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Small perturbations of convective singular eigenvalue problems

Ahmed Alsaedi [a], Nikolaos S. Papageorgiou [b,c], Vicenţiu D. Rădulescu [c,d,e,f,g,*]

[a] *Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

[b] *National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece*

[c] *Department of Mathematics, University of Craiova, 200585 Craiova, Romania*

[d] *Faculty of Applied Mathematics, AGH University of Kraków, 30-059 Kraków, Poland*

[e] *Brno University of Technology, Faculty of Electrical Engineering and Communication, Technická 3058/10, Brno 61600, Czech Republic*

[f] *Simion Stoilow Institute of Mathematics of the Romanian Academy, 010702 Bucharest, Romania*

[g] *Scientific Research Center, Baku Engineering University, AZ0102 Baku, Azerbaijan*

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ABSTRACT

We consider a nonlinear Dirichlet problem with gradient dependence. The features of this paper are twofold: (i) the problem is driven by a general nonlinear nonhomogeneous differential operator with Uhlenbeck–Lieberman structure; (ii) the reaction blows-up at the origin and it is gradient dependent. Using a topological approach based on fixed point theory, we show that for all small values of $\lambda > 0$ there are "eigenvalues" of the problem with smooth corresponding eigenfunctions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear eigenvalue problem

$$\begin{cases} -\operatorname{div} a(Du(z)) = \lambda f(z, u(z), Du(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \; u > 0, \lambda > 0. \end{cases} \qquad (\mathcal{P}_\lambda)$$

In this problem, $a : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous and strictly monotone function (hence $a(\cdot)$ is also maximal monotone), which satisfies certain other regularity conditions listed in hypotheses H_0 below. These hypotheses provide a broad analytical framework, in which we can fit many operators of interest such as the p-Laplacian and the (p, q)-Laplacian. These hypotheses permit the use of the global regularity theory (regularity up to the boundary $\partial\Omega$) of Lieberman [1], which makes possible the use of a variety of powerful analytical tools (see also [2], [3], [4], [5]).

In the reaction (right-hand side of problem $(\mathcal{P}_\lambda)$), the singular term is not decoupled from the perturbation, which is the case in almost all "singular" papers in the literature, where the reaction is of the form $\lambda[u^{-\eta} + f(z, u, Du)]$ (see Papageorgiou–Rădulescu–Repovs [6,7] and the references therein). So, the formulation here permits a reaction of the form $g(z, u, Du)u^{-\eta}$, where

* Corresponding author at: Faculty of Applied Mathematics, AGH University of Kraków, 30-059 Kraków, Poland.
E-mail addresses: aalsaedi@kau.edu.sa (A. Alsaedi), npapg@math.ntua.gr (N.S. Papageorgiou), radulescu@agh.edu.pl (V.D. Rădulescu).

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$g(z,x,y)$ is a Carathéodory function. The presence of the gradient in the reaction means that the problem $(\mathcal{P}_\lambda)$ is not variational. Consequently, our approach will be topological and it uses the fixed point theory. More precisely, we will use the Leray–Schauder alternative principle to establish the existence of a smooth solution (eigenfunction) for problem $(\mathcal{P}_\lambda)$, where $\lambda > 0$ is small.

Singular problems with convection were studied by Faraci–Puglisi [8] (semilinear problems driven by the Laplacian) and by Liu–Motreanu–Zeng [9], Papageorgiou–Rădulescu–Repovs [6], Papageorgiou–Scapellato [10] (equations driven by the p-Laplacian). We also mention the very recent work of Ozturk–Papageorgiou [11], where the problem is driven by a nonlinear, nonhomogeneous differential operator, like the one used here. In all the above works, the singular term is decoupled from the perturbation and the hypotheses on the perturbation are more restrictive.

2. Hypotheses and auxiliary properties

Let $\beta \in C^1(0,\infty)$ with $\beta(t) > 0$ for all $t > 0$ and suppose that

$$0 < c \le \frac{\beta'(t)t}{\beta(t)} \le \hat{c} \text{ and } c_0 t^{p-1} \le \beta(t) \le c_1(t^{s-1} + t^{p-1}) \text{ for all } t > 0 \text{ and some } c_0, c_1 > 0,\ 1 < s < p.$$

The hypotheses on the function $a : \mathbb{R}^N \to \mathbb{R}^N$ are the following:
H_0: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and
(i) $a_0 \in C^1(0,\infty)$, $(0,\infty) \ni t \mapsto a_0(t)t$ is strictly increasing, $a_0(t)t \to 0^+$ as $t \to 0^+$ and $\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1$;
(ii) $|\nabla a(y)| \le c_2 \frac{\beta(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \setminus \{0\}$, some $c_2 > 0$;
(iii) $\frac{\beta(|y|)}{|y|}|\xi|^2 \le (\nabla a(y)\xi, \xi)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$.

