MULTIPLE SOLUTIONS WITH SIGN INFORMATION FOR NONAUTONOMOUS, NONCOERCIVE (p, q)-EQUATIONS

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Abstract. We consider a nonlinear Dirichlet problem driven by the nonautonomous (p, q)-Laplacian with a $(p-1)$–linear reaction which makes the energy functional of the problem noncoercive. Using variational tools from the critical point theory, together with truncation and comparison techniques and critical groups, we show that the problem has at least five nontrivial smooth solutions, all with sign information (two positive, two negative and the fifth nodal).

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonautonomous, Dirichlet (p,q)-equation

$$\left\{ \begin{array}{l} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = f(z,u(z)), \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, 1 < q < p. \end{array} \right\} \tag{1.1}$$

For $a \in L^\infty(\Omega)$ and $r \in (1,\infty)$, by Δ_r^a we denote the nonautonomous r-Laplace differential operator (weighted r-Laplacian) defined by

$$\Delta_r^a u(z) = \operatorname{div}(a(z)|Du|^{r-2}Du).$$

The interest in the study of this type of problem is twofold. On the one hand, there are physical motivations, since the double phase operator has been applied to describe steady-state solutions of reaction-diffusion problems in biophysics, plasma physics, and chemical reaction analysis. The prototype equation for these models can be written in the form

$$u_t = \Delta_p^{a_1} u(z) + \Delta_q^{a_2} u(z) + f(z,u(z)).$$

In this framework, the function u generally stands for a concentration, the term $\Delta_p^{a_1} u(z) + \Delta_q^{a_2} u(z)$ corresponds to the diffusion with coefficient $a_1(z)|Du|^{p-2} + a_2(z)|Du|^{q-2}$, while $f(z,u)$ represents the reaction term related to source and loss processes; see Cherfils & Il'yasov [4] and Singer [23]. On the other hand, such operators provide a valuable framework for explaining the behavior of highly anisotropic materials whose hardening properties, which are linked to the exponent governing the propagation of the gradient variable, differ considerably with the point, where the modulating coefficient $a(z)$ dictates the geometry of a composite made by two different materials.

Equation in (1.1) is driven by the sum of two such operators with distinct exponents and possibly distinct weights. The reaction (right-hand side) of (1.1) is a Carathéodory function $f(z,x)$ (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z,x)$ is continuous). We assume that asymptotically as $x \to \pm\infty$, the quotient $\frac{f(z,x)}{|x|^{p-2}x}$ stays above the principal eigenvalue $\hat{\lambda}_1^{a_1}(p) > 0$ of $(-\Delta_p^{a_1}, W_0^{1,p}(\Omega))$. We allow only partial interaction with $\hat{\lambda}_1^{a_1}(p) > 0$ (nonuniform nonresonance). Such a behavior of $f(z,x)$, makes the energy functional of (1.1) unbounded from below (noncoercive). In general, the noncoercive case is more delicate than the coercive one, since the direct method of the calculus of variations is no longer applicable.

The double-phase problem (1.1) is motivated by numerous models arising in mathematical physics. For instance, we can refer to the following Born-Infeld equation [3] that appears in electromagnetism:

$$-\text{div}\left(\frac{\nabla u}{(1-2|\nabla u|^2)^{1/2}}\right) = h(u) \text{ in } \Omega.$$

Indeed, by the Taylor formula, we have for all $|x| < 1$

$$(1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3}{2\cdot 2^2}x^2 + \frac{5!!}{3!\cdot 2^3}x^3 + \cdots + \frac{(2n-3)!!}{(n-1)!2^{n-1}}x^{n-1} + \cdots .$$

Taking $x = 2|\nabla u|^2$ and adopting the first order approximation, we obtain problem (P_λ) for $p = 4$ and $q = 2$. Furthermore, the n-th order approximation problem is driven by the multi-phase differential operator

$$-\Delta u - \Delta_4 u - \frac{3}{2}\Delta_6 u - \cdots - \frac{(2n-3)!!}{(n-1)!}\Delta_{2n} u.$$

We also refer to the following fourth-order relativistic operator

$$u \mapsto \text{div}\left(\frac{|\nabla u|^2}{(1-|\nabla u|^4)^{3/4}}\,\nabla u\right),$$

which describes large classes of phenomena arising in relativistic quantum mechanics. Again, by Taylor's formula, we have

$$x^2(1-x^4)^{-3/4} = x^2 + \frac{3x^6}{4} + \frac{21x^{10}}{32} + \cdots .$$

This shows that the fourth-order relativistic operator can be approximated by the following autonomous double phase operator

$$u \mapsto \Delta_4 u + \frac{3}{4}\Delta_8 u.$$

In problem (1.1), we have the sum of two such operators and so the left-hand side of problem (1.1) is not homogeneous. In fact, the differential operator $u \mapsto -\Delta_p^{a_1} u - \Delta_q^{a_2} u$ driving problem (1.1) is related to the so-called "double-phase" integral functional given by

$$u \mapsto \int_\Omega (a_1(z)|Du|^p + a_2(z)|Du|^q)\, dz.$$

Such functionals were first investigated by Marcellini [12] and Zhikov [24], in the context of problems of the calculus of variations and of nonlinear elasticity for strongly anisotropic materials. For such problems, there is no global (that is, up to the boundary) regularity theory. There are only interior regularity results primarily due to Marcellini and coworkers and to Mingione and coworkers. We mention the papers of Marcellini [13] and Baroni-Colombo-Mingione [2] and the references therein. An informative survey of the recent developments on the subject can be found in Mingione-Rădulescu [14]. The lack of global regularity theory eliminates from consideration many of the tools used in the study of balanced (p, q)-equations.

Using variational tools from the critical point theory, together with suitable truncation and comparison techniques and critical groups, we show that problem (1.1) has at least five nontrivial smooth solutions, all with sign information(two positive, two negative and a fifth which is nodal(sign changing)). Such multiplicity results for noncoercive problems, were proved by Papageorgiou-Rădulescu [16] (autonomous (p, q)-equations), Liu-Papageorgiou [9, 10] and Papageorgiou-Scapellato [20] (nonautonomous (p, q) -equations). In the aforementioned works, the reaction $f(z, \cdot)$ is $(p-1)$-superlinear and they produce three solutions without providing sign information for all of them (see [16, Theorem 4.14]). In contrast here $f(z, \cdot)$ is $(p-1)$-linear as $x \pm \infty$ and we generate five nontrivial smooth solutions all with sign information.

2. Mathematical Background and Hypotheses

In the study of (1.1), the main function spaces, are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\bar{\Omega}) = \left\{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\right\}$. The Poincaré inequality, implies that on $W_0^{1,p}(\Omega)$, we can use the equivalent norm

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Moreover, the Banach space $C_0^1(\bar{\Omega})$ is ordered with positive (order) cone

$$C_+ = \left\{u \in C_0^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\right\},$$

where $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

We impose the following conditions on the weight functions a_1, a_2, H_0 : $a_1, a_2 \in C^{0,1}(\bar{\Omega})$ with $0 < \hat{c} \leq a_1(z)$, $0 \leq a_2(z)$ for all $z \in \bar{\Omega}$, $a_2 \not\equiv 0$.

