C}$. Obviously, $\mathcal{J}_{\lambda,R} \in C^1(E_{\lambda,R}, \mathbb{R})$ and its critical point $u_{\lambda,R}$ solves the following auxiliary equation:

$$\begin{cases} -(\nabla + \mathrm{i}A(x))^2 u + (\lambda Z(x) + \mathcal{V}(x) + 1)u \\ \quad = g_2(x,u) - \vartheta F_1'(u), & x \in B_R(0), \\ u = 0, & \text{on } \partial B_R(0). \end{cases} \tag{3.1}$$

Note that $g_2(x,s) = \vartheta F_2'(s) + |s|^{2^*-2}s$ for all $s \in [0, a_0]$ and a critical point $u_{\lambda,R}$ of $\mathcal{J}_{\lambda,R}(u)$ is a solution of the following equation:

$$\begin{cases} -(\nabla + \mathrm{i}A(x))^2 u + (\lambda Z(x) + \mathcal{V}(x))u \\ \quad = \vartheta u \log |u|^2 + |u|^{2^*-2}u & \text{in } B_R(0), \\ u = 0 & \text{on } \partial B_R(0), \end{cases} \tag{3.2}$$

if and only if $|u_{\lambda,R}(x)| \le a_0$ and $x \in B_R(0) \backslash \Omega'_\Gamma$.

Now we verify that $\mathcal{J}_{\lambda,R}$ satisfies the mountain pass geometry.

Lemma 3.1. *For all $\lambda \ge 1$, the functional $\mathcal{J}_{\lambda,R}$ satisfies the following conditions*:

(i) *there exist $\alpha, \varrho > 0$ such that $\mathcal{J}_{\lambda,R}(u) \ge \alpha$ for any $u \in E_{\lambda,R}$ with $\|u\|_{\lambda,R} = \varrho$;*
(ii) *there exists $e \in E_{\lambda,R}$ with $\|e\|_{\lambda,R} > \varrho$ such that $\mathcal{J}_{\lambda,R}(e) < 0$.*

Proof. (i) By (f_1)–(f_2), Sobolev inequality and the fact $G_2(x,t) \le \vartheta F_2(t) + \frac{1}{2^*}|t|^{2^*}$ for all $x \in \mathbb{R}^N$, $t > 0$, we have

$$\begin{aligned} \mathcal{J}_{\lambda,R}(u) = \frac{1}{2} \int_{B_R(0)} (|\nabla_A u|^2 + (\lambda Z(x) + \mathcal{V}(x) + 1)|u|^2) dx \\ + \vartheta \int_{B_R(0)} F_1(u) dx - \int_{B_R(0)} G_2(x,u) dx \\ \ge \frac{1}{2}\|u\|_{\lambda,R}^2 - \int_{B_R(0)} G_2(x,u) dx \\ \ge \frac{1}{2}\|u\|_{\lambda,R}^2 - \vartheta \int_{B_R(0)} F_2(u) dx - \frac{1}{2^*} \int_{B_R(0)} |u|^{2^*} dx \\ \ge \frac{1}{2}\|u\|_{\lambda,R}^2 - \vartheta C \|u\|_{\lambda,R}^p - C_1 \|u\|_{\lambda,R}^{2^*} \\ > 0, \end{aligned}$$

where $C_1 > 0$ and $\|u\|_{\lambda,R} = \varrho$ sufficiently small.

(ii) Fixing $0 < \omega \in C_0^\infty(\Omega_\Gamma)$, by (2.2), we have

$$\begin{aligned}\mathcal{J}_{\lambda,R}(s\omega) &= \frac{s^2}{2}\|\omega\|_{\lambda,R}^2 + \frac{s^2}{2}\||\omega|\|_{2,R}^2 + \vartheta\int_{B_R(0)} F_1(s\omega)dx - \int_{B_R(0)} G_2(x,s\omega)dx \\ &= s^2\mathcal{J}_{\lambda,R}(\omega) + \vartheta\int_{B_R(0)}[F_1(s\omega) - s^2F_1(\omega)]dx \\ &\quad + \int_{B_R(0)}[s^2G_2(x,\omega) - G_2(x,s\omega)]dx \\ &= s^2\mathcal{J}_{\lambda,R}(\omega) + \vartheta\int_{B_R(0)}\left(\frac{s^2}{2}|\omega|^2\log|\omega|^2 - \frac{s^2}{2}|\omega|^2\log s^2|\omega|^2\right)dx \\ &= s^2\left[\int_{B_R(0)}\frac{\vartheta}{2}|\omega|^2\log|\omega|^2dx\right. \\ &\quad \left.+\left(\mathcal{J}_{\lambda,R}(\omega) - \vartheta\int_{B_R(0)}\frac{1}{2}|\omega|^2\log s^2|\omega|^2dx\right)\right] \to -\infty\end{aligned}$$

as $s \to +\infty$. Consequently, there exists $s_0 > 0$ (independent of $\lambda > 0$ and $R > 0$) large enough such that $\mathcal{J}_{\lambda,R}(s_0\omega) < 0$, i.e. $e = s_0\omega$. The proof of Lemma 3.1 is completed. □

By Lemma 3.1 and a variant of mountain pass theorem without the Palais–Smale condition (see [43, Theorem 2.9]), we obtain that the mountain pass level connected with $\mathcal{J}_{\lambda,R}$, denoted by $c_{\lambda,R}$, is given by

$$c_{\lambda,R} = \inf_{\gamma\in\Gamma_{\lambda,R}}\max_{t\in[0,1]}\mathcal{J}_{\lambda,R}(\gamma(t)),$$

where $\Gamma_{\lambda,R} = \{\gamma \in C([0,1], E_{\lambda,R}) : \gamma(0) = 0 \text{ and } \mathcal{J}_{\lambda,R}(\gamma(1)) < 0\}$. Furthermore, with the aid of Lemma 3.1,

$$c_{\lambda,R} \geq \alpha > 0$$

for any $\lambda \geq 1$ and $R > 0$ large enough.

Lemma 3.2. *Let $\{v_n\}$ be a $(PS)_{c_{\lambda,R}}$ sequence for $\mathcal{J}_{\lambda,R}$, then the sequence $\{v_n\}$ is bounded in $E_{\lambda,R}$.*

Proof. Since $\{v_n\}$ is a $(PS)_{c_{\lambda,R}}$ sequence for $\mathcal{J}_{\lambda,R}$, we have that

$$\mathcal{J}_{\lambda,R}(v_n) - \frac{1}{2}\langle\mathcal{J}'_{\lambda,R}(v_n), v_n\rangle \leq c_{\lambda,R} + 1 + o_n(1)\|v_n\|_{\lambda,R} \tag{3.3}$$

for large n. From the definition of $F_1(t)$, we obtain $1 < \frac{F_1'(t)t}{F_1(t)} \leq 2$ for $t > 0$. Together with $\theta > 2$, Lemma 2.1 and

$$G_2(x,t) \leq \frac{b_0}{2\zeta}|t|^2$$

for all $x \in B_R(0)\backslash\Omega'_\Gamma$, we have

$$\begin{aligned}
&\mathcal{J}_{\lambda,R}(v_n) - \frac{1}{\theta}\langle \mathcal{J}'_{\lambda,R}(v_n), v_n\rangle \\
&\quad \geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|v_n\|_{\lambda,R} + \vartheta\int_{B_R(0)}\left(F_1(v_n) - \frac{1}{\theta}F_1'(v_n)\overline{v_n}\right)dx \\
&\qquad + \int_{B_R(0)}\left(\frac{1}{\theta}G_2'(x,v_n)\overline{v_n} - G_2(x,v_n)\right)dx \\
&\quad \geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|v_n\|_{\lambda,R} + \int_{B_R(0)\backslash\Omega'_\Gamma}\left(\frac{1}{\theta}G_2'(x,v_n)\overline{v_n} - G_2(x,v_n)\right)dx \\
&\quad \geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|v_n\|_{\lambda,R} - \int_{B_R(0)\backslash\Omega'_\Gamma} G_2(x,v_n)dx \\
&\quad \geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|v_n\|_{\lambda,R} - \frac{b_0}{2\zeta}\int_{B_R(0)\backslash\Omega'_\Gamma} |v_n|^2dx \\
&\quad \geq \left(\frac{1}{2}-\frac{1}{\theta}\right)(\|v_n\|_{\lambda,R} - b_0\||v_2|\|_2^2) \\
&\quad \geq \left(\frac{1}{2}-\frac{1}{\theta}\right)a_0\|v_n\|^2_{\lambda,R},
\end{aligned} \tag{3.4}$$

which implies that $\{v_n\}$ is bounded in $E_{\lambda,R}$. □

Next, fixed $j \in \Gamma$, we denote by c_j the minimax level of mountain pass theorem with the functional $\mathcal{E}_j : H_A^{0,1}(\Omega_j) \to \mathbb{R}$, defined by

$$\mathcal{E}_j(u) = \frac{1}{2}\int_{\Omega_j}(|\nabla_A u|^2 + \mathcal{V}(x)|u|^2)dx - \frac{\vartheta}{2}\int_{\Omega_j}|u|^2\log|u|^2dx - \frac{1}{2^*}\int_{\Omega_j}|u|^{2^*}dx.$$

If u is a critical point of the energy functional $\mathcal{E}_j$, we say that u is a weak solution of the following equation:

$$\begin{cases} -\Delta_A u + \mathcal{V}(x)u = \vartheta u\log|u|^2 + |u|^{2^*-2}u, & x \in \Omega_j,\ \ j \in \Gamma, \\ u > 0, & x \in \Omega_j, \\ u|_{\partial\Omega_j} = 0. \end{cases} \tag{3.5}$$

In order to prove Theorem 1.1, the methods we will use include the comparison between some energy levels of the functional corresponding to Eq. (1.1) and the energy levels of other auxiliary equation associated with Eq. (1.1), as well as the exploration of the behavior of $(PS)_c$ sequences.

In this respect, we give detailed proof of the following results.

Lemma 3.3. *There exists $\vartheta^* > 0$ such that, for all $\vartheta \geq \vartheta^*$, we have*

$$c_j \in \left(0, \frac{1}{k+1}\left(\frac{1}{2} - \frac{1}{\theta}\right) a_0 S^{N/2}\right) \quad \textit{for all } j \in \{1, 2, \ldots, k\}.$$

Proof. For each $j \in \{1, 2, \ldots, k\}$, we fix a non-negative function $\psi_j \in H_A^{0,1}(\Omega_j)\backslash\{0\}$. We see that there exists $\mathfrak{t}_{\vartheta,j} \in (0, +\infty)$ such that

$$c_j \leq \mathcal{E}_j(\mathfrak{t}_{\vartheta,j}\psi_j) = \max_{\mathfrak{t}\geq 0} \mathcal{E}_j(\mathfrak{t}\psi_j).$$

Therefore, we obtain

$$\begin{aligned} &\mathfrak{t}_{\vartheta,j}^2 \int_{\Omega_j} (|\nabla_A \psi_j|^2 + (\mathcal{V}(x) + 1)|\psi_j|^2) dx \\ &\quad = \vartheta \mathfrak{t}_{\vartheta,j} \int_{\Omega_j} F_2'(\mathfrak{t}_{\vartheta,j}\psi_j)\overline{\psi_j} dx - \vartheta \mathfrak{t}_{\vartheta,j} \int_{\Omega_j} F_1'(\mathfrak{t}_{\vartheta,j}\psi_j)\overline{\psi_j} dx \\ &\qquad + \mathfrak{t}_{\vartheta,j}^{2^*} \int_{\Omega_j} |\psi_j|^{2^*} dx. \end{aligned} \tag{3.6}$$

Taking the limit as $\mathfrak{t}_{\vartheta,j} \to \infty$, together with (f_4) and

$$F_1'(t) \leq C(1 + |t|), \tag{3.7}$$

we deduce that

$$\begin{aligned} &\int_{\Omega_j} (|\nabla_A \psi_j|^2 + (\mathcal{V}(x))|\psi_j|^2) dx \\ &\quad \geq \int_{\Omega_j} \frac{F_2'(\mathfrak{t}_{\vartheta,j}\psi_j)}{\mathfrak{t}_{\vartheta,j}\psi_j})|\psi_j|^2 dx - C\int_{\Omega_j} \left(\frac{1}{t_{\vartheta,j}}|\psi_j| + |\psi_j|^2\right) dx \to +\infty, \end{aligned}$$

which is impossible. Therefore, $\{\mathfrak{t}_{\vartheta,j}\}$ is bounded. Moreover, there exists a sequence $\vartheta \to \infty$ satisfying that $\mathfrak{t}_{\vartheta,j} \to \mathfrak{t}_0 \geq 0$. Consequently, there exists some $C > 0$ such that

$$\mathfrak{t}_{\vartheta,j}^2 \int_{\Omega_j} (|\nabla_A \psi_j|^2 + \mathcal{V}(x)|\psi_j|^2) dx \leq C.$$

If we suppose that $\mathfrak{t}_0 > 0$, then by the first equality of (3.6), we obtain

$$\lim_{\vartheta\to\infty} \mathfrak{t}_{\vartheta,j}^2 \int_{\Omega_j} (|\nabla_A \psi_j|^2 + \mathcal{V}(x)|\psi_j|^2) dx = \infty,$$

which implies a contradiction. Thus, we have $\mathfrak{t}_0 = 0$, and so $\mathfrak{t}_{\vartheta,j} \to 0$ as $\vartheta \to +\infty$. By this fact, we have

$$\mathcal{E}_j(\mathfrak{t}_{\vartheta,j}\psi_j) \to 0 \quad \text{as } \vartheta \to +\infty,$$

whence it follows that there exists $\vartheta^* > 0$ such that

$$c_j \in \left(0, \frac{1}{k+1}\left(\frac{1}{2} - \frac{1}{\theta}\right) a_0 S^{N/2}\right)$$

for all $j \in \{1, 2, \ldots, k\}$ and all $\vartheta \in [\vartheta^*, +\infty)$. □

Remark 3.1. In particular, it follows from Lemma 3.3 that

$$\sum_{j=1}^{k} c_j \in \left(0, \left(\frac{1}{2}-\frac{1}{\theta}\right) a_0 S^{N/2}\right). \tag{3.8}$$

The above result is very important to obtain the following result.