We set $G_0(t) = \int_0^t a_0(s)s\,ds$, $t \ge 0$. On account of hypotheses H_0, $G_0(\cdot)$ is strictly convex and strictly increasing. We introduce $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Evidently, $G(\cdot)$ is differentiable and convex. We have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \text{ for all } y \in \mathbb{R}^N \setminus \{0\},\ \nabla G(0) = 0.$$

The convexity of $G(\cdot)$ implies that

$$G(y) \le (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N. \tag{1}$$

Hypotheses H_0 imply the following properties for the function $y \mapsto a(y)$ (see Papageorgiou–Rădulescu [12]).

Lemma 1. *If hypotheses H_0 hold, then the following properties are fulfilled:*
(a) the function $y \mapsto a(y)$ is continuous, strictly monotone (thus, maximal monotone too) and coercive;
(b) $|a(y)| \le c_3(|y|^{s-1} + |y|^{p-1})$ for all $y \in \mathbb{R}^N$, some $c_3 > 0$;
(c) $\frac{c_0}{p-1}|y|^p \le (a(y), y)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N$.

Using this lemma and (1), we obtain the following bilateral growth conditions for the primitive $G(\cdot)$:

$$\frac{c_0}{p(p-1)}|y|^p \le G(y) \le c_4(|y|^s + |y|^p) \text{ for all } y \in \mathbb{R}^N, \text{some } c_4 > 0. \tag{2}$$

The hypotheses on the reaction $f(z,x,y)$ are the following:
H_1: $f : \Omega \times (0,\infty) \times \mathbb{R}^N \to [0,\infty)$ is a Carathéodory function such that
(i) $0 < \gamma_1 \le \liminf_{x\to 0^+} f(z,x,y)x^\eta \le \limsup_{x\to 0^+} f(z,x,y)x^\eta \le \gamma_2$ uniformly for a.a. $z \in \Omega$, all $|y| \le \delta$ with $\delta > 0$;
(ii) $0 \le f(z,x,y) \le \hat{a}(z)[1 + x^{-\eta} + x^{r-1} + |y|^{\theta-1}]$ for a.a. $z \in \Omega$, all $x > 0$, all $y \in \mathbb{R}^N$, with $\hat{a} \in L^\infty(\Omega)$, $p < r < p^*$, $\theta - 1 < p$ and $0 < \eta < 1$.

In the study of singular problems, a useful tool is the following Hardy inequality (see Papageorgiou–Winkert [13, p. 682]).

Proposition 1. *Let $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \Omega$. Then*

$$\left\|\frac{h}{\hat{d}}\right\|_p \le c^* \|Dh\|_p \text{ for all } h \in W_0^{1,p}(\Omega),\ 1 < p < \infty, \text{ some } c^* > 0.$$

Finally, let $V : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega)$ $(p' = p/(p-1))$ denote the nonlinear operator

$$\langle V(u), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\,dz \text{ for all } u, h \in W_0^{1,p}(\Omega). \tag{3}$$

This operator has the following properties (see [13, p. 665]).

Proposition 2. *The operator $V(\cdot)$ defined by (3) is bounded (maps bounded sets to bounded sets), continuous, strictly monotone (thus, maximal monotone too), coercive and of type $(S)_+$, that is,*

"if $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$".

On account of hypothesis $H_1(i)$, we can find $0 < \hat{\gamma}_0 < \gamma_0$ and $\gamma_1 < \hat{\gamma}_1$ such that

$$\hat{\gamma}_0 x^{-\eta} \le f(z,x,y) \le \hat{\gamma}_1 x^{-\eta} \text{ for a.a. } z \in \Omega, \text{ all } 0 < x \le \delta, \text{ all } |y| \le \delta. \tag{4}$$

Consider the positive (order) cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : 0 \le u(z) \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by $\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\right\}$, where $n(\cdot)$ is the outward unit normal on $\partial\Omega$.

