EIGENVALUE PROBLEMS WITH UNBALANCED GROWTH

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Abstract. We consider a nonlinear eigenvalue problem driven by the nonautonomous (p, q)-Laplacian with unbalanced growth. Using suitable Rayleigh quotients and varational tools, we show that the problem has a continuous spectrum which is an upper half line and we also show a nonexistence result for a lower half line.

1. Introduction

In [1] the authors studied a nonlinear eigenvalue problem driven by an anisotropic differential operator and proved the existence of a continuous spectrum. At the end of the paper, they mentioned as an interesting research direction, the extension of their work to unbalanced growth problems (double phase equations). In this paper, we present such an extension.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We study the following nonlinear eigenvalue problem

$$\left\{ \begin{array}{l} -\Delta_p^\alpha u(z) - \Delta_q u(z) = \lambda f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ \lambda \in \mathbb{R}, \ 1 < q < p < N. \end{array} \right\} \qquad (\mathcal{P}_\lambda)$$

In this problem $\alpha \in C^{0,1}(\bar{\Omega})$, $\alpha(z) > 0$ for all $z \in \Omega$ and by Δ_p^α we denote the weighted p-Laplacian defined by $\Delta_p^\alpha u = \operatorname{div}(\alpha(z)|Du|^{p-2}Du)$. We do not assume that $\min_{\bar{\Omega}} \alpha > 0$. So, the density function

$$\theta(z, t) = \alpha(z)t^p + t^q, \text{ for all } z \in \Omega, \text{ all } t \geq 0$$

associated with the differential operator of $(\mathcal{P}_\lambda)$ has unbalanced growth, namely we have

$$t^q \leq \theta(z, t) \leq c_0(1 + t^p) \text{ for all } z \in \Omega, \text{ all } t \geq 0, \text{ some } c_0 > 0.$$

So, $\theta(z, t)$ is trapped between two different powers of t. This has significant implications on the structure of the problem. A first implication is that the classical Lebesgue and Sobolev spaces are not suitable to study problem $(\mathcal{P}_\lambda)$ and we have to pass to generalized Orlicz spaces. A second implication is that for such problems we do not have a global (up to the boundary) regularity theory and this means that we do not have at our disposal some powerful tools, which are readily available for balanced growth problems. So, we do not have a nonlinear Hopf maximum principle, strong comparison results (see Papageorgiou-Rădulescu-Repovs [7]) and the equivalence of local Sobolev and Hölder minimizers (see García Azorero-Peral Alonso-Manfredi [3]). This makes the study of unbalanced growth problems more difficult. Recently, Papageorgiou-Pudełko-Rădulescu [6], developed the spectral properties of the operator $u \to -\Delta_p^\alpha u$ with Dirichlet boundary condition. Using their results, we are able to establish a continuous spectrum for $(\mathcal{P}_\lambda)$, extending the work of [1] to unbalanced growth problems. We point out that (H4) in [1], excludes from consideration (p, q)-equations.

2. Mathematical background

The analysis of problem $(\mathcal{P}_\lambda)$ requires the use of generalized Orlicz spaces. For a comprehensive presentation of the theory of these spaces, we refer to the book of Harjulehto-Hästö [4].

Let $L^0(\Omega)$ be the linear space of all measurable functions $u : \Omega \to \mathbb{R}$. We identify two such functions which differ only on a Lebesgue null set. Recall that $\theta(z, t) = \alpha(z)t^p + t^q$ for all $z \in \Omega$, all $t \geq 0$. The Lebesgue-Orlicz space $L^\theta(\Omega)$ is defined by

$$L^\theta(\Omega) = \left\{ u \in L^0(\Omega) : \ \rho_\theta(u) = \int_\Omega \theta(z, |u|)\mathrm{d}z < \infty \right\}.$$

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The integral functional $\rho_\theta(\cdot)$ is known as the modular function corresponding to θ. We equip $L^\theta(\Omega)$ with the so-called Luxemburg norm $\|\cdot\|_\theta$ defined by

$$\|u\|_\theta = \inf\left\{\mu > 0 : \rho_\theta\left(\frac{u}{\mu}\right) \leq 1\right\}.$$