Lemma 3.4. *Let* (Z_1)*–*(Z_4) *and* (f_1)*–*(f_4) *be satisfied. For any* $\lambda \geq 1$, $\mathcal{J}_\lambda$ *satisfies the* $(PS)_c$ *condition with*

$$c \in \left(0, \left(\frac{1}{2}-\frac{1}{\theta}\right) a_0 S^{N/2}\right).$$

Proof. Let $\{u_n\} \subset H^1_A(\mathbb{R}^N)$ be a $(PS)_c$-sequence, that is,

$$\mathcal{J}_\lambda(u_n) \to c \quad \text{and} \quad \mathcal{J}'_\lambda(u_n) \to 0.$$

From Lemma 3.2 and taking the limit as $R \to \infty$, we see that $\{u_n\}$ is bounded in $H^1_A(\mathbb{R}^N)$. By diamagnetic inequality, the boundedness of $\{|u_n|\}$ is obtained in $H^1(\mathbb{R}^N)$. Then, for some subsequence, there exists $u \in H^1_A(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in E_λ and $H^1_A(\mathbb{R}^N)$.

Now we claim that

$$\int_{\mathbb{R}^N} |u_n|^{2^*} dx \to \int_{\mathbb{R}^N} |u|^{2^*} dx \quad \text{as } n \to \infty. \tag{3.9}$$

In order to verify that this claim holds, we assume that

$$|\nabla|u_n||^2 \rightharpoonup |\nabla|u||^2 + \mu \quad \text{and} \quad |u_n|^{2^*} \rightharpoonup |u|^{2^*} + \nu \quad (\text{weak}^* \text{ sense of measures}).$$

Using the concentration compactness principle [27, Lemma 1.2], there exist a countable index set I, sequences $\{x_j\} \subset \mathbb{R}^N$, $\{\mu_j\}, \{\nu_j\} \subset (0, \infty)$ such that

$$\nu = \sum_{j\in I} \delta_{x_j}\nu_j, \quad \mu \geq \sum_{j\in I} \delta_{x_j}\mu_j \quad \text{and} \quad \mu_j \geq S\nu_j^{2/2^*} \tag{3.10}$$

for all $j \in I$, where δ_{x_j} are Dirac measures at x_j and μ_j, ν_j are constants.

Now, let x_j be a singular point of the measures μ and ν. We define a cut off function $\varphi_\varepsilon(x) \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\varphi_\varepsilon(x) = 1$ on $B(x_j, \varepsilon)$, $\varphi_\varepsilon(x) = 0$ in $\mathbb{R}^N \backslash B(x_j, 2\varepsilon)$ and $|\nabla\varphi_\varepsilon| \leq 2/\varepsilon$ in $\mathbb{R}^N$. Due to the boundedness of $\{u_n\varphi_\varepsilon\}$ in $H^1_A(\mathbb{R}^N)$ and φ_ε takes values in $\mathbb{R}$, we see that

$$\langle \mathcal{J}'_\lambda(u_n), u_n\varphi_\varepsilon\rangle \to 0$$

and

$$\overline{\nabla_A(u_n\varphi_\varepsilon)} = \mathrm{i}\overline{u_n}\nabla\varphi_\varepsilon + \varphi_\varepsilon\overline{\nabla_A u_n}.$$

Consequently,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla_A u_n|^2 \varphi_\varepsilon dx &+ \int_{\mathbb{R}^N} (\lambda Z(x) + \mathcal{V}(x) + 1)|u_n|^2 \varphi_\varepsilon dx \\ &= -\mathrm{Re}\left(\int_{\mathbb{R}^N} \mathrm{i}\overline{u_n} \nabla_A u_n \nabla \varphi_\varepsilon dx\right) + \vartheta \int_{\mathbb{R}^N} (F_2'(u_n) u_n \varphi_\varepsilon - F_1'(u_n) u_n \varphi_\varepsilon) dx \\ &\quad + \int_{\mathbb{R}^N} |u_n|^{2^*} \varphi_\varepsilon dx + o_n(1). \end{aligned} \tag{3.11}$$

By the Hölder's inequality, we deduce that

$$\limsup_{n\to\infty} \left| \mathrm{Re}\left(\int_{\mathbb{R}^N} \mathrm{i}\overline{u_n} \nabla_A u_n \nabla \varphi_\varepsilon dx\right)\right| = 0.$$

By diamagnetic inequality (see (2.1)), we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla |u_n||^2 \varphi_\varepsilon dx &\le \vartheta \int_{\mathbb{R}^N} (F_2'(u_n) u_n \varphi - F_1'(u_n) u_n \varphi_\varepsilon) dx \\ &\quad + \int_{\mathbb{R}^N} |u_n|^{2^*} \varphi_\varepsilon dx + o_n(1). \end{aligned} \tag{3.12}$$

Consequently, by the fact that $u_n \to u$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for all $s \in [1, 2^*)$ and φ has compact support, by (f_2) and (3.7), we can obtain that

$$\lim_{\varepsilon\to 0^+} \lim_{n\to\infty} \int_{\mathbb{R}^N} F_1'(u_n) u_n \varphi_\varepsilon dx = \lim_{\varepsilon\to 0^+} \lim_{n\to\infty} \int_{\mathbb{R}^N} F_2'(u_n) u_n \varphi_\varepsilon dx = 0, \tag{3.13}$$

$$\lim_{\varepsilon\to 0^+} \lim_{n\to\infty} \int_{\mathbb{R}^N} (\lambda Z(x) + \mathcal{V}(x) + 1)|u_n|^2 \varphi_\varepsilon dx = 0, \tag{3.14}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla |u_n||^2 \varphi_\varepsilon dx = \int_{\mathbb{R}^N} \varphi_\varepsilon d\mu \ge \int_{\mathbb{R}^N} |\nabla |u||^2 \varphi_\varepsilon dx + \mu_j \tag{3.15}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*} \varphi_\varepsilon dx = \int_{\mathbb{R}^N} \varphi_\varepsilon d\nu + \nu_j. \tag{3.16}$$

Inserting (3.13)–(3.16) into (3.12), and letting $\varepsilon \to 0$ and $n \to \infty$, we get

$$\mu_j \le \nu_j.$$

Together with (3.10), we have

$$\nu_j \ge S \nu_j^{2/2^*},$$

which implies

$$\text{(I)} \quad \nu_j = 0 \quad \text{or} \quad \text{(II)} \quad \nu_j \ge S^{N/2}.$$

In order to obtain the possible concentration of mass at infinity, we will apply the concentration compactness principle at infinity [11]. In the same way, we also define a cut off function $\varphi_R \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\varphi_R(x) = 0$ on $|x| < R$ and $\varphi_R(x) = 1$ on $|x| > R+1$. Note that $\{u_n\varphi_R\}$ is bounded in $H_A^1(\mathbb{R}^N)$ and φ_R takes values in $\mathbb{R}$, we can obtain that $\langle \mathcal{J}_\lambda'(u_n), u_n\varphi_R\rangle \to 0$, i.e.

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla_A u_n|^2\varphi_R dx + \int_{\mathbb{R}^N} (\lambda Z(x) + \mathcal{V}(x) + 1)|u_n|^2\varphi_R dx \\ = -\mathrm{Re}\left(\int_{\mathbb{R}^N} \mathrm{i}\overline{u_n}\nabla_A u_n \nabla\varphi_R dx\right) + \vartheta\int_{\mathbb{R}^N} (F_2'(x,u_n)u_n\varphi_R \\ - F_1'(x,u_n)u_n\varphi_R)dx + \int_{\mathbb{R}^N} |u_n|^{2^*}\varphi_R dx + o_n(1). \qquad (3.17)\end{aligned}$$

By Hölder's inequality again, we know

$$-\lim_{R\to\infty}\lim_{n\to\infty} \mathrm{Re}\left(\int_{\mathbb{R}^N} \mathrm{i}\overline{u_n}\nabla_A u_n\nabla\varphi_R dx\right) = 0.$$

By (3.7) and (f_2), we can obtain

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} F_1'(u_n)u_n\varphi_R dx = \lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} F_2'(u_n)u_n\varphi_R dx = 0, \qquad (3.18)$$

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} (\lambda Z(x) + \mathcal{V}(x) + 1)|u_n|^2\varphi_R dx = 0, \qquad (3.19)$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla|u_n||^2\varphi_R dx = \int_{\mathbb{R}^N}\varphi_R d\mu \geq \int_{\mathbb{R}^N} |\nabla|u||^2\varphi_R dx + \mu_j \qquad (3.20)$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{2^*}\varphi_R dx = \int_{\mathbb{R}^N}\varphi_R d\nu + \nu_j. \qquad (3.21)$$

Inserting (3.18)–(3.21) into (3.12), taking the limits as $R \to \infty$ and $n \to \infty$, we obtain

$$\mu_\infty \leq \nu_\infty.$$

Therefore, we have $\nu_\infty \geq S\nu_\infty^{2/2^*}$, which yields

$$\text{(III)} \quad \nu_\infty = 0 \quad \text{or} \quad \text{(IV)} \quad \nu_\infty \geq S^{N/2}.$$

In what follows, we claim that (II) and (IV) cannot occur. If case (II) is true, for some $i \in I$, using the proof Lemma 3.2 once more, we have

$$c + o_n(1) = \mathcal{J}_\lambda(u_n) \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) a_0\|u_n\|_\lambda.$$

Since

$$\int_{\mathbb{R}^N} ((\lambda Z(x) + \mathcal{V}(x))|u_n|^2)dx \geq 0,$$

it follows from the diamagnetic inequality that

$$\left(\frac{1}{2}-\frac{1}{\theta}\right)a_0\int_{\mathbb{R}^N}|\nabla|u_n||^2dx \le \left(\frac{1}{2}-\frac{1}{\theta}\right)a_0\int_{\mathbb{R}^N}|\nabla_A u_n|^2dx$$
$$\le \left(\frac{1}{2}-\frac{1}{\theta}\right)a_0\|u_n\|_\lambda^2 \le c+o_n(1)$$

and then

$$\left(\frac{1}{2}-\frac{1}{\theta}\right)a_0\mu_j \le c \quad \text{for all } n\in\mathbb{N}. \tag{3.22}$$

Recalling that $\mu_j \ge S\nu_j^{2/2^*}$, from (3.10) and (3.22) we obtain

$$c \ge \left(\frac{1}{2}-\frac{1}{\theta}\right)a_0 S^{N/2},$$

which yields a contradiction. Consequently, $\nu_j = 0$ for all $j\in I$. Similarly, we may verify that (IV) cannot occur for each j. Therefore, (3.9) is true.

In the following, we assume that $\{u_n\}\subset E_\lambda$ is a $(PS)_c$ sequence for $\mathcal{J}_\lambda$. By Lemma 3.2, we obtain that the boundedness of $\{u_n\}$ in E_λ. Without loss of generality, we may suppose that there exist $u\in E_\lambda$ and a subsequence $\{u_n\}$ such that

$$u_n \rightharpoonup u \quad \text{in } E_\lambda,$$
$$u_n \to u \quad \text{in } L^s(B_R(0)), \quad \forall\, s\in[1,2^*),$$
$$u_n \to u \quad \text{a.e. in } \mathbb{R}^N.$$

By (f_2), (3.7) and Lebesgue's Theorem, we infer

$$\int_{\mathbb{R}^N}F_2'(u_n)\overline{u_n}dx \to \int_{\mathbb{R}^N}F_2'(u)\overline{u}dx$$

and

$$\int_{\mathbb{R}^N}F_1'(u_n)\overline{u_n}dx \to \int_{\mathbb{R}^N}F_1'(u)\overline{u}dx.$$

Furthermore, combined with (3.9), we have

$$\int_{\mathbb{R}^N}G_2'(u_n)\overline{u_n}dx \to \int_{\mathbb{R}^N}G_2'(u)\overline{u}dx.$$

Thus, from Brézis–Lieb Lemma [9], we have

$$o(1)\|u_n\|_\lambda = \langle\mathcal{J}_\lambda'(u_n),u_n\rangle = \|u_n\|_\lambda^2 + \||u|\|_2^2 + \vartheta\int_{\mathbb{R}^N}F_1'(u_n)dx - \int_{\mathbb{R}^N}G_2'(x,u_n)u_ndx$$
$$= \|u_n-u\|_\lambda^2 + \|u\|_\lambda^2 + \||u|\|_2^2 + \vartheta\int_{\mathbb{R}^N}F_1'(u)dx - \int_{\mathbb{R}^N}G_2'(x,u)udx$$
$$= \|u_n-u\|_\lambda^2 + o(1)\|u\|_\lambda,$$

here we use $\mathcal{J}_\lambda'(u) = 0$. Thus, we conclude that $\{u_n\}$ strongly converges to u in E_λ. This completes the proof of Lemma 3.4. □

Lemma 3.5. *Equation* (3.1) *has a nontrivial solution* $u_{\lambda,R} \in E_{\lambda,R}$ *such that* $\mathcal{J}_{\lambda,R}(u_{\lambda,R}) = c_{\lambda,R}$, *where* $c_{\lambda,R}$ *is the mountain pass level connected with* $\mathcal{J}_{\lambda,R}$.

