ANISOTROPIC DOUBLE PHASE ELLIPTIC INCLUSION SYSTEMS WITH LOGARITHMIC PERTURBATION AND MULTIVALUED CONVECTIONS

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Abstract. In this paper, we investigate a class of variable exponent double phase elliptic inclusion systems involving anisotropic partial differential operators with logarithmic perturbation as well as two fully coupled multivalued terms, one of them is defined in the domain and the other is defined on the boundary, respectively. Firstly, under the suitable coercive conditions, the existence of a weak solution for the double phase elliptic inclusion systems is verified via applying a surjectivity theorem concerning multivalued pseudomonotone operators. Then, when the elliptic inclusion system is considered in non-coercive framework, we employ the sub-supersolution method to establish the existence and compactness results. Finally, we deliver several solvability properties of some special cases with respect to the elliptic inclusion system under consideration via constructing proper sub- and super-solutions.

Keywords: Elliptic inclusion systems, Variable exponents double phase operator with logarithmic perturbation, Subsolution and supersolution, Surjectivity theorem, Multivalued convection term, Existence and compactness property.

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1. Introduction

This paper is concerned with the existence and compactness properties to the following variable exponents double phase elliptic inclusion system: Find $\sigma = (\sigma_1, \sigma_2) \in K := (K_1, K_2)$ such that

$$\begin{cases} 0 \in A_1(\sigma_1) + \partial I_{K_1}(\sigma_1) + \mathcal{F}_1(\sigma_1, \sigma_2) + \mathcal{F}_{\Gamma_1}(\sigma_1, \sigma_2) & \text{in } W^{1,\mathcal{G}_1}(\Omega)^*, \\ 0 \in A_2(\sigma_2) + \partial I_{K_2}(\sigma_2) + \mathcal{F}_2(\sigma_1, \sigma_2) + \mathcal{F}_{\Gamma_2}(\sigma_1, \sigma_2) & \text{in } W^{1,\mathcal{G}_2}(\Omega)^*, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N (N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, for each $i = 1, 2$ the part Γ_i is a relatively open subset of $\partial\Omega$, $\Gamma_0^i = \partial\Omega \backslash \Gamma_i$ is such that $\partial\Omega = \Gamma_i \cap \Gamma_0^i$, for each $i = 1, 2$ K_i denotes a closed convex subset to $U^{\Gamma_0^i}$. Here, we let $U^{\Gamma_0^i}$ be a closed subspace of $W^{1,\mathcal{G}_i}(\Omega)$ given as

$$U^{\Gamma_0^i} = \left\{ \sigma \in W^{1,\mathcal{G}_i}(\Omega) \ : \ \sigma|_{\Gamma_0^i} = 0 \right\}, \tag{1.2}$$

I_{K_i} stands for the indicate function of K_i, ∂I_{K_i} is the corresponding subdifferential in convex analysis sense, $W^{1,\mathcal{G}_i}(\Omega)^*$ is the dual space of the Musielak-Orlicz Sobolev space $W^{1,\mathcal{G}_i}(\Omega)$ (defined in Section 2). The lower order multivalued operator $\mathcal{F}_i$ which depends on the gradient of solutions (called multivalued convection term) is generated by the corresponding multivalued function $f_i \colon \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}} \backslash \{\emptyset\}$, moreover, $\mathcal{F}_{\Gamma_i}$ is formulated by the boundary multivalued function $f_{\Gamma_i} \colon \Gamma_i \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$. Furthermore, the nonlinear and nonhomogeneous partial differential operator $A_i \colon W^{1,\mathcal{G}_i}(\Omega) \to W^{1,\mathcal{G}_i}(\Omega)^*$ given in (1.1) is formulated as

$$A_i v := -\text{div} \left(\frac{\mathcal{G}_i'(x, |\nabla v|)}{|\nabla v|} \nabla v \right) \text{ for all } v \in W^{1,\mathcal{G}_i}(\Omega), \tag{1.3}$$

in which the functional $\mathcal{G}_i \colon \Omega \times [0, +\infty) \to [0, +\infty)$ is defined by

$$\mathcal{G}_i(x, t) = \left[t^{p_i(x)} + \mu_i(x) t^{q_i(x)} \right] \log(e + \alpha t) \text{ for all } x \in \Omega \text{ and for all } t \in [0, +\infty) \tag{1.4}$$

where $p_i, q_i \in C(\overline{\Omega})$ fulfilling $1 < p_i(x) < N$, $p_i(x) < q_i(x)$, $0 \leq \mu_i(\cdot) \in L^1(\Omega)$, $\alpha \geq 0$ and e denotes the Euler constant. Note that $\mathcal{G}_i$ possesses unbalanced growth, i.e., for $0 \leq \mu_i(\cdot) \in L^\infty(\Omega)$ and any $\varepsilon > 0$

one can find constants $s_1, s_2 > 0$ satisfying the following inequalities

$$t^{p_i(x)} \leq \mathcal{G}_i(x,t) \leq s_1 t^{q_i(x)+\varepsilon} + s_2 \text{ for a.a. } x \in \Omega, \text{ all } t \in [0,+\infty).$$

Let p_i^* and $(p_i)_*$ denote the critical Sobolev exponents of p_i with $1 < p_i(x) < N$ for all $x \in \overline{\Omega}$, in the domain and boundary respectively, which are defined as

$$p_i^*(x) := \frac{Np_i(x)}{N - p_i(x)} \quad \text{and} \quad (p_i)_*(x) := \frac{(N-1)p_i(x)}{N - p_i(x)}. \tag{1.5}$$

The basic assumptions with respect to problem (1.1) are imposed blow:

(H0) For each $i = 1, 2$ fixed, $p_i, q_i \in C(\overline{\Omega})$ satisfying $1 < p_i(x) < N$ as well as $p_i(x) < q_i(x) < p_i^*(x)$ for all $x \in \overline{\Omega}$, also, $0 \leq \mu_i(\cdot) \in L^\infty(\Omega)$.

(H1) For each $i = 1, 2$ fixed, $f_i \colon \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ and $f_{\Gamma_i} : \Gamma_i \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ are graph measurable functions such that, for a.a. $x \in \Omega$, $f_i(x,\cdot,\cdot,\cdot,\cdot) : \mathbb{R}^{2N+2} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ is upper semicontinuous, and for a.a. $x \in \Gamma_i$, $f_{\Gamma_i}(x,\cdot,\cdot) : \mathbb{R}^2 \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ is upper semicontinuous.

Moreover, some local growth conditions for f_i and f_{Γ_i} will be made later (see (H2) in Section 3 and (H3) in Section 4).

One of the main characteristics of problem (1.1) is the presence of the nonlinear and nonhomogeneous partial differential operator (1.3) with variable exponents and logarithmic perturbation. It can observe that when $\alpha = 0$, differential operator (1.3) reduces to the following variable exponents double phase differential operator:

$$-\text{div}\left(|\nabla\sigma|^{p_i(x)-2}\nabla\sigma + \mu_i(x)|\nabla\sigma|^{q_i(x)-2}\nabla\sigma\right).$$

If p, q are two constants, then the above differential operator becomes the classical double phase differential operator

$$\text{div}\left(|\nabla\sigma|^{p-2}\nabla\sigma + \mu(x)|\nabla\sigma|^{q-2}\nabla\sigma\right). \tag{1.6}$$

which was initially introduced by Zhikov ([56]) who used the corresponding integral functional

$$\sigma \mapsto \int_\Omega \left(|\nabla\sigma|^p + \mu(x)|\nabla\sigma|^q\right) dx. \tag{1.7}$$

for studying the complicated mechanical models with respect to strongly anisotropic materials. The primary advantage of such integral functional is that it can precisely describe the phenomenon that the energy density changes its ellipticity and growth properties according to the point in the domain. In fact, we call (1.6) (or its integral functional (1.7)) as double phase operators, because operator (1.3) exhibits the p growth when $\mu(x) = 0$ and the q growth when $\mu(x) > 0$. On the other hand, Zhikov [57, 58] found that the double phase operators also demonstrates Lavrentiev's phenomena, which leads to some classical method cannot able applied to study double phase problems.

After the work of Zhikov, double phase problem has been becoming a hot-pint and challenging topic, due to its wide application in physics and engineering, for example, in material science, non-Newton fluid and population dynamic problems. Therefore, some important and impressive results have been obtained, for instance, Hästö–Ok [30] established the local $C^{1,\alpha}$-regularity of local minimizers to the functional with unbalanced (p,q)-growth condition, and Beck-–Mingione [8] studied nonuniformly elliptic problems and proved several regularity results and prior estimates of solutions. As for papers involving logarithmic double phase operators we refer to Arora–Crespo-Blanco–Winkert [2, 3] and Vetro–Winkert [49], who dealt with the following the logarithmic double phase type operator

$$\text{div}\left(|\nabla\sigma|^{p(x)-2}\nabla\sigma + \vartheta(x)\left[\log(e+|\nabla\sigma|) + \frac{|\nabla\sigma|}{q(x)(e+|\nabla\sigma|)}\right]|\nabla\sigma|^{q(x)-2}\nabla\sigma\right),$$

and established existence and multiplicity results of various problems driven by the above operator under different setting. For more results concerning the research of double phase problems readers can refer to Baroni–Colombo–Mingione [4, 5], Byun–Ok–Song [10], Fuchs–Mingione [22], Marcellini [42, 43], Baharouni et al. [6, 7], Benci–D'Avenlia–Fortunato–Pisani [9], Liu–Dai [35], Zeng–Bai–Gasiński–Winkert [52], Zhang–Zhang–Rădulescu [55], Amoroso et al. [1], Baroni–Colombo–Mingione

[4], Filippis–Mingione[14], Fuchs–Mingione [22], Ho–Kim–Zhang [31], Marcellini–Papi [38], Moussaoui et al. [41], Ragusa–Tachikawa [45], Xiang et al. [51], and so on.

Moreover, we mention that the current work is inspired by the following researches. Recently, Carl–Le–Winkert [12] studied the following nonlinear double phase equations without logarithmic perturbation

$$v \in K : 0 \in \mathcal{A}v + \partial I_K(v) + \mathcal{F}(v) + \mathcal{F}_\Gamma(v)$$

which is driven by the variable exponents double phase operator:

$$\mathcal{A}v := \operatorname{div}\left(|\nabla v|^{p(x)-2}\nabla v + \mu(x)|\nabla v|^{q(x)-2}\nabla v\right) \tag{1.8}$$

and established the existence and uniqueness results by utilizing some critical properties for operator (1.8) given in Crespo-Blanco et al. [13]. After that, Liu–Lu–Vetro [36] extended the results of Carl–Le–Winkert [12] to following double phase elliptic inclusion:

$$0 \in \mathcal{B}v + \partial I_K(v) + \mathcal{F}(v) + \mathcal{F}_\Gamma(v)$$

in which $\mathcal{B}$ denotes the perturbed nonlinear and nonhomogeneous partial differential operator (introduced by Vetro–Zeng [50]):

$$\mathcal{B}v := -\operatorname{div}\left(\frac{\mathcal{G}_L'(x,|\nabla v|)}{|\nabla v|}\nabla v\right) \text{ for all } v \in W^{1,\mathcal{G}_L}(\Omega), \tag{1.9}$$

with

$$\mathcal{G}_L(x,t) = [t^p + \mu(x)t^q]\log(e+t) \text{ for all } x \in \Omega, \text{ and for all } t \in [0,+\infty). \tag{1.10}$$

Therefore, motivated by the above works, we concentrate on the variable exponent double phase elliptic inclusion systems (1.1), which is a generalization of the above researches. As we know, this is the first work dealing with double phase inclusion systems with variable exponent, logarithmic perturbation and convection terms.

The second feature with respect to problem (1.1) is that our problem can be deemed an anisotropic nonlinear nonlinear obstacle system with bilateral constraints where constraint set K_i is given by

$$K_i = \left\{\sigma \in W^{1,\mathcal{G}_i}(\Omega) \ : \ \sigma(x) \geq \pi(x) \text{ a.a. in } \Omega\right\}, \tag{1.11}$$

where $\pi\colon \Omega \to \mathbb{R}$ is a given obstacle function. However, such model is useful for studying various multi-body contact problems with multivalued and nonsmooth constitutive laws, namely, nonsmooth and nonconvex elliptic systems which are coupled by several variational inequalities or hemivariational inequalities (see Stefan [48], Lions [34], Duvaut and Lions [17], Rodrigues [47] and Zeng et al. [52, 53]).

Another challenging of problem (1.1) is that it involves two fully coupled multivalued convection terms $\mathcal{F}_1$ and $\mathcal{F}_2$ (defined in the domain Ω), as well as two fully coupled boundary multivalued functions $\mathcal{F}_{\Gamma_1}$ and $\mathcal{F}_{\Gamma_2}$. Obviously, it is one of difficulty to study problem (1.1) that we have to overcome the influence of coupled construct. It is well-known that equations with multivalued functions can be widely applied to a plenty of practical problems, such as frictional contact problems with multivalued constitutive laws (see for example Panagiotopoulos [42, 43] as well as Carl and Le [11]). On the other hand, the effect of convection may occur spontaneously in a single or multiphase fluid flow due to the combined effects of influence of body forces on a fluid (generally gravity and density) and material heterogeneity, and the convection terms (depending on the gradient of solutions) could describe the convection effect of different fluids flow well (see for instance, Dupaigne–Ghergu–Rădulescu [16], El Manouni–Marino–Winkert [18], Faraci–Motreanu–Puglisi [20], Figueiredo–Madeira [21], Gasiński–Papageorgiou [23], Gasiński–Winkert [24], Guarnotta–Livrea–Winkert [25], Guarnotta et al. [26, 27, 28], Liu–Motreanu–Zeng [37], Motreanu–Tornatore [39], Motreanu–Vetro–Vetro [40], Papageorgiou–Rădulescu–Repovš [44], and Rădulescu–Vetro [46]). However, due to the appearance of the convection term, corresponding problem becomes nonvariational, that is, the standard variational tools and relevant theory can not be applied to deal with the corresponding energy functionals, so we have to make use of the nonvariational tools to solve our problem. This is another challenging of the current paper.

Finally, we point out that the elliptic inclusion systems (1.1) contain several interesting and challenging problems as special cases, and some of them have not been studied yet.