With this norm, $L^\theta(\Omega)$ becomes a separable, reflexive Banach space (in fact it is uniformly convex, since $\theta(z,\cdot)$ is uniformly convex). Using $L^\theta(\Omega)$, we can define the corresponding Sobolev-Orlicz space $W^{1,\theta}(\Omega)$ by

$$W^{1,\theta}(\Omega) = \left\{u \in L^\theta(\Omega) : |Du| \in L^\theta(\Omega)\right\}.$$

Here Du denotes the weak gradient of u. We equip $W^{1,\theta}(\Omega)$ with the norm $\|\cdot\|_{1,\theta}$ defined by

$$\|u\|_{1,\theta} = \|u\|_\theta + \|Du\|_\theta, \text{ for all } u \in W^{1,\theta}(\Omega),$$

with $\|Du\|_\theta = \||Du|\|_\theta$. Also, let

$$W_0^{1,\theta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\theta}}$$

with $C_c^\infty(\Omega)$ being the space of C^∞-functions with compact support. If we assume that $p < q^* = \frac{Nq}{N-q}$, then on $W_0^{1,\theta}(\Omega)$, the Poincaré inequality holds, that is, there exists $\hat{c} = \hat{c}(\Omega)$ such that $\|u\|_\theta \leq \hat{c}\|Du\|_\theta$, for all $u \in W_0^{1,\theta}(\Omega)$ (see [2, Proposition 2.18]). Therefore on $W_0^{1,\theta}(\Omega)$ we can consider the equivalent norm $\|\cdot\|$ defined by $\|u\| = \|Du\|_\theta$, for all $u \in W_0^{1,\theta}(\Omega)$. By $\hookrightarrow$ we denote a continuous embedding and by $\overset{c}{\hookrightarrow}$ a compact embedding.

Proposition 1. (a) $L^\theta(\Omega) \hookrightarrow L^r(\Omega)$, $W_0^{1,\theta}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ *for all* $1 \leq r \leq q$;
(b) $W_0^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ *for all* $1 \leq r \leq q^*$, $W_0^{1,\theta}(\Omega) \overset{c}{\hookrightarrow} L^r(\Omega)$ *for all* $0 \leq r < q^*$;
(c) $L^p(\Omega) \hookrightarrow L^\theta(\Omega)$.

Also there is a close relation between the the modular function $\rho_\theta(\cdot)$ and the norm $\|\cdot\|$.

Proposition 2. (a) $\|u\| = \mu > 0 \ \Leftrightarrow\ \rho_\theta\left(\frac{Du}{\mu}\right) = 1$;
(b) $\|u\| < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow$ $\rho_\theta(Du) < 1$ (*resp.* $= 1, > 1$);
(c) $\|u\| < 1 \ \Rightarrow\ \|u\|^p \leq \rho_\theta(Du) \leq \|u\|^q$;
(d) $\|u\| > 1 \ \Rightarrow\ \|u\|^q \leq \rho_\theta(Du) \leq \|u\|^p$;
(e) $\|u\| \to 0$ (*resp.* $\to +\infty$) $\Leftrightarrow$ $\rho_\theta(Du) \to 0$ (*resp.* $\to +\infty$).

Let $\beta \in L^1_{\text{loc}}(\Omega)$. We say that $\beta(\cdot)$ is a weight function, if $\beta(z) > 0$ for a.a. $z \in \Omega$. Consider $r \in (1,\infty)$ and by $|\cdot|_N$ denote the Lebesgue measure on $\mathbb{R}^N$. We say that the weight function $\beta(\cdot)$ belongs to the class Muckenhoupt $\mathcal{A}_r$ (denoted by $\beta \in \mathcal{A}_r$), if the following holds

$$\sup_B \left[\frac{1}{|B|_N}\int_B \beta(z)\mathrm{d}z\right]\left[\frac{1}{|B|_N}\int_B \beta(z)^{\frac{1}{1-r}}\mathrm{d}z\right]^{r-1} < \infty,$$

where the supremum is taken over all balls $B \subseteq \Omega$.

Now we introduce our hypotheses on the weight $\alpha(\cdot)$ and the exponents p, q.

H_0 : $\alpha \in C^{0,1}(\bar{\Omega}) \cap \mathcal{A}_p$, $\alpha(z) > 0$ for all $z \in \Omega$, $1 < q < p < N$, $\frac{p}{q} < 1 + \frac{1}{N}$.