We define the operators

$$A_p^{a_1} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \left(p' = \frac{p}{p-1}\right),$$

$$A_q^{a_2} : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega) = W_0^{1,q}(\Omega)^* \left(q' = \frac{q}{q-1}\right)$$

by setting

$$\langle A_p^{a_1}(u), h\rangle = \int_\Omega a_1(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\,dz \text{ for all } u, h \in W_0^{1,p}(\Omega),$$

$$\langle A_q^{a_2}(u), h\rangle = \int_\Omega a_2(z)|Du|^{q-2}(Du, Dh)_{\mathbb{R}^N}\,dz \text{ for all } u, h \in W_0^{1,q}(\Omega).$$

Recalling that $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$ (since $q < p$), we have that $W^{-1,q'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ and so we can define the operator

$$V = A_p^{a_1} + A_q^{a_2} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega).$$

Using the properties of the operators $A_p^{a_1}, A_q^{a_2}$(see, for example, Gasiński-Papageorgiou [5, p.279]), we have the following properties for the nonlinear operator $V(\cdot)$.

Proposition 2.1. *The operator $V(\cdot)$ is bounded (maps bounded sets to bounded sets), continuous, strictly monotone(thus $V(\cdot)$ is maximal monotone), coercive and of type $(S)_+$, that is, it has the following property*

"if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$."

Remark 2.2. *Since $V(\cdot)$ is maximal monotone, coercive, it is surjective (see Papageorgiou-Rădulescu-Repovš* [18, p.135]*).*

For $a \in C^{0,1}(\bar\Omega)$ with $0 < \hat c \leq a(z)$ for all $z \in \bar\Omega$, $1 < r < \infty$ and $\vartheta \in L^\infty(\Omega)\backslash\{0\}$, $\vartheta(z) \geq 0$ for a.a $z \in \Omega$, we consider the following nonlinear eigenvalue problem

$$\left\{\begin{array}{l} -\Delta_p^r u(z) = \hat\lambda\vartheta(z)|u(z)|^{r-2}u(z), \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{array}\right\} \tag{2.1}$$

From Liu-Papageorgiou [11], we know that (2.1) admits a smallest eigenvalue $\hat\lambda_1^a(r) > 0$ which admits the following variational characterization

$$\hat\lambda_1^a(r,\vartheta) = \inf\left\{\frac{\int_\Omega a(z)|Du|^r\,dz}{\int_\Omega \vartheta(z)|u|^r\,dz} : u \in W_0^{1,r}(\Omega), u \neq 0\right\}. \tag{2.2}$$

This eigenvalue is isolated, simple (that is, if $\hat u, \hat v$ are eigenfunctions corresponding to $\hat\lambda_1^a(r,\vartheta) > 0$, then $\hat u = \mu\hat v$ for some $\mu \in \mathbb{R}\backslash\{0\}$). The infimum in (2.2) is realized on the corresponding one-dimensional eigenspace of $\hat\lambda_1^a(r,\vartheta)$ and it is clear from (2.2) that the elements of this eigenspace have fixed sign. By $\hat u_1(r,\theta)$ we denote the positive, L^r-normalized (that is, $\|\hat u_1(r,\theta)\|_r = 1$) eigenfunction corresponding to $\hat\lambda_1^a(r,\vartheta)$. All eigenfunctions corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1^a(r)$ are nodal. From the global nonlinear regularity theory of Lieberman [8], we know that the eigenfunctions of (2.1) belong in $C_0^1(\bar{\Omega})$. Therefore $\hat{u}_1(r,\vartheta) \in C_+\backslash\{0\}$ and then the nonlinear Hopf maximum principle of Pucci-Serrin [22, p.120], implies that $\hat{u}_1(r,\vartheta) \in \operatorname{int} C_+$. As a function of $\vartheta(\cdot)$, the principal eigenvalue has the following strict monotonicity property.

Proposition 2.3. *If $\vartheta_1, \vartheta_2 \in L^\infty(\Omega)$, $0 \leq \vartheta_1(z) \leq \vartheta_2(z)$ for a.a $z \in \Omega$, $\vartheta_1 \neq 0$, $\vartheta_1 \neq \vartheta_2$, then $\hat{\lambda}_1^a(r,\vartheta_2) < \hat{\lambda}_1^a(r,\vartheta_1)$. If $\vartheta(z) = 1$ for a.a $z \in \Omega$, then $\hat{\lambda}_1^a(r,\vartheta) = \hat{\lambda}_1^a(r)$.*

If $u : \Omega \to \mathbb{R}$ is a measurable function, then we set

$$u^\pm = \max\{\pm u, 0\}$$

and we have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,p}(\Omega)$, then $u^\pm \in W_0^{1,p}(\Omega)$.

Let X be a Banach space and $\varphi \in C^1(X)$. We say that $\varphi(\cdot)$ satisfies the "C-condition", if it has the following property:

"Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence."

We introduce the following sets

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}(\text{the critical set of } \varphi(\cdot)),$$

$$\varphi^c = \{u \in X : \varphi(u) \leq c\} \text{ for all } c \in \mathbb{R}.$$

If $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, by $H_k(Y_1, Y_2)$ we denote the k-th singular homology group for the pair (Y_1, Y_2) with integer coefficients. If $u \in K_\varphi$ is isolated and $c = \varphi(u)$, then the critical groups of $\varphi(\cdot)$ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U\backslash\{u\}) \text{ for all } k \in \mathbb{N}_0,$$

where U is a neighborhood of u such that $\varphi^c \cap U \cap K_\varphi = \{u\}$. The excision property of singular homology, implies that this definition is independent of the choice of the isolating neighborhood U.

Finally, we introduce the hypotheses on the reaction $f(z,x)$.

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) for every $\rho > 0$, there exists a function $a_\rho \in L^\infty(\Omega)$ such that $|f(z,x)| \leq a_\rho(z)$ for a.a. $z \in \Omega$, all $|x| \leq \rho$;

(ii) there exist functions $\vartheta_0, \vartheta_1 \in L^\infty(\Omega)$ such that $\hat{\lambda}_1^{a_1}(p) \leq \vartheta_0(z)$ for a.a. $z \in \Omega$, $\vartheta_0 \not\equiv \hat{\lambda}_1^a(p)$,

$$\vartheta_0(z) \leq \liminf_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \leq \limsup_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \leq \vartheta_1(z)$$

uniformly for a.a. $z \in \Omega$;

(iii) there exist $\tau \in (1,q)$ and $\delta > 0$ such that $0 < \beta_0 \leq \liminf_{x \to 0} \frac{f(z,x)}{|x|^{\tau-2}x}$ uniformly for a.a. $z \in \Omega$, $f(z,x)x \leq \tau F(z,x)$ for a.a. $z \in \Omega$, all $|x| \leq \delta$, with $F(z,x) = \int_0^x f(z,s)\,ds$;

(iv) there exist $\mu_1, \mu_2 > 0$ and $\hat{\xi} > 0$ such that

$$f(z,\mu_1) \leq -\hat{\beta} < 0 < \hat{\beta} \leq f(z,\mu_2)$$

for a.a. $z \in \Omega$ and for a.a. $z \in \Omega$, the function

$$x \to f(z,x) + \hat{\xi}|x|^{p-2}x$$

is nondecreasing on $[-\mu_2, \mu_1]$.

Remark 2.4. *On account of hypotheses $H_1(iii)$, we have $f(z,0) = 0$ for a.a. $z \in \Omega$.*

3. Solutions of constant sign

First we produce positive solutions. To this end, we introduce the C^1-functional $\varphi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_+(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q\,dz - \int_\Omega F(z,u^+)\,dz$$

for all $u \in W_0^{1,p}(\Omega)$.