Special case 1.1. For $i = 1,2$, let $\Gamma_i = \partial\Omega$ (i.e., $\Gamma_0^i = \emptyset$, $U^{\Gamma_0^i} = W^{1,\mathcal{G}_i}(\Omega)$) and K_i is given by (1.11), then problem (1.1) can be rewritten as the multivalued obstacle Neumann boundary elliptic inclusion systems:

$$
\begin{aligned}
-\operatorname{div}\left(\frac{\mathcal{G}_1'(x,|\nabla\sigma_1|)}{|\nabla\sigma_1|}\nabla\sigma_1\right) + f_1(x,\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2) &\ni 0 \quad \text{in } \Omega,\\
-\operatorname{div}\left(\frac{\mathcal{G}_2'(x,|\nabla\sigma_2|)}{|\nabla\sigma_2|}\nabla\sigma_2\right) + f_2(x,\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2) &\ni 0 \quad \text{in } \Omega,\\
\sigma_1(x) &\geq \pi_1(x) \quad \text{in } \Omega,\\
\sigma_2(x) &\geq \pi_2(x) \quad \text{in } \Omega,\\
-\frac{\partial\sigma_1}{\partial\nu_{A_1}} &\in f_{\Gamma_1}(x,\sigma_1,\sigma_2) \quad \text{on } \partial\Omega,\\
-\frac{\partial\sigma_2}{\partial\nu_{A_2}} &\in f_{\Gamma_2}(x,\sigma_1,\sigma_2) \quad \text{on } \partial\Omega,
\end{aligned} \tag{1.12}
$$

in which ν is the outward unit normal on Γ_i

$$\frac{\partial\sigma_i}{\partial\nu_{A_i}} = \left(\frac{\mathcal{G}_i'(x,|\nabla\sigma_i|)}{|\nabla\sigma_i|}\nabla\sigma_i\right)\cdot\nu.$$

Special case 1.2. For $i = 1,2$, let $\Gamma_i = \partial\Omega$ (that is, $\Gamma_0^i = \emptyset$, $U^{\Gamma_0^i} = W^{1,\mathcal{G}_i}(\Omega)$) and $K_i = W^{1,\mathcal{G}_i}(\Omega)$, then problem (1.1) can be rewritten as the multivalued Neumann boundary elliptic inclusion systems:

$$
\begin{aligned}
-\operatorname{div}\left(\frac{\mathcal{G}_1'(x,|\nabla\sigma_1|)}{|\nabla\sigma_1|}\nabla\sigma_1\right) + f_1(x,\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2) &\ni 0 \quad \text{in } \Omega,\\
-\operatorname{div}\left(\frac{\mathcal{G}_2'(x,|\nabla\sigma_2|)}{|\nabla\sigma_2|}\nabla\sigma_2\right) + f_2(x,\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2) &\ni 0 \quad \text{in } \Omega,\\
-\frac{\partial\sigma_1}{\partial\nu_{A_1}} &\in f_{\Gamma_1}(x,\sigma_1,\sigma_2) \quad \text{on } \partial\Omega,\\
-\frac{\partial\sigma_2}{\partial\nu_{A_2}} &\in f_{\Gamma_2}(x,\sigma_1,\sigma_2) \quad \text{on } \partial\Omega.
\end{aligned} \tag{1.13}
$$

Special case 1.3. For $i = 1,2$, let $\Gamma_0^i = \partial\Omega$ (i.e., $\Gamma_i = \emptyset$, $U^{\Gamma_0^i} = W_0^{1,\mathcal{G}_i}(\Omega)$) and K_i is given by (1.11), then problem (1.1) can be rewritten as the multivalued Dirichlet obstacle elliptic inclusion systems:

$$
\begin{aligned}
-\operatorname{div}\left(\frac{\mathcal{G}_1'(x,|\nabla\sigma_1|)}{|\nabla\sigma_1|}\nabla\sigma_1\right) + f_1(x,\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2) &\ni 0 \quad \text{in } \Omega,\\
-\operatorname{div}\left(\frac{\mathcal{G}_2'(x,|\nabla\sigma_2|)}{|\nabla\sigma_2|}\nabla\sigma_2\right) + f_2(x,\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2) &\ni 0 \quad \text{in } \Omega,\\
\sigma_1(x) &\geq \pi_1(x) \quad \text{in } \Omega,\\
\sigma_2(x) &\geq \pi_2(x) \quad \text{in } \Omega,\\
\sigma_1(x) = \sigma_2(x) &= 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{1.14}
$$

Special case 1.4. For $i = 1,2$, let $\Gamma_0^i = \partial\Omega$ (i.e., $\Gamma_i = \emptyset$ and $U^{\Gamma_0^i} = W_0^{1,\mathcal{G}_i}(\Omega)$) and $K_i = W_0^{1,\mathcal{G}_i}(\Omega)$, then problem (1.1) can be rewritten as the multivalued Dirichlet elliptic inclusion systems:

$$
\begin{aligned}
-\operatorname{div}\left(\frac{\mathcal{G}_1'(x,|\nabla\sigma_1|)}{|\nabla\sigma_1|}\nabla\sigma_1\right) + f_1(x,\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2) &\ni 0 \quad \text{in } \Omega,\\
-\operatorname{div}\left(\frac{\mathcal{G}_2'(x,|\nabla\sigma_2|)}{|\nabla\sigma_2|}\nabla\sigma_2\right) + f_2(x,\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2) &\ni 0 \quad \text{in } \Omega,\\
\sigma_1(x) = \sigma_2(x) &= 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{1.15}
$$

The remaining sections of this paper are organized as follows. In Section 2, the useful definitions and results related to the generalized Lebesgue spaces and Musielak-Orlicz spaces generated by double phase partial differential operator involving logarithmic perturbation (1.3) in variable exponents setting will be given. Also, we introduce the concepts of weak solutions, sub- and supersolutions to elliptic inclusion systems (1.1). Under a certain coercive condition, Section 3 is devoted to show the existence of weak solutions to elliptic inclusion systems (1.1) with the help of a surjectivity theorem of multivalued pseudomonotone operators. However, when (1.1) is considered in noncoercive framework, the existence and compactness results could be obtain by employing the sub- and supersolution method and the theory of nonsmooth analysis, in Section 4. While in Section 5, we make a further discussion for Special case 1.1, and finally obtain the existence for a solution via constructing suitable sub- and supersolutions.

2. Preliminaries

In this section, we review some basic notations also some useful results about the variable exponents Lebesgue space, most of them are given by Diening–Harjulehto–Hästö-Růžička [15], Fan–Zhao [19], Harjulehto–Hästö [29] and Kováčik–Rákosník [32]. Also, for some vital and useful properties for the logarithmic variable exponents double phase operator as well as the corresponding Musielak-Orlicz Sobolev spaces can be found in [54].

Let $\Omega \subseteq \mathbb{R}^N (N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and introduce the notation $C_+(\overline{\Omega})$ given as

$$C_+(\overline{\Omega}) := \{h \in C(\overline{\Omega}) : 1 < h(x) \text{ for all } x \in \overline{\Omega}\}.$$

For any $r \in C_+(\overline{\Omega})$, we give the definition of r^- and r^+ as

$$r^- := \min_{x\in\overline{\Omega}} r(x) \quad \text{ and } \quad r^+ := \max_{x\in\overline{\Omega}} r(x),$$

and $r' \in C_+(\overline{\Omega})$ is the conjugate variable exponent for r, namely,

$$\frac{1}{r(x)} + \frac{1}{r'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

In the sequel, we denote by $M(\Omega)$ the space of all measurable functions $\sigma : \Omega \to \mathbb{R}$. For $r \in C_+(\overline{\Omega})$ the corresponding variable exponent Lebesgue space is given by

$$L^{r(x)}(\Omega) = \left\{\sigma \in M(\Omega) : \int_\Omega |\sigma|^{r(x)} dx < \infty\right\},$$

the modular function is formulated as follows

$$\varrho_{r(\cdot)}(\sigma) = \int_\Omega |\sigma|^{r(x)} dx \quad \text{for all } \sigma \in L^{r(x)}(\Omega) \text{ where } r \in C_+(\overline{\Omega}).$$

It is well-known that $L^{r(x)}(\Omega)$ endowed the Luxemburg norm defined as

$$\|\sigma\|_{r(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|\sigma|}{\lambda}\right)^{r(x)} dx \leq 1\right\}$$

is a separable and reflexive Banach space. Meanwhile, by applying the $(N-1)$-dimensional Hausdorff surface measure, the variable exponent boundary Lebesgue space $(L^{r(\cdot)}(\partial\Omega), \|\sigma\|_{r(\cdot),\partial\Omega})$ can be defined in a similar way. Moreover, $L^{r'(x)}(\Omega)$ is the dual space of $L^{r(x)}(\Omega)$ and the Hölder type inequality given blow is valid

$$\int_\Omega |\sigma v| dx \leq \left[\frac{1}{r^-} + \frac{1}{(r')^-}\right] \|\sigma\|_{r(\cdot)}\|v\|_{r'(\cdot)} \leq 2\|\sigma\|_{r(\cdot)}\|v\|_{r'(\cdot)}$$

for all $\sigma \in L^{r(x)}(\Omega)$, all $v \in L^{r'(x)}(\Omega)$. Furthermore, if $r_1, r_2 \in C_+(\overline{\Omega})$ satisfying $r_1(x) \leq r_2(x)$ for all $x \in \overline{\Omega}$, then there holds:

$$L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega).$$

Based on the notations and definitions concerning variable exponent Lebesgue space, the definition for the variable exponent Sooblev space can be given as follows

$$W^{1,r(\cdot)}(\Omega)=\left\{\sigma\in L^{r(x)}(\Omega):|\nabla\sigma|\in L^{r(x)}(\Omega)\right\}$$

equipped with the norm

$$\|\sigma\|_{1,r(\cdot)}=\|\sigma\|_{r(\cdot)}+\|\nabla\sigma\|_{r(\cdot)},$$

with $\|\nabla\sigma\|_{r(\cdot)}=\||\nabla\sigma|\|_{r(\cdot)}$. Moreover, we introduce a subspace of $W^{1,r(\cdot)}(\Omega)$, that is,

$$W_0^{1,r(\cdot)}(\Omega)=\overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,r(\cdot)}}.$$

Actually, it can show that $W^{1,r(\cdot)}(\Omega)$ as well as $W_0^{1,r(\cdot)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces. Also, in $W_0^{1,r(\cdot)}(\Omega)$, the following Poincaré inequality is available

$$\|\sigma\|_{r(\cdot)}\le c_0\|\nabla\sigma\|_{r(\cdot)}\quad\text{for all }\sigma\in W_0^{1,r(\cdot)}(\Omega),$$

where the constant $c_0>0$. Therefore, the following norms are equivalent in space $W_0^{1,r(\cdot)}(\Omega)$:

$$\|\sigma\|_{1,r(\cdot),0}=\|\nabla\sigma\|_{r(\cdot)}\quad\text{for all }\sigma\in W_0^{1,r(\cdot)}(\Omega).$$

Next, Let us see the definitions of Musielak-Orlicz Lebesgue and Sobolev spaces generalized by functionals $\mathcal{G}_1$ and $\mathcal{G}_2$ (see [54] and Harjulehto-Hästö [29] for more details). Let $i=1,2$, then the Musielak-Orlicz space $L^{\mathcal{G}_i}(\Omega)$ with respect to functional $\mathcal{G}_i$ (given by (1.4)) is formulated as

$$L^{\mathcal{G}_i}(\Omega)=\{\sigma\in M(\Omega):\rho_{\mathcal{G}_i}(\sigma)<+\infty\}.$$

According to [54], we see that under assumptions (H0), $\mathcal{H}_i$ is a N-function (refer to [54, Definition 2.7] for its precise definition), and the modular function associated to $\mathcal{G}_i$ is defined by

$$\rho_{\mathcal{G}_i}(\sigma)=\int_\Omega\mathcal{G}_i(x,|\sigma|)dx\text{ for all }\sigma\in L^{\mathcal{G}_i}(\Omega),$$

also, $L^{\mathcal{G}_i}(\Omega)$ endowed with the so-called Luxemburg norm turns out to be a reflexive and separable Banach space (proved in [54])

$$\|\sigma\|_{\mathcal{G}_i}=\inf\left\{\lambda>0:\rho_{\mathcal{G}_i}\left(\frac{\sigma}{\lambda}\right)\le 1\right\}.$$

The following proposition indicates the relation between modular $\rho_{\mathcal{G}_i}$ and the Luxemburg norm $\|\cdot\|_{\mathcal{G}_i}$.

Proposition 2.1. [54, Proposition 2.20] *Let $i=1,2$, if hypotheses (H0) hold, and modular function $\rho_{\mathcal{G}_i}$ is formulated by*

$$\rho_{\mathcal{G}_i}=\int_\Omega\left[|\sigma|^{p_i(x)}+\mu(x)|\sigma|^{q_i(x)}\right]\log(e+\alpha|\sigma|)dx\quad\textit{for all }\sigma\in L^{\mathcal{G}_i}(\Omega),$$

then it has

(i) $\|\sigma\|_{\mathcal{G}_i}=\lambda\iff\rho_{\mathcal{G}_i}(\frac{\sigma}{\lambda})=1$ *with* $\sigma\neq0$*;*
(ii) $\|\sigma\|_{\mathcal{G}_i}<1(resp.=1,>1)\iff\rho_{\mathcal{G}_i}(\sigma)<1(resp.=1,>1)$*;*
(iii) $\|\sigma\|_{\mathcal{G}_i}<1\implies\|\sigma\|_{\mathcal{G}_i}^{q_i^++1}\le\rho_{\mathcal{G}_i}(\sigma)\le\|\sigma\|_{\mathcal{G}_i}^{p_i^-}$*;*
(iv) $\|\sigma\|_{\mathcal{G}_i}>1\implies\|\sigma\|_{\mathcal{G}_i}^{p_i^-}\le\rho_{\mathcal{G}_i}(\sigma)\le\|\sigma\|_{\mathcal{G}_i}^{q_i^++1}$*;*
(v) $\|\sigma_n\|_{\mathcal{G}_i}\to0\iff\rho_{\mathcal{G}_i}(\sigma_n)\to0$*;*
(vi) $\|\sigma_n\|_{\mathcal{G}_i}\to\infty\iff\rho_{\mathcal{G}_i}(\sigma_n)\to\infty$*;*
(vii) $\|\sigma_n\|_{\mathcal{G}_i}\to1\iff\rho_{\mathcal{G}_i}(\sigma_n)\to1$*;*
(viii) $\sigma_n\to\sigma\in L^{\mathcal{G}_i}(\Omega)\implies\rho_{\mathcal{G}_i}(\sigma_n)\to\rho_{\mathcal{G}_i}(\sigma)$.

Furthermore, we consider the Musielak-Orlicz Sobolev space $W^{1,\mathcal{G}_i}(\Omega)$ formulated by $\mathcal{G}_i$:

$$W^{1,\mathcal{G}_i}(\Omega)=\left\{\sigma\in L^{\mathcal{G}_i}(\Omega):|\nabla\sigma|\in L^{\mathcal{G}_i}(\Omega)\right\},$$

endowed with the norm

$$\|\sigma\|_{1,\mathcal{G}_i}=\|\sigma\|_{\mathcal{G}_i}+\|\nabla\sigma\|_{\mathcal{G}_i},\tag{2.1}$$

with $\|\nabla\sigma\|_{\mathcal{G}_i} = \||\nabla\sigma|\|_{\mathcal{G}_i}$. Note that $W^{1,\mathcal{G}_i}(\Omega)$ is a reflexive and separable Banach space as well (prove by [54]). Furthermore, we denote the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{G}_i}(\Omega)$ by $W_0^{1,\mathcal{G}_i}(\Omega)$, which is a closed subspace of $W^{1,\mathcal{G}_i}(\Omega)$. Utilizing the Poincaré inequality (given by [54, Proposition 2.23]), we have the following equivalent norm in $W_0^{1,\mathcal{G}_i}(\Omega)$

$$\|\sigma\|_{1,\mathcal{G}_i,0} = \|\nabla\sigma\|_{\mathcal{G}_i} \text{ for all } \sigma \in W_0^{1,\mathcal{G}_i}(\Omega).$$

In addition, we define the equivalent norm of space $W^{1,\mathcal{G}_i}(\Omega)$, i.e.

$$\|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}} := \inf\left\{\lambda > 0 : \hat{\rho}_{\mathcal{G}_i}\left(\frac{\sigma}{\lambda}\right) \le 1\right\},$$

in which the modular function is formulated as

(2.2)

$$\hat{\rho}_{\mathcal{G}_i}(\sigma) = \int_\Omega \left(|\nabla\sigma|^{p_i(x)} + \mu(x)|\nabla\sigma|^{q_i(x)}\right)\log(e+\alpha|\nabla\sigma|)dx + \int_\Omega \left(|\sigma|^{p_i(x)} + \mu(x)|\sigma|^{q_i(x)}\right)\log(e+\alpha|\sigma|)dx$$

for $\sigma \in W^{1,\mathcal{G}_i}(\Omega)$.

The next proposition reveals the relationship of modular $\hat{\rho}_{\mathcal{G}_i}$ and norm $\|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}}$ which is demonstrated by [54, Proposition 2.21].