Remark 1. The last inequality implies that $p < q^*$.

We set $\theta_0(z,t) = \alpha(z)t^p$ for all $z \in \Omega$, all $t \geq 0$. We can define the generalized Orlicz spaces $L^{\theta_0}(\Omega)$ and $W_0^{1,\theta_0}(\Omega)$. They coincide with the weighted spaces $L^p(\Omega,\alpha)$ and $W_0^{1,p}(\Omega,\alpha)$. Moreover, from Papageorgiou-Rădulescu-Zhang [8, Lemma 2], we know that $W_0^{1,\theta_0}(\Omega) \hookrightarrow L^{\theta_0}(\Omega)$ compactly.

Consider the following nonlinear eigenvalue problem:

$$-\Delta_p^\alpha u(z) = \hat{\lambda}\alpha(z)|u(z)|^{p-2}u(z) \text{ in } \Omega,\ u\big|_{\partial\Omega} = 0.$$

According to [6], the problem has a smallest eigenvalue $\hat{\lambda}_1^\alpha > 0$ which satisfies

$$\hat{\lambda}_1^\alpha = \inf\left\{\frac{\rho_{\theta_0}(Du)}{\rho_{\theta_0}(u)} : u \in W_0^{1,\theta_0}(\Omega),\ u \neq 0\right\} = \inf\left\{\rho_{\theta_0}(Du) : \rho_{\theta_0}(u) = 1\right\}. \tag{1}$$

In addition, we have

- $\hat{\lambda}_1^\alpha > 0$ is isolated in the spectrum $\hat{\sigma}_\rho^\alpha$ of $(-\Delta_p^\alpha, W_0^{1,\theta_0}(\Omega))$.
- $\hat{\lambda}_1^\alpha > 0$ is simple.

- The eigenfunctions of $\hat{\lambda}_1^\alpha$ have fixed sign (see (1)). In fact, this is the only eigenvalue with eigenfunctions of constant sign.

Let $\hat{u}_1$ be the L^{θ_0}-normalized positive eigenfunction for $\hat{\lambda}_1^\alpha$. Then

$$\hat{u}_1 \in W_0^{1,\theta_0}(\Omega) \cap L^\infty(\Omega), \quad \|\hat{u}_1\|_{\theta_0} = 1 \text{ and } 0 \prec \hat{u}_1.$$

Here if $u \in L^0(\Omega)$, then we write $0 \prec u$ if for all $K \subseteq \Omega$ compact we have $0 < c_K \leq u(z)$ for a.a. $z \in K$. Evidently, such a function satisfies $0 < u(z)$ for a.a. $z \in \Omega$.

Our hypotheses on the reaction $f(z,x)$ are the following:

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ ia a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$, $0 < f(z,x)x$ for a.a. $z \in \Omega$, all $x \neq 0$ and there exist $\tau \in (q,p)$ and $\hat{c} > 0$ such that if $F(\xi,x) = \int_0^x f(z,s)\mathrm{d}s$, then $F(z,x) \leq f(z,x)x \leq \hat{c}\alpha(z)|x|^\tau$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$.

3. Continuous Spectrum

We introduce the following two quantities

$$\lambda^* = \inf\left\{ \frac{\frac{1}{p}\rho_{\theta_0}(Du) + \frac{1}{q}\|Du\|_q^q}{\int_\Omega F(z,u)\mathrm{d}z} : u \in W_0^{1,\theta_0}(\Omega),\ u \neq 0 \right\}, \tag{2}$$

$$\lambda_* = \inf\left\{ \frac{\rho_{\theta_0}(Du) + \|Du\|_q^q}{\int_\Omega f(z,u)u\mathrm{d}z} : u \in W_0^{1,\theta_0}(\Omega),\ u \neq 0 \right\}. \tag{3}$$

Proposition 3. *If hypotheses H_0, H_1 hold, then $0 < \frac{\lambda_*}{p} < \lambda^*$ and there exists $\bar{u} \in W_0^{1,\theta_0}(\Omega) \setminus \{0\}$ such that $\lambda_* = \frac{\rho_{\theta_0}(D\bar{u})}{\int_\Omega f(z,\bar{u})\bar{u}\mathrm{d}z}$.*