Proposition 3.1. *If hypotheses $H_0, H_1(i), (iii), (iv)$ hold, then problem* (1.1) *admits a positive solution $u_0 \in \operatorname{int} C_+$, which is a local minimizer of $\varphi_+(\cdot)$.*

Proof. We introduce the Carathéodory function $g_+(z,x)$ defined by

$$g_+(z,x) = \begin{cases} f(z,x^+) & \text{if } x \leq \mu_1, \\ f(z,\mu_1) & \text{if } \mu_1 < x. \end{cases} \tag{3.1}$$

We set $G_+(z,x) = \int_0^x g_+(z,s)\,ds$ and consider the C^1-functional $\psi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q\,dz - \int_\Omega G_+(z,u)\,dz$$

for all $u \in W_0^{1,p}(\Omega)$.

From (3.1), it is clear that $\psi_+(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we see that $\psi_+(\cdot)$ is sequentially weakly lower semicontinuous. Therefore by the Weierstrass-Tonelli theorem, there exists $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\psi_+(u_0) = \inf\left\{\psi_+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{3.2}$$

Hypotheses $H_1(iii)$ implies that we can find $\beta_1 \in (0, \beta_0)$ and $0 < \hat{\delta} \leq \min\{\mu_1, \delta\}$ such that

$$\frac{\beta_1}{\tau} x^\tau \leq F(z, x) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \hat{\delta}. \tag{3.3}$$

Let $u \in \operatorname{int} C_+$ and choose $t \in (0,1)$ small such that

$$0 \leq tu(z) \leq \hat{\delta} \text{ for all } z \in \bar{\Omega}. \tag{3.4}$$

Using (3.3) and (3.4), we obtain

$$\begin{aligned}\psi_+(tu) &\leq \frac{t^p}{p}\|a_1\|_\infty \|Du\|_p^p + \frac{t^q}{q}\|a_2\|_\infty \|Du\|_q^q - \frac{\beta_1 t^\tau}{\tau}\|u\|_\tau^\tau \\ &\leq c_1 t^q - c_2 t^\tau \text{ for some } c_1, c_2 > 0. (\text{recall } t \in (0,1), q < p)\end{aligned}$$

Since $\tau < q$ (see hypothesis $H_1(iii)$), we see that by choosing $t \in (0,1)$ even smaller if necessary, we will have

$$\begin{aligned}&\psi_+(tu) < 0, \\ &\Rightarrow \psi_+(u_0) < \psi_+(0) = 0 (\text{see } (3.2)), \\ &\Rightarrow u_0 \neq 0.\end{aligned}$$

From (3.2), we have

$$\begin{aligned}&\langle \psi_+'(u_0), h\rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega) \\ &\Rightarrow \langle V(u_0), h\rangle = \int_\Omega g_+(z, u_0) h \, dz \text{ for all } h \in W_0^{1,p}(\Omega).\end{aligned} \tag{3.5}$$

In (3.5) first we choose the test function $h = -u_0^- \in W_0^{1,p}(\Omega)$. We obtain

$$\begin{aligned}&\hat{c}\|Du_0^-\|_p^p \leq 0, \\ &\Rightarrow u_0 \geq 0, \ u_0 \neq 0.\end{aligned}$$

Next in (3.5), we choose the test function $(u_0-\mu_1)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\langle V(u_0),(u_0-\mu_1)^+\rangle &= \int_\Omega f(z,\mu_1)(u_0-\mu_1)^+\,dz \\ &\le 0 = \langle V(\mu),(u_0-\mu_1)^+\rangle (\text{see hypotheses } H_1(iv))\end{aligned}$$

Hence

$$u_0 \le \mu_1.$$

We have proved that

$$u_0 \in [0,\mu_1] = \left\{h \in W_0^{1,p}(\Omega) : 0 \le h(z) \le \mu_1 \text{ for a.a. } z \in \Omega\right\},\ u_0 \neq 0. \quad (3.6)$$

Then (3.1), (3.5) and (3.6) imply that $u_0 \in W_0^{1,p}(\Omega)$ is a positive solution of (1.1). Theorem 7.1, p.286 of Ladyzhenskaya-Ural'tseva [7] implies that $u_0 \in L^\infty(\Omega)$. Subsequently, the global nonlinear regularity theory of Lieberman [8], implies that $u_0 \in C_+\backslash\{0\}$.

Using hypothesis $H_1(iv)$, we have

$$\begin{aligned}&-\Delta_p^{a_1}u_0 - \Delta_q^{a_2}u_0 + \hat{\xi}u_0^{p-1} \\ &= f(z.u_0) + \hat{\xi}u_0^{p-1} \ge 0 \text{ in } \Omega, \\ &\Rightarrow \Delta_p^{a_1}u_0 + \Delta_q^{a_1}u_0 \le \hat{\xi}u_0^{p-1} \text{ in } \Omega.\end{aligned}$$

Invoking Lemma 1 of Liu-Papageorgiou [10], we infer that

$$u_0 \in \operatorname{int} C_+.$$

Moreover, we have

$$\begin{aligned}&-\Delta_p^{a_1}u_0 - \Delta_q^{a_2}u_0 + \hat{\xi}u_0^{p-1} \\ &= f(z,u_0) + \hat{\xi}u_0^{p-1} \\ &\le f(z,\mu_1) + \hat{\xi}\mu_1^{p-1} (\text{see (3.6) and hypothesis } H_1(iv)) \\ &\le -\hat{\beta} + \hat{\xi}\mu_1^{p-1} \text{ (see hypothesis } H_1(iv)) \\ &\le -\Delta_p^{a_1}\mu_1 - \Delta_q^{a_2}\mu_1 + \hat{\xi}\mu_1^{p-1} \text{ in } \Omega.\end{aligned} \quad (3.7)$$

Since $\hat{\beta} > 0$, from (3.7) and Proposition A4 of Papageorgiou-Rădulescu-Zhang [19], we infer that

$$u_0(z) < \mu_1 \text{ for all } z \in \bar{\Omega}.$$

Therefore we conclude that

$$u_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[0,\mu_1] = \operatorname{int}\{[0,\mu_1] \cap C_0^1(\bar{\Omega})\}. \quad (3.8)$$

From (3.1), we see that

$$\begin{aligned}
&\varphi_+|_{[0,\mu_1]} = \psi_+|_{[0,\mu_1]},\\
&\Rightarrow u_0 \in \operatorname{int} C_+ \text{ is a local } C_0^1(\bar{\Omega})\text{-minimizer of } \varphi_+(\cdot) (\text{see } (3.8)),\\
&\Rightarrow u_0 \in \operatorname{int} C_+ \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_+(\cdot)\\
&\quad (\text{see [19], Proposition A3}).
\end{aligned}$$

This completes the proof. ☐

In the next proposition, we compute the critical groups of $\varphi_+(\cdot)$ at $u \equiv 0$.

Proposition 3.2. *If hypotheses $H_0, H_1(i), (iii)$ hold, then $C_k(\varphi_+, 0) = 0$ for all $k \in \mathbb{N}_0$.*

Proof. Hypotheses $H_1(i), (iii)$ imply that we can find $c_3, c_4 > 0$ such that

$$c_3 x^\tau - c_4 x^p \leq f(z,x) \text{ for a.a. } z \in \Omega, \text{all } x \geq 0. \tag{3.9}$$

Then for $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ and $t \in (0,1]$, we have

$$\varphi_+(tu) \leq c_5 t^q \left[\|Du\|_p^p + \|Du\|_q^q + \|u\|_p^p\right] - c_6 t^\tau \|u\|_\tau^\tau$$

for some $c_5, c_6 > 0$ (see (3.9)).