Proof. By Lemmas 3.1 and 3.4, we obtain the existence of a nontrivial solution $u_{\lambda,R}$. □

In what follows, for each $R > 0$, we are interested in the behavior of a $(PS)_{\infty,R}$ sequence for $\mathcal{J}_{\lambda,R}$, i.e. a sequence $\{u_n\} \subset H^1_A(B_R(0))$ satisfying

$$u_n \in E_{\lambda_n,R} \quad \text{and} \quad \lambda_n \to \infty,$$

$$\mathcal{J}_{\lambda_n,R}(u_n) \to c_{\lambda_n,R}, \quad \|\mathcal{J}'_{\lambda_n,R}(u_n)\| \to 0.$$

Lemma 3.6. *Let* $\{u_n\} \subset H^1_A(B_R(0))$ *be a* $(PS)_{\infty,R}$ *sequence. Then for some subsequence* $\{u_n\}$, *there exists* $u \in H^1_A(B_R(0))$ *such that*

$$u_n \rightharpoonup u \quad \text{in } H^1_A(B_R(0)).$$

Moreover,

(i) $\|u_n - u\|_{\lambda_n,R} \to 0$, *and hence,*

$$u_n \to u \quad \text{in } H^1_A(B_R(0)).$$

(ii) $u \equiv 0$ *in* $B_R(0)\backslash\Omega_\Gamma$ *and* u *is a solution of*

$$\begin{cases} -(\nabla + iA(x))^2 u + \mathcal{V}(x) = \vartheta u \log|u|^2 + |u|^{2^*-2}u & \text{in } \Omega_\Gamma, \\ u = 0 & \text{on } \partial\Omega_\Gamma. \end{cases} \tag{3.23}$$

(iii) u_n *also satisfies*

$$\lambda_n \int_{B_R(0)} Z(x)|u_n|^2 dx \to 0,$$

$$\|u_n\|^2_{\lambda_n,B_R(0)\backslash\Omega_\Gamma} \to 0,$$

$$\|u_n\|^2_{\lambda_n,\Omega'_j} \to \int_{\Omega_j} (|\nabla_A u|^2 + \mathcal{V}(x)|u|^2)dx \quad \text{for all } j \in \Gamma.$$

Proof. From Lemma 3.2, there exists $D > 0$ satisfying

$$\|u_n\|^2_{\lambda_n,R} \le D, \quad \forall n \in \mathbb{N}.$$

Thus, $\{u_n\}$ is bounded in $H^1_A(B_R(0))$. Moreover, we may suppose that there exists $u \in H^1_A(B_R(0))$ such that

$$u_n \rightharpoonup u \quad \text{in } H^1_A(B_R(0))$$

and

$$u_n(x) \to u(x) \quad \text{a.e. in } B_R(0).$$

Fixing $\mathcal{C}_m = \left\{x \in B_R(0) : Z(x) \geq \frac{1}{m}\right\}$, we have

$$\int_{\mathcal{C}_m} |u_n|^2 dx \leq \frac{m}{\lambda_n} \int_{B_R(0)} \lambda_n Z(x)|u_n|^2 dx.$$

So

$$\int_{\mathcal{C}_m} |u_n|^2 dx \leq \frac{m}{\lambda_n} \|u_n\|^2_{\lambda_n,R}.$$

Using the Fatou's lemma, we deduce that

$$\int_{\mathcal{C}_m} |u|^2 dx = 0$$

for all $m \in \mathbb{N}$. Therefore, $u(x) = 0$ on $\bigcup_{m=1}^{+\infty} \mathcal{C}_m = B_R(0)\backslash\overline{\Omega}$ and $u|_{\Omega_j} \in H^1_A(\Omega_j)$, $j \in \{1, \ldots, k\}$.

In the following, we shall verify that (i)–(iii) are satisfied.

(i) Note that $u = 0$ in $B_R(0)\backslash\overline{\Omega}$ and $\langle \mathcal{J}'_{\lambda,R}(u_n), u_n - u\rangle = \langle \mathcal{J}'_{\lambda,R}(u), u_n - u\rangle = o_n(1)$, we obtain

$$\int_{B_R(0)} (|\nabla_A(u_n - u)|^2 + (\lambda_n Z(x) + \mathcal{V}(x) + 1)|u_n - u|^2) dx \to 0,$$

which implies $u_n \to u$ in $H^1_A(B_R(0))$.

(ii) By the facts that $u \in H^1_A(B_R(0))$ and $u = 0$ in $B_R(0)\backslash\overline{\Omega}$, we obtain $u \in H^1_A(\Omega)$ or $u|_{\Omega_j} \in H^1_A(\Omega_j)$ for $j = 1, \ldots, k$. Moreover, from the facts that $u_n \to u$ in $H^1_A(B_R(0))$ and $\langle \mathcal{J}'_{\lambda_n,R}(u_n), \varphi\rangle \to 0$ as $n \to +\infty$ for each $\varphi \in C_0^\infty(\Omega_\Gamma)$, we know

$$\mathrm{Re}\left(\int_{\Omega_\Gamma} (\nabla_A u \overline{\nabla_A \varphi} + (\mathcal{V}(x) + 1)u\overline{\varphi}) dx + \vartheta \int_{\Omega_\Gamma} F'_1(u)\overline{\varphi} dx \right.$$
$$\left. - \vartheta \int_{\Omega_\Gamma} F'_2(u)\overline{\varphi} dx - \int_{\Omega_\Gamma} |u|^{2^*-2} u\overline{\varphi} dx\right) = 0,$$

so $u|_{\Omega_\Gamma}$ solves Eq. (3.23). In addition, we infer that

$$\int_{\Omega_j} (|\nabla_A u|^2 + (\mathcal{V}(x) + 1)u^2) dx + \mathrm{Re}\left(\int_{\Omega_j} \vartheta F'_1(u)\overline{u} dx - \int_{\Omega_j} \widetilde{F}'_2(u)\overline{u} dx\right) = 0$$

for each $j \in \{1, 2, \ldots, k\}\backslash\Gamma$. By $F'_1(t)t \geq 0$, Lemma 2.1 and $\widetilde{F}'_2(t)t \leq b_0|t|^2$ for all $t \in \mathbb{R}^+$, we deduce

$$a_0\|u\|^2_{\lambda,\Omega_j} \leq \|u\|^2_{\lambda,\Omega_j} - b_0\||u|\|^2_{\lambda,\Omega_j}$$
$$\leq \|u\|^2_{\lambda,\Omega_j} - \mathrm{Re}\left(\int_{\Omega_j} \widetilde{F}'_2(u)\overline{u} dx\right) \leq 0.$$

Therefore, $u|_{\Omega_j} = 0$ for $j \in \{1, 2, \ldots, k\}\backslash\Gamma$. This implies $u = 0$ in $B_R(0) \setminus \Omega_\Gamma$.

In the following, we verify that (iii) holds. From (i), we deduce that

$$\begin{aligned}\int_{B_R(0)} \lambda_n Z(x)|u_n|^2 dx &= \int_{B_R(0)\setminus\Omega_\Gamma} \lambda_n Z(x)|u_n|^2 dx \\ &= \int_{B_R(0)\setminus\Omega_\Gamma} \lambda_n Z(x)|u_n - u|^2 dx \\ &\leq \|u_n - u\|^2_{\lambda_n,R} \to 0 \end{aligned} \tag{3.24}$$

as $n \to \infty$.

Furthermore, invoking (i) and (ii), we see

$$\|u_n\|^2_{\lambda_n,B_R(0)\setminus\Omega_\Gamma} \to 0$$

and

$$\|u_n\|^2_{\lambda_n,\Omega_j'} \to \int_{\Omega_j} (|\nabla_A u|^2 + \mathcal{V}(x)|u|^2)dx \quad \text{for all } j \in \Gamma.$$

Indeed, by $u = 0$ in $B_R(0)\setminus\overline{\Omega}, u = 0$ in Ω_j for $j \in \{1, 2, \ldots, k\}\setminus\Gamma$ and $u_n \to u$ in $H^1_A(B_R(0))$, we have $\|u_n\|^2_{\lambda_n,B_R(0)\setminus\Omega_\Gamma} \to 0$. It follows from (3.24) that the last conclusion holds. □

Using the similar arguments to Lemmas 3.6 and 3.2, we also have the following result which will be used in Sec. 3.

Lemma 3.7. *Assume that $u_n \in E_{\lambda_n,R_n}$ is a $(PS)_{\infty,R_n}$ sequence with $R_n \to +\infty$, i.e.*

$$u_n \in E_{\lambda_n,R_n} \quad \textit{and} \quad \lambda_n \to \infty,$$

$$\mathcal{J}_{\lambda_n,R_n}(u_n) \to c \quad \textit{with } c \in \left(\frac{1}{2} - \frac{1}{\theta}\right) a_0 S^{N/2}, \quad \|\mathcal{J}'_{\lambda_n,R_n}(u_n)\| \to 0.$$

Then for some subsequence, still denoted by $\{u_n\}$, there exists $u \in H^1_A(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \textit{in } H^1_A(\mathbb{R}^N).$$

Moreover,

(i) $\|u_n - u\|_{\lambda_n,R_n} \to 0$, *and further*

$$u_n \to u \quad \textit{in } H^1_A(\mathbb{R}^N).$$

(ii) $u \equiv 0$ *in* $\mathbb{R}^N\setminus\Omega_\Gamma$ *and* u *solves the following equation:*

$$\begin{cases} -(\nabla + \mathrm{i}A(x))^2 u + \mathcal{V}(x)u = \vartheta u \log |u|^2 + |u|^{2^*-2}u & \textit{in } \Omega_\Gamma, \\ u = 0 & \textit{on } \partial\Omega_\Gamma. \end{cases}$$

(iii) u_n *also satisfies*

$$\lambda_n \int_{B_{R_n}(0)} Z(x)|u_n|^2 dx \to 0,$$

$$\|u_n\|^2_{\lambda_n, B_{R_n}(0)\setminus\Omega_\Gamma} \to 0,$$

$$\|u_n\|^2_{\lambda_n,\Omega_j'} \to \int_{\Omega_j} (|\nabla u|^2 + \mathcal{V}(x)|u|^2)dx \quad \textit{for all } j \in \Gamma.$$

Proof. Invoking the boundedness of $\mathcal{J}_{\lambda_n,R_n}(u_n)$ in E_{λ_n,R_n}, we see that $\{u_n\}$ is also bounded in E_{λ_n,R_n}. Therefore, we may suppose that there exists $u \in H^1_A(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{in } H^1_A(\mathbb{R}^N),$$

$$u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N$$

and $u(x) = 0$ on $\mathbb{R}^N\setminus\overline{\Omega}$.