Proof. Given $\varepsilon > 0$, we can find $u \in W_0^{1,\theta}(\Omega) \setminus \{0\}$ such that

$$\begin{aligned}\lambda^* + \varepsilon \geq & \frac{\frac{1}{p}\rho_{\theta_0}(Du) + \frac{1}{q}\|Du\|_q^q}{\int_\Omega F(z,u)\mathrm{d}z} \quad (\text{see (2)}) > \frac{\frac{1}{p}[\rho_{\theta_0}(Du) + \|Du\|_q^q]}{\int_\Omega F(z,u)\mathrm{d}z} \quad (\text{since } q < p, u \neq 0) \\ > & \frac{1}{p}\frac{\rho_{\theta_0}(Du) + \|Du\|_q^q}{\int_\Omega f(z,u)u\mathrm{d}z} \quad (\text{see hypotheses } H_1) > \frac{\lambda_*}{p} \quad (\text{see (3)}).\end{aligned}$$

Let $\varepsilon \to 0^+$ to conclude that $\frac{\lambda_*}{p} < \lambda^*$. Next, let $u \in W_0^{1,\theta}(\Omega) \setminus \{0\}$ such that

$$\frac{\rho_{\theta_0}(Du) + \|Du\|_q^q}{\int_\Omega f(z,u)u\mathrm{d}z} \leq \lambda_* + 1 \quad (\text{see (3)}).$$

For such a $u \in W_0^{1,\theta}(\Omega) \setminus \{0\}$, we have

$$\begin{aligned}\lambda_* + 1 \geq & \frac{\rho_{\theta_0}(Du) + \|Du\|_q^q}{\int_\Omega f(z,u)u\mathrm{d}z} \geq \frac{\|Du\|_q^q}{\hat{c}\int_\Omega \alpha(z)|u|^\tau\mathrm{d}z} \quad (\text{see hypotheses } H_1) \\ \geq & c_1 \frac{\|Du\|_q^q}{\|u\|_\tau^\tau} \quad \text{for some } c_1 > 0 \quad (\text{see hypotheses } H_0) \\ \geq & \frac{c_2}{\|Du\|_q^{\tau - q}} \quad \text{for some } c_2 > 0 \ (\text{by } W_0^{1,q}(\Omega) \hookrightarrow L^\tau(\Omega),\ \text{since } q < \tau < p < q^*).\end{aligned}$$

It follows that

$$\frac{c_2}{\lambda_* + 1} \leq \|Du\|_q^{\tau - q}. \tag{4}$$

Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega) \setminus \{0\}$ be a sequence such that

$$\frac{\rho_\theta(Du_n)}{\int_\Omega f(z,u_n)u_n\mathrm{d}z} \downarrow \lambda_* \text{ as } n \to \infty.$$

We may assume that

$$\frac{\rho_\theta(Du_n)}{\int_\Omega f(z,u_n)u_n\mathrm{d}z} = \frac{\rho_{\theta_0}(Du_n) + \|Du_n\|_q^q}{\int_\Omega f(z,u_n)u_n\mathrm{d}z} \leq \lambda_* + 1 \text{ for all } n \in \mathbb{N}$$

$$\Rightarrow \frac{\rho_\theta(Du_n)}{\hat{c}\int_\Omega \alpha(z)|u_n|^\tau\mathrm{d}z} \leq \lambda_* + 1 \text{ for all } n \in \mathbb{N} \ (\text{see hypotheses } H_1). \tag{5}$$

By Hölder's inequality with $r = \frac{p}{\tau}$ and $r' = \frac{p}{p-\tau}$,

$$\int_\Omega \alpha(z)|u_n|^\tau \mathrm{d}z = \int_\Omega \alpha(z)^{1-\frac{\tau}{p}}\alpha(z)^{\frac{\tau}{p}}|u_n|^\tau \mathrm{d}z \le c_3\rho_{\theta_0}(u_n)^{\frac{\tau}{p}} \text{ for some } c_3>0, \text{ all } n\in\mathbb{N}. \tag{6}$$