We consider the following two purely singular problems:

$$-\operatorname{div} a(Du(z)) = \lambda\hat{\gamma}_0 u(z)^{-\gamma} \text{ in } \Omega,\ u|_{\partial\Omega} = 0,\ \lambda > 0,\ u > 0, \tag{5}$$

$$-\operatorname{div} a(Dy(z)) = \lambda\hat{\gamma}_1 y(z)^{-\gamma} \text{ in } \Omega,\ y|_{\partial\Omega} = 0,\ \lambda > 0,\ y > 0. \tag{6}$$

Proposition 3. *If hypotheses H_0 hold, then problems (5) and (6) have unique solutions $\overline{u}_\lambda \in \operatorname{int} C_+$ and $\overline{y}_\lambda \in \operatorname{int} C_+$ respectively, such that $0 \le \overline{u}_\lambda \le \overline{y}_\lambda$ and $\overline{y}_\lambda \to 0$ in $C_0^1(\overline{\Omega})$ as $\lambda \to 0^+$.*

Proof. The existence and uniqueness of the solutions $\overline{u}_\lambda$ (for problem (5)) and $\overline{y}_\lambda$ (for problem (6)) follow from Papageorgiou–Rădulescu–Repovs [7, Proposition 11]. □

On account of Proposition 3, we can find $\hat{\lambda} > 0$ such that

$$\|\overline{y}_\lambda\|_\infty \le \delta,\ \|D\overline{u}_\lambda\|_\infty,\ \|D\overline{y}_\lambda\|_\infty \le \delta \text{ for all } 0 < \lambda \le \hat{\lambda}. \tag{7}$$

Then, from (4), we have for all $0 < \lambda \le \hat{\lambda}$

$$-\operatorname{div} a(D\overline{u}_\lambda) = \lambda\hat{\gamma}_0 \overline{u}_\lambda^{-\eta} \le \lambda f(z,\overline{u}_\lambda, D\overline{u}_\lambda) \text{ in } \Omega, \tag{8}$$

$$-\operatorname{div} a(D\overline{y}_\lambda) = \lambda\hat{\gamma}_1 \overline{y}_\lambda^{-\eta} \ge \lambda f(z,\overline{y}_\lambda, D\overline{y}_\lambda) \text{ in } \Omega. \tag{9}$$

We introduce the truncation operator $\hat{\tau} : L^p(\Omega) \to L^p(\Omega)$ defined by

$$\hat{\tau}(z) = \begin{cases} \overline{u}_\lambda(z) \text{ if } u(z) < \overline{u}_\lambda(z) \\ u(z) \quad \text{if } \overline{u}_\lambda(z) \le u(z) \le \overline{y}_\lambda(z) \\ \overline{y}_\lambda(z) \ \text{ if } \overline{y}_\lambda(z) < u(z). \end{cases} \tag{10}$$

We know (see [13, p. 382]) that for all $u \in W_0^{1,p}(\Omega)$

$$D\hat{\tau}(z) = \begin{cases} D\overline{u}_\lambda(z) \text{ if } u(z) < \overline{u}_\lambda(z) \\ Du(z) \quad \text{if } \overline{u}_\lambda(z) \le u(z) \le \overline{y}_\lambda(z) \\ D\overline{y}_\lambda(z) \ \text{ if } \overline{y}_\lambda(z) < u(z). \end{cases} \tag{11}$$

Therefore $\hat{\tau} : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ and it is continuous (see Proposition 4.1 of Ozturk–Papageorgiou [11]).

Let $v \in C_0^1(\overline{\Omega})$ and consider another auxiliary Dirichlet problem

$$\begin{cases} -\operatorname{div} a(Du(z)) = \lambda f(z, \hat{\tau}(v)(z), D\hat{\tau}(v)(z)) \ \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ u > 0, 0 < \lambda \le \hat{\lambda}. \end{cases} \tag{12}$$

Proposition 4. *If hypotheses H_0, H_1 hold and $0 < \lambda \le \hat{\lambda}$, then problem (12) has a unique solution $k_\lambda(v) \in \operatorname{int} C_+$.*