Since $\tau < q$, we see that we can find $t' \in (0,1]$ such that

$$\varphi_+(tu) < 0 \text{ for all } t \in (0,t'). \tag{3.10}$$

Let $u \in W_0^{1,p}(\Omega)\backslash\{0\}$, $u \geq 0$, $0 < \|u\| \leq 1$, $\varphi_+(u) = 0$. Then

$$\begin{aligned}
&\frac{d}{dt}\varphi_+(tu)_{|t=1} = \langle \varphi_+'(u), u\rangle (\text{using the chain rule})\\
&= \int_\Omega a_1(z)|Du|^p\,dz + \int_\Omega a_2(z)|Du|^q\,dz - \int_\Omega f(z,u^+)u^+\,dz\\
&\geq \left[1 - \frac{\tau}{p}\right]\hat{c}\|Du\|_p^p - \int_\Omega \left[f(z,u^+)u^+ - \tau F(z,u^+)\right]dz (\text{since } \varphi_+(u) = 0)\\
&\geq \left[1 - \frac{\tau}{p}\right]\hat{c}\|Du\|_p^p + \int_{\{u^+ > \delta\}} \left[\tau F(z,u^+) - f(z,u^+)u^+\right]dz\\
&(\text{see hypothesis } H_1(iii))\\
&\geq c_7\|u\|^p - c_8\|u\|^r \text{ for some } c_7, c_8 > 0 \text{ with } r > p.
\end{aligned}$$

Since $p < r$, for $\rho \in (0,1)$ small, we see that

$$\frac{d}{dt}\varphi_+(tu)|_{t=1} > 0 \text{ for all } u \in W_0^{1,p}(\Omega), 0 < \|u\| \leq \rho, \varphi_+(u) = 0. \tag{3.11}$$

We will show that for such a $u \in W_0^{1,p}(\Omega)\backslash\{0\}$, we have

$$\varphi_+(tu) \le 0 \text{ for all } t \in [0,1]. \tag{3.12}$$

Evidently (3.12) is true if $t=0$, $t=1$. Arguing by contradiction, suppose that we can find $t_0 \in (0,1)$ such that $\varphi_+(t_0,u) > 0$. Recall that $\varphi_+(u)=0$. We set

$$\begin{aligned} &t^* = \min\{t \in [t_0,1] : \varphi_+(tu)=0\} > t_0 > 0, \\ &\Rightarrow \varphi_+(tu) > 0 \text{ for all } t \in [t_0,t^*). \end{aligned} \tag{3.13}$$

Set $y = t^* u$. Then

$$0 < \|y\| = t^*\|u\| \le \|u\| \le \rho \text{ and } \varphi_+(y) = 0.$$

Then according to (3.11), we have

$$\frac{d}{dt}\varphi_+(ty)|_{t=1} > 0. \tag{3.14}$$

Form (3.13), we have

$$\begin{aligned} &\varphi_+(y) = \varphi_+(t^*u) = 0 < \varphi_+(tu) \text{ for all } t \in [t_0,t^*), \\ &\Rightarrow \frac{d}{dt}\varphi_+(ty)|_{t=1} = t^*\frac{d}{dt}\varphi_+(tu)|_{t=t^*} \\ &= t^* \lim_{t\to(t^*)^-} \frac{\varphi_+(tu)}{t-t^*} \le 0. \end{aligned} \tag{3.15}$$

Comparing (3.14) and (3.15), we have a contradiction. This proves that relation (3.12) is true.

Let

$$\bar{B}_\rho = \left\{u \in W_0^{1,p}(\Omega) : \|u\| \le \rho\right\}.$$

We may assume that K_{φ_+} is finite (otherwise, since $u \in K_{\varphi_+} \Rightarrow u \ge 0$, we see that we already have an infinity of positive solutions of (1.1) which are smooth by [8] and so we are done). Taking $\rho \in (0,1)$ even smaller, we may have $K_{\varphi_+} \cap \bar{B}_\rho = \{0\}$. We consider the deformation $h_+ : [0,1] \times (\varphi_+^0 \cap \bar{B}_\rho) \to \varphi_+^0 \cap \bar{B}_\rho$ defined by

$$h_+(t,u) = (1-t)u \text{ for all } t \in [0,1], \text{ all } u \in \varphi_+^0 \cap \bar{B}_\rho \text{ (see (3.12))}.$$

It follows that $\varphi_+^0 \cap \bar{B}_\rho$ is contractible.

Consider $u \in \bar{B}_\rho$ such that $\varphi_+(u) > 0$. We will show that there exists an unique $t(u) \in (0,1)$ such that

$$\varphi_+(t(u)u) = 0. \tag{3.16}$$

From (3.10) and Bolzano's theorem, we know that such $t(u) \in (0,1)$ exists. We need to show uniqueness. So, suppose we can find $0 < \hat{t}_1 = t_1(u) < \hat{t}_2 = t_2(u) < 1$ such that

$$\varphi_+(\hat{t}_1 u) = \varphi_+(\hat{t}_2 u) = 0. \tag{3.17}$$

From (3.12), we know that

$$\varphi_+(t\hat{t}_2 u) \leq 0 \text{ for all } 0 \leq t \leq 1. \tag{3.18}$$

Then (3.17) and (3.18), imply that $t := \frac{\hat{t}_1}{\hat{t}_2} \in (0,1)$ is a maximizer of the fiber function $t \to \varphi_+(t\hat{t}_2 u)$. Therefore,

$$\begin{aligned} 0 &= \frac{d}{dt}\varphi_+(t\hat{t}_2 u)|_{t=\frac{\hat{t}_1}{\hat{t}_2}} \\ &= \frac{d}{dt}\varphi_+(t\hat{t}_1 u)|_{t=1}. \end{aligned} \tag{3.19}$$

Since $\varphi_+(\hat{t}_1 u) = 0$(see (3.17)), equality (3.17) contradicts (3.11). This proves the uniqueness of $t(u) \in (0,1)$ in (3.16).

Thus, we have

$$\varphi_+(tu) < 0 \text{ for all } t \in (0, t(u)),$$

and

$$\varphi_+(tu) > 0 \text{ for all } t \in (t(u), 1].$$

We introduce the map $\gamma_+ : \bar{B}_\rho \backslash \{0\} \to (0,1]$ defined by

$$\gamma_+(u) = \begin{cases} 1 & \text{if } u \in \bar{B}_\rho \backslash \{0\}, \varphi_+(u) < 0, \\ t(u) & \text{if } u \in \bar{B}_\rho \backslash \{0\}, \varphi_+(u) > 0. \end{cases} \tag{3.20}$$

We claim that $\gamma_+(\cdot)$ is continuous. From (3.20), we see that it suffices to show continuity at $u \in \bar{B}_\rho \backslash \{0\}$ with $\varphi_+(u) = 0$. Therefore, we consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \bar{B}_\rho \backslash \{0\}$ with $\varphi_+(u_n) = 0$ for all $n \in \mathbb{N}$ such that $u_n \to u$ in $W_0^{1,p}(\Omega)$. From (3.20), we see that we may assume that $\varphi_+(u_n) > 0$ for all $n \in \mathbb{N}$. We argue by contradiction. So, suppose that at least for a subsequence, we have

$$\begin{aligned}
& t(u_n) \le \tilde{t} < 1 \text{ for all } n \in \mathbb{N}, \\
& \Rightarrow \varphi_+(tu_n) > 0 \text{ for all } t \in (\tilde{t}, 1], \\
& \Rightarrow \varphi_+(tu) \ge 0 \text{ for all } t \in (\tilde{t}, 1], \\
& \Rightarrow \varphi_+(tu) = 0 \text{ for all } t \in (\tilde{t}, 1] \\
& \quad \text{(note that } \varphi_+(u) = 0 \text{ and see (3.12))}, \\
& \Rightarrow \frac{d}{dt}\varphi_+(tu)|_{t=1} = 0,
\end{aligned}$$

which contradicts (3.11). This proves the continuity of $\gamma_+(\cdot)$.