Using (6) in (5), we obtain

$$\frac{\rho_\theta(Du_n)}{c_3\rho_{\theta_0}(u_n)^{\frac{\tau}{p}}} \le \lambda_*+1 \text{ for all } n\in\mathbb{N}, \Rightarrow \frac{\hat{\lambda}_1^\alpha}{c_3}\rho_{\theta_0}(u_n)^{1-\frac{\tau}{p}} \le \lambda_*+1 \text{ (see (1))},$$

$$\Rightarrow \{\rho_{\theta_0}(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+ \text{ is bounded (recall that } \tau<p). \tag{7}$$

We return to (5) and use (6) and (7) to conclude that $\{\rho_\theta(Du_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ is bounded, hence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega)$ is bounded (see Proposition 2). So, we may assume that

$$u_n \xrightarrow{w} \bar{u} \text{ in } W_0^{1,\theta}(\Omega),\ u_n\to\bar{u} \text{ in } L^{\theta_0}(\Omega) \text{ and in } L^\tau(\Omega) \tag{8}$$
$$\text{(since } W_0^{1,\theta}(\Omega)\hookrightarrow W_0^{1,\theta_0}(\Omega)\xrightarrow{c} L^{\theta_0}(\Omega) \text{ and see Proposition 1).}$$

Suppose that $\bar{u}=0$. We have

$$\begin{aligned}\rho_\theta(Du_n) \le&[\lambda_*+1]\int_\Omega f(z,u_n)u_n\mathrm{d}z \text{ for all } n\in\mathbb{N}\\ \le&[\lambda_*+1]\hat{c}\int_\Omega \alpha(z)|u_n|^\tau\mathrm{d}z \text{ for all } n\in\mathbb{N} \text{ (see hypotheses } H_1)\\ \le& c_4\rho_{\theta_0}(u_n)^{\frac{\tau}{p}} \text{ for some } c_4>0, \text{ all } n\in\mathbb{N} \text{ (see (6))}\end{aligned}$$

$$\Rightarrow \quad \rho_\theta(Du_n)\to 0 \text{ (see (8) and recall } \bar{u}=0), \Rightarrow \quad u_n\to 0 \text{ in } W_0^{1,\theta}(\Omega) \text{ as } n\to\infty,$$

which contradicts (4). Therefore $\bar{u}\ne 0$. From (8) and the sequential weak lower semicontinuity of $\rho_\theta(\cdot)$, we have $\frac{\rho_\theta(D\bar{u})}{\int_\Omega f(z,\bar{u})\bar{u}\mathrm{d}z} \le \lambda_*$. By (3) and since $\bar{u}\ne 0$, we conclude that $\lambda_* = \frac{\rho_\theta(D\bar{u})}{\int_\Omega f(z,\bar{u})\bar{u}\mathrm{d}z} > 0$. ☐

Proposition 4. *If hypotheses H_0, H_1 hold, then*

$$\lim_{u\to 0}\frac{\frac{1}{p}\rho_{\theta_0}(Du)+\frac{1}{q}\|Du\|_q^q}{\int_\Omega F(z,u)\mathrm{d}z} = +\infty.$$

Proof. For $u\in W_0^{1,\theta}(\Omega)\setminus\{0\}$, we have

$$\begin{aligned}\frac{\frac{1}{p}\rho_{\theta_0}(Du)+\frac{1}{q}\|Du\|_q^q}{\int_\Omega F(z,u)\mathrm{d}z} \ge& \frac{1}{p}\frac{\rho_{\theta_0}(Du)+\|Du\|_q^q}{\int_\Omega F(z,u)\mathrm{d}z} \text{ (since } q<p) \ge \frac{1}{p}\frac{\rho_{\theta_0}(Du)+\|Du\|_q^q}{\int_\Omega f(z,u)u\mathrm{d}z} \text{ (see hypotheses } H_1)\\ \ge&\frac{1}{pc_5}\frac{\|Du\|_q^q}{\|u\|_\tau^\tau} \text{ for some } c_5>0 \text{ (see hypotheses } H_1)\\ \ge&\frac{c_6}{\|Du\|_p^{\tau-q}} \text{ for some } c_6>0 \text{ (recall that } W_0^{1,q}(\Omega)\hookrightarrow L^\tau(\Omega)).\end{aligned} \tag{9}$$

If $u\to 0$ in $W_0^{1,\theta}(\Omega)$, then $\|Du\|_q\to 0$ and so from (9) we conclude the proof. ☐