Proposition 2.2. *If hypotheses (H0) hold, and $\hat{\rho}_{\mathcal{G}_i}$ is given by (2.2), then for each $\sigma \in W^{1,\mathcal{G}_i}(\Omega)$ we see that*

(i) $\|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}} = \lambda \iff \hat{\rho}_{\mathcal{G}_i}(\frac{\sigma}{\lambda}) = 1$ *with* $\sigma \ne 0$;
(ii) $\|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}} < 1(resp. = 1, > 1) \iff \hat{\rho}_{\mathcal{G}_i}(\sigma) < 1(resp. = 1, > 1)$;
(iii) $\|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}} < 1 \implies \|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}}^{q_i^+ + 1} \le \hat{\rho}_{\mathcal{G}_i}(\sigma) \le \|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}}^{p_i^-}$;
(iv) $\|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}} > 1 \implies \|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}}^{p_i^-} \le \hat{\rho}_{\mathcal{G}_i}(\sigma) \le \|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}}^{q_i^+ + 1}$;
(v) $\|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}} \to 0 \iff \hat{\rho}_{\mathcal{G}_i}(\sigma) \to 0$;
(vi) $\|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}} \to \infty \iff \hat{\rho}_{\mathcal{G}_i}(\sigma) \to \infty$;
(vii) $\|\sigma\|_{\hat{\rho}_{\mathcal{G}_i}} \to 1 \iff \hat{\rho}_{\mathcal{G}_i}(\sigma) \to 1$;
(viii) $\sigma_n \to \sigma$ *in* $W^{1,\mathcal{G}_i}(\Omega) \implies \hat{\rho}_{\mathcal{G}_i}(\sigma_n) \to \hat{\rho}_{\mathcal{G}_i}(\sigma)$.

The following embedding results can be directly found in [54, Propositions 2.22 and 2.23].

Proposition 2.3. *For $i = 1, 2$, if hypotheses (H0) hold, then*

(i) $L^{\mathcal{G}_i}(\Omega) \hookrightarrow L^{r_i(\cdot)}(\Omega)$, $W^{1,\mathcal{G}_i}(\Omega) \hookrightarrow W^{1,r_i(\cdot)}(\Omega)$, *and* $W_0^{1,\mathcal{G}_i}(\Omega) \hookrightarrow W_0^{1,r_i(\cdot)}(\Omega)$ *for all* $r_i \in C(\overline{\Omega})$ *such that* $1 \le r_i(x) \le p_i(x)$ *for all* $x \in \Omega$;
(ii) $W^{1,\mathcal{G}_i}(\Omega) \hookrightarrow\hookrightarrow L^{r_i(\cdot)}(\Omega)$ *and* $W_0^{1,\mathcal{G}_i}(\Omega) \hookrightarrow\hookrightarrow L^{r_i(\cdot)}(\Omega)$ *for all* $r \in C(\overline{\Omega})$ *such that* $1 \le r_i(x) < p_i^*(x)$ *for all* $x \in \overline{\Omega}$;
(iii) $W^{1,\mathcal{G}_i}(\Omega) \hookrightarrow\hookrightarrow L^{r_i(\cdot)}(\partial\Omega)$ *and* $W_0^{1,\mathcal{G}_i}(\Omega) \hookrightarrow\hookrightarrow L^{r_i(\cdot)}(\partial\Omega)$ *for all* $r \in C(\overline{\Omega})$ *such that* $1 \le r_i(x) < (p_i)_*(x)$ *for all* $x \in \overline{\Omega}$;
(iv) $W^{1,\mathcal{G}_i}(\Omega) \hookrightarrow\hookrightarrow L^{\mathcal{G}_i}(\Omega)$.

Take $y^+ = \max\{y, 0\}$ and $y^- = -\min\{y, 0\}$ for any $y \in \mathbb{R}$, define $\sigma^\pm(\cdot) = [\sigma(\cdot)]^\pm$ for any function $\sigma : \Omega \to \mathbb{R}$. According to [54, Proposition 2.24], we infer the following results.

Proposition 2.4. *Let $\sigma \in W^{1,\mathcal{G}_i}(\Omega)$, $v \in W_0^{1,\mathcal{G}_i}(\Omega)$ and $\{\sigma_n\} \subset W^{1,\mathcal{G}_i}(\Omega)$. If (H0) hold, then we have*

(i) $\pm\sigma^\pm \in W^{1,\mathcal{G}_i}(\Omega)$ *with* $\nabla(\pm\sigma^\pm) = \nabla\sigma 1_{\{\pm\sigma>0\}}$;
(ii) *if* $\sigma_n \to \sigma$ *in* $W^{1,\mathcal{G}_i}(\Omega)$, *then* $\pm\sigma_n^\pm \to \pm\sigma^\pm$ *in* $W^{1,\mathcal{G}_i}(\Omega)$;
(iii) $\pm v^\pm \in W_0^{1,\mathcal{G}_i}(\Omega)$.

In the sequel, we denote by $W^{1,\mathcal{G}_i}(\Omega)^*$ and $W_0^{1,\mathcal{G}_i}(\Omega)^*$ the dual spaces of $W^{1,\mathcal{G}_i}(\Omega)$ and $W_0^{1,\mathcal{G}_i}(\Omega)$, respectively. For a given Banach space X, let X^* be the corresponding dual space, we define $\mathcal{K}(X^*)$ as

$$\mathcal{K}(X^*) = \{U \subset X^* : U \ne \emptyset, U \text{ is closed and convex}\}.$$

Definition 2.5. If X is a real reflexive Banach space, X^* is the dual space of X and $\langle\cdot,\cdot\rangle$ denotes their duality pairing. Then, operator $B : X \to X^*$ is called

(i) completely continuous iff $\sigma_n \rightharpoonup \sigma$ in X implies $B\sigma_n \to B\sigma$ in X^*.
(ii) to satisfy the (S_+)-property if $\sigma_n \rightharpoonup \sigma$ in X and $\limsup_{n\to\infty}\langle B\sigma_n, \sigma_n - \sigma\rangle \le 0$ imply $\sigma_n \to \sigma$ in X.

For $i = 1, 2$, we define operator $A_i : X \to X^*$ as:

$$\langle A_i(\sigma_i), v_i\rangle_{\mathcal{G}_i} := \int_\Omega \frac{\mathcal{G}_i'(x, |\nabla\sigma_i|)}{|\nabla\sigma_i|}\nabla\sigma_i \cdot \nabla v_i dx, \tag{2.3}$$

for all $\sigma_i, v_i \in W^{1,\mathcal{G}_i}(\Omega)$, here $X = W^{1,\mathcal{G}_i}(\Omega)$ or $X = W_0^{1,\mathcal{G}_i}(\Omega)$ with $\langle\cdot,\cdot\rangle$ being the dual pairing between X and X^*.

Referring to [54], we see that for each $i = 1, 2$, A_i gets the following properties.

Proposition 2.6. *For $i = 1, 2$, if hypotheses (H1) hold, then A_i (given by (2.3)) is continuous, bounded, strictly monotone (thus maximal monotone) and satisfies (S_+) property.*

Next, we recall some important properties for multivalued operators that will be used in the proof of our main results.

Definition 2.7. Let X be a real reflexive Banach space, X^* be its dual space and $\langle\cdot,\cdot\rangle$ denote their duality pairing. Then operator $B : X \to 2^{X^*}$ is called

(i) pseudomonotone iff
  (a) the set $B(\sigma)$ is nonempty, bounded, closed and convex for all $\sigma \in X$;
  (b) B is upper semicontinuous from each finite dimensional subspace of X to the weak topology on X^*;
  (c) $(\sigma_n) \subset X$ with $\sigma_n \rightharpoonup \sigma$, and $\sigma_n^* \in B(\sigma_n)$ being such that $\limsup\langle\sigma_n^*, \sigma_n - \sigma\rangle \le 0$, imply there exists $\sigma^*(v) \in B(\sigma)$ such that
  $$\liminf\langle\sigma_n^*, \sigma_n - v\rangle \ge \langle\sigma^*(v), \sigma - v\rangle.$$
  for each element $v \in X$.
(ii) generalized pseudomonotone iff $(\sigma_n) \subset X$ and $(\sigma_n^*) \subset X^*$ with $\sigma_n^* \in B(\sigma_n)$ being such that $\sigma_n \rightharpoonup \sigma$ in X, $\sigma_n^* \rightharpoonup \sigma^*$ in X^* and $\limsup\langle\sigma_n^*, \sigma_n - \sigma\rangle \le 0$ imply that the element σ^* lies in $B(\sigma)$ and
$$\langle\sigma_n^*, \sigma_n\rangle \to \langle\sigma^*, \sigma\rangle.$$
(iii) coercive iff for $\sigma \in X$ satisfying $\|\sigma\|_X \to \infty$, there hold
$$\frac{\inf\{\langle\sigma^*, \sigma\rangle : \sigma^* \in B(\sigma)\}}{\|\sigma\|_X} \to +\infty.$$

Remark 2.8. $B : X \to 2^{X^*}$ being a pseudomonotone implies that B is generalized pseudomonotone. Moreover, if B is maximal monotone with $D(B) = X$, we infer that B is pseudomonotone.

The next proposition gives the sufficient conditions to guarantee that a generalized pseudomonotone multivalued operator becomes a pseudomonotone multivalued operator, see also [11, Proposition 2.18].

Proposition 2.9. *If X is a real reflexive Banach space, and operator $B : X \to 2^{X^*}$ satisfies the following conditions:*

(i) *For each $\sigma \in X$ the set $B(\sigma)$ is a nonempty, closed, and convex in X^*;*
(ii) *$B : X \to 2^{X^*}$ is bounded;*
(iii) *$B : X \to 2^{X^*}$ is generalized pseudomonotone.*

Then operator $B : X \to 2^{X^*}$ is pseudomonotone.

Assume $l_j \in C(\overline{\Omega})$ with $(l_j)_- \ge 1$ for $j = 0, 1, \cdots, m$. Let F be a function from $\Omega \times \mathbb{R}^m$ into $2^{\mathbb{R}}$. For each measurable function $\sigma = (\sigma_1, \cdots, \sigma_m) : \Omega \to \mathbb{R}^m$, we consider the multivalued function $\Omega \ni x \mapsto F(x, \sigma_1(x), \cdots, \sigma_m(x)) = F(x, \sigma(x)) \in 2^{\mathbb{R}}$ and denote $\tilde{F}(\sigma) = \{v \in M(\Omega) : v(x) \in F(x, \sigma(x))$ for a.a. $x \in \Omega\}$. The following theorem references [11, Theorem 7.8].

Theorem 2.10. *Assume F satisfies the following conditions:*

(i) *For a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^m$, $F(x,\xi)$ is closed and nonempty.*
(ii) *F is superpositionally measurable, i.e., if $\sigma \in [M(\Omega)]^m$, then $F(\cdot,\sigma(\cdot)) : \Omega \to 2^{\mathbb{R}}$ is measurable.*
(iii) *For a.e. $x \in \Omega$, the function $\mathbb{R}^m \ni \xi \mapsto F(x,\xi)$ is Hausdorff upper semicontinuous (h-u.s.c. for short).*
(iv) *There exist $a \in L^{l_0(\cdot)}(\Omega)$ and $b > 0$ such that*

$$|v| \le a(x) + b\sum_{i=1}^{m} |\xi_i|^{\frac{l_i(x)}{l_0(x)}}, \tag{2.4}$$

for a.e. $x \in \Omega$, all $v \in F(x,\xi)$.

Thus, for each $u \in \prod_{i=1}^m L^{l_i(\cdot)}(\Omega)$, $\tilde{F}(\sigma)$ is a nonempty and closed subset of $L^{l_0(\cdot)}(\Omega)$, and $\tilde{F} : \sigma \mapsto \tilde{F}(\sigma)$ is h-u.s.c. from $\prod_{i=1}^m L^{l_i(\cdot)}(\Omega)$ to $2^{L^{l_0(\cdot)}(\Omega)}$.

In addition, let $B_R(0) := \{\sigma \in X \mid \|\sigma\|_X < R\}$ denote an open ball with center 0 and radius $R > 0$, we review the following surjective theorem, see [33, Theorem 2.2].

Theorem 2.11. *Assume X is a real reflexive Banach space, $F : D(F) \subset X \to 2^{X^*}$ is a maximal monotone operator and $G : D(G) = X \to 2^{X^*}$ is a bounded multi-valued pseudomonotone operator, and $L \in X^*$. If there exist $\sigma_0 \in X$ and $R \ge \|\sigma_0\|_X$ satisfying $D(F) \cap B_R(0) \neq \emptyset$ and*

$$\langle \xi + \eta - L, \sigma - \sigma_0 \rangle_{X^* \times X} > 0$$

for all $\sigma \in D(F)$ with $\|\sigma\|_X = R$, all $\xi \in F(\sigma)$ and all $\eta \in G(\sigma)$, then the following inclusion

$$F(\sigma) + G(\sigma) \ni L$$

has a solution in $D(F)$, namely, $F + G$ is surjective.

Finally, we define the following notations for some sets and function spaces:

$$\begin{aligned} K &:= K_1 \times K_2, \\ \mathcal{L} &:= L^{\mathcal{G}_1}(\Omega) \times L^{\mathcal{G}_2}(\Omega), \\ L^{r_1(\cdot),r_2(\cdot)}(\Omega) &:= L^{r_1(\cdot)}(\Omega) \times L^{r_2(\cdot)}(\Omega), \\ L^{\iota_1(\cdot),\iota_2(\cdot)}(\partial\Omega) &:= L^{\iota_1(\cdot)}(\partial\Omega) \times L^{\iota_2(\cdot)}(\partial\Omega), \\ \mathcal{W} &:= W^{1,\mathcal{G}_1}(\Omega) \times W^{1,\mathcal{G}_2}(\Omega). \end{aligned}$$

Obviously, $\mathcal{L}$, $L^{r_1(\cdot),r_2(\cdot)}(\Omega)$, $L^{\iota_1(\cdot),\iota_2(\cdot)}(\partial\Omega)$ and $\mathcal{W}$ endowed with norms

$$\begin{aligned} \|\sigma\|_{\mathcal{L}} &= \|\sigma\|_{\mathcal{G}_1} + \|\sigma\|_{\mathcal{G}_2}, \\ \|\sigma\|_{L^{r_1(\cdot),r_2(\cdot)}(\Omega)} &= \|\sigma\|_{r_1(\cdot)} + \|\sigma\|_{r_2(\cdot)}, \\ \|\sigma\|_{L^{\iota_1(\cdot),\iota_2(\cdot)}(\partial\Omega)} &= \|\sigma\|_{\iota_1(\cdot),\partial\Omega} + \|\sigma\|_{\iota_2(\cdot),\partial\Omega}, \\ \|\sigma\|_{\mathcal{W}} &= \|\sigma\|_{1,\mathcal{G}_1} + \|\sigma\|_{1,\mathcal{G}_2}, \end{aligned}$$

respectively, become reflexive and separable Banach spaces. Referring to Proposition 2.2 we get:

$$\mathcal{W} \hookrightarrow\hookrightarrow \mathcal{L}, \quad \mathcal{W} \hookrightarrow\hookrightarrow L^{p_1(\cdot),p_2(\cdot)}(\Omega), \quad \mathcal{W} \hookrightarrow L^{q_1(\cdot),q_2(\cdot)}(\Omega), \quad \mathcal{W} \hookrightarrow\hookrightarrow L^{p_1(\cdot),p_2(\cdot)}(\partial\Omega). \tag{2.5}$$

We end this section by introducing the weak solutions, subsolutions and supersoltions of problem (1.1).