We introduce the deformation $\eta_+ : \bar{B}_\rho \backslash \{0\} \to (\varphi_+^0 \cap \bar{B}_\rho)\backslash\{0\}$ defined by

$$\eta_+(u) = \begin{cases} u & \text{if } u \in \bar{B}_\rho\backslash\{0\}, \varphi_+(u) \le 0, \\ \gamma_+(u)u & \text{if } u \in \bar{B}_\rho\backslash\{0\}, \varphi_+(u) > 0. \end{cases} \tag{3.21}$$

On account of the continuity of $\gamma_+(\cdot)$, we have the continuity of $\eta_+(\cdot)$. From (3.21), we see that

$$\eta_+|_{(\varphi_+^0 \cap \bar{B}_\rho)\backslash\{0\}} = \mathrm{id}|_{(\varphi_+^0 \cap \bar{B}_\rho)\backslash\{0\}}.$$

Hence it follows that $(\varphi_+^0 \cap \bar{B}_\rho)\backslash\{0\}$ is a retract of $\bar{B}_\rho\backslash\{0\}$ and the latter is contractible. Therefore $(\varphi_+^0 \cap \bar{B}_\rho)\backslash\{0\}$ is contractible too. So, from Papageorgiou-Rădulescu-Repovš [18, p.469], we have

$$\begin{aligned}
& H_k(\varphi_+^0 \cap \bar{B}_\rho, (\varphi_+^0 \cap \bar{B}_\rho)\backslash\{0\}) \text{ for all } k \in \mathbb{N}_0, \\
& \Rightarrow C_k(\varphi_+, 0) = 0 \text{ for all } k \in \mathbb{N}_0.
\end{aligned}$$

The proof is now complete. □

Remark 3.3. *We mention that the first such computation of critical groups at zero for functionals with a concave term (see hypothesis $H_1(iii)$), was done by Moroz* [15]*, in the context of semilinear equations driven by the Laplacian. The hypotheses of Moroz* [15] *were stronger and required that the concave term to be global. We mention also Proposition 3.7 of* [16].

Proposition 3.4. *If hypotheses H_0, H_1 hold, then the functional $\varphi_+(\cdot)$ satisfies the C-condition.*

Proof. Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that $\{\varphi_+(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\varphi_+'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{3.22}$$

From (3.22), we have

$$\left|\langle V(u_n), h\rangle - \int_\Omega f(z, u_n^+)h\,dz\right| \leq \frac{\epsilon_n\|h\|}{1+\|u_n\|} \tag{3.23}$$

for all $h \in W_0^{1,p}(\Omega)$ with $\epsilon_n \to 0^+$. In (3.23), we choose the test function $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}&\hat{c}\|Du_n^-\|_p^p \leq \epsilon_n \text{ for all } n \in \mathbb{N},\\ &\Rightarrow u_n^- \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty.\end{aligned} \tag{3.24}$$

Suppose that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is not bounded. By passing to a subsequence if necessary, we may assume that

$$\|u_n^+\| \to \infty. \tag{3.25}$$

We set $y_n := \frac{u_n^+}{\|u_n^+\|}, n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega), y_n \to y \text{ in } L^p(\Omega), y \geq 0. \tag{3.26}$$

We multiply (3.23) with $\frac{1}{\|u_n^+\|^{p-1}}$ and use (3.24). We obtain

$$\langle A_p^{a_1}(y_n), h\rangle + \frac{1}{\|u_n^+\|^{p-q}}\langle A_q^{a_2}(y_n), h\rangle \leq \epsilon_n' + \int_\Omega \frac{f(z, u_n^+)}{\|u_n^+\|^{p-1}}h\,dz \tag{3.27}$$

for all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$, with $\epsilon_n' \to 0^+$.

Hypotheses $H_1(i), (ii)$ imply that

$$|f(z,x)| \leq c_7\left[1 + |x|^{p-1}\right] \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_7 > 0.$$

It follows that

$$\left\{\frac{f(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\right\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega) \text{ is bounded.}$$

The reflexivity of $L^{p'}(\Omega)$, the Eberlein-Smulian theorem and hypothesis $H_1(ii)$, imply that

$$\frac{f(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \hat{\vartheta}(\cdot)y^{p-1} \text{ in } L^{p'}(\Omega), \tag{3.28}$$

where $\hat{\vartheta} \in L^\infty(\Omega)$, $\vartheta_0(z) \leq \hat{\vartheta}(z) \leq \vartheta_1(z)$ for a.a. $z \in \Omega$(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16).

We return to (3.27) and use the test function $h = y_n - y \in W_0^{1,p}(\Omega)$. Using (3.25), (3.26), (3.27), in the limit as $n \to \infty$, we obtain

$$\begin{aligned}&\lim_{n\to\infty} \langle A_p^{a_1}(y_n), y_n - y\rangle = 0\\ &\Rightarrow y_n \to y \text{ in } W_0^{1,p}(\Omega), \|y\| = 1, y \geq 0 (\text{by the } (S)_+\text{-property of } A_p^{a_1}(\cdot)).\end{aligned} \tag{3.29}$$

If in (3.27), we pass to the limit as $n \to \infty$ and use (3.25), (3.28) and (3.29), we obtain

$$\begin{aligned}&\langle A_p^{a_1}(y), h\rangle = \int_\Omega \hat{\vartheta}(z) y^{p-1} h \, dz \text{ for all } h \in W_0^{1,p}(\Omega),\\ &\Rightarrow -\Delta_p^{a_1} y(z) = \vartheta(z) y(z)^{p-1} \text{ in } \Omega, \ y|_{\partial\Omega} = 0.\end{aligned} \tag{3.30}$$

From Proposition 2.3, we have

$$\hat{\lambda}_1^{a_1}(p, \hat{\vartheta}) < \hat{\lambda}_1^{a_1}(p, \hat{\lambda}_1^{a_1}(p)) = 1.$$

Then from (3.30) and since $y \neq 0$(see (3.29)), we infer that $y(\cdot)$ must be nodal, contradicting (3.26). This proves that

$$\begin{aligned}&\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded},\\ &\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded(see (3.20))}.\end{aligned}$$

We may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega), u_n \to u \text{ in } L^p(\Omega). \tag{3.31}$$

We return to (3.23), choose the test function $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.31). Then

$$\begin{aligned}&\lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0,\\ &\Rightarrow u_n \to u \text{ in } W_0^{1,p}(\Omega) (\text{see Proposition 2.1}),\\ &\Rightarrow \varphi_+(\cdot) \text{ satisfies the } C\text{-condition}.\end{aligned}$$

This completes the proof. □

We can have a second positive solution distinct from u_0.