Proposition 5. *If hypotheses H_0, H_1 hold, then*

$$\lim_{\|u\|\to\infty}\frac{\frac{1}{p}\rho_{\theta_0}(Du)+\frac{1}{q}\|Du\|_q^q}{\int_\Omega F(z,u)\mathrm{d}z} = +\infty.$$

Proof. For $u\in W_0^{1,\theta}(\Omega)\setminus\{0\}$, we have

$$\begin{aligned}\frac{\frac{1}{p}\rho_{\theta_0}(Du)+\frac{1}{q}\|Du\|_q^q}{\int_\Omega F(z,u)\mathrm{d}z} \ge& \frac{1}{p}\frac{\rho_\theta(Du)}{\int_\Omega f(z,u)u\mathrm{d}z} \ge \frac{1}{p\hat{c}}\frac{\rho_\theta(Du)}{\int_\Omega \alpha(z)|u|^\tau\mathrm{d}z} \text{ (see hypotheses } H_1)\\ =&c_7\frac{\rho_\theta(Du)}{\int_\Omega\alpha(z)|u|^\tau\mathrm{d}z} \text{ (for some } c_7>0) \ge c_7(\hat{\lambda}_1^\alpha)^{\frac{\tau}{p}}\frac{\rho_\theta(Du)}{\rho_{\theta_0}(Du)^{\frac{\tau}{p}}} \ge c_7(\hat{\lambda}_1^\alpha)^{\frac{\tau}{p}}\rho_\theta(Du)^{1-\frac{\tau}{p}}.\end{aligned}$$

Since $\tau<p$ and using Proposition 2, we conclude the proof. ☐

Proposition 6. *If hypotheses H_0, H_1 hold, then there exists $\hat{u}\in W_0^{1,\theta}(\Omega)\setminus\{0\}$ such that*

$$\lambda^* = \frac{\frac{1}{p}\rho_{\theta_0}(D\hat{u})+\frac{1}{q}\|D\hat{u}\|_q^q}{\int_\Omega F(z,\hat{u})\mathrm{d}z}.$$

Proof. Consider a sequence $\{u_n\} \subseteq W_0^{1,\theta}(\Omega) \setminus \{0\}$ such that

$$\frac{\frac{1}{p}\rho_{\theta_0}(Du_n) + \frac{1}{q}\|Du_n\|_q^q}{\int_\Omega F(z,u_n)\mathrm{d}z} \downarrow \lambda^* \quad \text{as} \quad n \to \infty. \tag{10}$$

From Proposition 5 it follows that $\{u_n\} \subseteq W_0^{1,\theta}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} \hat{u} \text{ in } W_0^{1,\theta}(\Omega),\ u_n \to \hat{u} \text{ in } L^\tau(\Omega) \quad \text{(see Proposition 1).} \tag{11}$$

Then by (11) and the sequential weak lower semicontinuity of $\rho_{\theta_0}(\cdot)$ and of $\|u\|_{1,q}$, we obtain

$$\frac{\frac{1}{p}\rho_{\theta_0}(D\hat{u}) + \frac{1}{q}\|D\hat{u}\|_q^q}{\int_\Omega F(z,\hat{u})\mathrm{d}z} \le \lambda^*. \tag{12}$$

If we show that $\hat{u} \neq 0$, then on account of (2), we will have equality in (12) and so we are done. By contradiction, assume that $\hat{u} = 0$. From (10) we see that given $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that

$$\frac{1}{p}\rho_{\theta_0}(Du_n) + \frac{1}{q}\|Du_n\|_q^q \le [\lambda^* + \varepsilon]\int_\Omega F(z,u_n)\mathrm{d}z \Rightarrow \frac{1}{p}\rho_\theta(Du_n) \le [\lambda^* + \varepsilon]\int_\Omega F(z,u_n)\mathrm{d}z \text{ for all } n \ge n_0,$$

$$\Rightarrow\ \rho_\theta(Du_n) \to 0 \ \text{ (see (12) and since } \hat{u} = 0 \text{ and } F(z,0) = 0), \Rightarrow\ u_n \to 0 \text{ in } W_0^{1,\theta}(\Omega) \text{ (see Proposition 2).}$$