Definition 2.12. A function $\sigma = (\sigma_1,\sigma_2) \in K$ is called a weak solution of problem (1.1), if it fulfills the following conditions: For $i = 1,2$, there exist $\tau_i \in C(\overline{\Omega})$ and $\theta_i \in C(\Gamma_i)$ with $1 < \tau_i(x) < p_i^*(x)$ for all $x \in \overline{\Omega}$ as well as $1 < \theta_i(x) < (p_i)_*(x)$ for all $x \in \Gamma_i$ and $\eta_i \in L^{\tau_i'(\cdot)}(\Omega)$, $\zeta_i \in L^{\theta_i'(\cdot)}(\Gamma_i)$ with $\eta_i(x) \in \mathcal{F}_i(\sigma)(x) := f_i(x,\sigma_1(x),\sigma_2(x),\nabla\sigma_1(x),\nabla\sigma_2(x))$ for a.a. $x \in \Omega$ and $\zeta_i(x) \in \mathcal{F}_{\Gamma_i}(\sigma)(x) := f_{\Gamma_i}(x,\sigma_1(x),\sigma_2(x))$ for a.a. $x \in \Gamma_i$ are such that:

$$\int_\Omega \frac{\mathcal{G}_1'(x,|\nabla\sigma_1|)}{|\nabla\sigma_1|}\nabla\sigma_1 \cdot \nabla(v_1 - \sigma_1)\mathrm{d}x + \int_\Omega \eta_1(v_1-\sigma_1)\mathrm{d}x + \int_{\Gamma_1} \zeta_1(v_1-\sigma_1)\mathrm{d}\varsigma \ge 0 \tag{2.6}$$

and

$$\int_\Omega \frac{\mathcal{G}_2'(x,|\nabla\sigma_2|)}{|\nabla\sigma_2|}\nabla\sigma_2\cdot\nabla(v_2-\sigma_2)\mathrm{d}x+\int_\Omega \eta_2(v_2-\sigma_2)\mathrm{d}x+\int_{\Gamma_2}\zeta_2(v_2-\sigma_2)\mathrm{d}\varsigma\geq 0 \tag{2.7}$$

for all $(v_1,v_2)\in K$.

Note that the boundary integral $\int_{\Gamma_i}\zeta_i(v_i-\sigma_i)\mathrm{d}\varsigma$ means that

$$\int_{\Gamma_i}\zeta_i\left(\mathcal{I}_{\theta_i(\cdot)}v_i|_{\Gamma_i}-\mathcal{I}_{\theta_i(\cdot)}\sigma_i|_{\Gamma_i}\right)\mathrm{d}\varsigma,$$

where $\mathcal{I}_{\theta_i(\cdot)}:W^{1,\mathcal{G}_i}(\Omega)\to L^{\theta_i(\cdot)}(\partial\Omega)$ is the trace operator and $\mathcal{I}_{\theta_i(\cdot)}v_i|_{\Gamma_i}$ is the restriction of $\mathcal{I}_{\theta_i(\cdot)}v_i$ to Γ_i.

Definition 2.13. Let $i=1,2$, a function $\underline{\sigma}=(\underline{\sigma}_1,\underline{\sigma}_2)\in\mathcal{W}$ is called a subsolution of problem (1.1), if there exist $\tau_i\in C(\overline{\Omega})$ and $\theta_i\in C(\Gamma_i)$ with $1<\tau_i(x)<p_i^*(x)$ for all $x\in\overline{\Omega}$ as well as $1<\theta_i(x)<(p_i)_*(x)$ for all $x\in\Gamma_i$ and $\eta_i\in L^{\tau_i'(\cdot)}(\Omega)$, $\zeta_i\in L^{\theta_i'(\cdot)}(\Gamma_i)$ fulfilling the conditions:

(i) $\underline{\sigma}_i\vee K_i\subset K_i$;
(ii) $\underline{\eta}_1(x)\in f_1(x,\underline{\sigma}_1(x),z_2(x),\nabla\underline{\sigma}_1(x),\nabla z_2(x))$ for a.a. $x\in\Omega$, $\underline{\zeta}_1(x)\in f_{\Gamma_1}(x,\underline{\sigma}_1(x),z_2(x))$ for a.a. $x\in\Gamma_1$ and $\underline{\eta}_2(x)\in f_2(x,z_1(x),\underline{\sigma}_2(x),\nabla z_1(x),\nabla\underline{\sigma}_2(x))$ for a.a. $x\in\Omega$, $\underline{\zeta}_2(x)\in f_{\Gamma_2}(x,z_1(x),\underline{\sigma}_2(x))$ for a.a. $x\in\Gamma_2$;
(iii) the inequality holds

$$\begin{aligned}&\int_\Omega \frac{\mathcal{G}_1'(x,|\nabla\underline{\sigma}_1|)}{|\nabla\underline{\sigma}_1|}\nabla\underline{\sigma}_1\cdot\nabla(v_1-\underline{\sigma}_1)\mathrm{d}x+\int_\Omega \underline{\eta}_1(v_1-\underline{\sigma}_1)\mathrm{d}x+\int_{\Gamma_1}\underline{\zeta}_1(v_1-\underline{\sigma}_1)\mathrm{d}\varsigma\\&+\int_\Omega \frac{\mathcal{G}_2'(x,|\nabla\underline{\sigma}_2|)}{|\nabla\underline{\sigma}_2|}\nabla\underline{\sigma}_2\cdot\nabla(v_2-\underline{\sigma}_2)\mathrm{d}x+\int_\Omega \underline{\eta}_2(v_2-\underline{\sigma}_2)\mathrm{d}x+\int_{\Gamma_2}\underline{\zeta}_2(v_2-\underline{\sigma}_2)\mathrm{d}\varsigma\geq 0\end{aligned}$$

for all $(v_1,v_2)\in(\underline{\sigma}_1,\underline{\sigma}_2)\wedge(K_1,K_2)$, all $(z_1,z_2)\in\mathcal{W}$ satisfying $\underline{\sigma}_i\leq z_i\leq\overline{\sigma}_i$.

We point out that if $\sigma=(\sigma_1,\sigma_2)$ belongs to a nonempty set $K=(K_1,K_2)$, then $\sigma\vee K$ states $\sigma\vee k=(\sigma_1+(k_1-\sigma_2)^+,\sigma_2+(k_2-\sigma_2)^+)$ for all $k=(k_1,k_2)\in K$. Likewise, $\sigma\wedge K$ states $\sigma\wedge k=(\sigma_1-(\sigma_1-k_1)^+,\sigma_2-(\sigma_2-k_2)^+)$ for all $k\in K$.

Definition 2.14. Let $i=1,2$, a function $\overline{\sigma}=(\overline{\sigma}_1,\overline{\sigma}_2)\in\mathcal{W}$ is called a supersolution of problem (1.1), if there exist $\tau_i\in C(\overline{\Omega})$ and $\theta_i\in C(\Gamma_i)$ with $1<\tau_i(x)<p_i^*(x)$ for all $x\in\overline{\Omega}$ as well as $1<\theta_i(x)<(p_i)_*(x)$ for all $x\in\Gamma_i$ and $\eta_i\in L^{\tau_i'(\cdot)}(\Omega)$, $\zeta_i\in L^{\theta_i'(\cdot)}(\Gamma_i)$ fulfilling the conditions:

(i) $\overline{\sigma}_i\wedge K_i\subset K_i$;
(ii) $\overline{\eta}_1(x)\in f_1(x,\overline{\sigma}_1(x),z_2(x),\nabla\overline{\sigma}_1(x),\nabla z_2(x))$ for a.a. $x\in\Omega$, $\overline{\zeta}_1(x)\in f_{\Gamma_1}(x,\overline{\sigma}_1(x),z_2(x))$ for a.a. $x\in\Gamma_1$ and $\overline{\eta}_2(x)\in f_2(x,z_1(x),\overline{\sigma}_2(x),\nabla z_1(x),\nabla\overline{\sigma}_2(x))$ for a.a. $x\in\Omega$, $\overline{\zeta}_2(x)\in f_{\Gamma_2}(x,z_1(x),\overline{\sigma}_2(x))$ for a.a. $x\in\Gamma_2$;
(iii) the inequality holds

$$\begin{aligned}&\int_\Omega \frac{\mathcal{G}_1'(x,|\nabla\overline{\sigma}_1|)}{|\nabla\overline{\sigma}_1|}\nabla\overline{\sigma}_1\cdot\nabla(v_1-\overline{\sigma}_1)\mathrm{d}x+\int_\Omega \overline{\eta}_1(v_1-\overline{\sigma}_1)\mathrm{d}x+\int_{\Gamma_1}\overline{\zeta}_1(v_1-\overline{\sigma}_1)\mathrm{d}\varsigma\\&+\int_\Omega \frac{\mathcal{G}_2'(x,|\nabla\overline{\sigma}_2|)}{|\nabla\overline{\sigma}_2|}\nabla\overline{\sigma}_2\cdot\nabla(v_2-\overline{\sigma}_2)\mathrm{d}x+\int_\Omega \overline{\eta}_2(v_2-\overline{\sigma}_2)\mathrm{d}x+\int_{\Gamma_2}\overline{\zeta}_2(v_2-\overline{\sigma}_2)\mathrm{d}\varsigma\geq 0\end{aligned}$$

for all $(v_1,v_2)\in(\overline{\sigma}_1,\overline{\sigma}_2)\vee(K_1,K_2)$, all $(z_1,z_2)\in\mathcal{W}$ satisfying $\underline{\sigma}_i\leq z_i\leq\overline{\sigma}_i$.

If $\underline{\sigma}=(\underline{\sigma}_1,\underline{\sigma}_2)$ and $\overline{\sigma}=(\overline{\sigma}_1,\overline{\sigma}_2)$ is a pair of sub- and supersolution, then we say that the order interval $[\underline{\sigma},\overline{\sigma}]=[\underline{\sigma}_1,\overline{\sigma}_1]\times[\underline{\sigma}_2,\overline{\sigma}_2]$ is a trapping region with

$$[\underline{\sigma}_i,\overline{\sigma}_i]=\left\{\sigma\in W^{1,\mathcal{G}_i}(\Omega):\underline{\sigma}_i\leq\sigma\leq\overline{\sigma}_i\ \text{ a.e. in }\ \Omega\right\}.$$

3. Existence results in coercive setting

In this section, we focus on the situation that problem (1.1) fulfills a mild coercive condition. This permits us to utilize the surjectivity theorem, Theorem 2.11, for the purpose of showing the existence of weak solutions.

So, we first make the following assumptions for problem (1.1) :

(H2) For $i=1,2$, one can find $r_i\in C(\overline{\Omega}), \iota_i\in C(\Gamma)$ such that $1<r_i(x)<p_i^*(x)$ for a.a. $x\in\overline{\Omega}$, $1<\iota_i(x)<(p_i)_*(x)$ for a.a. $x\in\Gamma_i$, $\beta_i^\Omega,\gamma_i\geq 0$, $\beta_i^{\Gamma_i}\geq 0$ and nonnegative functions $\alpha_i^\Omega\in L^{r_i'(\cdot)}(\Omega)$, $\alpha_i^{\Gamma_i}\in L^{\iota_i'(\cdot)}(\Gamma_i)$ satisfying:

$$\sup\{|\eta_1|:\eta_1\in f_1(x,y_1,y_2,\varphi_1,\varphi_2)\}\leq \alpha_1^\Omega(x)+\beta_1^\Omega\left(|y_1|^{r_1(x)-1}+|y_2|^{\frac{r_2(x)}{r_1'(x)}}\right)+\gamma_1\left(|\varphi_1|^{\frac{p_1(x)}{r_1'(x)}}+|\varphi_2|^{\frac{p_2(x)}{r_1'(x)}}\right),$$

as well as

$$\sup\{|\eta_2|:\eta_2\in f_2(x,y_1,y_2,\varphi_1,\varphi_2)\}\leq \alpha_2^\Omega(x)+\beta_2^\Omega\left(|y_1|^{\frac{r_1(x)}{r_2'(x)}}+|y_2|^{r_2(x)-1}\right)+\gamma_2\left(|\varphi_1|^{\frac{p_1(x)}{r_2'(x)}}+|\varphi_2|^{\frac{p_2(x)}{r_2'(x)}}\right),$$

for a.a. $x\in\Omega$, for all $y_i\in\mathbb{R}$ and for all $\varphi_i\in\mathbb{R}^N$, and

$$\sup\{|\zeta_1|:\zeta_1\in f_{\Gamma_1}(x,y_1,y_2)\}\leq\alpha_1^{\Gamma_1}(x)+\beta_1^{\Gamma_1}\left(|y_1|^{\iota_1(x)-1}+|y_2|^{\frac{\iota_2(x)}{\iota_1'(x)}}\right)\text{ for a.a. } x\in\Gamma_1 \text{ and all } y_1,y_2\in\mathbb{R},$$

as well as

$$\sup\{|\zeta_2|:\zeta_2\in f_{\Gamma_2}(x,y_1,y_2)\}\leq\alpha_2^{\Gamma_2}(x)+\beta_2^{\Gamma_2}\left(|y_1|^{\frac{\iota_1(x)}{\iota_2'(x)}}+|y_2|^{\iota_2(x)-1}\right)\text{ for a.a. } x\in\Gamma_2 \text{ and all } y_1,y_2\in\mathbb{R}.$$

In the sequel, for $i=1,2$, we denote by $\mathcal{I}_{r_i(\cdot)}:W^{1,\mathcal{G}_i}(\Omega)\to L^{r_i(\cdot)}(\Omega)$ the embedding operator and denote by $\mathcal{I}_{\iota_i(\cdot)}:W^{1,\mathcal{G}_i}(\Omega)\to L^{\iota_i(\cdot)}(\Gamma_i)$ the trace operator. Taking (H2), Proposition 2.3(ii) and (iii) into account, we get the compactness of both $\mathcal{I}_{r_i(\cdot)}$ and $\mathcal{I}_{\iota_i(\cdot)}$. Denote the corresponding adjoint operators of $\mathcal{I}_{r_i(\cdot)}$ and $\mathcal{I}_{\iota_i(\cdot)}$ by $\mathcal{I}^*_{r_i(\cdot)}:L^{r_i'(\cdot)}(\Omega)\to W^{1,\mathcal{G}_i}(\Omega)^*$ and $\mathcal{I}^*_{\iota_i(\cdot)}:L^{\iota_i'(\cdot)}(\Gamma_i)\to W^{1,\mathcal{G}_i}(\Omega)^*$, respectively. Take any $(v_1,v_2)\in M(\Omega)\times M(\Omega)$ and $\varphi_i\in[M(\Omega)]^N$ with $i=1,2$, we define $\tilde{f}_i(\cdot,v_1,v_2,\varphi_1,\varphi_2)$ as the set of measurable selections of $f_i(\cdot,v_1,v_2,\varphi_1,\varphi_2)$, namely,

$$\tilde{f}_i(v_1,v_2,\varphi_1,\varphi_2)=\{\eta_i\in M(\Omega)\,:\,\eta_i(x)\in f_i(x,v_1(x),v_2(x),\varphi_1(x),\varphi_2(x))\text{ for a.a. } x\in\Omega\}.$$

Since assumption (H1) hold, the above set is nonempty. Analogously, for any $(v_1,v_2)\in M(\Gamma_1)\times M(\Gamma_2)$, we denote the set of measurable selections of $f_{\Gamma_i}(\cdot,v_1,v_2)$ by

$$\tilde{f}_{\Gamma_i}(v_1,v_2)=\{\zeta_i\in M(\Gamma_i)\mid\zeta_i(x)\in f_{\Gamma_i}(x,v_1(x),v_2(x))\text{ for a.a. } x\in\Gamma_i\}.$$

Similarly, the above set is nonempty thanks for hypotheses (H1). Since (H2) hold true, let $i=1,2$, then for any $v_i\in L^{r_i(\cdot)}(\Omega)$ and $\varphi_i\in[L^{p_i(\cdot)}(\Omega)]^N$ we can show that $\tilde{f}_i(v_1,v_2,\varphi_1,\varphi_2)\subset L^{r_i'(\cdot)}(\Omega)$. Likewise, for any $v_i\in L^{\iota_i(\cdot)}(\Gamma_i)$, we get $\tilde{f}_{\Gamma_i}(v_1,v_2)\subset L^{\iota_i'(\cdot)}(\Gamma_i)$.