Proposition 3.5. *If hypotheses H_0, H_1 hold, then problem* (1.1) *admits a second positive solution $\hat{u} \in int C_+$, $\hat{u} \neq u_0$.*

Proof. Recall that without any loss of generality, we assume that K_{φ_+} is finite. Let $u_0 \in \operatorname{int} C_+$ be the first positive solution of (1.1) produced in Proposition 3.1. We know that u_0 is a local minimizer of $\varphi_+(\cdot)$. Using

Theorem 5.7.6 of Papageorgiou-Rădulescu-Repovš [18, p.449], we can find $\rho \in (0,1)$ small such that

$$\varphi_+(u_0) < \inf\{\varphi_+(u) : \|u - u_0\| = \rho\} = m_+. \tag{3.32}$$

Hypotheses $H_1(i), (ii)$ imply that given $\epsilon > 0$, we can find $c_8 = c_8(\epsilon) > 0$ such that

$$\frac{\vartheta_0(z) - \epsilon}{p}|x|^p - c_8 \leq F(z,x) \text{ for a.a. } z \in \Omega, \text{all } x \in \mathbb{R}. \tag{3.33}$$

Let $\hat{u}_1 = \hat{u}_1(\rho) \in \operatorname{int} C_+$ be the positive principal eigenfunction of $(-\Delta_p^{a_1}, W_0^{1,p}(\Omega))$ with $\|\hat{u}_1\|_p = 1$. For $t > 0$, we have

$$\varphi_+(t\hat{u}_1) \leq \frac{t^p}{p}\int_\Omega \left[\hat{\lambda}_1^{a_1}(p) - \vartheta_0(z)\right]\hat{u}_1^p\,dz + \frac{t^p}{p}\epsilon + \frac{t^q}{q}\int_\Omega a_2(z)|D\hat{u}_1|^q\,dz + c_9 \tag{3.34}$$

for some $c_9 > 0$(see (3.33)).

Note that

$$k_0 = \int_\Omega \left[\vartheta_0(z) - \hat{\lambda}_1^{a_1}(p)\right]\hat{u}_1^p\,dz > 0.$$

So, if we choose $\epsilon \in (0, k_0)$, then from (3.34), we have

$$\begin{aligned} \varphi_+(t\hat{u}) &\leq -\frac{t^p}{p}[k_0 - \epsilon] + c_{10}t^q + c_9 \text{ for some } c_{10} > 0 \\ &= c_{10}t^q - c_{11}t^p + c_9 \text{ for some } c_{11} > 0, \\ &\Rightarrow \varphi_+(t\hat{u}_1) \to -\infty \text{ as } t \to +\infty (\text{recall } q < p). \end{aligned} \tag{3.35}$$

From Proposition 3.4, we know that

$$\varphi_+(\cdot) \text{ satisfies the } C\text{-condition}. \tag{3.36}$$

From (3.32), (3.35), (3.36) and the mountain pass theorem, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\varphi_+} \text{ and } m_+ \leq \varphi_+(\hat{u}). \tag{3.37}$$

Then we have

$$\begin{aligned} &\langle \varphi_+'(\hat{u}), h\rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega), \\ &\Rightarrow \langle V(\hat{u}), h\rangle = \int_\Omega f(z, \hat{u}^+)h\,dz \text{ for all } h \in W_0^{1,p}(\Omega). \end{aligned}$$

We choose the test function $h = -\hat{u}^- \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} &\hat{c}\|D\hat{u}^-\|_p^p \leq 0, \\ &\Rightarrow \hat{u} \geq 0. \end{aligned}$$

So, $\hat{u}$ is a nonnegative solution of (1.1) and as before the nonlinear regularity theory of Lieberman [8] implies that $\hat{u} \in C_+$. Moreover, from (3.32) and (3.37), we have $\hat{u} \neq u_0$. If we show that $\hat{u} \neq 0$, then this is the second positive solution of (1.1).

We know that $\hat{u} \in K_{\varphi_+}$ is of mountain pass type. So, from Theorem 6.5.8, p. 527, of Papageorgiou-Rădulescu-Repovš [18], we have

$$C_1(\varphi_+, \hat{u}) \neq 0. \tag{3.38}$$

On the other hand, from Proposition 3.4, we know that

$$C_k(\varphi_+, 0) = 0 \text{ for all } k \in \mathbb{N}_0. \tag{3.39}$$

Comparing (3.38) and (3.39), we conclude that $\hat{u} \neq 0$. Hypotheses H_1, imply that for all $\rho > 0$, we can find $\hat{\xi}_\rho > 0$ such that

$$f(z,x) + \hat{\xi}_\rho x^{p-1} \geq 0 \text{ for a.a. } z \in \Omega, \text{all } 0 \leq x \leq \rho.$$

Let $\rho = \|\hat{u}\|_\infty$ and $\hat{\xi}_\rho > 0$ as above. Then

$$\begin{aligned}& -\Delta_p^{a_1}\hat{u} - \Delta_q^{a_2}\hat{u} + \hat{\xi}_\rho \hat{u}^{p-1} \geq 0 \text{ in } \Omega,\\ &\Rightarrow \hat{u} \in \operatorname{int} C_+ (\text{see [10]}).\end{aligned}$$

The proof is now complete. □

Similarly working on the negative semi-axis, we can generate two negative solutions

$$v_0, \hat{v} \in -\operatorname{int} C_+, v_0 \neq \hat{v}.$$

We can state the following multiplicity theorem for the constant sign solutions of problem (1.1).

Theorem 3.6. *If hypotheses H_0, H_1 hold, then problem* (1.1) *has at least four nontrivial smooth constant sign solutions*

$$\begin{aligned}&u_0, \hat{u} \in \operatorname{int} C_+, u_0 \neq \hat{u},\\ &v_0, \hat{v} \in -\operatorname{int} C_+, v_0 \neq \hat{v}.\end{aligned}$$

Remark 3.7. *The solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -int\ C_+$ are local minimizers of the energy functional $\varphi \in C^1(W_0^{1,p}(\Omega))$ of problem* (1.1) *defined by*

$$\varphi(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q\,dz - \int_\Omega F(z,u)\,dz$$

for all $u \in W_0^{1,p}(\Omega)$.

If $q = 2$, then we can relax hypothesis $H_1(iii)$ and assume the existence of $\mu_1, \mu_2 > 0$ such that

$$f(z, \mu_2) \leq 0 \leq f(z, \mu_1) \text{ for a.a. } z \in \Omega.$$

This means that we fit in our framework also reactions satisfying the sign condition. Note that in this case, the differential operator is described by the map $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ given by

$$a(z, y) = a_1(z)|y|^{p-2}y + a_2(z)y \text{ for all } z \in \Omega, \text{all } y \in \mathbb{R}^N, \text{ with } 2 < p.$$

Evidently $a(z, \cdot) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and

$$\nabla_y a(z, y) = |y|^{p-2} \left[a_1(z)\text{id} + (p-2)\frac{y \otimes y}{|y|^2} \right] + a_2(z)\text{id},$$
$$\Rightarrow (\nabla_y a(z, y)\xi, \xi)_{\mathbb{R}^N} \geq \hat{c}|\xi|^2 \text{ for all } \xi \in \mathbb{R}^N.$$

So, we can use the tangency principle of Pucci-Serrin [22, p.35], and deduce that

$$-\mu_2 < v_0(z) < 0 < u_0(z) < \mu_1 \text{ for all } z \in \Omega.$$

Therefore, Proposition 3.1 remains valid.