But then on account of Proposition 4, we have $\frac{\frac{1}{p}\rho_{\theta_0}(Du_n)+\frac{1}{q}\|Du_n\|_q^q}{\int_\Omega F(z,u_n)\mathrm{d}z} \to +\infty$ as $n \to +\infty$, which contradicts (10). Therefore $\hat{u} \neq 0$ and so from (12) and (2), we conclude the proof. □

Proposition 7. *If hypotheses H_0, H_1 hold and $\lambda > \lambda^*$, then λ is an eigenvalue of problem $(\mathcal{P}_\lambda)$ with eigenfunctions in $W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Consider the C^1-functional $\varphi_\lambda : W_0^{1,\theta}(\Omega) \to \mathbb{R}$ defined by $\varphi_\lambda(u) = \frac{1}{p}\rho_{\theta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \lambda F(z,u)\mathrm{d}z$. Since $\tau < p$, a straightforward computation shows that $\varphi_\lambda(\cdot)$ is coercive. Also using Proposition 1, we show that $\varphi_\lambda(\cdot)$ is sequentially weak lower semicontinuous. So, by the Weierstrass-Tonelli Theorem, we can find $u_0 \in W_0^{1,\theta}(\Omega)$ such that $\varphi_\lambda(u_0) = \inf\{\varphi_\lambda(u) : u \in W_0^{1,\theta}(\Omega)\}$. Let $\hat{u} \in W_0^{1,\theta}(\Omega) \setminus \{0\}$ be the eigenfunction for $\lambda^* > 0$ produced in Proposition 6. Then $\varphi_\lambda(u_0) \le \varphi_\lambda(\hat{u}) < \frac{1}{p}\rho_{\theta_0}(D\hat{u}) + \frac{1}{q}\|D\hat{u}\|_q^q - \lambda^* \int_\Omega F(z,\hat{u})\mathrm{d}z$ (since $\lambda > \lambda^*$) $= 0 = \varphi_\lambda(0) \Rightarrow\ u_0 \neq 0$. It follows that

$$\langle \varphi_\lambda'(u_0), h\rangle = 0 \text{ for all } h \in W_0^{1,\theta}(\Omega), \Rightarrow\ u_0 \text{ is an eigenfunction for } \lambda, \Rightarrow\ \lambda \text{ is an eigenvalue.} \tag{13}$$

We show that $u_0 \in L^\infty(\Omega)$. Let $k > 1$ and set $U_k = \{z \in \Omega : |u(z)| > k\}$. Let $k_0 \in \mathbb{N}$ be such that

$$\|(|u_0| - k)^+\| \le 1 \quad \text{for all } k \ge k_0. \tag{14}$$

In (13) we choose the test function $h = (|u_0| - k)^+ \in W_0^{1,\theta}(\Omega)$. Then

$$\begin{aligned} \|(|u_0| - k)^+\|^p \le& \rho_\theta(D(|u_0| - k)^+) \qquad \text{(see (14) and Proposition 2)} \\ =& \lambda \int_\Omega f(z,u_0)(|u_0| - k)^+\mathrm{d}z \le \lambda \int_\Omega \alpha(z)((|u_0| - k)^+)^\tau \mathrm{d}z \qquad \text{(see hypotheses } H_1). \end{aligned} \tag{15}$$

Let $\eta \in (p, q^*)$ (since $p < q^*$, see H_0). By Proposition 1, $\chi_{u_k} \in L^{\frac{\eta}{\eta-\tau}}(\Omega)$, $(|u_0| - k)^+ \in L^{\frac{\eta}{\tau}}(\Omega)$. Then from (15) and Hölder's inequality combined with $W_0^{1,\theta}(\Omega) \hookrightarrow L^\eta(\Omega)$ (cf. Proposition 1), we have

$$\|(|u_0| - k)^+\|^p \le c_{10}|U_k|_N^{\frac{\eta-\tau}{\eta}} \|(|u_0| - k)^+\|^\tau \text{ for some } c_{10} > 0 \Rightarrow \|(|u_0| - k)^+\|^{p-\tau} \le c_{10}|U_k|_N^{\frac{\eta-\tau}{\eta}}. \tag{16}$$