Proof. We rewrite problem (12) as the following equivalent abstract equation

$$Vu = \lambda N_f(\hat{\tau}(v)) \text{ in } W_0^{1,p}(\Omega),$$

where $N_f(y)(\cdot) = f(\cdot, y(\cdot), Dy(\cdot))$ for all $y \in W_0^{1,p}(\Omega)$ (the Nemytskii map corresponding to the function f). By (10), (11), (4) and (7), for every $h \in W_0^{1,p}(\Omega)$ we have

$$|\int_\Omega f(z, \hat{\tau}(v), D\hat{\tau}(v))hz| \le \int_{v<\overline{u}_\lambda} f(z,\overline{u}_\lambda, D\overline{u}_\lambda)|h|dz + \int_{\overline{u}_\lambda \le v \le \overline{y}_\lambda} f(z,v,Dv)|h|dz + \int_{\overline{y}_\lambda < v} f(z,\overline{y}_\lambda, D\overline{y}_\lambda)|h|dz. \tag{13}$$

Since $\overline{u}_\lambda$, $\overline{y}_\lambda \in \operatorname{int} C_+$, combining Lemma 2.3 of Guo–Webb [14] with Proposition 1, we obtain that

$$\int_{v<\overline{u}_\lambda} f(z,\overline{u}_\lambda, D\overline{u}_\lambda)|h|dz \le c_9\|h\| \text{ for some } c_9 > 0 \tag{14}$$

$$\int_{\overline{y}_\lambda<v} f(z,\overline{y}_\lambda, D\overline{y}_\lambda)|h|dz \le c_{10}\|h\| \text{ for some } c_{10} > 0. \tag{15}$$

Finally, by Hypothesis $H_1(i)$ and since $v \in C_0^1(\overline{\Omega})$, we have for some $c_{11}, c_{12} > 0$

$$\int_{\bar{u}_\lambda \le v \le \bar{y}_\lambda} f(z, v, Dv)|h|dz \le c_{11}[\int_{\bar{u}_\lambda \le v \le \bar{y}_\lambda} (1 + |Dv|^{p-1})|h|dz + \int_{\bar{u}_\lambda \le v \le \bar{y}_\lambda} \bar{u}_\lambda^{-\eta}|h|dz] \le c_{12}\|h\|\,. \tag{16}$$

We return to (13) and use relations (14), (15), (16). Then

$$|\int_\Omega f(z, \hat{\tau}(v), D\hat{\tau}(v))hz| \le c_{13}\|h\| \text{ for some } c_{13} > 0,$$

hence $N_f(\hat{\tau}(v)) \in W_0^{1,p}(\Omega)^* \cap L^s(\Omega)$ with $s \in [1, 1/\eta)$ (see Lazer–McKenna [15]). From Proposition 2, we know that $V(\cdot)$ is continuous, strictly monotone, coercive, thus surjective (see Papageorgiou–Winkert [13, p. 576]). Therefore we can find $k_\lambda(v) \in W_0^{1,p}(\Omega)$ such that $V(k_\lambda(v)) = \lambda N_f(\hat{\tau}(v))$ in $W_0^{1,p}(\Omega)^*$. The strict monotonicity of $V(\cdot)$ implies that this solution is unique. Note that for some $c_{14}, c_{15} > 0$ we have

$$0 \le N_f(\hat{\tau}(v)) \le c_{14}\bar{u}_\lambda^{-\eta} \text{ (see (4), (7))} \le c_{15}\hat{d}^{-\eta} \text{ see Guo–Webb[14].}$$

Invoking Theorem 1.7 of Giacomoni–Kumar–Sreenadh [16], he obtain that $k_\lambda(v) \in C_+ \setminus \{0\}$, hence $k_\lambda(v) \in \operatorname{int} C_+$ (see Pucci–Serrin [17, p. 120]). □

We can define the solution map $k_\lambda : C_0^1(\overline{\Omega}) \to \operatorname{int} C_+$ for problem (12).

Proposition 5. *If hypotheses H_0, H_1 hold and $0 < \lambda \le \hat{\lambda}$, then the solution map $k_\lambda : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ is compact.*

Proof. We first show the continuity of $k_\lambda(\cdot)$. Suppose that $v_n \to v$ in $C_0^1(\overline{\Omega})$ as $n \to \infty$. Let $u_n = k_\lambda(v_n)$. Then

$$V(u_n) = \lambda N_f(\hat{\tau}(v_n)) \text{ in } W_0^{1,p}(\Omega)^*. \tag{17}$$

Then, by (10) and (11), there exists $c_{16} > 0$ such that

$$\frac{c_0}{p-1}\|u_n\|^p \le \int_\Omega \lambda f(z, \hat{\tau}(v_n), D\hat{\tau}(v_n))u_n dz \le c_{16}\|u_n\| \text{ for all } n \in \mathbb{N},$$

hence $(u_n) \subset W_0^{1,p}(\Omega)$ is bounded. We can assume that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega),\ u_n \to u \text{ in } L^r(\Omega). \tag{18}$$