Next we will show that problem (1.1) has a smallest positive solution and a biggest negative solution (extremal constant sign solutions). In Section 4, we will use them to generate a nodal (sign changing) solution.

Hypotheses $H_1(i), (ii), (iii)$ imply that we can find $c_{12}, c_{13} > 0$ such that

$$c_{12}|x|^\tau - c_{13}|x|^p \leq f(z, x)x \text{ for a.a. } z \in \Omega, \text{all } x \in \mathbb{R}. \tag{3.40}$$

This unilateral growth condition on $f(z, \cdot)$, leads to the following auxiliary Dirichlet problem

$$\left\{ \begin{array}{l} -\Delta_p^{a_1} u - \Delta_q^{a_2} u = c_{12}|u|^{\tau-2}u - c_{13}|u|^{p-2}u, \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{array} \right\} \tag{3.41}$$

Reasoning as in the proof of Proposition 3.1 of Liu-Papageorgiou [10], we obtain the following result.

Proposition 3.8. *If hypotheses H_0 hold and $1 < \tau < q < p$, then problem* (3.41) *has an unique positive solution $\bar{u} \in int C_+$ and since the problem is odd, $\bar{v} = -\bar{u} \in -int C_+$ is the unique negative solution of* (3.41).

Let S_+(resp., S_-) be the set of positive (resp., negative) solutions of problem (1.1). From Theorem 3.6, we know that

$$\emptyset \neq S_+ \subseteq \text{int}\, C_+ \text{ and } \emptyset \neq \mathcal{S}_- \subseteq -\text{int}\, C_+.$$

Proposition 3.9. *If hypotheses H_0, H_1 hold, then $\bar{u} \le u$ for all $u \in S_+$ and $v \le \bar{v}$ for all $v \in S_-$.*

Proof. Let $u \in S_+ \subseteq \operatorname{int} C_+$ and introduce the Carathéodory function $e_+(z,x)$ defined by

$$e_+(z,x) = \begin{cases} c_{12}(x^+)^{\tau-1} - c_{13}(x^+)^{p-1} & \text{if } x \le u(z), \\ c_{12}u(z)^{\tau-1} - c_{13}u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \tag{3.42}$$

We set $E_+(z,x) = \int_0^x e_+(z,s)\,ds$ and consider the C^1-functional $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_+(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q\,dz - \int_\Omega E_+(z,u)\,dz$$

for all $u \in W_0^{1,p}(\Omega)$.

Hypotheses H_0 and (3.42), imply that $\sigma_+(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we see that $\sigma_+(\cdot)$ is sequentially weakly lower semicontinuous. Therefore by the Weierstrass-Tonelli theorem, we can find $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\sigma_+(\tilde{u}) = \inf\left\{\sigma_+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{3.43}$$

Since $\tau < q < p$, we infer that

$$\begin{aligned} &\sigma_+(\tilde{u}) < 0 = \sigma_+(0), \\ &\Rightarrow \tilde{u} \ne 0. \end{aligned}$$

From (3.43), we have

$$\begin{aligned} &\langle \sigma_+'(\tilde{u}), h\rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega), \\ &\Rightarrow \langle V(\tilde{u}), h\rangle = \int_\Omega e_+(z,\tilde{u})h\,dz \text{ for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.44}$$

First we choose the test function $h = -\tilde{u}^- \in W_0^{1,p}(\Omega)$ and obtain $\tilde{u} \ge 0$, $\tilde{u} \ne 0$. Next in (3.44), we use the test function $h(\tilde{u} - u)^+ \in W_0^{1,p}(\Omega)$ and have

$$
\begin{aligned}
&\langle V(\tilde{u}), (\tilde{u}-u)^+\rangle \\
&= \int_\Omega \left[c_{12}u^{\tau-1} - c_{13}u^{p-1}\right](\tilde{u}-u)^+\,dz (\text{see } (3.42)) \\
&\le \int_\Omega f(z,u)(\tilde{u}-u)^+\,dz (\text{see } (3.40)) \\
&= \langle V(u), (\tilde{u}-u)^+\rangle (\text{since } u \in S_+), \\
&\Rightarrow \tilde{u} \le u (\text{see Proposition 2.1}).
\end{aligned}
$$

We have proved that

$$
\begin{aligned}
&\tilde{u} \in [0,u] = \left\{h \in W_0^{1,p}(\Omega) : 0 \le h(z) \le u(z) \text{ for a.a. } z \in \Omega\right\},\ \tilde{u} \ne 0, \\
&\Rightarrow \tilde{u} = \bar{u} (\text{see } (3.42), (3.44), \text{ and Proposition 3.8}), \\
&\Rightarrow \bar{u} \le u \text{ for all } u \in S_+.
\end{aligned}
$$

Similarly, we show that

$$
v \le \bar{v} \text{ for all } v \in S_-,
$$

which completes the proof. □

Using these bounds, we can generate the extremal constant sign solutions for problem (1.1).

Proposition 3.10. *If hypotheses H_0, H_1 hold, then problem* (1.1) *has a smallest positive solution $u_* \in S_+$, that is, $u_* \le u$ for all $u \in S_+$ and a biggest negative solution $v_* \in S_-$, that is, $v \le v_*$ for all $v \in S_-$.*

Proof. From Papageorgiou-Rădulescu-Repovš [17], we know that S_+ is downward directed, that is, if $u_1, u_2 \in S_+$, then there exists $u \in S_+$ such that $u \le u_1, u \le u_2$. Then invoking Theorem 5.109 of Hu-Papageorgiou [6, p.308], we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_+$ such that

$$
\inf S_+ = \inf_{n\in\mathbb{N}} u_n.
$$

We have

$$
\langle V(u_n), h\rangle = \int_\Omega f(z,u_n)h\,dz \text{ for all } h \in W_0^{1,p}(\Omega), \text{all } n \in \mathbb{N}, \tag{3.45}
$$

$$
\bar{u} \le u_n \le u_1 \text{ for all } n \in \mathbb{N} (\text{see Proposition } (3.9)). \tag{3.46}
$$

In (3.45), we choose the test function $h = v_n \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} &\hat{c}\|Du_n\|_p^p \leq c_{14} \text{ for some } c_{14} > 0, \text{all } n \in \mathbb{N} \\ &\quad \text{(see (3.46) and hypotheses } H_1(i),(ii)), \\ &\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \end{aligned}$$

We may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W_0^{1,p}(\Omega),\ u_n \to u_* \text{ in } L^p(\Omega). \tag{3.47}$$

In (3.45), we choose the test function $h = u_n - u_* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.47). Then

$$\begin{aligned} &\lim_{n\to\infty} \langle V(u_n), u_n - u_*\rangle = 0, \\ &\Rightarrow u_n \to u_* \text{ in } W_0^{1,p}(\Omega) \text{(see Proposition 2.1).} \end{aligned} \tag{3.48}$$

If in (3.45), we pass to the limit as $n \to \infty$ and use (3.48), then

$$\begin{aligned} &\langle V(u_*), h\rangle = \int_\Omega f(z,u_*)h\,dz \text{ for all } h \in W_0^{1,p}(\Omega),\ \bar{u} \leq u_*, \\ &\Rightarrow u_* \in S_+,\ v_* = \inf S_+. \end{aligned}$$

Similarly working with S_-, we produce $v_* \in S_-$ such that

$$v \leq v_* \text{ for all } v \in S_-.$$

Note that S_- is upward directed(that is, if $v_1, v_2 \in S_-$, then there exists $v \in S_-$ such that $v_1 \leq v, v_2 \leq v$). □

4. Nodal Solutions

In this section, using the extremal constant sign solutions $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$, we generate a nodal solution. The idea is the following. We truncate $f(z,x)$ from below at $v_*(z)$ and from above at $u_*(z)$. This way the corresponding energy functional has nontrivial critical points in the order interval

$$[v_*, u_*] = \left\{h \in W_0^{1,p}(\Omega) : v_*(z) \leq h(z) \leq u_*(z) \text{ for a.a. } z \in \Omega\right\}.$$

Evidently any such nontrivial critical point distinct from u_* and v_*, is a nodal(sign changing) solution of (1.1).