Furthermore, let $i=1,2$, under the conditions of (H2), for any $(v_1,v_2,\nabla v_1,\nabla v_2)\in L^{r_1(\cdot)}(\Omega)\times L^{r_2(\cdot)}(\Omega)\times[L^{p_1(\cdot)}(\Omega)]^N\times[L^{p_2(\cdot)}(\Omega)]^N$ (resp. $(v_1,v_2)\in L^{\iota_1(\cdot)}(\Gamma_1)\times L^{\iota_2(\cdot)}(\Gamma_2)$), we have $\tilde{f}_i(v_1,v_2,\nabla v_1,\nabla v_2)\subset L^{r_i'(\cdot)}(\Omega)$ (resp. $\tilde{f}_{\Gamma_i}(v_1,v_2)\subset L^{\iota_i'(\cdot)}(\Gamma_i)$). Now, we defined the following mappings $\mathcal{F}_i=\mathcal{I}^*_{r_i(\cdot)}\circ\tilde{f}_i:\mathcal{W}\to 2^{W^{1,\mathcal{G}_i}(\Omega)^*}$ (resp. $\mathcal{F}_{\Gamma_i}=\mathcal{I}^*_{\iota_i(\cdot)}\circ\tilde{f}_{\Gamma_i}\circ\mathcal{I}_{\iota_i(\cdot)}:\mathcal{W}\to 2^{W^{1,\mathcal{G}_i}(\Omega)^*}$), that is, $\mathcal{F}_i(v_1,v_2)=\{\hat{\eta}_i\in W^{1,\mathcal{G}_i}(\Omega)^*:\hat{\eta}_i\in\tilde{f}_i(v_1,v_2,\nabla v_1,\nabla v_2)\}$ (resp. $\mathcal{F}_{\Gamma_i}(v_1,v_2)=\{\hat{\zeta}_i\in W^{1,\mathcal{G}_i}(\Omega)^*\,:\,\hat{\zeta}_i\in\tilde{f}_{\Gamma_i}(v_1,v_2)\}$) where $i=1,2$.

Let $A(\sigma)=(A_1(\sigma_1),A_2(\sigma_2))$ with $A_i(i=1,2)$ given by (2.3). According to Proposition 2.6 we see that $A:\mathcal{W}\to\mathcal{W}^*$ is continuous, bounded, strictly monotone and of type (S_+). In the sequel, define $\mathcal{F}=(\mathcal{F}_1,\mathcal{F}_2)$ and $\mathcal{F}_\Gamma=(\mathcal{F}_{\Gamma_1},\mathcal{F}_{\Gamma_2})$.

Next, we will verify that $A+\mathcal{F}+\mathcal{F}_\Gamma$ is pseudomonotone, and the proof is motivated by Carl-Le [11].

Proposition 3.1. *Assume hypotheses (H0), (H1) and (H2) are satisfied, then operator $A+\mathcal{F}+\mathcal{F}_\Gamma$ turns out to be bounded and pseudomonotone from $\mathcal{W}$ into $\mathcal{K}(\mathcal{W}^*)$.*

Proof. For $i=1,2$, hypotheses (H1) ensure that f_i is graph measurable, so $\tilde{f}_i : L^{r_1(\cdot)}(\Omega)\times L^{r_2(\cdot)}(\Omega)\times [L^{p_1(\cdot)}(\Omega)]^N\times[L^{p_2(\cdot)}(\Omega)]^N \to L^{r_i'(\cdot)}(\Omega)$ is well defined. Moreover, since f_i is a multivalued upper semicontinuous function, we deduce that $f_i(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)$ is a closed interval within $\mathbb{R}$, and then $\tilde{f}_i(\sigma_1,\sigma_2,\varphi_1,\varphi_2)$ is convex. By applying (H2), we see that for any $\eta_i\in\tilde{f}_i(\sigma_1,\sigma_2,\varphi_1,\varphi_2)$ and fixed $(\sigma_1,\sigma_2,\varphi_1,\varphi_2)\in L^{r_1(\cdot)}(\Omega)\times L^{r_2(\cdot)}(\Omega)\times[L^{p_1(\cdot)}(\Omega)]^N\times[L^{p_2(\cdot)}(\Omega)]^N$ there holds
(3.1)

$$\sup\{|\eta_i| : \eta_i\in f_i(x,y_1,y_2,\varphi_1,\varphi_2)\}\le \alpha_i^{\Omega}(x)+\beta_i^{\Omega}\left(|y_1|^{r_1(x)-1}+|y_2|^{\frac{r_2(x)}{r_i'(x)}}\right)+\gamma_i\left(|\varphi_1|^{\frac{p_1(x)}{r_i'(x)}}+|\varphi_2|^{\frac{p_2(x)}{r_i'(x)}}\right)$$

for a.a. $x\in\Omega$, thus $\tilde{f}_i(\sigma_1,\sigma_2,\varphi_1,\varphi_2)$ is bounded in $L^{r_i'(\cdot)}(\Omega)$. Let $\{\eta_{n,i}\}\subset\tilde{f}_i(\sigma_1,\sigma_2,\varphi_1,\varphi_2)$ be such that $\eta_{n,i}\to\eta_i$ in $L^{r_i'(\cdot)}(\Omega)$ (thus $\eta_{n,i}\to\eta_i$ a.a. in Ω), note that $\eta_{n,i}(x)\in f_i(x,\sigma_1(x),\sigma_2(x),\varphi_1(x),\varphi_2(x))$ for a.a. $x\in\Omega$ and for all $n\in\mathbb{N}$, in addition, $f_i(x,\sigma_1(x),\sigma_2(x),\varphi_1(x),\varphi_2(x))$ is closed in $\mathbb{R}$, then we know that $\eta_i(x)\in f_i(x,\sigma_1(x),\sigma_2(x),\varphi_1(x),\varphi_2(x))$. Recalling the definition of $\tilde{f}_i(\sigma_1,\sigma_2,\varphi_1,\varphi_2)$ we can observe that it is closed. Moreover, we show $\tilde{f}_i : L^{r_1(\cdot)}(\Omega)\times L^{r_2(\cdot)}(\Omega)\times[L^{p_1(\cdot)}(\Omega)]^N\times[L^{p_2(\cdot)}(\Omega)]^N\to L^{r_i'(\cdot)}(\Omega)$ is bounded. Let D be a bounded set in $L^{r_1(\cdot)}(\Omega)\times L^{r_2(\cdot)}(\Omega)\times[L^{p_1(\cdot)}(\Omega)]^N\times[L^{p_2(\cdot)}(\Omega)]^N$, (3.1) implies that set $\tilde{f}_i(D)$ is bounded in $L^{r_i'(\cdot)}(\Omega)$, so we get the boundedness of $\tilde{f}_i$. Likewise, we deduce that $\tilde{f}_{\Gamma_i}(\sigma_1,\sigma_2)$ is a closed, bounded and convex subset in $L^{\iota_i'(\cdot)}(\Gamma_i)$. Furthermore, $\tilde{f}_{\Gamma_i} : L^{\iota_1(\cdot),\iota_2(\cdot)}\to\mathcal{K}(L^{\iota_i'(\cdot)}(\Gamma_i))$ is bounded. On these basis, combining the boundedness of $\mathcal{I}_{r_i(\cdot)}$, $\mathcal{I}^*_{r_i(\cdot)}, \mathcal{I}_{\iota_i(\cdot)}$ and $\mathcal{I}^*_{\iota_i(\cdot)}$ we obtain the boundedness of $\mathcal{F}_i$ and $\mathcal{F}_{\Gamma_i}$, thus $\mathcal{F}$ and $\mathcal{F}_\Gamma$. Furthermore, using the boundedness of A given by Proposition 2.6, we infer that $A+\mathcal{F}$ and $A+\mathcal{F}+\mathcal{F}_\Gamma$ are bounded.

To demonstrate $A+\mathcal{F}$ is pseudomonotone, we first verify the generalized pseudomonotonicity of $A+\mathcal{F}:\mathcal{W}\to\mathcal{K}(\mathcal{W}^*)$. Let $\{\sigma_n\}\subset\mathcal{W}$ and $\{\sigma_n^*\}\subset\mathcal{W}^*$ (note that $\sigma_n=(\sigma_{n,1},\sigma_{n,2}),\sigma_n^*=(\sigma_{n,1}^*,\sigma_{n,2}^*)$) be such that

$$(\sigma_{n,1},\sigma_{n,2})\rightharpoonup(\sigma_1,\sigma_2)\text{ in }\mathcal{W}, \tag{3.2}$$

$$(\sigma_{n_1}^*,\sigma_{n_2}^*)\rightharpoonup(\sigma_1^*,\sigma_2^*)\text{ in }\mathcal{W}^*, \tag{3.3}$$

$$(\sigma_{n,1}^*,\sigma_{n,2}^*)\in(A+\mathcal{F})(\sigma_{n,1},\sigma_{n,2})\text{ for all }n\in\mathbb{N},$$

and

$$\limsup_{n\to\infty}\langle(\sigma_{n,1}^*,\sigma_{n,2}^*),(\sigma_{n,1},\sigma_{n,2})-(\sigma_1,\sigma_2)\rangle\le 0. \tag{3.4}$$

By the definition of generalized pseudomonotone to multivalued operators (recall Definition 2.7(ii)) we need to verify

$$(\sigma_1^*,\sigma_2^*)\in(A+\mathcal{F})(\sigma_1,\sigma_2), \tag{3.5}$$

and

$$\langle(\sigma_{n,1}^*,\sigma_{n,2}^*),(\sigma_{n,1},\sigma_{n,2})\rangle\to\langle(\sigma_1^*,\sigma_2^*),(\sigma_1,\sigma_2)\rangle. \tag{3.6}$$

First, for all $n\in\mathbb{N}$, let

$$(\sigma_{n,1}^*,\sigma_{n,2}^*)=(l_{n,1}^*,l_{n,2}^*)+(\eta_{n,1}^*,\eta_{n,2}^*) \tag{3.7}$$

where

$$(l_{n,1}^*,l_{n,2}^*)\in A(\sigma_{n,1},\sigma_{n,2})\text{ and }(\eta_{n,1}^*,\eta_{n,2}^*)\in\mathcal{F}(\sigma_{n,1},\sigma_{n,2}). \tag{3.8}$$

Utilizing the boundedness of $\{(\sigma_{n,1},\sigma_{n,2})\}$ in $\mathcal{W}$ together with the boundedness of $A,\mathcal{F}$ we get

$$(l_{n,1}^*,l_{n,2}^*)\rightharpoonup(l_{0,1}^*,l_{0,2}^*),\quad(\eta_{n,1}^*,\eta_{n,2}^*)\rightharpoonup(\eta_{0,1}^*,\eta_{0,2}^*),\text{ in }\mathcal{W}^*, \tag{3.9}$$

in the sense of subsequence. Applying (3.3), (3.7) and (3.9) we obtain

$$(\sigma_1^*,\sigma_2^*)=(l_{0,1}^*,l_{0,2}^*)+(\eta_{0,1}^*,\eta_{0,2}^*). \tag{3.10}$$

In addition, recalling the definition of $\mathcal{F}_i$, we can find $\eta_{n,i}\in\tilde{f}_i(\sigma_{n,1},\sigma_{n,2},\nabla\sigma_{n,1},\nabla\sigma_{n,2})$ such that $(\eta_{n,1}^*,\eta_{n,2}^*)=(\mathcal{I}^*_{r_1(\cdot)}\eta_{n,1},\mathcal{I}^*_{r_2(\cdot)}\eta_{n,2})$, for all $n\in\mathbb{N}$. Obviously, $\{(\sigma_{n,1},\sigma_{n,2})\}$ is bounded in $\mathcal{W}$, we see that $(\sigma_{n,1},\sigma_{n,2},\nabla\sigma_{n,1},\nabla\sigma_{n,2})$ is bounded in $L^{r_1(\cdot)}(\Omega)\times L^{r_2(\cdot)}(\Omega)\times[L^{p_1(\cdot)}(\Omega)]^N\times[L^{p_2(\cdot)}(\Omega)]^N$, along with condition (F2) we get the boundedness of $\{(\eta_{n,1},\eta_{n,2})\}$ in $L^{r_1'(\cdot),r_2'(\cdot)}(\Omega)$. Hence, $\{(\eta^*_{n,1},\eta^*_{n,2}) : n\in\mathbb{N}\}$ can be shown to be a relatively compact set in $\mathcal{W}^*$ because of the compactness of $\mathcal{I}^*_{r_1(\cdot)}$ and $\mathcal{I}^*_{r_2(\cdot)}$. Together with the weak convergence of $(\eta_{n,1},\eta_{n,2})$ in (3.9) we see that

$$(\eta^*_{n,1},\eta^*_{n,2}) \to (\eta^*_{0,1},\eta^*_{0,2}) \text{ in } \mathcal{W}^*. \tag{3.11}$$

So, $\langle(\eta^*_{n,1},\eta^*_{n,2}),(\sigma_{n,1},\sigma_{n,2})-(\sigma_1,\sigma_2)\rangle \to 0$ as $n\to\infty$, applying (3.4) we see that

$$\limsup\langle(l^*_{n,1},l^*_{n,2}),(\sigma_{n,1},\sigma_{n,2})-(\sigma_1,\sigma_2)\rangle \le 0. \tag{3.12}$$

The latter joining with the maximal monotonicity of A_i (recall Theorem 2.6) we see that A_i is generalized pseudomonotone, hence $A=(A_1,A_2)$ is generalized pseudomonotone. Then, taking (3.2), (3.8), (3.9) and (3.12) into account and utilizing the (S_+)-property as well as generalized pseudomonotonicity of A we obtain

$$(l_{0,1},l_{0,2})\in A(\sigma_1,\sigma_2), \quad \langle(l^*_{n,1},l^*_{n,2}),(\sigma_{n,1},\sigma_{n,2})\rangle \to \langle(l^*_{0,1},l^*_{0,2}),(\sigma_1,\sigma_2)\rangle \tag{3.13}$$

and

$$(\sigma_{n,1},\sigma_{n,2}) \to (\sigma_1,\sigma_2) \text{ in } \mathcal{W}. \tag{3.14}$$

Therefore, $(\sigma_{n,1},\sigma_{n,2},\nabla\sigma_{n,1},\nabla\sigma_{n,2}) \to (\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2)$ in $L^{r_1(\cdot)}(\Omega)\times L^{r_2(\cdot)}(\Omega)\times[L^{p_1(\cdot)}(\Omega)]^N\times[L^{p_2(\cdot)}(\Omega)]^N$. Combining hypotheses (F1), (F2) with Theorem 2.10 we can verify that $\tilde{f}_i(\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2)$ is Hausdorff upper semicontinuous, thus $h^*(\tilde{f}_i(\sigma_{n,1},\sigma_{n,2},\nabla\sigma_{n,1},\nabla\sigma_{n,2}),\tilde{f}_i(\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2)) \to 0$. It follows that

$$\inf_{\substack{\omega_1^*\in\tilde{f}_1(\sigma_{n,1},\sigma_{n,2},\nabla\sigma_{n,1},\nabla\sigma_{n,2})\\ \omega_2^*\in\tilde{f}_2(\sigma_{n,1},\sigma_{n,2},\nabla\sigma_{n,1},\nabla\sigma_{n,2})}} \|(\eta_{n,1},\eta_{n,2})-(\omega_1^*,\omega_2^*)\|_{L^{r_1'(\cdot),r_2'(\cdot)}(\Omega)} \to 0,$$

where $\eta_{n,i}\in\tilde{f}_i(\sigma_{n,1},\sigma_{n,2},\nabla\sigma_{n,1},\nabla\sigma_{n,2})$. Then, for $i=1,2$, we can find $\{\omega^*_{n,i}\}\subset\tilde{f}_i(\sigma_{n,1},\sigma_{n,2},\nabla\sigma_{n,1},\nabla\sigma_{n,2})$ fulfilling

$$\|(\eta_{n,1},\eta_{n,2})-(\omega^*_{n,1},\omega^*_{n,2})\|_{L^{r_1'(\cdot),r_2'(\cdot)}(\Omega)} \to 0. \tag{3.15}$$