(i) From Lemma 3.4, we obtain

$$\mathrm{Re}\left(\int_{\mathbb{R}^N} G_2'(x,u_n)\overline{\varpi}dx\right) \to \mathrm{Re}\left(\int_{\mathbb{R}^N} G_2'(x,u)\overline{\varpi}dx\right), \quad \forall \varpi \in C_0^\infty(\mathbb{R}^N),$$

$$\mathrm{Re}\left(\int_{\mathbb{R}^N} G_2'(x,u_n)\overline{u_n}dx\right) \to \mathrm{Re}\left(\int_{\mathbb{R}^N} G_2'(x,u)\overline{u}dx\right)$$

and

$$\int_{\mathbb{R}^N} G_2(x,u_n) \to \int_{\mathbb{R}^N} G_2'(x,u)dx.$$

By $\lim_{n\to\infty}\langle \mathcal{J}'_{\lambda_n,R_n}(u_n), \varpi\rangle = 0$ for all $\varpi \in C_0^\infty(\mathbb{R}^N)$ and the boundedness of $\{u_n\}$ in E_{λ_n,R_n}, we know

$$\mathrm{Re}\left(\int_{\mathbb{R}^N} (\nabla_A u\overline{\nabla_A \varpi} + (\mathcal{V}(x)+1)u\overline{\varpi})dx + \int_{\mathbb{R}^N} F_1'(u)\overline{\varpi}dx\right)$$
$$= \mathrm{Re}\left(\int_{\mathbb{R}^N} G_2'(x,u)\overline{\varpi}dx\right).$$

Therefore, one has

$$\mathrm{Re}\left(\int_{\mathbb{R}^N} (|\nabla_A u|^2 + (\mathcal{V}(x)+1)|u|^2)dx + \int_{\mathbb{R}^N} F_1'(u)\overline{u}dx\right) = \mathrm{Re}\left(\int_{\mathbb{R}^N} G_2'(x,u)\overline{u}dx\right).$$

Together with the fact $\lim_{n\to\infty}\langle \mathcal{J}'_{\lambda_n,R_n}(u_n), u_n\rangle = 0$, i.e.

$$\int_{\mathbb{R}^N} (|\nabla_A u_n|^2 + (\lambda_n Z(x) + \mathcal{V}(x)+1)|u_n|^2)dx + \int_{\mathbb{R}^N} F_1'(u_n)\overline{u_n}dx$$
$$= \int_{\mathbb{R}^N} G_2'(x,u_n)\overline{u_n}dx + o_n(1),$$

we have

$$\lim_{n\to+\infty}\left(\int_{\mathbb{R}^N}(|\nabla_A u_n|^2+(\lambda_n Z(x)+\mathcal{V}(x)+1)|u_n|^2)dx+\int_{\mathbb{R}^N}F_1'(u_n)\overline{u_n}dx\right)$$
$$=\int_{\mathbb{R}^N}(|\nabla_A u|^2+(\mathcal{V}(x)+1)|u|^2)dx+\int_{\mathbb{R}^N}F_1'(u)\overline{u}dx,$$

up to subsequence if necessary, we have

$$u_n\to u \quad \text{in } H_A^1(\mathbb{R}^N),\quad \lambda_n\int_{\mathbb{R}^N}Z(x)|u_n|^2dx\to 0,$$
$$\int_{\mathbb{R}^N}\mathcal{V}(x)|u_n|^2dx\to\int_{\mathbb{R}^N}\mathcal{V}(x)|u|^2dx$$

and

$$F_1'(u_n)\overline{u_n}\to F_1'(u)\overline{u}\quad \text{in } L^1(\mathbb{R}^N).$$

Since F_1 is convex, even and $F(0)=0$, we know that $F_1'(t)\overline{t}\geq F_1(t)\geq 0$ for all $t\in\mathbb{C}$. Hence, using Lebesgue dominated convergence theorem, it holds

$$F_1(u_n)\to F_1(u)\quad \text{in } L^1(\mathbb{R}^N).$$

Therefore,

$$\|u_n-u\|^2_{\lambda_n,R_n}=\int_{\mathbb{R}^N}|\nabla_A u_n-\nabla_A u|^2dx+\int_{\mathbb{R}^N}(\mathcal{V}(x)+1)|u_n-u|^2dx$$
$$+\lambda_n\int_{\mathbb{R}^N}Z(x)|u_n|^2dx\to 0$$

as $n\to\infty$. Therefore, (i) holds. By the arguments similar to Lemma 3.6, we have that (ii) and (iii) hold. □

In the following, we shall consider the boundedness outside Ω'_Γ for the solutions of Eq. (3.1).

Lemma 3.8. *Let $u_{\lambda,R}$ be a nontrivial solution of Eq. (3.1) satisfying*

$$\sup_{\lambda\geq 1}(\mathcal{J}_{\lambda,R}(u_{\lambda,R}))<\left(\frac{1}{2}-\frac{1}{\theta}\right)a_0S^{N/2}$$

for $R>0$ large enough. Then there exists $D>0$ independent of $\lambda\geq 1$ and $R>0$, and $R^*>0$ such that

$$|||u_{\lambda,R}|||_{\infty,R}\leq D,\quad \forall\,\lambda\geq 1,\ \ R\geq R^*.$$

In particular, $u_{\lambda,R}$ solves original Eq. (1.1).

Proof. Consider $\lambda \geq 1, L > 0$ and $\beta > 1$, and let

$$|u_{L,\lambda}| := \begin{cases} |u_{\lambda,R}| & \text{if } |u_{\lambda,R}| \leq L, \\ L & \text{if } |u_{\lambda,R}| > L, \end{cases}$$

$$|z_{L,\lambda}| = |u_{L,\lambda}|^{2(\beta-1)}|u_{\lambda,R}| \quad \text{and} \quad \omega_{L,\lambda} = |u_{\lambda,R}||u_{L,\lambda}|^{\beta-1}.$$

Since $u_{\lambda,R}$ is a nontrivial solution to Eq. (3.1), we have

$$-(\nabla + \mathrm{i}A(x))^2 u_{\lambda,R} + (\lambda Z(x) + \mathcal{V}(x) + 1)u_{\lambda,R} = g_2(x, u_{\lambda,R}) - \vartheta F_1'(u_{\lambda,R}).$$

With the aid of Kato's inequality

$$\Delta|u_{\lambda,R}| \geq \mathrm{Re}\left(\frac{\overline{u_{\lambda,R}}}{|u_{\lambda,R}|}(\nabla + \mathrm{i}A(x))^2 u_{\lambda,R}\right),$$

we obtain that

$$-\Delta|u_{\lambda,R}| + (\lambda Z(x) + 1)|u_{\lambda,R}| \leq g_2(x, |u_{\lambda,R}|) - \vartheta F_1'(|u_{\lambda,R}|), \quad x \in \mathbb{R}^N.$$

Taking $z_{L,\lambda}$ as a test function in the inequality above, we have

$$\begin{aligned} &\int_{B_R(0)} |u_{L,\lambda}|^{2(\beta-1)}|\nabla|u_{\lambda,R}||^2 dx + 2(\beta-1) \\ &\quad \times \int_{B_R(0)} |u_{L,\lambda}|^{2\beta-3}|u_{\lambda,R}|\nabla|u_{\lambda,R}|\nabla|u_{L,\lambda}|dx \\ &\quad + \int_{B_R(0)} (\lambda Z(x) + \mathcal{V}(x) + 1)|u_{L,\lambda}|^{2(\beta-1)}|u_{\lambda,R}|^2 dx \\ &\quad + \vartheta \int_{B_R(0)} F_1'(|u_{\lambda,R}|)|u_{L,\lambda}|^{2(\beta-1)}|u_{\lambda,R}|dx \\ &\leq \int_{B_R(0)} G_2'(x, |u_{\lambda,R}|)|u_{L,\lambda}|^{2(\beta-1)}|u_{\lambda,R}|dx. \end{aligned} \tag{3.25}$$

Fixed $\vartheta > 0$. According to the definition of G_2 and (f_2), we infer that

$$G_2'(x,t) \leq \vartheta F_2'(t) + |t|^{2^*-1} \leq C\vartheta t^{p-1} + |t|^{2^*-1} \tag{3.26}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$ and $p \in (2, 2^*)$. Hence, by (3.25) and (3.26), we deduce

$$\begin{aligned} &\int_{B_R(0)} (|\nabla\omega_{L,\lambda}|^2 + |\omega_{L,\lambda}|^2)dx \\ &\leq C\vartheta \int_{B_R(0)} |u_{\lambda,R}|^p|u_{L,\lambda}|^{2(\beta-1)}dx + \int_{B_R(0)} |u_{\lambda,R}|^{2^*}|u_{L,\lambda}|^{2(\beta-1)}dx \\ &= C\vartheta \int_{B_R(0)} |u_{\lambda,R}|^{p-2}|\omega_{L,\lambda}|^2 dx + \int_{B_R(0)} |u_{\lambda,R}|^{2^*-2}|\omega_{L,\lambda}|^{2(\beta-1)}dx. \end{aligned} \tag{3.27}$$

Invoking the Hölder's inequality, we see that

$$\int_{B_R(0)} |u_{\lambda,R}|^{p-2}|\omega_{L,\lambda}|^2 dx \leq C\beta^2 \left(\int_{B_R(0)} |u_{\lambda,R}|^p dx\right)^{\frac{p-2}{p}} \left(\int_{B_R(0)} |\omega_{L,\lambda}|^p dx\right)^{\frac{2}{p}} \tag{3.28}$$

and

$$\int_{B_R(0)} |u_{\lambda,R}|^{2^*-2}|\omega_{L,\lambda}|^2 dx \leq C\beta^2 \left(\int_{B_R(0)} |u_{\lambda,R}|^{2^*} dx\right)^{\frac{2^*-2}{2^*}} \left(\int_{B_R(0)} |\omega_{L,\lambda}|^{2^*} dx\right)^{\frac{2}{2^*}}. \tag{3.29}$$

Moreover, it follows from Sobolev inequality that

$$\left(\int_{B_R(0)} |\omega_{L,\lambda}|^{2^*} dx\right)^{\frac{2}{2^*}} \leq C\int_{B_R(0)} \left(|\nabla\omega_{L,\lambda}|^2 + |\omega_{L,\lambda}|^2\right) dx. \tag{3.30}$$

By (3.27)–(3.30), Sobolev inequality and the boundedness of $\{u_{\lambda,R}\}$ in $E_{\lambda,R}$, there exists a constant $\tilde{C} > 0$ such that

$$\begin{aligned} \left(\int_{B_R(0)} |\omega_{L,\lambda}|^{2^*} dx\right)^{\frac{2}{2^*}} &\leq C\int_{B_R(0)} (|\nabla\omega_{L,\lambda}|^2 + |\omega_{L,\lambda}|^2) dx \\ &\leq C\beta^2 \left\{\vartheta \left(\int_{B_R(0)} |u_{\lambda,R}|^p dx\right)^{\frac{p-2}{p}} \left(\int_{B_R(0)} |\omega_{L,\lambda}|^p dx\right)^{\frac{2}{p}} \right. \\ &\quad \left. + \left(\int_{B_R(0)} |u_{\lambda,R}|^{2^*} dx\right)^{\frac{2^*-2}{p}} \left(\int_{B_R(0)} |\omega_{L,\lambda}|^{2^*} dx\right)^{\frac{2}{2^*}} \right\} \\ &\leq \tilde{C}\beta^2 \left(\vartheta \left(\int_{B_R(0)} |\omega_{L,\lambda}|^p dx\right)^{\frac{2}{p}} \right. \\ &\quad \left. + \left(\int_{B_R(0)} |\omega_{L,\lambda}|^{2^*} dx\right)^{\frac{2}{2^*}} \right). \end{aligned}$$

Taking $\tilde{C}\beta^2 \in (0,1)$, we have

$$\left(\int_{B_R(0)} |\omega_{L,\lambda}|^{2^*} dx\right)^{\frac{2}{2^*}} \leq \tilde{C}\beta^2\vartheta \left(\int_{B_R(0)} |\omega_{L,\lambda}|^p dx\right)^{\frac{2}{p}}.$$

By the Fatou's Lemma for variable L, we obtain

$$\left(\int_{B_R(0)} |u_\lambda|^{2^*\beta} dx\right)^{\frac{2}{2^*}} \leq \tilde{C}\beta^2\vartheta \left(\int_{B_R(0)} |u_\lambda|^{p\beta} dx\right)^{\frac{2}{p}}.$$

Therefore,

$$\left(\int_{B_R(0)} |u_\lambda|^{2^*\beta} dx\right)^{\frac{1}{2^*\beta}} \leq \tilde{C}^{\frac{1}{2\beta}}\beta^{\frac{1}{\beta}}\vartheta^{\frac{1}{2\beta}} \left(\int_{B_R(0)} |u_\lambda|^{p\beta} dx\right)^{\frac{1}{p\beta}}. \tag{3.31}$$

Since $\mathcal{J}_{\lambda,R}(u_{\lambda,R})$ is bounded in $E_{\lambda,R}$ and $u_{\lambda,R}$ solves Eq. (3.1), and by the arguments similar to Lemma 3.2, there exists $C > 0$ satisfying

$$\|u_{\lambda,R}\|_{\lambda,R} \leq C$$

for $\lambda \geq 1$ and $R > 0$ large enough. Passing the limits as $\lambda_n \to +\infty$ and $R_n \to +\infty$, we see that u_{λ_n,R_n} satisfies the assumption of Lemma 3.7. Therefore, $u_{\lambda_n,R_n} \to u$ in $H^1_A(\mathbb{R}^N)$. Now, by $2 < p < 2^*$, the boundedness of $\|u_{\lambda_n,R_n}\|_{L^{2^*}(\mathbb{R}^N)}$ in $\mathbb{R}$, a well-known iteration argument (see [5, Lemma 3.10]) and (3.31), there exists a constant $D_1 > 0$ such that

$$\||u_{\lambda_n,R_n}|\|_{L^\infty(\mathbb{R}^N)} \leq D_1, \quad \forall n \in \mathbb{N}.$$

Therefore, the proof of Lemma 3.8 is completed. □

Lemma 3.9. *Assume that $u_{\lambda,R}$ is a nontrivial solution of Eq. (3.1) satisfying*

$$\sup_{\lambda\geq 1}(\mathcal{J}_{\lambda,R}(u_{\lambda,R})) < \left(\frac{1}{2} - \frac{1}{\theta}\right) a_0 S^{N/2}.$$

Then there exist $\lambda' > 1$ and $R' > 0$ satisfying

$$\||u_{\lambda,R}|\|_{\infty,B_R(0)\setminus\Omega'_\Gamma} \leq a_0,$$

for any $\lambda \geq \lambda'$ and $R \geq R'$. Moreover, $u_{\lambda,R}$ solves original Eq. (3.2) for any $\lambda \geq \lambda'$ and $R \geq R'$.