On (17) we act with $u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (18). Then $\lim_{n\to\infty}\langle V(u_n), u_n - u\rangle = 0$, hence $u_n \to u$ in $W_0^{1,p}(\Omega)$ (see Proposition 2). Thus, if we pass to the limit in (17) as $n \to \infty$ and we use the continuity of $\hat{\tau}(\cdot)$, we obtain that $V(u) = \lambda N_f(\hat{\tau}(v))$ in $W_0^{1,p}(\Omega)^*$, thus $u = k_\lambda(v)$. We conclude that $k_\lambda(v_n) \to k_\lambda(v)$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$, so $k_\lambda : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ is continuous.

Let $B \subseteq C_0^1(\overline{\Omega})$ be bounded. With the same arguments as above, we show that $k_\lambda(B) \subseteq L^\infty(\Omega)$ is bounded. Then Theorem 1.7 of Giacomoni–Kumar–Sreenadh [16] implies that

$$\overline{k_\lambda(B)}^{C_0^1(\overline{\Omega})} \subseteq C_0^1(\overline{\Omega}) \text{ is compact.}$$

We conclude that the map $k_\lambda(\cdot)$ is compact. □

We introduce the set $E_{k_\lambda} = \{u \in C_0^1(\overline{\Omega}) : u = tk_\lambda(u),\ 0 < t < 1\}$.

Proposition 6. *If hypotheses H_0, H_1 hold and $0 < \lambda \le \hat{\lambda}$, then $E_{k_\lambda} \subseteq C_0^1(\overline{\Omega})$ is bounded.*

Proof. Let $u \in E_{k_\lambda}$. We have

$$\frac{1}{t}u = k_\lambda(u)\ (0 < t < 1)$$
$$\Rightarrow -\operatorname{div} a(\frac{1}{t}Du) = \lambda f(z, \hat{\tau}(u), D\hat{\tau}(u)) \text{ in } \Omega.$$

Acting with $u \in W_0^{1,p}(\Omega)$ and using Lemma 1, we obtain

$$\frac{1}{t^{p-1}}\frac{c_0}{p-1}\|u\|^p \le c_{17}\|u\| \text{ for some } c_{17} > 0 \text{ (see (10), (11))}$$
$$\Rightarrow E_{k_\lambda} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

Then, by Proposition 4 of Papageorgiou–Rădulescu [18], we obtain that $E_{k_\lambda} \subseteq L^\infty(\Omega)$ is bounded. Next, by Theorem 1.7 of Giacomoni–Kumar–Sreenadh [16] (see also Papageorgiou–Rădulescu [18, Theorem 4]), we conclude that $E_{k_\lambda} \subseteq C_0^1(\overline{\Omega})$ is bounded. □

3. Case of small perturbations

The following theorem is the main result of this paper and it establishes the existence of solutions to problem $(\mathcal{P}_\lambda)$ in the case of small perturbations of the reaction term. In particular, the following property establishes that the nonlinear singular eigenvalue problem $(\mathcal{P}_\lambda)$ has a continuous family of eigenvalues. For the definition of *eigenvalues* in the context of *nonlinear* eigenvalue problems we refer to Fučik–Nečas–Souček–Souček [19, p. 117].

Theorem 1. *If hypotheses H_0 and H_1 hold and $0 < \lambda \leq \hat{\lambda}$, then problem $(\mathcal{P}_\lambda)$ has a solution $\overline{u}_\lambda \in \operatorname{int} C_+$.*

Proof. Propositions 5, 6 and the Leray–Schauder alternative principle (see Papageorgiou–Winkert [13, p. 634]) imply that there exists $u_\lambda \in W_0^{1,p}(\Omega)$ such that $u_\lambda = k_\lambda(u_\lambda)$, hence

$$\langle V(u_\lambda), h\rangle = \int_\Omega \lambda f(z, \hat{\tau}(u_\lambda), D\hat{\tau}(u_\lambda))h dz \text{ for all } h \in W_0^{1,p}(\Omega). \tag{19}$$

In (19) we first use the test function $h = (u_\lambda - \overline{y}_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\langle V(u_\lambda), (u_\lambda - \overline{y}_\lambda)^+\rangle &= \int_\Omega \lambda f(z, \overline{y}_\lambda, D\overline{y}_\lambda)(u_\lambda - \overline{y}_\lambda)^+ dz \text{ (see (9), (10))}\\ &\leq \int_\Omega \lambda \hat{\gamma}_1 \overline{y}_\lambda^{-\eta}(u_\lambda - \overline{y}_\lambda)^+ dz = \langle V(\overline{y}_\lambda), (u_\lambda - \overline{y}_\lambda)^+\rangle \text{ (see Proposition 3).}\end{aligned}$$

Thus, by Proposition 2, we deduce that $u_\lambda \leq \overline{y}_\lambda$.