Moreover, since $\tilde{f}_i(\sigma_{n,1},\sigma_{n,2},\nabla\sigma_{n,1},\nabla\sigma_{n,2})$ is bounded, there exists some $(\omega^*_{0,1},\omega^*_{0,2})\in L^{r_1'(\cdot),r_2'(\cdot)}(\Omega)$ such that

$$(\omega^*_{n,1},\omega^*_{n,2}) \rightharpoonup (\omega^*_{0,1},\omega^*_{0,2}) \tag{3.16}$$

in $L^{r_1'(\cdot),r_2'(\cdot)}(\Omega)$. This associating the fact that $\tilde{f}_i(\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2)$ is closed and convex (hence weakly closed) in $L^{r_i'(\cdot)}(\Omega)$ implies $\omega^*_{0,i}\in\tilde{f}_i(\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2)$. Combining (3.15) with (3.16) we get

$$(\eta_{n,1},\eta_{n,2}) \rightharpoonup (\omega^*_{0,1},\omega^*_{0,2}) \text{ in } L^{r_1'(\cdot),r_2'(\cdot)}(\Omega).$$

Using the compactness of $\mathcal{I}^*_{r_i(\cdot)}$ again, we have

$$(\eta^*_{n,1},\eta^*_{n,2}) = \left(\mathcal{I}^*_{r_1(\cdot)}\eta_{n,1},\mathcal{I}^*_{r_2(\cdot)}\eta_{n,2}\right) \to \left(\mathcal{I}^*_{r_1(\cdot)}\omega^*_{0,1},\mathcal{I}^*_{r_2(\cdot)}\omega^*_{0,2}\right) \text{ in } \mathcal{W}^*.$$

Together with (3.11), there holds

$$\begin{aligned}(\eta^*_{0,1},\eta^*_{0,2}) &= \left(\mathcal{I}^*_{r_1(\cdot)}\omega^*_{0,1},\mathcal{I}^*_{r_2(\cdot)}\omega^*_{0,2}\right)\\ &\in \left(\mathcal{I}^*_{r_1(\cdot)}\tilde{f}_1(\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2),\mathcal{I}^*_{r_2(\cdot)}\tilde{f}_2(\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2)\right) = \mathcal{F}(\sigma_1,\sigma_2).\end{aligned} \tag{3.17}$$

Furthermore, from (3.11) and (3.14) we have

$$\langle(\eta^*_{n,1},\eta^*_{n,2}),(\sigma_{n,1},\sigma_{n,2})\rangle \to \langle(\eta^*_{0,1},\eta^*_{0,2}),(\sigma_1,\sigma_2)\rangle. \tag{3.18}$$

Taking (3.7), (3.10), (3.13), (3.17) and (3.18) into account we clarify (3.5) and (3.6). So, we obtain that $A+\mathcal{F}$ is a multivalued generalized pseudomonotone operator. Therefore, along with the boundedness of $A+\mathcal{F}$ we can show it is pseudomonotone. In addition, similar to the proof of [12, Proposition 3.1] one can prove that $\mathcal{F}_\Gamma$ is pseudomontone, hence, $A+\mathcal{F}+\mathcal{F}_\Gamma$ is bounded and pseudomonotone. $\square$

By the pseudomonotonicity of $A+\mathcal{F}+\mathcal{F}_\Gamma$, we are going to invoke the surjective theorem Theorem 2.11 for getting the following existence result to elliptic inclusion systems (1.1).

Theorem 3.2. *Let* (H0), (H1) *and* (H2) *hold true. Suppose the next coercivity condition:*

there are $\sigma_0 \in K$ and $R \geq \|\sigma_0\|_{\mathcal{W}}$ satisfying $K \cap B_R(0) \neq \emptyset$ and

$$\langle Au + \eta^* + \zeta^*, \sigma - \sigma_0 \rangle > 0, \tag{3.19}$$

for all $\sigma \in K$ with $\|\sigma\|_{\mathcal{W}} = R$, for all $\eta^* \in \mathcal{F}(\sigma)$ and for all $\zeta^* \in \mathcal{F}_\Gamma(\sigma)$.

Then there exists at least one weak solution $(\sigma_1, \sigma_2) \in K = (K_1, K_2)$ of (1.1).

Proof. According to Proposition 3.1, we get the boundedness and pseudomonotonicity for $A+\mathcal{F}+\mathcal{F}_\Gamma : \mathcal{W} \to 2^{\mathcal{W}^*}$. Since set $K = (K_1, K_2)$ is closed and convex, then mapping $\partial I_K = (\partial I_{K_1}, \partial I_{K_2})$ is maximal monotone from $\mathcal{W}$ to $2^{\mathcal{W}^*}$. Hence, invoking Theorem 2.11 we prove that there exists at least one solution of (1.1) under coercivity condition (3.19). □

The following corollary is a result of Theorem 3.2.

Corollary 3.3. *Let hypotheses (H0), (H1) and (H2) be satisfied. Take $\sigma_0 \in K$ and suppose the following coercivity condition:*

$$\lim_{\substack{\|\sigma\|_{\mathcal{W}} \to \infty \\ \sigma \in K}} \left[\inf_{\substack{\eta^* \in \mathcal{F}(\sigma) \\ \zeta^* \in \mathcal{F}_\Gamma(\sigma)}} \langle A\sigma + \eta^* + \zeta^*, \sigma - \sigma_0 \rangle \right] = \infty,$$

then (1.1) *possesses at least one solution.*

4. Existence results in Noncoercive framework

In this section, we concentrate on the existence of a weak solution for problem (1.1) under the noncoercive framework. To prove that, we will employ the sub and supersolution method associating the theory of nonsmooth analysis and truncation techniques for obtaining the existence as well as compactness results of solutions to problem (1.1). In the remaining parts, C stands for a constant that could change from line to line.

We make the following assumptions.

(H3) Denote by $\underline{\sigma} = (\underline{\sigma}_1, \underline{\sigma}_2)$ and $\overline{\sigma} = (\overline{\sigma}_1, \overline{\sigma}_2)$ a pair of subsolution and supersolution for (1.1) with $\underline{\sigma}_i \leq \overline{\sigma}_i$ for $i = 1, 2$. Fixing $i = 1, 2$, one can find $\tau_i \in C(\overline{\Omega}), \theta_i \in C(\Gamma_i)$ such that $1 < \tau_i(x) < p_i^*(x)$ for a.a. $x \in \overline{\Omega}$, $1 < \theta_i(x) < (p_i)_*(x)$ for a.a. $x \in \Gamma_i$, $\kappa_i \geq 0$ and $g_i^\Omega \in L^{\tau_i'(\cdot)}(\Omega), g_i^{\Gamma_i} \in L^{\theta_i'(\cdot)}(\Gamma_i)$ such that

$$\sup\{|\eta_1| : \eta_1 \in f_1(x, y_1, y_2, \varphi_1, \varphi_2)\} \leq g_1^\Omega(x) + \kappa_1 \left(|\varphi_1|^{\frac{p_1(x)}{\tau_1'(x)}} + |\varphi_2|^{\frac{p_2(x)}{\tau_1'(x)}} \right),$$

$$\sup\{|\eta_2| : \eta_2 \in f_2(x, y_1, y_2, \varphi_1, \varphi_2)\} \leq g_2^\Omega(x) + \kappa_2 \left(|\varphi_1|^{\frac{p_1(x)}{\tau_2'(x)}} + |\varphi_2|^{\frac{p_2(x)}{\tau_2'(x)}} \right),$$

for a.a. $x \in \Omega$, for all $y_i \in [\underline{\sigma}_i, \overline{\sigma}_i]$ as well as for all $\varphi_i \in \mathbb{R}^N$, also

$$\sup\{|\zeta_1| : \zeta_1 \in f_{\Gamma_1}(x, y_1, y_2)\} \leq g_1^{\Gamma_1}(x) \text{ or a.a. } x \in \Gamma_1, \text{ all } y_i \in [\underline{\sigma}_i, \overline{\sigma}_i],$$

$$\sup\{|\zeta_2| : \zeta_2 \in f_{\Gamma_2}(x, y_1, y_2)\} \leq g_2^{\Gamma_2}(x) \text{ or a.a. } x \in \Gamma_2, \text{ all } y_i \in [\underline{\sigma}_i, \overline{\sigma}_i].$$

The following results in this section are stated by the following theorem.

Theorem 4.1. *Suppose (H0), (H1), and (H3) hold true. Then, there exists a solution $\sigma = (\sigma_1, \sigma_2)$ to problem* (1.1) *such that*

$$\underline{\sigma}_i \leq \sigma_i \leq \overline{\sigma}_i \text{ in } \Omega$$

with $i = 1, 2$.

Proof. For any $i=1,2$, let τ_i, θ_i, $\underline{\sigma}$ and $\overline{\sigma}$ satisfy (H3). In the meantime, let τ_i, θ_i be the exponents and $\underline{\eta}_i$, $\underline{\zeta}_i$, $\overline{\eta}_i$, $\overline{\zeta}_i$ be the functions given in Definitions 2.13 and 2.14 with $\underline{\sigma}=(\underline{\sigma}_1,\underline{\sigma}_2)$ and $\overline{\sigma}=(\overline{\sigma}_1,\overline{\sigma}_2)$, respectively.

Let $i=1,2$, we introduce the following truncation operators $T_i: W^{1,\mathcal{G}_i}(\Omega)\to W^{1,\mathcal{G}_i}(\Omega)$

$$T_i(\sigma_i)(x)=\begin{cases}\overline{\sigma}_i(x) & \text{if } \sigma_i(x)>\overline{\sigma}_i(x),\\ \sigma_i(x) & \text{if } \underline{\sigma}_i(x)\le\sigma_i(x)\le\overline{\sigma}_i(x),\\ \underline{\sigma}_i(x) & \text{if } \sigma_i(x)<\underline{\sigma}_i(x),\end{cases}\tag{4.1}$$

$J_i: W^{1,\mathcal{G}_i}(\Omega)\times\mathbb{R}^N\to[L^{p_i(\cdot)}(\Omega)]^N$

$$J_i(\sigma_i,\varphi_i)(x)=\begin{cases}\nabla\overline{\sigma}_i(x) & \text{if } \sigma_i(x)>\overline{\sigma}_i(x),\\ \varphi_i(x) & \text{if } \underline{\sigma}_i(x)\le\sigma_i(x)\le\overline{\sigma}_i(x),\\ \nabla\underline{\sigma}_i(x) & \text{if } \sigma_i(x)<\underline{\sigma}_i(x),\end{cases}\tag{4.2}$$

and $J_i': W^{1,\mathcal{G}_i}(\Omega)\times\mathbb{R}^N\to[L^{p_i(\cdot)}(\Omega)]^N$

$$\tilde{J}_i(\sigma_i,\varphi_i)(x)=\begin{cases}\nabla\overline{\sigma}_i(x) & \text{if } \sigma_i(x)\ge\overline{\sigma}_i(x),\\ \varphi_i(x) & \text{if } \underline{\sigma}_i(x)<\sigma_i(x)<\overline{\sigma}_i(x),\\ \nabla\underline{\sigma}_i(x) & \text{if } \sigma_i(x)\le\underline{\sigma}_i(x).\end{cases}\tag{4.3}$$

For the convenience, we denote $J_i(\sigma_i,\varphi_i)(x)$ (*resp.* $\tilde{J}_i(\sigma_i,\varphi_i)(x)$) by $J_i\varphi_i$ (*resp.* $\tilde{J}_i\varphi_i$) in the sequel.

As before, we mention that $\overline{\eta}_1,\underline{\eta}_1\in L^{\tau_1(\cdot)}(\Omega)$ and $\overline{\zeta}_1,\underline{\zeta}_1\in L^{\theta_1(\cdot)}(\Gamma_1)$ are the functions given in Definitions 2.13 and 2.14 with $\underline{\sigma}=(\underline{\sigma}_1,\underline{\sigma}_2)$ and $\overline{\sigma}=(\overline{\sigma}_1,\overline{\sigma}_2)$ such that $\overline{\eta}_1\in f_1(x,\overline{\sigma}_1,T_2\sigma_2,\nabla\overline{\sigma}_1,\tilde{J}_2\varphi_2)$, $\underline{\eta}_1\in f_1(x,\underline{\sigma}_1,T_2\sigma_2,\nabla\underline{\sigma}_1,\tilde{J}_2\varphi_2)$ as well as $\overline{\zeta}_1\in f_{\Gamma_1}(x,\overline{\sigma}_1,T_2\sigma_2)$, $\underline{\zeta}_1\in f_{\Gamma_1}(x,\underline{\sigma}_1,T_2\sigma_2)$. Now, we construct the following truncation function $f_1^0:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\to 2^{\mathbb{R}}$:

$$f_1^0(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)=\begin{cases}\{\overline{\eta}_1\} & \text{if } \sigma_1>\overline{\sigma}_1,\\ \operatorname{conv}[f_1(x,\overline{\sigma}_1,T_2\sigma_2,\nabla\overline{\sigma}_1,\tilde{J}_2\varphi_2)\cup f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2) & \\ \qquad\cup f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)] & \text{if } \sigma_1=\overline{\sigma}_1,\\ \operatorname{conv}[f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)\cup f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)] & \text{if } \underline{\sigma}_1<\sigma_1<\overline{\sigma}_1,\\ \operatorname{conv}[f_1(x,\underline{\sigma}_1,T_2\sigma_2,\nabla\underline{\sigma}_1,\tilde{J}_2\varphi_2)\cup f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2) & \\ \qquad\cup f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)] & \text{if } \sigma_1=\underline{\sigma}_1,\\ \{\underline{\eta}_1\} & \text{if } \sigma_1<\underline{\sigma}_1,\end{cases}\tag{4.4}$$

where $\operatorname{conv}(S)$ is the convex hull of $S\subset\mathbb{R}$. and $f_1^{0\Gamma_1}:\Omega\times\mathbb{R}\times\mathbb{R}\to 2^{\mathbb{R}}$ as

$$f^0_{\Gamma_1}(x,\sigma_1,\sigma_2)=\begin{cases}\{\overline{\zeta}_1\} & \text{if } \sigma_1>\overline{\sigma}_1,\\ f_{\Gamma_1}(x,\sigma_1,T_2\sigma_2) & \text{if } \underline{\sigma}_1\le\sigma_1\le\overline{\sigma}_1,\\ \{\underline{\zeta}_1\} & \text{if } \sigma_1<\underline{\sigma}_1.\end{cases}\tag{4.5}$$

Likewise, we can define the truncation functions $f_2^0(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)$ and $f^0_{\Gamma_2}(x,\sigma_1,\sigma_2)$.

Next, we are going to verify that f_i^0 and $f^0_{\Gamma_i}$ satisfy hypotheses (H2). Using the same procedure of to the proof of [11, Theorem 7.13], one can show f_1^0 is graph measurable as well as $f_1^0(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)\in\mathcal{K}(\mathbb{R})$ for a.e. $x\in\Omega$ and for all $(\sigma_1,\sigma_2,\varphi_1,\varphi_2)\in\mathbb{R}^{2N+2}$. It remains to demonstrate that $f_1^0(x,\cdot,\cdot,\cdot,\cdot)$ is upper semicontinuous for all $(\sigma_1,\sigma_2,\varphi_1,\varphi_2)$ in $\mathbb{R}^{2N+2}$. Let $x\in\Omega$ be any point fulfilling $\overline{\eta}_1(x)\in f_1(x,\overline{\sigma}_1(x),T_2\sigma_2(x),\nabla\overline{\sigma}_1(x),\tilde{J}_2\varphi_2(x))$ and $\underline{\eta}_1(x)\in f_1(x,\underline{\sigma}_1(x),T_2\sigma_2(x),\nabla\underline{\sigma}_1(x),\tilde{J}_2\varphi_2(x))$. Suppose that $\mathcal{V}\subset\mathbb{R}$ is an open set being such that $f_1^0(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)\subset\mathcal{V}$. We need to find $\delta>0$ satisfying: for $(v_1,v_2,l_1,l_2)\in\mathbb{R}^{2N+2}$ with $|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta$, there holds $f_1^0(x,v_1,v_2,l_1,l_2)\subset\mathcal{V}$. By the upper semicontinuity of $f_1(x,\cdot,\cdot,\cdot,\cdot)$ we can select $\overline{\eta}_1$ to be the function given in Definition 2.13 with respect to supersolution $\overline{\sigma}=(\overline{\sigma}_1,\overline{\sigma}_2)$ satisfying $\overline{\eta}_1\in f_1(x,\overline{\sigma}_1,T_2\sigma_2,\nabla\overline{\sigma}_1,\tilde{J}_2\varphi_2)\cap$

$f_1(x,\overline{\sigma}_1,T_2v_2,\nabla\overline{\sigma}_1,\tilde{J}_2l_2)$ if $|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta$ with $\delta>0$ small enough. Now, we discuss the following several cases.