Proof. Choose $R_0 > 0$ large enough such that $\overline{\Omega'_\Gamma} \subset B_{R_0}(0)$ and fix a neighborhood $\mathcal{D}$ of $\partial\Omega'_\Gamma$ satisfying

$$\mathcal{D} \subset B_{R_0}(0)\setminus\Omega_\Gamma,$$

together with Moser's iteration, there exists $C > 0$ independent of λ such that

$$\||u_{\lambda,R}|\|_{L^\infty(\partial\Omega'_\Gamma)} \leq C\||u_{\lambda,R}|\|_{L^{2^*}(\mathcal{D})}$$

for any $R \geq R_0$. Passing the limits as $\lambda_n \to +\infty$ and $R_n \to +\infty$ and using Lemma 3.7, we have $u_{\lambda_n,R_n} \to 0$ in $H^1_A(B_{R_n}(0)\setminus\Omega_\Gamma)$ for some subsequence, and then $u_{\lambda_n,R_n} \to 0$ in $H^1_A(B_{R_0}(0)\backslash\Omega_\Gamma)$, so

$$\||u_{\lambda_n,R_n}|\|_{L^{2^*}(\mathcal{D})} \to 0 \quad \text{as } n \to \infty.$$

Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\||u_{\lambda_n,R_n}|\|_{L^\infty(\partial\Omega'_\Gamma)} \le a_0, \quad \forall\, n \ge n_0.$$

Now, for $n \ge n_0$ we set $\hat{u}_{\lambda_n,R_n} : B_{R_n}(0)\backslash\Omega'_\Gamma \to \mathbb{C}$ by

$$|\hat{u}_{\lambda_n,R_n}(x)| = (|u_{\lambda_n,R_n}(x)| - a_0)^+.$$

Therefore, $\hat{u}_{\lambda_n,R_n}(x) \in H^1_A(B_{R_n}(0)\backslash\Omega'_\Gamma)$.

In what follows, we shall verify that $\hat{u}_{\lambda_n,R_n}(x) = 0$ in $B_{R_n}(0)\backslash\Omega'_\Gamma$. From this fact, we have

$$\||u_{\lambda_n,R_n}|\|_{\infty,B_{R_n}(0)\backslash\Omega'_\Gamma} \le a_0.$$

Indeed, choosing $\hat{u}_{\lambda_n,R_n}$ as a test function and extending $\hat{u}_{\lambda_n,R_n}(x) = 0$ in Ω'_Γ, we obtain

$$\begin{aligned}
&\mathrm{Re}\Bigg(\int_{B_{R_n}(0)\backslash\Omega'_\Gamma} \nabla_A u_{\lambda_n,R_n}\overline{\nabla_A \hat{u}_{\lambda_n,R_n}}dx \\
&\qquad + \int_{B_{R_n}(0)\backslash\Omega'_\Gamma} (\lambda_n Z(x) + \mathcal{V}(x) + 1)\, u_{\lambda_n,R_n}\overline{\hat{u}_{\lambda_n,R_n}}dx\Bigg) \\
&\quad \le \mathrm{Re}\Bigg(\int_{B_{R_n}(0)\backslash\Omega'_\Gamma} \widetilde{F}'_2(u_{\lambda_n,R_n})\overline{\hat{u}_{\lambda_n,R_n}}dx\Bigg).
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{B_{R_n}(0)\backslash\Omega'_\Gamma} \nabla_A u_{\lambda_n,R_n}\overline{\nabla_A \hat{u}_{\lambda_n,R_n}}dx = \int_{B_{R_n}(0)\backslash\Omega'_\Gamma} |\nabla \hat{u}_{\lambda_n,R_n}|^2 dx, \\
&\mathrm{Re}\Bigg(\int_{B_{R_n}(0)\backslash\Omega'_\Gamma} (\lambda_n Z(x) + \mathcal{V}(x) + 1) u_{\lambda_n,R_n}\overline{\hat{u}_{\lambda_n,R_n}}dx\Bigg) \\
&\quad = \mathrm{Re}\Bigg(\int_{(B_{R_n}(0)\backslash\Omega'_\Gamma)_+} (\lambda_n Z(x) + \mathcal{V}(x) + 1)(\hat{u}_{\lambda_n,R_n} + a_0)\overline{\hat{u}_{\lambda_n,R_n}}dx\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Re}\Bigg(\int_{B_{R_n}(0)\backslash\Omega'_\Gamma} \widetilde{F}'_2(u_{\lambda_n,R_n})\overline{\hat{u}_{\lambda_n,R_n}}dx\Bigg) \\
&\quad = \mathrm{Re}\Bigg(\int_{(B_{R_n}(0)\backslash\Omega'_\Gamma)_+} \frac{\widetilde{F}'_2(u_{\lambda_n,R_n})}{u_{\lambda_n,R_n}}(\hat{u}_{\lambda_n,R_n} + a_0)\overline{\hat{u}_{\lambda_n,R_n}}dx\Bigg),
\end{aligned}$$

where

$$(B_{R_n}(0)\backslash\Omega'_\Gamma)_+ = \{x \in B_{R_n}(0)\backslash\Omega'_\Gamma : |u_{\lambda_n,R_n}(x)| > a_0\}.$$

By the facts above, we deduce

$$\mathrm{Re}\left(\int_{(B_{R_n}(0)\setminus\Omega'_\Gamma)_+}\left((\lambda_n Z(x)+\mathcal{V}(x)+1)-\frac{\widetilde{F}'_2(u_{\lambda_n,R_n})}{u_{\lambda_n,R_n}}\right)(\hat{u}_{\lambda_n,R_n}+a_0)\overline{\hat{u}_{\lambda_n,R_n}}dx\right) + \int_{B_{R_n}(0)\setminus\Omega'_\Gamma}|\nabla\hat{u}_{\lambda_n,R_n}|^2dx \le 0.$$

It follows from the definition of $\widetilde{F}'_2$, (Z_3) and $\zeta > 1$, we have

$$(\lambda_n Z(x)+\mathcal{V}(x)+1)-\frac{\widetilde{F}'_2(u_{\lambda_n,R_n})}{u_{\lambda_n,R_n}} \ge b_0\left(1-\frac{1}{\zeta}\right)+1>0 \quad \text{in } (B_{R_n}(0)\setminus\Omega'_\Gamma)_+.$$

Therefore, $|\hat{u}_{\lambda_n,R_n}| = 0$ in $(B_{R_n}(0)\setminus\Omega'_\Gamma)_+$ and $B_{R_n}(0)\setminus\Omega'_\Gamma$. Moreover, there exist $\lambda' > 0$ and $R' > 0$ such that

$$\||u_{\lambda,R}|\|_{\infty,B_R(0)\setminus\Omega'_\Gamma} \le a_0$$

for any $\lambda \ge \lambda'$ and $R \ge R'$. Therefore, we finish the proof of Lemma 3.9. □

4. Minimax Level

In this section, for any $\lambda \ge 1$ and $j \in \Gamma$, we consider the following two functionals:

$$\mathcal{E}_j(u) = \frac{1}{2}\int_{\Omega_j}(|\nabla_A u|^2+(\mathcal{V}(x)+1)|u|^2)dx - \frac{1}{2^*}\int_{\Omega_j}|u|^{2^*}dx - \frac{\vartheta}{2}\int_{\Omega_j}|u|^2\log|u|^2dx$$

and

$$\mathcal{E}_{\lambda,j}(u) = \frac{1}{2}\int_{\Omega'_j}(|\nabla_A u|^2+(\lambda Z(x)+\mathcal{V}(x)+1)|u|^2)dx - \frac{1}{2^*}\int_{\Omega_j}|u|^{2^*}dx - \frac{\vartheta}{2}\int_{\Omega'_j}|u|^2\log|u|^2dx,$$

which are related to the following logarithmic equations:

$$\begin{cases} -(\nabla+\mathrm{i}A(x))^2u+\mathcal{V}(x)u = |u|^{2^*-2}u+\vartheta u\log|u|^2 & \text{in } \Omega_j, \\ u=0 & \text{on } \partial\Omega_j \end{cases} \tag{4.1}$$

and

$$\begin{cases} -(\nabla+\mathrm{i}A(x))^2u+(\lambda Z(x)+\mathcal{V}(x))u \\ \quad = |u|^{2^*-2}u+\vartheta u\log|u|^2 & \text{in } \Omega'_j, \\ \dfrac{\partial u}{\partial\eta}=0 & \text{on } \partial\Omega'_j. \end{cases} \tag{4.2}$$

Clearly, $\mathcal{E}_j$ and $\mathcal{E}_{\lambda,j}$ satisfy the mountain pass geometry. Due to the boundedness of Ω_j and Ω'_j, $\mathcal{E}_j$ and $\mathcal{E}_{\lambda,j}$ satisfy the $(PS)_c$ condition, together with the same arguments in Sec. 3, then there exist two nontrivial functions $w_j \in H_A^1(\Omega_j)$ and $w_{\lambda,j} \in H_A^1(w_j')$ satisfying

$$\mathcal{E}_j(w_j) = c_j, \quad \mathcal{E}_{\lambda,j}(w_{\lambda,j}) = c_{\lambda,j} \quad \text{and} \quad \mathcal{E}_j'(w_j) = \mathcal{E}_{\lambda,j}'(w_{\lambda,j}) = 0,$$

where

$$c_j = \inf_{\gamma\in\Upsilon_j} \max_{t\in[0,1]} \mathcal{E}_j(\gamma(t)),$$
$$c_{\lambda,j} = \inf_{\gamma\in\Upsilon_{\lambda,j}} \max_{t\in[0,1]} \mathcal{E}_{\lambda,j}(\gamma(t))$$

and

$$\Upsilon_j = \{\gamma \in C([0,1], H_A^{0,1}(\Omega_j)) : \gamma(0) = 0 \text{ and } \mathcal{E}_j(\gamma(1)) < 0\},$$
$$\Upsilon_{\lambda,j} = \{\gamma \in C([0,1], H_A^1(\Omega_j')) : \gamma(0) = 0 \text{ and } \mathcal{E}_{\lambda,j}(\gamma(1)) < 0\}.$$

In fact, a simple computation implies

$$c_j = \inf_{u\in\mathcal{M}_j} \mathcal{E}_j(u),$$
$$c_{\lambda,j} = \inf_{u\in\mathcal{M}_j'} \mathcal{E}_{\lambda,j}(u),$$

where

$$\mathcal{M}_j = \{u \in H_A^{0,1}(\Omega_j)\backslash\{0\} : \mathcal{E}_j'(u)u = 0\}$$

and

$$\mathcal{M}_j' = \{u \in H_A^1(\Omega_j')\backslash\{0\} : \mathcal{E}_{\lambda,j}'(u)u = 0\}.$$

In addition, by a direct computation, there exists $\kappa > 0$ satisfying: if $u \in \mathcal{M}_j$ for any $j \in \Gamma$, then

$$\|u\|_j > \kappa, \tag{4.3}$$

where $\|\cdot\|_j$ is defined by

$$\|u\|_j^2 = \int_{\Omega_j} (|\nabla_A u|^2 + \mathcal{V}(x)|u|^2)dx.$$

Particularly, it follows from $w_j \in \mathcal{M}_j$ that $\|w_{\lambda,j}\|_j > \kappa$, where $w_{\lambda,j} = w_j|_{\Omega_j}$ for all $j \in \Gamma$.

Moreover, we obtain the following important result.

Lemma 4.1. *For $j \in \Gamma$, the following properties are satisfied:*

(i) $0 < c_{\lambda,j} \le c_j$ *for all* $\lambda \ge 1$.
(ii) c_j ($c_{\lambda,j}$, *respectively*) *is a least energy level for* $\mathcal{E}_j(u)$ ($\mathcal{E}_{\lambda,j}(u)$, *respectively*).
(iii) $c_{\lambda,j} \to c_j$ *as* $j \to \infty$.