Proposition 4.1. *If hypotheses H_0, H_1 hold, then problem* (1.1) *admits a nodal solution $y_0 \in [v_*, u_*] \cap C_0^1(\bar{\Omega})$.*

Proof. Implementing the strategy outlined above, we introduce the Carathéodory function $l(z,x)$ defined by

$$l(z,x) = \begin{cases} f(z, v_*(z)) & \text{if } x < v_*(z) \\ f(z,x) & \text{if } v_*(z) \le x \le u_*(z) \\ f(z, u_*(z) & \text{if } u_*(z) < x. \end{cases} \tag{4.1}$$

Also we introduce the positive and negative truncations of $l(z,\cdot)$ namely the Carathéodory functions $l_\pm(z,x) := l(z, \pm x^\pm)$. We set

$$L(z,x) = \int_0^x l(z,s)\,ds \text{ and } L_\pm(z,x) = \int_0^x l_\pm(z,s)\,ds$$

and consider the C^1-functions $k, k_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$k(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q\,dz - \int_\Omega L(z,u)\,dz,$$

$$k_\pm(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q\,dz - \int_\Omega L_\pm(z,u)\,dz.$$

From (4.1) and the nonlinear regularity theory of Lieberman [8], we obtain

$$K_k \subseteq [v_*, u_*] \cap C_0^1(\bar{\Omega}),\ K_{k_+} \subseteq [0, u_*] \cap C_+,\ K_{k_-} \subseteq [v_*, 0] \cap (-C_+).$$

Exploiting the extremality of u_*, v_*, we infer that

$$K_k \subseteq [v_*, u_*] \cap C_0^1(\bar{\Omega}),\ K_{k_+} = [0, u_*],\ K_{k_-} = [v_*, 0]. \tag{4.2}$$

We claim that u_* and v_* are local minimizers of $k(\cdot)$. Clearly, $k_+(\cdot)$ is coercive and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_* \in W_0^{1,p}(\Omega)$ such that

$$k_+(\tilde{u}_*) = \inf\left\{k_+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{4.3}$$

Let $u \in C_+\backslash\{0\}$. Using Proposition 4.1.22 of [18, p.274], we can find $t \in (0,1)$ small such that

$$0 \le tu(z) \le \min\{u_*(z), \delta\} \text{ for all } z \in \bar{\Omega}.$$

Using hypothesis $H_1(iii)$ and since $\tau < q < p$, we obtain

$$\begin{aligned} & k_+(tu) < 0, \\ & \Rightarrow k_+(\tilde{u}_*) < 0 = k_+(0)(\text{see } (4.3)), \\ & \Rightarrow \tilde{u}_* \neq 0. \end{aligned} \tag{4.4}$$

We know that $\tilde{u}_* \in K_{k_+}$(see (4.3)). Hence from (4.2) and (4.4) it follows that $\tilde{u}_* = u_* \in \operatorname{int} C_+$. Note that

$$k|_{C_+} = k_+|_{C_+}.$$

It follows that u_* is a local $C_0^1(\bar{\Omega})$-minimizer of $k(\cdot)$.

$$\Rightarrow u_* \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } k(\cdot)$$
$$\text{(see Papageorgiou-Rădulescu-Zhang [19], Proposition A3).}$$

Similarly for $v_* \in -\operatorname{int} C_+$, using $k_-(\cdot)$. This proves the claim.

The functional $k(\cdot)$ is coercive(see (4.1)). Therefore Proposition 5.1.15, p. 369, of [18], implies that $k(\cdot)$ satisfies the C-condition. We may assume without any loss of generality that

$$k(v_*) \leq k(u_*) \text{ and } K_k \text{ is finite (see (4.2)).}$$

So, by Theorem 5.7.6 of [18, p.449], we take $\rho \in (0, \|v_*\|)$ small such that

$$k(v_*) \leq k(u_*) < \inf\{k(u) : \|u - u_*\| = \rho\}. \tag{4.5}$$

We apply the mountain pass theorem and we find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_k \subseteq [v_*, u_*] \cap C_0^1(\bar{\Omega})\text{(see (4.2))},$$
$$y_0 \notin \{v_*, u_*\}\text{(see (4.5))}.$$

We know that

$$C_1(k, y_0) \neq 0\text{(see [18, p.527])}. \tag{4.6}$$

Also since $0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[v_*, u_*]$, using a standard argument involving the homotopy invariance of critical groups, we show that

$$C_l(k, 0) = C_l(\varphi, 0) \text{ for all } l \in \mathbb{N}_0 \text{ (recall } \varphi \text{ is the energy of (1.1))},$$
$$\Rightarrow C_l(k, 0) = 0 \text{ for all } l \in \mathbb{N}_0 \tag{4.7}$$
$$\text{(by Proposition 3.2 and since } C_l(\varphi, 0) = C_l(\varphi_+, 0) \text{ for all } l \in \mathbb{N}_0).$$

Comparing (4.6) and (4.7), we conclude that $y_0 \neq 0$. So $y_0 \in C_0^1(\bar{\Omega})$ is a nodal solution of problem (1.1). □

Remark 4.2. *As before if $q = 2$, using the tangency principle of Pucci-Serrin* [22, p.36]*, we can say that $y_0 \in int_{C_0^1(\bar{\Omega})}[v_*, u_*]$.*

Finally, we can state the following multiplicity theorem for problem (1.1). Note that we provide sign information for all solutions produced.

Theorem 4.3. *If hypotheses H_0, H_1 hold, then problem* (1.1) *has at least five nontrivial smooth solutions*

$$\begin{aligned}
&u_0, \hat{u} \in \operatorname{int} C_+,\ u_0 \leq \hat{u},\ u_0 \neq \hat{u}, \\
&v_0, \hat{v} \in -\operatorname{int} C_+,\ \hat{v} \leq v_0,\ v_0 \neq \hat{v}, \\
&y_0 \in [v_0, u_0] \cap C_0^1(\bar{\Omega}) \quad \textit{nodal.}
\end{aligned}$$

Remark 4.4. *It will be interesting to extend this work to problems driven by the double phase differential operator*

$$u \to -\Delta_p^a u - \Delta_q u$$

with $a \in C^{0,1}(\Omega)\backslash\{0\}$, $a(z) \geq 0$ for all $z \in \bar{\Omega}$. For such problems there is no global regularity theory. So, many of the tools used here are no longer available and we have to come up with different methods and techniques. A first result in this direction for parametric double phase problem, can be found in the recent work of Papageorgiou-Zhang-Zhang [21].

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DECLARATIONS

Conflict of interest. The authors declare that there is no conflict of interest. We also declare that this manuscript has no associated data.

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