Similarly, if in (19) we choose the test function $h = (\overline{u}_\lambda - u_\lambda)^+$, then from (5) and (7), we infer that $\overline{u}_\lambda \leq u_\lambda$. So, we have proved that $\overline{u}_\lambda \leq u_\lambda \leq \overline{y}_\lambda$, thus u_λ is a solution of $(\mathcal{P}_\lambda)$. The nonlinear regularity theory implies that $u_\lambda \in \operatorname{int} C_+$. □

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Data availability

No data was used for the research described in the article.

References

[1] G.M. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations, Comm. Partial Differential Equations 16 (1991) 311–361.
[2] V. Benci, P. D'Avenia, D. Fortunato, L. Pisani, Solitons in several space dimensions: Derrick's problem and infinitely many solutions, Arch. Ration. Mech. Anal. 154 (4) (2000) 297–324.
[3] L. Cherfils, Y. Il'yasov, On the stationary solutions of generalized reaction diffusion equations with $p\&q$-Laplacian, Commun. Pure Appl. Anal. 4 (1) (2005) 9–22.
[4] T. Roubicek, Nonlinear Partial Differential Equations with Applications, Birkhäuser, Basel, 2013.
[5] K. Uhlenbeck, Regularity for a class of nonlinear elliptic systems, Acta Math. 138 (3–4) (1977) 219–240.
[6] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovs, Positive solutions for nonlinear Neumann problems with singular terms and convection, J. Math. Pures Appl. 136 (2020) 1–21.
[7] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovs, Nonlinear nonhomogeneous singular problems, Calc. Var. Partial Differential Equations 59 (9) (2020).
[8] F. Faraci, D. Puglisi, A singular semilinear problem with dependence on the gradient, J. Differential Equations 260 (4) (2016) 3327–3349.
[9] Z. Liu, D. Motreanu, S. Zeng, Positive solutions for nonlinear singular elliptic equations of p-Laplacian type with dependence on the gradient, Calc. Var. Partial Differential Equations 58 (1) (2019) 22, Paper No. 28.
[10] N.S. Papageorgiou, A. Scapellato, Nonlinear singular problems with convection, J. Differential Equations 296 (2021) 493–511.
[11] E. Ozturk, N.S. Papageorgiou, Nonhomogeneous singular problems with convection, J. Fixed Point Theory Appl. 26 (4) (2024) 14, Paper No. 62.
[12] N.S. Papageorgiou, V.D. Rădulescu, Coercive and noncoercive nonlinear Neumann problems with indefinite potential, Forum Math. 28 (3) (2016) 545–571.
[13] N.S. Papageorgiou, P. Winkert, Applied Nonlinear Functional Analysis, De Gruyter, Berlin, 2024.
[14] Z.M. Guo, J.R.L. Webb, Uniqueness of positive solutions for quasilinear elliptic equations when a parameter is large, Proc. Roy. Soc. Edinburgh Sect. A 124 (1994) 189–198.
[15] A.C. Lazer, P.J. McKenna, On a singular nonlinear elliptic boundary-value problem, Proc. Amer. Math. Soc. 111 (3) (1991) 721–730.
[16] J. Giacomoni, D. Kumar, K. Sreenadh, Sobolev and Hölder regularity results for some singular nonhomogeneous quasilinear problems, Calc. Var. Partial Differential Equations 60 (3) (2021) 33, Paper No. 121.
[17] P. Pucci, J. Serrin, The Maximum Principle, Birkhäuser, Basel, 2007.
[18] N.S. Papageorgiou, V.D. Rădulescu, Some useful tools in the study of nonlinear elliptic problems, Expo. Math. 42 (6) (2024) 27, Paper No. 125616.
[19] S. Fučik, J. Nečas, J. Souček, V. Souček, Spectral Analysis of Nonlinear Operators, in: Lecture Notes in Mathematics, vol. 346, Springer-Verlag, Berlin-New York, 1973.