Proof. By the arguments similar to Lemma 5.1 in [24], we can obtain the conclusions of Lemma 4.1, so we omit them here. □

In the following, $c_\Gamma = \sum_{j=1}^{l} c_j$ with $c_\Gamma \in (0, (\frac{1}{2} - \frac{1}{\theta})a_0 S^{N/2})$ and $\mathcal{T} > 0$ is a constant sufficiently large, which does not depend on λ and $R > 0$ large enough, such that

$$0 < \left\langle \mathcal{E}_j^0{}' \left(\frac{1}{\mathcal{T}} \omega_j \right), \frac{1}{\mathcal{T}} \omega_j \right\rangle, \quad \langle \mathcal{E}_j'(\mathcal{T}\omega_j), \mathcal{T}\omega_j \rangle < 0, \quad \forall\, j \in \Gamma. \tag{4.4}$$

Therefore, it follows from the definition of c_j that

$$\max_{s \in [1/\mathcal{T}^2, 1]} \mathcal{E}_j(s\mathcal{T}\omega_j) = c_j$$

for all $j \in \Gamma$. Without loss of generality, we consider $\Gamma = \{1, 2, \ldots, l\}$ with $l \leq k$ and fix

$$\gamma_0(s_1, s_2, \ldots, s_l)(x) = \sum_{j=1}^{l} s_j \mathcal{T} \omega_j(x), \quad \forall\, (s_1, s_2, \ldots, s_l) \in [1/\mathcal{T}^2, 1]^l,$$

$$\Gamma_* = \{\gamma \in C([1/\mathcal{T}^2, 1]^l, E_{\lambda,R} \backslash \{0\}) : \gamma = \gamma_0 \text{ on } \partial([1/\mathcal{T}^2, 1]^l)\}$$

and

$$b_{\lambda,R,\Gamma} = \inf_{\gamma \in \Gamma_*} \max_{(s_1, s_2, \ldots, s_l) \in [1/\mathcal{T}^2, 1]^l} \mathcal{J}_{\lambda,R}(\gamma(s_1, s_2, \ldots, s_l)).$$

Note that $\gamma_0 \in \Gamma_*$, then $\Gamma_* \neq \emptyset$ and $b_{\lambda,R,\Gamma}$ is well defined.

Lemma 4.2. *For each $\gamma \in \Gamma_*$, there exists $(t_1, t_2, \ldots, t_l) \in [1/\mathcal{T}^2, 1]^l$ satisfying*

$$\mathcal{E}_{\lambda,j}'(\gamma(t_1, \ldots, t_l))\gamma(t_1, \ldots, t_l) = 0 \quad \text{for } j \in \{1, \ldots, l\}.$$

Proof. Let $\gamma \in \Gamma_*$, we consider the map $\hat{\gamma} : [1/\mathcal{T}^2, 1]^l \to \mathbb{R}^l$ defined by

$$\hat{\gamma}(s_1, \ldots, s_l) = (\mathcal{E}_{\lambda,1}'(\gamma(s_1, \ldots, s_l))\gamma(s_1, \ldots, s_l), \ldots, \mathcal{E}_{\lambda,l}'$$
$$(\gamma(s_1, \ldots, s_l))\gamma(s_1, \ldots, s_l)).$$

For any $(s_1, \ldots, s_l) \in \partial([1/\mathcal{T}^2, 1]^l)$, we have

$$\gamma(s_1, \ldots, s_l) = \gamma_0(s_1, \ldots, s_l).$$

Hence, with the aid of (4.4) and Miranda's theorem [30], we complete the proof of Lemma 4.2. □

By the same arguments as that of [24, Lemma 5.3], we have the following results.

Lemma 4.3. *The following facts are satisfied*:

(a) *For any $\lambda \geq 1$ and $R > 0$ large enough*, $\sum_{j=1}^{l} c_{\lambda,j} \leq b_{\lambda,R,\Gamma} \leq c_\Gamma$.
(b) *For $\gamma \in \Gamma_*$ and $(s_1, \ldots, s_l) \in \partial([1/\mathcal{T}^2, 1]^l)$*,

$$\mathcal{J}_{\lambda,R}(\gamma(s_1, \ldots, s_l)) < c_\Gamma, \quad \forall\, \lambda > 0.$$

Lemma 4.4. *The following facts are true*:

(a) $b_{\lambda,R,\Gamma}$ *is a critical value of* $\mathcal{J}_{\lambda,R}$ *for* $\lambda \geq 1$ *and* $R > 0$ *large enough.*
(b) $b_{\lambda,R,\Gamma} \to c_\Gamma$, *when* $\lambda \to +\infty$ *uniformly for* $R > 0$ *large enough.*

Proof. By the arguments similar to that of [24, Corollary 5.1], we can finish the proof of Lemma 4.4, so we omit these here. □

5. Uniform Estimates

First of all, we define $\mathcal{J}_{\lambda,R}^{c_\Gamma}$ and Σ by

$$\mathcal{J}_{\lambda,R}^{c_\Gamma} := \{u \in E_{\lambda,R} : \mathcal{J}_{\lambda,R}(u) \leq c_\Gamma\}$$

and

$$\Sigma := \left\{u \in E_{\lambda,R} : \|u\|_{\lambda,\Omega_j'} > \frac{\kappa}{2\mathcal{T}},\ \forall\, j \in \Gamma\right\},$$

where κ and $\mathcal{T}$ are fixed in (4.3) and (4.4), respectively. Fixing $\varsigma = \frac{\kappa}{8\mathcal{T}}$ and $\mu > 0$, we define

$$\begin{aligned} A_{\mu,R}^{\lambda} = \{u \in \Sigma_{2\varsigma} : \mathcal{J}_{\lambda,B_R(0)\setminus\Omega_\Gamma'}(u) \geq 0, \|u\|_{\lambda,B_R(0)\setminus\Omega_\Gamma}^2 \leq \mu, |\mathcal{E}_{\lambda,j}(u) - c_j| \\ \leq \mu,\ \forall\, j \in \Gamma\}, \end{aligned}$$

where Σ_r denotes the set

$$\Sigma_r = \left\{u \in E_{\lambda,R} : \inf_{v\in\Sigma} \|u - v\|_{\lambda,\Omega_j'} \leq r,\ \forall\, j \in \Gamma\right\} \quad \text{for } r > 0.$$

By $w = \sum_{j=1}^{l} w_j \in A_{\mu,R}^{\lambda} \cap \mathcal{J}_{\lambda,R}^{c_\Gamma}$, we obtain that $A_{\mu,R}^{\lambda} \cap \mathcal{J}_{\lambda,R}^{c_\Gamma} \neq \emptyset$.

Now, we shall show uniform estimate of $\|\mathcal{J}_{\lambda,R}'(u)\|$ in the set $(A_{2\mu,R}^{\lambda}\setminus A_{\mu,R}^{\lambda}) \cap \mathcal{J}_{\lambda,R}^{c_\Gamma}$.

Lemma 5.1. *For each* $\mu > 0$, *there exist* $\lambda_* > 1, R^* > 0$ *large enough and* $\sigma_0 > 0$ *independent of* λ *and* $R > 0$ *sufficient large such that*

$$\|\mathcal{J}_{\lambda,R}'(u)\| \geq \sigma_0 \quad \textit{for } \lambda \geq \lambda_*, \quad R \geq R^* \quad \textit{and} \quad u \in (A_{2\mu,R}^{\lambda}\setminus A_{\mu,R}^{\lambda}) \cap \mathcal{J}_{\lambda,R}^{c_\Gamma}.$$

Proof. Arguing by contradiction, we assume that there exist $\lambda_n, R_n \to \infty$ and $u_n \in (A_{2\mu,R_n}^{\lambda_n}\setminus A_{\mu,R_n}^{\lambda_n}) \cap \mathcal{J}_{\lambda_n,R_n}^{c_\Gamma}$ such that

$$\|\mathcal{J}_{\lambda_n,R_n}'(u_n)\| \to 0.$$

Since $u_n \in A_{2\mu,R_n}^{\lambda_n}$, we know that $\{\|u_n\|_{\lambda_n,R_n}\}$ and $\mathcal{J}_{\lambda_n,R_n}(u_n)$ are both bounded in E_{λ_n,R_n}. By Lemma 3.7, we may extract a subsequence such that $u_n \to u$ in

$H^1_A(\Omega_\Gamma)$ and u is a solution of Eq. (4.1). Moreover, we obtain

$$u_n \to u \quad \text{in } H^1_A(\mathbb{R}^N),$$

$$\|u_n\|^2_{\lambda_n, B_{R_n}(0)\setminus\Omega_\Gamma} \to 0 \quad \text{and} \quad \mathcal{J}_{\lambda_n,R_n}\{u_n\} \to \mathcal{E}_\Gamma(u) \in (-\infty, c_\Gamma].$$

Note that $\{u_n\} \subset \Sigma_{2\varsigma}$, it holds

$$\|u_n\|^2_{\lambda_n,\Omega'_j} > \frac{\kappa}{4\mathcal{T}}, \quad \forall\, j \in \Gamma.$$

Taking the limit as $n \to +\infty$, we obtain that

$$\|u\|^2_j \geq \frac{\kappa}{4\mathcal{T}} > 0, \quad \forall\, j \in \Gamma.$$

This implies $u|_{\Omega_j} \neq 0, j = 1, \ldots, l$, and $\mathcal{E}'_\Gamma(u) = 0$. Using (4.3), we obtain

$$\|u\|^2_j > \frac{\kappa}{2\mathcal{T}} > 0, \quad \forall\, j \in \Gamma.$$

Therefore, $\mathcal{E}_\Gamma(u) \geq c_\Gamma$. On the other hand, by the facts that $\mathcal{E}_{\lambda_n,R_n}(u_n) \leq c_\Gamma$ and $\mathcal{J}_{\lambda_n,R_n}(u_n) \to \mathcal{E}_\Gamma(u)$ as $n \to +\infty$, we obtain $\mathcal{E}_\Gamma(u) = c_\Gamma$. Hence, for n sufficiently large,

$$\|u_n\|^2_j > \frac{\kappa}{2\mathcal{T}}, \quad |\mathcal{J}_{\lambda_n,R_n}(u_n) - c_\Gamma| \leq \mu$$

for any $j \in \Gamma$. Consequently, $u_n \in A^{\lambda_n}_{\mu,R_n}$ for large n, which contradicts $u_n \in (A^{\lambda_n}_{2\mu,R_n} \setminus A^{\lambda_n}_{\mu,R_n})$. This completes the proof of Lemma 5.1. □

Now, we define μ_0 and μ_* as follows:

$$\min_{t\in\partial[1/\mathcal{T}^2,1]^l} |\mathcal{E}_\Gamma(\gamma_0(t)) - c_\Gamma| = \mu_0 > 0$$

and

$$\mu_* = \min\{\mu_1, \kappa, \ell/2\},$$

where $\varsigma = \frac{\kappa}{8\mathcal{T}}$ is given before and $\ell > \max\{\|w_j\|_{H^1_0(\Omega_j t)} : j = 1, \ldots, l\}$. For each $t > 0$, we also define

$$\mathcal{B}^\lambda_t := \{u \in E_\lambda(B_R(0)) : \|u\|_{\lambda,R} \leq t\}.$$

Lemma 5.2. *Let $\mu \in (0, \mu_*), \lambda_* > 1$ and $R^* > 0$ sufficiently large as given in Lemma 5.1. Then there exists a nontrivial solution $u_{\lambda,R}$ of Eq. (3.1) such that $u_\lambda \in A^\lambda_{\mu,R} \cap \mathcal{J}^{c_\Gamma}_{\lambda,R} \cap \mathcal{B}^\lambda_{\ell+1}$ for $\lambda \geq \lambda_*$ and $R \geq R^*$.*

Proof. Arguing by contradiction, we suppose that there exist no critical points for the functional $\mathcal{J}_{\lambda,R}(u)$ in $A^\lambda_{\mu,R} \cap \mathcal{J}^{c_\Gamma}_{\lambda,R} \cap \mathcal{B}^\lambda_{\ell+1}$ for $\lambda \geq \lambda_*$. Since $\mathcal{J}_{\lambda,R}$ satisfies the (PS) condition, there exists a constant $\tilde{d}_\lambda > 0$ such that

$$\|\mathcal{J}'_{\lambda,R}(u)\| \geq \tilde{d}_\lambda$$

for all $u \in A^{\lambda}_{\mu,R} \cap \mathcal{J}^{c_\Gamma}_{\lambda,R} \cap \mathcal{B}^{\lambda}_{\ell+1}$. By Lemma 5.1, we obtain

$$\|\mathcal{J}'_{\lambda,R}(u)\| \geq \sigma_0 \quad \text{for all } u \in (A^{\lambda}_{2\mu,R} \backslash A^{\lambda}_{\mu,R}) \cap \mathcal{J}^{c_\Gamma}_{\lambda,R},$$

where $\sigma_0 > 0$ is independent of λ. Now, we define $\Phi : E_{\lambda,R} \to \mathbb{R}$, which is a continuous functional such that

$$\begin{aligned} &\Phi(u) = 1 && \text{for } u \in A^{\lambda}_{3\mu/2,R} \cap \Upsilon_{\varsigma} \cap \mathcal{B}^{\lambda}_{\ell}, \\ &\Phi(u) = 0 && \text{for } u \notin A^{\lambda}_{2\mu,R} \cap \Upsilon_{2\varsigma} \cap \mathcal{B}^{\lambda}_{\ell+1}, \\ &0 \leq \Phi(u) \leq 1 && \text{for all } u \in E_{\lambda,R} \end{aligned}$$

and $\mathcal{H} : \mathcal{J}^{c_\Gamma}_{\lambda,R} \to E_{\lambda}(B_R(0))$ is a function given by