- If $\sigma_1>\overline{\sigma}_1$, then $f_1^0(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)=\overline{\eta}(x)\in\mathcal{V}$. Select $\delta\in(0,\sigma_1-\overline{\sigma}_1(x))$, if $|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta$ we see that $v_1>\overline{\sigma}_1$. Hence, $f_1^0(x,v_1,v_2,l_1,l_2)=\{\overline{\eta}(x)\}=f_1^0(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)\subset\mathcal{V}$. If $\sigma_1<\underline{\sigma}_1$, the proof is similar.
- When $\underline{\sigma}_1<\sigma_1<\overline{\sigma}_1$, we have $f_1^0(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)=\text{conv}[f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)\cup f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)]\subset\mathcal{V}$, so $f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)\subset\mathcal{V}$ and $f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)\subset\mathcal{V}$. Let $\delta_1=\min\{\sigma_1-\underline{\sigma}_1,\overline{\sigma}_1-\sigma_1\}>0$. If $|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta_1$, then we have $\underline{\sigma}_1<v_1<\overline{\sigma}_1$. Hence, $f_1^0(x,v_1,v_2,l_1,l_2)=\text{conv}[f_1(x,v_1,T_2v_2,l_1,\tilde{J}_2l_2)\cup f_1(x,v_1,T_2v_2,l_1,J_2l_2)]$. By the upper semicontinuity of $f_1(x,\cdot,\cdot,\cdot,\cdot)$, we can find δ_2 such that if $|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta_2$ and $f_1(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)\subset\mathcal{V}$, then it holds $f_1(x,v_1,v_2,l_1,l_2)\subset\mathcal{V}$. Under the situation that $\underline{\sigma}_1<v_1<\overline{\sigma}_1$, we consider some subcases:
  - (i) If the case $\sigma_2>\overline{\sigma}_2$ happens. Let $\delta_3\in(0,\sigma_2-\overline{\sigma}_2)$ and $\delta=\min\{\delta_1,\delta_2,\delta_3\}$, thus $v_2>\overline{\sigma}_2$. So, it yields $|(\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)-(v_1,T_2v_2,l_1,\tilde{J}_2l_2)|=|(\sigma_1,\overline{\sigma}_2,\varphi_1,\nabla\overline{\sigma}_2)-(v_1,\overline{\sigma}_2,l_1,\nabla\overline{\sigma}_2)|\le|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta\le\delta_2$. Similarly, it is valid $|(\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)-(v_1,T_2v_2,l_1,J_2l_2)|<\delta_2$. Taking the upper semicontinuity of $f_1(x,\cdot,\cdot,\cdot,\cdot)$ into account, we get $f_1^0(x,v_1,v_2,l_1,l_2)=\text{conv}[f_1(x,v_1,T_2v_2,l_1,\tilde{J}_2l_2)\cup f_1(x,v_1,T_2v_2,l_1,J_2l_2)]\subset\mathcal{V}$. When $\sigma_2>\underline{\sigma}_2$, the proof is similar.
  - (ii) Under the assumption $\sigma_2=\overline{\sigma}_2=\underline{\sigma}_2$, let $\delta=\min\{\delta_1,\delta_2\}$. We could show that $v_2=\overline{\sigma}_2$ or $v_2>\overline{\sigma}_2$. If $v_2=\overline{\sigma}_2$, then one has $|(\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)-(v_1,T_2v_2,l_1,\tilde{J}_2l_2)|=|(\sigma_1,\overline{\sigma}_2,\varphi_1,\nabla\overline{\sigma}_2)-(v_1,\overline{\sigma}_2,l_1,\nabla\overline{\sigma}_2)|\le|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta\le\delta_2$, and $|(\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)-(v_1,T_2v_2,l_1,J_2l_2)|=|(\sigma_1,\overline{\sigma}_2,\varphi_1,\varphi_2)-(v_1,\overline{\sigma}_2,l_1,l_2)|\le|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta_2$, thus $f_1^0(x,v_1,v_2,l_1,l_2)\subset\mathcal{V}$. When $v_2>\overline{\sigma}_2$, it implies $|(\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)-(v_1,T_2v_2,l_1,\tilde{J}_2l_2)|=|(\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)-(v_1,T_2v_2,l_1,J_2l_2)|=|(\sigma_1,\overline{\sigma}_2,\varphi_1,\nabla\overline{\sigma}_2)-(v_1,\overline{\sigma}_2,l_1,\nabla\overline{\sigma}_2)|\le|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta\le\delta_2$, which indicates that $f_1^0(x,v_1,v_2,l_1,l_2)\subset\mathcal{V}$.
  - (iii) Suppose that $\sigma_2=\overline{\sigma}_2>\underline{\sigma}_2$, let $\delta_4\in(0,\sigma_2-\underline{\sigma}_2)$. It could observe that it holds $v_2<\overline{\sigma}_2$, or $v_2=\overline{\sigma}_2$ or $v_2>\overline{\sigma}_2$. If $v_2<\overline{\sigma}_2$, let $\delta=\min\{\delta_1,\delta_2,\delta_4\}$, then $|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta$ implies $v_2>\underline{\sigma}_2$, i.e., $|(\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)-(v_1,T_2v_2,l_1,\tilde{J}_2l_2)|=|(\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)-(v_1,T_2v_2,l_1,J_2l_2)|=|(\sigma_1,\overline{\sigma}_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|=|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta\le\delta_2$. So, $f_1^0(x,v_1,v_2,l_1,l_2)\subset\mathcal{V}$. If $v_2=\overline{\sigma}_2$, from the proof of the case (ii), we can see that $|(\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)-(v_1,T_2v_2,l_1,\tilde{J}_2l_2)|<\delta_2$ and $|(\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)-(v_1,T_2v_2,l_1,J_2l_2)|<\delta_2$, so $f_1^0(x,v_1,v_2,l_1,l_2)\subset\mathcal{W}$. Likewise, if $v_2>\overline{\sigma}_2$, then $|(\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)-(v_1,T_2v_2,l_1,\tilde{J}_2l_2)|=|(\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)-(v_1,T_2v_2,l_1,J_2l_2)|<\delta_2$. Therefore, $f_1^0(x,v_1,v_2,l_1,l_2)\subset\mathcal{V}$. For the case $\sigma_2=\underline{\sigma}_2<\overline{\sigma}_2$, the proof is all most the same.
  - (iv) If $\underline{\sigma}_2<\sigma_2<\overline{\sigma}_2$ is true, then let $\delta=\min\{\delta_1,\delta_2,\overline{\sigma}_2-\sigma_2,\sigma_2-\underline{\sigma}_2\}$ and there hold $\underline{\sigma}_2<v_2<\overline{\sigma}_2$ and $|(\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)-(v_1,T_2v_2,l_1,\tilde{J}_2l_2)|=|(\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)-(v_1,T_2v_2,l_1,J_2l_2)|=|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta\le\delta_2$. This means $f_1^0(x,v_1,v_2,l_1,l_2)\subset\mathcal{V}$.
- Assume that $\sigma_1=\underline{\sigma}_1$, we know that $f_1^0(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)=\text{conv}[f_1(x,\overline{\sigma}_1,T_2\sigma_2,\nabla\overline{\sigma}_1,\tilde{J}_2\varphi_2)\cup f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)\cup f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)]$. There hold

$$f_1(x,\overline{\sigma}_1,T_2\sigma_2,\nabla\overline{\sigma}_1,\tilde{J}_2\varphi_2)\subset\mathcal{V}, \tag{4.6}$$

$$f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)\subset\mathcal{V}, \tag{4.7}$$

and

$$f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)\subset\mathcal{V}. \tag{4.8}$$

Then, we consider two subcases, that is, $\sigma_1=\overline{\sigma}_1(x)=\underline{\sigma}_1(x)$ or $\sigma_1=\overline{\sigma}_1(x)>\underline{\sigma}_1(x)$. On the one hand, if $\sigma_1=\overline{\sigma}_1(x)=\underline{\sigma}_1(x)$, by (4.6), (4.7) and (4.8) along with the upper semicontinuity of $f_1(x,\cdot,\cdot,\cdot,\cdot)$, employing the same method above one can find $\delta'>0$ such that

$|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta'$, namely, $|(x,\overline{\sigma}_1,T_2\sigma_2,\nabla\overline{\sigma}_1,\tilde{J}_2\varphi_2)-(x,\overline{\sigma}_1,T_2v_2,\nabla\overline{\sigma}_1,\tilde{J}_2l_2)|<\delta'$, $|(x,\sigma_1,T_2\sigma_2,\varphi_1,\tilde{J}_2\varphi_2)-(x,v_1,T_2v_2,l_1,\tilde{J}_2l_2)|<\delta'$ and $|(x,\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2)-(x,v_1,T_2v_2,l_1,J_2l_2)|<\delta'$. Hence, we have

$$f_1(x,\overline{\sigma}_1,T_2v_2,\nabla\overline{\sigma}_1,\tilde{J}_2l_2)\subset\mathcal{V}, \tag{4.9}$$

$$f_1(x,v_1,T_2v_2,l_1,\tilde{J}_2l_2)\subset\mathcal{V}, \tag{4.10}$$

and

$$f_1(x,v_1,T_2v_2,l_1,J_2l_2)\subset\mathcal{V}. \tag{4.11}$$

Let $0<\delta\le\delta'$ and $|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta$, if $v_1=\sigma_1=\overline{\sigma}_1(x)=\underline{\sigma}_1(x)$, we deduce $f_1^0(x,v_1,v_2,l_1,l_2)=\operatorname{conv}[f_1(x,\overline{\sigma}_1,T_2v_2,\nabla\overline{\sigma}_1,\tilde{J}_2l_2)\cup f_1(x,v_1,T_2v_2,l_1,\tilde{J}_2l_2)\cup f_1(x,v_1,T_2v_2,l_1,J_2l_2)]\subset\mathcal{V}$. If $v_1>\sigma_1=\overline{\sigma}_1(x)$, then $f_1^0(x,v_1,v_2,l_1,l_2)=\overline{\eta}_1\in f_1(x,\overline{\sigma}_1,T_2v_2,\nabla\overline{\sigma}_1,\tilde{J}_2l_2)\subset\mathcal{V}$. However, when $v_1<\sigma_1(=\underline{\sigma}_1)$, the desired conclusion can be obtained by using the same ways. On the other hand, if $\sigma_1=\overline{\sigma}_1(x)>\underline{\sigma}_1(x)$, let $\delta=\min\{\delta',\sigma_1-\underline{\sigma}_1\}>0$ and $|(\sigma_1,\sigma_2,\varphi_1,\varphi_2)-(v_1,v_2,l_1,l_2)|<\delta$, we see that $v_1>\underline{\sigma}_1$. In this situation, we can see that $v_1<\overline{\sigma}_1$, or $v_1=\overline{\sigma}_1$ or $v_1>\overline{\sigma}_1$. If $v_1<\overline{\sigma}_1$, then $f_1^0(x,v_1,v_2,l_1,l_2)=\operatorname{conv}[f_1(x,v_1,T_2v_2,l_1,\tilde{J}_2l_2)\cup f_1(x,v_1,T_2v_2,l_1,J_2l_2)]\subset\mathcal{V}$. If $v_1=\overline{\sigma}_1$, we use (4.9) and (4.10) to get $f_1^0(x,v_1,v_2,l_1,l_2)=\operatorname{conv}[f_1(x,\overline{\sigma}_1,T_2v_2,\nabla\overline{\sigma}_1,J_2l_2)\cup f_1(x,v_1,T_2v_2,l_1,\tilde{J}_2l_2)\cup f_1(x,v_1,T_2v_2,l_1,J_2l_2)]\subset\mathcal{V}$. When $v_1>\overline{\sigma}_1$, then $f_1^0(x,v_1,v_2,l_1,l_2)=\overline{\eta}_1\in f_1(x,\overline{\sigma}_1,T_2v_2,\nabla\overline{\sigma}_1,\tilde{J}_2l_2)\subset\mathcal{V}$. Under the condition $\sigma_1=\underline{\sigma}_1$, it could be proved directly by using the uppermicontinuity of f_1^0. Likewise, it could show the upper semicontinuity of $f_2^0,f_{\Gamma_1}^0$ and $f_{\Gamma_2}^0$.

Additionally, we discuss another truncation function $f_1^1:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\to2^{\mathbb{R}}$ which is defined by

$$f_1^1(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)=\begin{cases}\{\overline{\eta}_1\} & \text{if } \sigma_1>\overline{\sigma}_1,\\ f_1(x,\sigma_1,T_2\sigma_2,\varphi_1,J_2\varphi_2) & \text{if } \underline{\sigma}_1\le\sigma_1\le\overline{\sigma}_1,\\ \{\underline{\eta}_1\} & \text{if } \sigma_1<\underline{\sigma}_1.\end{cases} \tag{4.12}$$

Also, $f_2^1:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\to2^{\mathbb{R}}$ can be defined in a similar way. Despite f_i^1 is not upper semicontinuous, however, we could employ the similar arguments given in Theorem 7.13 of [11] to find that $f_i^0(x,\sigma_1(x),\sigma_2(x),\nabla\sigma_1(x),\nabla\sigma_2(x))=f_i^1(x,\sigma_1(x),\sigma_2(x),\nabla\sigma_1(x),\nabla\sigma_2(x))$ for a.a. $x\in\Omega$. So, we can replace $f_i^0(x,\sigma_1(x),\sigma_2(x),\nabla\sigma_1(x),\nabla\sigma_2(x))$ with $f_i^1(x,\sigma_1(x),\sigma_2(x),\nabla\sigma_1(x),\nabla\sigma_2(x))$ in the sequel, and vice-versa.

Invoking conditions (H3), we deduce that

$$\begin{aligned}&\sup\{|w|:w\in f_i^0(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)\}\\ \le&g_i^\Omega(x)+\kappa_i\left(|\varphi_1|^{\frac{p_1(x)}{\tau_i'(x)}}+|\varphi_2|^{\frac{p_2(x)}{\tau_i'(x)}}\right)+\kappa_i\sum_{k=1}^2\left(|\nabla\overline{\sigma}_k|^{\frac{p_k(x)}{\tau_i'(x)}}+|\nabla\underline{\sigma}_k|^{\frac{p_k(x)}{\tau_i'(x)}}\right)+|\overline{\eta}_i(x)|+|\underline{\eta}_i(x)|\\ \le&\gamma_i^\Omega(x)+C_{\kappa_i}\left(|\varphi_1|^{\frac{p_1(x)}{\tau_i'(x)}}+|\varphi_2|^{\frac{p_2(x)}{\tau_i'(x)}}\right)\end{aligned} \tag{4.13}$$

for a.a. $x\in\Omega$ and all $(\sigma_1,\sigma_2,\varphi_1,\varphi_2)\in\mathbb{R}^{2N+2}$, with $\gamma_i^\Omega(x)=g_i^\Omega(x)+\kappa_i\sum_{k=1}^2\left(|\nabla\overline{\sigma}_k|^{\frac{p_k(x)}{\tau_i'(x)}}+|\nabla\underline{\sigma}_k|^{\frac{p_k(x)}{\tau_i'(x)}}\right)+|\overline{\eta}_i(x)|+|\underline{\eta}_i(x)|$ which belongs to $L^{\tau_i'(\cdot)}(\Omega)$.