Let $m > k$. Then

$$\begin{aligned} (m-k)^\tau |U_m|_N^{\frac{\tau}{\eta}} \le& \left[\int_{U_m} ((|u_0| - k)^+)^\eta \mathrm{d}z\right]^{\frac{\tau}{\eta}} \le \left[\int_{U_k} ((|u_0| - k)^+)^\eta \mathrm{d}z\right]^{\frac{\tau}{\eta}} \quad (\text{since } U_m \subseteq U_k) \\ \le& c_{11}\|(|u_0| - k)^+\|^\tau \ \text{ for some } c_{12} > 0 \text{ (since } W_0^{1,\theta}(\Omega) \hookrightarrow L^\eta(\Omega)) \\ \Rightarrow (m-k)^{p-\tau}|U_m|_N^{\frac{p-\tau}{\eta}} \le& c_{12}\|(|u_0| - k)^+\|^{p-\tau} \text{ with } c_{12} \le c_{13}|U_k|_N^{\frac{\eta-\tau}{\eta}} \text{ for some } c_{13} > 0 \text{ (see (16)).} \end{aligned}$$

Hence $(m-k)^\eta |U_m|_N \le c_{14}|U_k|_N^{\frac{\eta-\tau}{p-\tau}}$ for some $c_{14} > 0$. Note that $\frac{\eta-\tau}{p-\tau} > 1$. So, by Lemma B.1 in [5, p.63], there exists $\hat{k} > k_0$ large such that $|U_{\hat{k}}|_N = 0$, hence $u_0 \in L^\infty(\Omega)$. □

In what follows, $V : W_0^{1,\theta}(\Omega) \to W_0^{1,\theta}(\Omega)^*$ is the nonlinear operator defined by $\langle V(u), h\rangle = \int_\Omega [a(z)|Du|^{p-2} + |Du|^{q-2}](Du, Dh)_{\mathbb{R}^N} dz$ for all $u, h \in W_0^{1,\theta}(\Omega)$.

Proposition 8. *If hypotheses H_0, H_1 hold and $\lambda \leq \frac{1}{p}\lambda_*$, then λ is not an eigenvalue of $(\mathcal{P}_\lambda)$.*

Proof. By contradiction, suppose that λ is an eigenvalue. Then we can find $u_\lambda \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega) \setminus \{0\}$ such that $\langle V(u_\lambda), h\rangle = \lambda \int_\Omega f(z,u_\lambda)h\mathrm{d}z$ for all $h \in W_0^{1,\theta}(\Omega)$. Choose $h = u_\lambda \in W_0^{1,\theta}(\Omega)$, hence $\rho_\theta(Du_\lambda) = \lambda \int_\Omega f(z,u_\lambda)u_\lambda \mathrm{d}z \leq \frac{1}{p}\lambda_* \int_\Omega f(z,u_\lambda)u_\lambda\mathrm{d}z$ (see hypotheses H_0), hence $\lambda_* \leq \frac{1}{p}\lambda_*$, a contradiction, since $p > 1$. Thus, λ is not an eigenvalue. $\square$

Summarizing, we can state the following theorem describing the spectrum of $(\mathcal{P}_\lambda)$.

Theorem 1. *If hypotheses H_0, H_1 hold, then*
- (a) *every $\lambda \geq \lambda^*$ is an eigenvalue of $(\mathcal{P}_\lambda)$ with eigenfunctions in $W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega)$;*
- (b) *every $\lambda \leq \frac{1}{p}\lambda_*$ is not an eigenvalue.*

Remark 2. (a) We do not know what can be said about $\lambda \in (\frac{\lambda_*}{p}, \lambda^*)$.
- (b) If $f(z,\cdot)$ is odd, then the eigenfunctions have constant sign and we may assume that they are nonnegative. If in addition we assume that for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function $x \mapsto f(z,x) + \hat{\xi}_\rho x^{\rho-1}$ is nondecreasing on $[0,\rho]$, then for every eigenfunction u, we have $0 \prec u$ (see [9, Proposition 2.4]). Thus, we can relax the hypothesis of $0 < f(z,x)x$ for a.a $z \in \Omega$, all $x \neq 0$, as follows: there exists $D \subseteq \Omega$ measurable with $|D|_N > 0$ such that $f(z,x)x > 0$ for all $z \in D$, all $x \neq 0$.
- (c) In the context of isotropic, balanced growth problems, our result is more general than that of [1] since it covers the important case of $(p,q)-$equations and our hypotheses on $f(z,x)$ are more general.

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