$$\mathcal{H}(u) := \begin{cases} -\Phi(u)\dfrac{Y(u)}{\|Y(u)\|}, & u \in A^{\lambda}_{2\mu,R} \cap \mathcal{B}^{\lambda}_{\ell+1}, \\ 0, & u \notin A^{\lambda}_{2\mu,R} \cap \mathcal{B}^{\lambda}_{\ell+1}, \end{cases}$$

where Y is a pseudo-gradient vector field for $\mathcal{J}_{\lambda,R}$ on $\mathscr{A} = \{u \in E_{\lambda,R} : \mathcal{J}'_{\lambda,R}(u) \neq 0\}$. Since $\mathcal{J}'_{\lambda,R}(u) \neq 0$ for $u \in A^{\lambda}_{2\mu,R} \cap \Phi^{c_\Gamma}_{\lambda,R}$, we know that $\mathcal{H}$ is well defined. By the following fact:

$$\|\mathcal{H}(u)\| \leq 1$$

for all $\lambda \geq \lambda_*$ and $u \in \Phi^{c_\Gamma}_{\lambda,R}$, we have

$$\frac{d}{dt}\mathcal{J}_{\lambda,R}(\eta(t,u)) \leq -\Phi(\eta(t,u))\|\mathcal{J}'_{\lambda,R}(\eta(t,u))\| \leq 0, \tag{5.1}$$

$$\left\|\frac{d\eta}{dt}\right\|_{\lambda} = \|\mathcal{H}(\eta)\|_{\lambda} \leq 1, \quad \eta(t,u) = u \quad \text{for all } t \geq 0 \quad \text{and}$$
$$u \in \mathcal{J}^{c_\Gamma}_{\lambda,R} \backslash (A^{\lambda}_{2\mu,R} \cap \mathcal{B}^{\lambda}_{r+1}), \tag{5.2}$$

where the deformation flow $\eta : [0,\infty) \times \mathcal{J}^{c_\Gamma}_{\lambda,R} \to \mathcal{J}^{c_\Gamma}_{\lambda,R}$ defined by

$$\frac{d\eta}{dt} = \mathcal{H}(\eta) \quad \text{and} \quad \eta(0,u) = u \in \mathcal{J}^{c_\Gamma}_{\lambda,R}.$$

Next, we consider two paths:

(1) The path $t \to \eta(t, \gamma_0(t))$, where $t = (t_1, \ldots, t_l) \in [1/\mathcal{T}^2, 1]^l$.
If $\mu \in (0, \mu_*)$, we obtain

$$\gamma_0(t) \notin A^{\lambda}_{2\mu,R} \quad \text{for all } t \in \partial([1/\mathcal{T}^2, 1]^l).$$

As $\mathcal{J}_{\lambda,R}(\gamma_0(t)) \leq c_\Gamma$ for any $t \in \partial([1/\mathcal{T}^2, 1]^l)$, by (5.2), we get

$$\eta(t, \gamma_0(t)) = \gamma_0(t) \quad \text{for all } t \in \partial([1/\mathcal{T}^2, 1]^l).$$

Hence, $\eta(t, \gamma_0(t)) \in \Gamma_*$ for all $t \geq 0$.

(2) The path $t \to \gamma_0(t)$, where $t = (t_1, \ldots, t_l) \in [1/\mathcal{T}^2, 1]^l$.

In view of $\text{supp}(t) \subset \overline{\Omega_\Gamma}$ for all $t \in [1/\mathcal{T}^2, 1]^l$, $\mathcal{J}_{\lambda,R}(\gamma_0(t))$ does not depend on $\lambda > 0$. Note that

$$\mathcal{J}_{\lambda,R}(\gamma_0(t)) \le c_\Gamma, \quad \forall\, t \in [1/\mathcal{T}^2, 1]^l$$

and

$$\mathcal{J}_{\lambda,R}(\gamma_0(t)) = c_\Gamma \Leftrightarrow t_j = 1/\mathcal{T}, \quad \forall\, j \in \Gamma.$$

Therefore,

$$\mathfrak{m}_0 := \sup\{\mathcal{J}_{\lambda,R}(u) : u \in \gamma_0([1/\mathcal{T}^2, 1]^l) \backslash A_\mu^\lambda\}$$

is independent of λ, $R > 0$ and $\mathfrak{m}_0 < c_\Gamma$. Moreover, we obtain that there exists a $\kappa_* > 0$ satisfying

$$|\mathcal{J}_{\lambda,R}(u) - \mathcal{J}_{\lambda,R}(v)| \le \kappa_* \|u - v\|_{\lambda,R}$$

for any $u, v \in \mathcal{B}_\ell^\lambda$.

In the following, we verify that if $\mathcal{T}_* > 0$ is large enough, it holds

$$\max_{t \in [1/\mathcal{T}^2, 1]^l} \mathcal{J}_\lambda(\eta(\mathcal{T}_*, \gamma_0(t))) < \max\left\{\mathfrak{m}_0, c_\Gamma - \frac{1}{2\kappa_*}\sigma_0\mu\right\}. \tag{5.3}$$

Indeed, write $u = \gamma_0(t)$, $t \in [1/\mathcal{T}^2, 1]^l$. If $u \notin A_{\mu,R}^\lambda$, by (5.2), we must have that

$$\mathcal{J}_{\lambda,R}(\eta(t, u)) \le \mathcal{J}_\lambda(\eta(0, u)) = \mathcal{J}_{\lambda,R}(u) \le \mathfrak{m}_0$$

for all $t \ge 0$. If $u \in A_{\mu,R}^\lambda$, let $\hat{\eta}(t) = \eta(t, u)$, $\hat{d}_\lambda := \min\{\tilde{d}_\lambda, \sigma_0\}$ and $\mathcal{T}_* = \frac{\sigma_0\mu}{2\kappa_*\hat{d}_\lambda} > 0$, now we consider the following cases:

(1) $\hat{\eta}(t) \in A_{3\mu/2,R}^\lambda \cap \Sigma_\varsigma \cap \mathcal{B}_\ell^\lambda$ for all $t \in [0, \mathcal{T}_*]$.
(2) $\hat{\eta}(t_0) \notin A_{3\mu/2,R}^\lambda \cap \Sigma_\varsigma \cap \mathcal{B}_\ell^\lambda$ for some $t_0 \in [0, \mathcal{T}_*]$.

If (1) is true, then $\Phi(\hat{\eta}(t)) \equiv 1$ and $\|\mathcal{J}'_{\lambda,R}(\hat{\eta}(t))\| \ge \hat{d}_\lambda$ for all $t \in [0, \mathcal{T}_*]$. By (5.1), we have

$$\begin{aligned}
\mathcal{J}_{\lambda,R}(\hat{\eta}(\mathcal{T}_*)) &= \mathcal{J}_{\lambda,R}(u) + \int_0^{\mathcal{T}_*} \frac{d}{ds}\mathcal{J}_{\lambda,R}(\hat{\eta}(s))ds \\
&\le c_\Gamma - \int_0^{\mathcal{T}_*} \hat{d}_\lambda ds \\
&= c_\Gamma - \hat{d}_\lambda \mathcal{T}_* \\
&\le c_\Gamma - \frac{\sigma_0\mu}{2\kappa_*}.
\end{aligned}$$

If (2) is true, the following cases should be considered.

(i) There exists $t_2 \in [0, \mathcal{T}_*]$ such that $\hat{\eta}(t_2) \notin \Sigma_\varsigma$.
In this case, we have

$$\|\hat{\eta}(t_2) - \hat{\eta}(t_1)\|_{\lambda,R} \ge \delta > \mu \quad \text{as } t_1 = 0,$$

since $\hat{\eta}(0) = u \in \Sigma$.

(ii) There exists $t_2 \in [0, \mathcal{T}_*]$ such that $\hat{\eta}(t_2) \notin \mathcal{B}_\ell^\lambda$.

In this case, for $t_1 = 0$, we obtain

$$\|\hat{\eta}(t_2) - \hat{\eta}(t_1)\|_{\lambda,R} \geq \ell > \mu,$$

because $\hat{\eta}(0) = u \in \mathcal{B}_\ell^\lambda$.

(iii) $\hat{\eta}(t) \notin \Sigma_\varsigma \cap \mathcal{B}_\ell^\lambda$, and there exist t_1 and t_2 satisfying $0 \leq t_1 < t_2 \leq \mathcal{T}_*$ such that $\hat{\eta}(t) \in A_{3\mu/2,R}^\lambda \backslash A_{\mu,R}^\lambda$ for all $t \in [t_1, t_2]$ with

$$|\mathcal{J}_{\lambda,R}(\hat{\eta}(t_1)) - c_\Gamma| = \mu \quad \text{and} \quad |\mathcal{J}_{\lambda,R}(\hat{\eta}(t_2)) - c_\Gamma| = \frac{3\mu}{2}.$$

According to the definition of κ_*,

$$\begin{aligned}\|\hat{\eta}(t_2) - \hat{\eta}(t)\|_{\lambda,R} &\geq \frac{1}{\kappa_*}|\mathcal{J}_{\lambda,R}(\hat{\eta}(t_2)) - \mathcal{J}_{\lambda,R}(\hat{\eta}(t_1))| \\ &\geq \frac{1}{\kappa_*}(|\mathcal{J}_{\lambda,R}(\hat{\eta}(t_2)) - c_{j_0}| - |\mathcal{J}_{\lambda,R}(\hat{\eta}(t_1)) - c_{j_0}|) \\ &\geq \frac{1}{2\kappa_*}\mu.\end{aligned}$$

By Mean Value Theorem and $t_2 - t_1 \geq \frac{1}{2\kappa_*}\mu$, we have

$$\begin{aligned}\mathcal{J}_{\lambda,R}(\hat{\eta}(\mathcal{T}_*)) &= \mathcal{J}_{\lambda,R}(u) + \int_0^{\mathcal{T}_*} \frac{d}{ds}\mathcal{J}_{\lambda,R}(\hat{\eta}(s))ds \\ &\leq \mathcal{J}_{\lambda,R}(u) - \int_0^{\mathcal{T}_*} \Phi(\hat{\eta}(s))\|\mathcal{J}'_{\lambda,R}(\hat{\eta}(s))\|ds \\ &\leq c_\Gamma - \int_{t_1}^{t_2} \sigma_0 ds \\ &= c_\Gamma - \sigma_0(t_2 - t_1) \\ &\leq c_\Gamma - \frac{\sigma_0\mu}{2\kappa_*},\end{aligned}$$

which yields that (5.3) is true.

Fixing $\widetilde{\eta}(t) = \eta(\mathcal{T}_*, \gamma_0(t))$, we have $\widetilde{\eta}(t) \in \Sigma_{2\varsigma}$ and $\widetilde{\eta}(t)(t)|_{\Omega'_j} \neq 0$ for all $j \in \Gamma$. Hence, $\widetilde{\eta}(t) \in \Gamma_*$ and

$$b_{\lambda,R,\Gamma} \leq \max_{s\in[1/\mathcal{T}^2,1]^l} \mathcal{J}_{\lambda,R}(\widetilde{\eta}(s)) \leq \max\left\{\mathfrak{m}_0, c_\Gamma - \frac{\sigma_0\mu}{2\kappa_*}\right\} < c_\Gamma.$$

However, by Lemma 4.4, $b_{\lambda,R,\Gamma} \to c_\Gamma$ as $\lambda \to \infty$ uniformly holds for $R > 0$ large enough, we can obtain a contradiction. Therefore, the proof of Lemma 5.2 is finished. □

6. Proof of Theorem 1.1

Using Lemma 5.2, for $\mu \in (0, \mu_*)$ and $\lambda_* > 1$, we can find a nontrivial solution $u_{\lambda,R}$ for Eq. (3.1) such that $u_{\lambda,R} \in A_{\mu,R}^\lambda \cap \mathcal{J}_{\lambda,R}^{c_\Gamma} \cap \mathcal{B}_{\ell+1}^\lambda$ for all $\lambda \geq \lambda_*$ and $R \geq R^*$.

Now fix $\lambda \geq \lambda_*$ and let $R_n \to +\infty$, then there exists a solution $u_{\lambda,n} = u_{\lambda,R_n}$ for Eq. (3.1) with

$$u_{\lambda,n} \in A^{\lambda}_{\mu,R_n} \cap \mathcal{J}^{c_\Gamma}_{\lambda,R_n} \cap \mathcal{B}^{\lambda}_{\ell+1}, \quad \forall\, n \in \mathbb{N}.$$

By the boundedness of $\{u_{\lambda,n}\}$ in $H^1_A(\mathbb{R}^N)$, we may suppose that for some $u_\lambda \in H^1_A(\mathbb{R}^N)$,

$$\begin{aligned} &\mathcal{J}_{\lambda,R_n}(u_{\lambda,n}) \to \tilde{d} \leq c_\Gamma, \\ &\qquad u_{\lambda,n} \to u_\lambda \quad \text{in } H^1_A(\mathbb{R}^N), \\ &\qquad u_{\lambda,n} \to u_\lambda \quad \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{for any } s \in [1, 2^*) \end{aligned}$$

and

$$u_{\lambda,n}(x) \to u_\lambda(x) \quad \text{a.e. } x \in \mathbb{R}^N.$$

By Lemma 3.9, we have

$$0 \leq |u_{\lambda,n}(x)| \leq a_0, \quad \forall\, x \in \mathbb{R}^N \backslash \Omega_\Gamma.$$

Therefore,

$$0 \leq |u_\lambda(x)| \leq a_0, \quad \forall\, x \in \mathbb{R}^N \backslash \Omega_\Gamma.$$

By the same arguments as Lemma 3.4, we can obtain the following lemma.

Lemma 6.1. *For $\lambda \geq 1$, $u_{\lambda,n} \to u_\lambda$ in $H^1_A(\mathbb{R}^N)$. Furthermore,*

$$F_1(u_{\lambda,n}) \to F_1(u_\lambda) \quad \text{and} \quad F_1'(u_{\lambda,n})u_{\lambda,n} \to F_1'(u_\lambda)u_\lambda \quad \text{in } L^1(\mathbb{R}^N).$$

By Lemma 6.1, we consider the energy functional $\mathcal{J}_\lambda : E_\lambda \to (-\infty, +\infty]$,

$$\begin{aligned} \mathcal{J}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla_A u|^2 + (\lambda Z(x) + \mathcal{V}(x) + 1)|u|^2)dx - \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \log |u|^2 dx \\ - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx. \end{aligned}$$

It is easy to see that u_λ is a critical point of $\mathcal{J}_\lambda$ satisfying

$$\begin{aligned} u_\lambda \in A^\lambda_\mu = \big\{ u \in (\Sigma_\infty)_{2\varsigma} : \mathcal{J}_{\lambda,\mathbb{R}^N\backslash\Omega'_\Gamma}(u) \geq 0, \|u\|^2_{\lambda,\mathbb{R}^N\backslash\Omega_\Gamma} \\ \leq \mu, |\mathcal{J}_{\lambda,j}(u) - c_j| \leq \mu,\ \forall\, j \in \Gamma \big\}, \end{aligned}$$

where

$$\Sigma_\infty = \left\{ u \in E_\lambda : \|u\|_{\lambda,\Omega'_j} > \frac{\kappa}{2\mathcal{T}},\ \forall\, j \in \Gamma \right\}$$

and

$$(\Sigma_\infty)_\ell = \left\{ u \in E_\lambda : \inf_{v \in \Sigma_\infty} \|u - v\|_{\lambda,\Omega'_j} \leq \ell,\ \forall\, j \in \Gamma \right\}.$$

Proof of Theorem 1.1. Let $\lambda_n \to +\infty$ and $\mu_n \in (0, \mu_*)$ with $\mu_n \to 0$, then there exists a solution $u_n \in A^{\lambda_n}_{\mu_n}$ of Eq. (1.1) with $\lambda = \lambda_n$. Therefore, $\{u_n\}$ is bounded in $H^1_A(\mathbb{R}^N)$ such that

(a) $\|\mathcal{J}'_{\lambda_n}(u_{\lambda_n})\| = 0$, $\forall\, n \in \mathbb{N}$;

(b) $\|u_{\lambda_n}\|_{\lambda_n,\mathbb{R}^N\backslash\Omega_\Gamma} \to 0$;

(c) $\mathcal{J}_{\lambda_n}(u_n) \to \tilde{d} \le c_\Gamma$, where

$$\|\mathcal{J}'_\lambda(u)\| = \sup\{\langle \mathcal{J}'_\lambda(u), \tilde{z}\rangle : \tilde{z} \in H^1_A(\mathbb{R}^N) \text{ and } \|\tilde{z}\|_\lambda \le 1\}.$$

Taking the arguments similar to Lemma 3.7, there exists $u \in H^1_A(\mathbb{R}^N)$ satisfying $u_{\lambda_n} \to u$ in $H^1_A(\mathbb{R}^N)$, and $u \equiv 0$ in $\mathbb{R}^N\backslash\Omega_\Gamma$ and u is a nontrivial solution of the following equation:

$$\begin{cases} -(\nabla + \mathrm{i}A(x))^2 u + \mathcal{V}(x)u = \vartheta u \log|u|^2 + |u|^{2^*-2}u & \text{in } \Omega_\Gamma, \\ u = 0 & \text{on } \partial\Omega_\Gamma, \end{cases} \tag{6.1}$$

which implies $\mathcal{J}_\Gamma(u) \ge c_\Gamma$. Moreover, note that $\mathcal{J}_{\lambda_n}(u_{\lambda_n}) \to \mathcal{E}_\Gamma(u)$, then $\mathcal{E}_\Gamma(u) = \tilde{d}$ and $\tilde{d} \ge c_\Gamma$. Due to $\tilde{d} \le c_\Gamma$, we obtain that $\mathcal{E}_\Gamma(u) = c_\Gamma$, which implies that u is a least energy solution for Eq. (6.1). This finishes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Since $\Omega = \cup_{j=1}^k \Omega_j$, where k is a finite positive integer, together with Lemma 3.3, we obtain that the conclusion of Theorem 1.2 holds. □

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References

[1] C. O. Alves and D. C. de Morais Filho, Existence and concentration of positive solutions for a Schrödinger logarithmic equation, *Z. Angew. Math. Phys.* **69** (2018) 1–22.

[2] C. O. Alves and C. Ji, Multi-bump positive solutions for a logarithmic Schrödinger equation with deepening potential well, *Sci. China Math.* **65** (2022) 1577–1598.

[3] C. O. Alves and C. Ji, Multi-peak positive solutions for a logarithmic Schrödinger equation via variational methods, *Israel J. Math.* **259** (2024) 835–885.

[4] C. O. Alves and C. Ji, Multiple normalized solutions to a logarithmic Schrödinger equation via Lusternik–Schnirelmann category, *J. Geom. Anal.* **34** (2024) 198.

[5] C. O. Alves and C. Ji, Existence and concentration of positive solutions for a logarithmic Schrödinger equation via penalization method, *Calc. Var. Partial Differ. Equ.* **59** (2020) 21.

[6] C. O. Alves and V. Ambrosio, Multiple Concentrating solutions for a fractional p-Laplacian logarithmic Schrödinger equation, *Anal. Appl.* **22** (2024) 311–349.

[7] A. V. Avdeenkov and K. G. Zloshchastiev, Quantum bose liquids with logarithmic nonlinearity: Self-sustainability and emergence of spatial extent, *J. Phys. B* **44** (2011) 195–303.

[8] I. Bialynicki–Birula and J. Mycielski, Nonlinear wave mechanics, *Ann. Phys.* **100** (1976) 62–93.

[9] H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88** (1983) 486–490.

[10] D. Cao and Z. Tang, Existence and uniqueness of multi-bump bound states of nonlinear Schrödinger equations with electromagnetic fields, *J. Differ. Equ.* **222** (2006) 381–424.

[11] J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var. Partial Differ. Equ.* **3** (1995) 493–512.

[12] P. d'Avenia and C. Ji, Multiplicity and concentration results for a magnetic Schrödinger equation with exponential critical growth in $\mathbb{R}^2$, *Int. Math. Res. Not.* **2022** (2022) 862–897.

[13] Y. H. Ding and K. Tanaka, Multiplicity of positive solutions of a nonlinear Schrödinger equation, *Manuscripta Math.* **112** (2003) 109–135.

[14] W. Feng, X. Tang and L. Zhang, Existence of a positive bound state solution for logarithmic Schrödinger equation, *J. Math. Anal. Appl.* **531** (2024) 127861.

[15] E. F. Hefter, Application of the nonlinear Schrödinger equation with a logarithmic inhomogeneous term to nuclear physics, *Phys. Rev. A* **32** (1985) 1201–1204.

[16] C. Ji and V. D. Rădulescu, Multiplicity and concentration of solutions to the nonlinear magnetic Schrödinger equation, *Calc. Var. Partial Differ. Equ.* **59** (2020) 115.

[17] C. Ji and A. Szulkin, A logarithmic Schrödinger equation with asymptotic conditions on the potential, *J. Math. Anal. Appl.* **437** (2016) 241–254.

[18] C. Ji and X. Ying, Existence and concentration of positive solution for a fractional logarithmic Schrödinger equation, *Differ. Integral Equ.* **35** (2022) 677–704.

[19] C. Ji and V. D. Rădulescu, Multi-bump solutions for the nonlinear magnetic Schrödinger equation with exponential critical growth in $\mathbb{R}^2$, *Manuscripta Math.* **164** (2021) 509–542.

[20] C. Ji, Y. Zhang and V. D. Rădulescu, Multi-bump solutions for the magnetic Schrödinger-Poisson system with critical growth, *Electron. J. Qual. Theory Differ. Equ.* (2022) 21.

[21] C. Ji and V. D. Rădulescu, Multi-bump solutions for the nonlinear magnetic Choquard equation with deepening potential well, *J. Differ. Equ.* **306** (2022) 251–279.

[22] Q. Li, K. Teng and X. Wu, Ground states for fractional Schrödinger equations with electromagnetic fields and critical growth, *Acta Math. Sci.* **40** (2020) 59–74.

[23] Q. Li, J. Nie and W. Wang, Nontrivial solutions for fractional Schrödinger equations with electromagnetic fields and critical or supercritical growth, *Qual. Theory Dyn. Syst.* **23** (2024) 65.

[24] S. Liang, D. Repovš and B. Zhang, On the fractional Schrödinger–Kirchhoff equations with electromagnetic fields and critical nonlinearity, *Comput. Math. Appl.* **75** (2018) 1778–1794.

[25] J. Liu and P. Pucci, Existence of solutions for a double-phase variable exponent equation without the Ambrosetti–Rabinowitz condition, *Adv. Nonlinear Anal.* **12** (2023) 20220292.

[26] X. Lin, X. Tang and N. Zhang, Mountain-pass-type solutions for Schrödinger equations in $\mathbb{R}^2$ with unbounded or vanishing potentials and critical exponential growth nonlinearities, *Adv. Nonlinear Anal.* **13** (2024) 20230127.

[27] P. L. Lions, The concentration compactness principle in the calculus of variations. The locally compact case. Part I and II, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984) 109–145, 223–283.

[28] G. Lauro, A note on a Korteweg fluid and the hydrodynamic form of the logarithmic Schrödinger equation, *Geophys. Astrophys. Fluid Dyn.* **102** (2008) 373–380.

Delete [33], because it is not used in this paper.

[29] E. H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics (American Mathematical Society, Providence, RI, 2001).
[30] C. Miranda, Un'osservazione su un teorema di Brouwer, *Boll. Unione. Mat. Ital.* **9** (1940) 5–7.
[31] G. Palatucci and A. Pisante, Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces, *Calc. Var. Partial Differ. Equ.* **50** (2014) 799–829.
[32] M. del Pino and P. L. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var. Partial Differ. Equ.* **4** (1996) 121–137.
[33] M. del Pino and J. Dolbeault, The optimal Euclidean L^p-Sobolev logarithmic inequality, *J. Funct. Anal.* **197** (2003) 151–161.
[34] P. Pucci and Y. Ye, Existence of nontrivial solutions for critical Kirchhoff-Poisson systems in the Heisenberg group, *Adv. Nonlinear Stud.* **22** (2022) 361–371.
[35] L. Shen and M. Squassina, Existence and concentration of normalized solutions for p-Laplacian equations with logarithmic nonlinearity, *J. Differ. Equ.* **421** (2025) 1–49.
[36] W. Shuai, Multiple solutions for logarithmic Schrödinger equations, *Nonlinearity* **32** (2019) 2201.
[37] Y. Song and S. Shi, Existence and multiplicity solutions for the p-fractional Schrödinger–Kirchhoff equations with electromagnetic fields and critical nonlinearity, *Complex Var. Elliptic Equ.* **64** (2019) 1163–1183.
[38] M. Squassina and A. Szulkin, Multiple solution to logarithmic Schrödinger equations with periodic potential, *Calc. Var. Partial Differ. Equ.* **54** (2015) 585–597.
[39] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3** (1986) 77–109.
[40] K. Tanaka and C. Zhang, Multi-bump solutions for logarithmic Schrödinger equations, *Calc. Var. Partial Differ. Equ.* **56** (2017) 1–35.
[41] J. Wang and Z. Yin, Multi-bump solutions for the nonlinear magnetic Schrödinger equation with logarithmic nonlinearity, *Math. Nachr.* **298** (2025) 328–355.
[42] L. Wang, K. Cheng and J. Wang, The multiplicity and concentration of positive solutions for the Kirchhoff-Choquard equation with magnetic fields, *Acta Math. Sci. Ser. B* **42** (2022) 1453–1484.
[43] M. Willem, *Minimax Theorems* (Birkhäuser, Boston, 1996).
[44] M. Xiang, P. Pucci, M. Squassina and B. Zhang, Nonlocal Schrödinger Kirchhoff equations with external magnetic field, *Discrete Contin. Dyn. Syst.* **37** (2017) 1631–1649.
[45] K. Yasue, Quantum mechanics of nonconservative systems, *Ann. Phys.* **114** (1978) 479–496.
[46] K. G. Zloshchastiev, Logarithmic nonlinearity in theories of quantum gravity: Origin of time and observational consequences, *Gravit. Cosmol.* **16** (2010) 288–297.