Analogously, there holds

$$\sup\{|w|:w\in f_{\Gamma_i}^0(x,\sigma_1,\sigma_2)\}\le g_i^{\Gamma_i}(x)+|\overline{\zeta}_i(x)|+|\underline{\zeta}_i(x)|=\gamma_i^{\Gamma_i}(x) \tag{4.14}$$

for a.a. $x\in\Gamma_i$ and for all $\sigma_i\in\mathbb{R}$, with $\gamma_i^{\Gamma_i}\in L^{\theta_i'(\cdot)}(\Gamma_i)$.

From above, we see that for $i=1,2$, f_i^0 and $f_{\Gamma_i}^0$ satisfy (H1) and (H2). Then, we define $\mathcal{F}_i^0(\sigma)=\mathcal{I}_{\tau_i(\cdot)}^*f_i^0(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)$, and $\mathcal{F}_{\Gamma_i}^0(\sigma)=\mathcal{I}_{\theta_i(\cdot)}^*f_{\Gamma_i}^0(x,\sigma_1,\sigma_2,\varphi_1,\varphi_2)\mathcal{I}_{\theta_i(\cdot)}$. By the proof of Proposition 3.1, we see that $\mathcal{F}_0=(\mathcal{F}_1^0,\mathcal{F}_2^0)$ and $\mathcal{F}_{0\Gamma}=(\mathcal{F}_{\Gamma_1}^0,\mathcal{F}_{\Gamma_2}^0):\mathcal{W}\to\mathcal{K}(\mathcal{W}^*)$ are bounded and pseudomonotone.

Moreover, for $i=1,2$ we give the truncation-regularization functions $b_i, d_i : \Omega\times\mathbb{R}\to\mathbb{R}$ formulated by

$$b_i(x,y)=\begin{cases}[y-\overline{\sigma}_i]^{q_i(x)-1}\log(e+\alpha|y|) & \text{if } y>\overline{\sigma}_i(x),\\ 0 & \text{if } \underline{\sigma}_i(x)\le y\le\overline{\sigma}_i(x),\\ -[\underline{\sigma}_i-y]^{q_i(x)-1}\log(e+\alpha|y|) & \text{if } y<\underline{\sigma}_i(x),\end{cases} \tag{4.15}$$

and

$$d_i(x,y)=\begin{cases}[y-\overline{\sigma}_i]^{\tau_i(x)-1} & \text{if } y>\overline{\sigma}_i(x),\\ 0 & \text{if } \underline{\sigma}_i(x)\le y\le\overline{\sigma}_i(x),\\ -[\underline{\sigma}_i-y]^{\tau_i(x)-1} & \text{if } y<\underline{\sigma}_i(x).\end{cases} \tag{4.16}$$

It is not hard to find a constant $c_{1,i}>0$ satisfying

$$|b_i(x,y)|\le c_{1,i}|y|^{q_i(x)-1}\log(e+\alpha|y|)+c_{1,i}\left(|\underline{\sigma}_i|^{q_i(x)-1}+|\overline{\sigma}_i|^{q_i(x)-1}\right)\log(e+\alpha|y|) \tag{4.17}$$

for a.a. $x\in\Omega$, all $y\in\mathbb{R}$. But, from Proposition 2.3, for $\mu_i(\cdot)\in L^\infty(\Omega)$ and $\varepsilon_i>0$ with $q_i(\cdot)+\varepsilon_i<p_i^*(\cdot)$ we see that, for $i=1,2$, $W^{1,\mathcal{G}_i}(\Omega)\hookrightarrow\hookrightarrow L^{q_i(\cdot)+\varepsilon_i}(\Omega)\hookrightarrow L^{\mathcal{G}_i}(\Omega)$. Set $\epsilon=\min\{\varepsilon_1,\varepsilon_2\}$, we can find $0<\eta_\epsilon<1$ small enough such that for all $y\in\mathbb{R}$ it holds that

$$\log(e+\alpha|y|)\le c_2+C_\alpha|y|^{\eta_\epsilon} \tag{4.18}$$

with some constant $c_2>0$. Moreover, by virtue of $\underline{\sigma}_i,\overline{\sigma}_i\in W^{1,\mathcal{G}_i}(\Omega)$ and $W^{1,\mathcal{G}_i}(\Omega)\hookrightarrow L^{\tau_i(\cdot)}(\Omega)$, we have

$$|d_i(x,y)|\le\omega_i(x)+c_{3,i}|y|^{\tau_i(x)-1}, \tag{4.19}$$

for a.a. $x\in\Omega$, all $y\in\mathbb{R}$, with $c_{3,i}>0$, $\omega_i\in L^{\tau_i'(\cdot)}(\Omega)$. Furthermore, there holds

$$\int_\Omega|\sigma_i|^{q_i(x)}\log(e+\alpha|\sigma_i|)dx\ge\int_\Omega|\sigma_i|^{p_i(x)}\log(e+\alpha|\sigma_i|)dx-C_\alpha|\Omega| \tag{4.20}$$

for all $\sigma_i\in L^{q_i(\cdot)}(\Omega)$. Therefore, we define the mapping $\mathcal{B}_i:\ L^{q_i(\cdot)+\varepsilon_i}(\Omega)\to L^{(q_i(\cdot)+\varepsilon_i)'}(\Omega)$ as

$$\langle\mathcal{B}_i(\sigma_i),v_i\rangle=\int_\Omega b_i(x,\sigma_i)v_i\,dx \text{ for all } \sigma_i,v_i\in L^{q_i(\cdot)+\varepsilon_i}(\Omega), \tag{4.21}$$

which is continuous and bounded. Along with the fact that $\mathcal{I}_{q_i(\cdot)}:W^{1,\mathcal{G}_i}(\Omega)\hookrightarrow L^{q_i(\cdot)+\varepsilon_i}(\Omega)$ is compact, we see that $\mathcal{I}^*_{q_i(\cdot)}\mathcal{B}_i\mathcal{I}_{q_i(\cdot)}:W^{1,\mathcal{G}_i}(\Omega)\to W^{1,\mathcal{G}_i}(\Omega)^*$ turns out to be bounded and completely continuous, which indicates the pseudomonotonicity and boundedness.

Similarly, we can define $\mathcal{D}_i:L^{\tau_i(\cdot)}(\Omega)\to L^{\tau_i'(\cdot)}(\Omega)$ as

$$\langle\mathcal{D}_i,v_i\rangle=\int_\Omega d_i(x,\sigma_i)v_i\,dx \text{ for all } \sigma_i,v_i\in L^{\tau_i(\cdot)}(\Omega), \tag{4.22}$$

which is also continuous and bounded. Analogously, since $\mathcal{I}_{\tau_i(\cdot)}:W^{1,\mathcal{G}_i}(\Omega)\hookrightarrow L^{\tau_i(\cdot)}(\Omega)$ is compact, we obtain the boundedness and pseudomonotonicity for $\mathcal{I}^*_{\tau_i(\cdot)}\mathcal{D}_i\mathcal{I}_{\tau_i(\cdot)}:W^{1,\mathcal{G}_i}(\Omega)\to W^{1,\mathcal{G}_i}(\Omega)^*$.

Invoking the same method as in the proof of [36, Theorem 3.4], for $z_i\in W^{1,\mathcal{G}_i}$ fixed, we have

$$\begin{aligned}\langle\mathcal{I}^*_{q_i(\cdot)}\mathcal{B}_i\mathcal{I}_{q_i(\cdot)}(\sigma_i),\sigma_i-z_i\rangle=&\langle\mathcal{B}_i(\sigma_i),\sigma_i-z_i\rangle_{L^{(q_i(\cdot)+\varepsilon_i)'}(\Omega),L^{q_i(\cdot)+\varepsilon_i}(\Omega)}\\ =&\int_\Omega b_i(x,\sigma_i)(\sigma_i-z_i)dx\\ \ge&a_{1,i}\int_\Omega\mathcal{G}_i(x,|\sigma_i|)dx-C,\ \text{ for all } \sigma_i\in W^{1,\mathcal{G}_i}(\Omega),\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}\langle \mathcal{I}^*_{\tau_i(\cdot)}\mathcal{D}_i\mathcal{I}_{\tau_i(\cdot)}(\sigma_i), \sigma_i - z_i\rangle =&\langle \mathcal{D}_i(\sigma_i), \sigma_i\rangle_{L^{\tau_i'(\cdot)}(\Omega), L^{\tau_i(\cdot)}(\Omega)}\\ =&\int_\Omega d_i(x,\sigma_i)(\sigma_i - z_i)dx\\ \geq& a_{2,i}\int_\Omega |\sigma_i|^{\tau_i(\cdot)}dx - C, \text{ for all } \sigma_i \in W^{1,\mathcal{G}_i}(\Omega).\end{aligned} \tag{4.24}$$

Define $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_i > 0 (i = 1, 2)$, then let $\mathcal{B}(\sigma) = \left(\mathcal{I}^*_{q_1(\cdot)}\mathcal{B}_1\mathcal{I}_{q_1(\cdot)}(\sigma_1), \mathcal{I}^*_{q_2(\cdot)}\mathcal{B}_2\mathcal{I}_{q_2(\cdot)}(\sigma_2)\right)$ and $\lambda\mathcal{D}(\sigma) = \left(\lambda_1\mathcal{I}^*_{\tau_1(\cdot)}\mathcal{D}_1\mathcal{I}_{\tau_1(\cdot)}(\sigma_1), \lambda_2\mathcal{I}^*_{\tau_2(\cdot)}\mathcal{D}_2\mathcal{I}_{\tau_2(\cdot)}(\sigma_2)\right)$. We set $A(\sigma) = (A_1(\sigma_1), A_2(\sigma_2))$ with A_i being given by (2.3). Applying Proposition 2.6, we see that $A : \mathcal{W} \to \mathcal{W}^*$ is continuous, bounded, strictly monotone and of type (S_+). Moreover, the following representations will be used in the sequel

$$\begin{aligned}\langle A(\sigma), v\rangle_{\mathcal{W}} &= \sum_{i=1}^2 \int_\Omega \frac{\mathcal{G}_i'(x, |\nabla\sigma_i|)}{|\nabla\sigma_i|}\nabla\sigma_i \cdot \nabla v_i dx,\\ \langle \mathcal{B}(\sigma), v\rangle_{\mathcal{W}} &= \sum_{i=1}^2 \int_\Omega b_i(x,\sigma_i)v_i dx,\\ \langle \mathcal{D}(\sigma), v\rangle_{\mathcal{W}} &= \sum_{i=1}^2 \int_\Omega d_i(x,\sigma_i)v_i dx.\end{aligned} \tag{4.25}$$

In what follows, set $\eta(x) = (\eta_1(x), \eta_2(x))$ as well as $\zeta(x) = (\zeta_1(x), \zeta_2(x))$. Next, for $i = 1, 2$, let us focus on the auxiliary variational inequality: find $\sigma = (\sigma_1, \sigma_2) \in K = (K_1, K_2)$, $\eta_i \in L^{\tau_i'(\cdot)}(\Omega), \zeta_i \in L^{\theta_i'(\cdot)}(\Gamma_i)$ such that

$$\begin{cases}\eta_i(x) \in f_i^0(x, \sigma_1(x), \sigma_2(x), \nabla\sigma_1(x), \nabla\sigma_2(x)) \text{ for a.a. } x \in \Omega\\ \zeta_i(x) \in f_{\Gamma_i}^0(x, \sigma_1(x), \sigma_2(x), \nabla\sigma_1(x), \nabla\sigma_2(x)) \text{ for a.a. } x \in \Gamma_i,\end{cases} \tag{4.26}$$

and

$$\begin{aligned}&\langle A\sigma, v-\sigma\rangle + \sum_{i=1}^2\int_\Omega \eta_i(v_i - \sigma_i)\mathrm{d}x + \sum_{i=1}^2\int_{\Gamma_i}\zeta_i(v_i-\sigma_i)\mathrm{d}\sigma + \langle \mathcal{B}(\sigma), v\rangle + \langle \lambda\mathcal{D}(\sigma), v\rangle\\ &= \sum_{i=1}^2 \int_\Omega \frac{\mathcal{G}_i'(x, |\nabla\sigma_i|)}{|\nabla\sigma_i|}\nabla\sigma_i \cdot \nabla v_i dx + \sum_{i=1}^2\int_\Omega \eta_i(v_i - \sigma_i)\mathrm{d}x + \sum_{i=1}^2\int_{\Gamma_i}\zeta_i(v_i-\sigma_i)\mathrm{d}\varsigma\\ &+ \sum_{i=1}^2\int_\Omega b_i(x,\sigma_i)v_i dx + \sum_{i=1}^2 \lambda_i\int_\Omega d_i(x,\sigma_i)v_i dx\\ &\geq 0 \quad \text{for all } (v_1, v_2) \in K.\end{aligned} \tag{4.27}$$

The above inequality equals to: find $\sigma \in K$ satisfying

$$\left\langle A\sigma + \tilde{\eta} + \tilde{\zeta} + \mathcal{B}(\sigma) + \lambda\mathcal{D}(\sigma), v - \sigma\right\rangle \geq 0 \quad \text{for all } v \in K,$$

in which $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2)$ with $\tilde{\eta}_i = \mathcal{I}^*_{\tau_i(\cdot)}\eta_i \in \mathcal{F}_i^0(\sigma) = \left[\mathcal{I}^*_{\tau_i(\cdot)}\tilde{f}_i^0\right](\sigma_1, \sigma_2, \nabla\sigma_1, \nabla\sigma_2)$ and $\tilde{\zeta} = (\tilde{\zeta}_1, \tilde{\zeta}_2)$ with $\tilde{\zeta}_i = \mathcal{I}^*_{\theta_i(\cdot)}\zeta_i\mathcal{I}_{\theta_i(\cdot)} \in \mathcal{F}^0_{\Gamma_i}(\sigma) = \left[\mathcal{I}^*_{\theta_i(\cdot)}\tilde{f}^0_{\Gamma_i}\mathcal{I}_{\theta_i(\cdot)}\right](\sigma_1, \sigma_2)$.

It is clear that solving the variational inequality (4.27) is equivalent to find $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_i \in D\left(\partial I_{K_i}\right)$, $\xi = (\xi_1, \xi_2)$ with $\xi_i \in \partial I_{K_i}(\sigma_i)$, and

$$\tilde{\eta}_i = \mathcal{I}^*_{\tau_i(\cdot)}\eta_i \in \mathcal{F}_i^0(\sigma), \quad \tilde{\zeta}_i = \mathcal{I}^*_{\theta_i(\cdot)}\zeta_i\mathcal{I}_{\theta_i(\cdot)} \in \mathcal{F}^0_{\Gamma_i}(\sigma)$$

fulfilling

$$\mathcal{A}(\sigma, \xi, \tilde{\eta}, \tilde{\zeta}) := A\sigma + \xi + \tilde{\eta} + \tilde{\zeta} + \mathcal{B}(\sigma) + \lambda\mathcal{D}(\sigma) = 0 \quad \text{in } \mathcal{W}^*. \tag{4.28}$$

For each $i=1,2$, ∂I_{K_i} is maximal monotone. Hence, $\partial I_K=(\partial I_{K_1},\partial I_{K_2})$ is maximal monotone and

$$A+\mathcal{F}_0+\mathcal{F}_{0\Gamma}+\mathcal{B}+\lambda\mathcal{D}:\mathcal{W}\to 2^{\mathcal{W}(\Omega)^*}$$

is bounded and pseudomonotone. By using Corollary 3.3, we can establish the existence result under the following coercive framework: there exists $z=(z_1,z_2)\in K$ such that

$$\lim_{\substack{\|\sigma\|_{\mathcal{W}}\to\infty\\ \sigma\in K}}\left[\inf_{\substack{\xi\in\partial I_K(\sigma)\\ \tilde{\eta}\in\mathcal{F}_0(\sigma)\\ \tilde{\zeta}\in\mathcal{F}_{0\Gamma}(\sigma)}}\left\langle\mathcal{A}\left(\sigma,\xi,\tilde{\eta},\tilde{\zeta}\right),\sigma-z\right\rangle\right]=\infty. \tag{4.29}$$

For any fixed $z\in K$ and all $\sigma\in K$, there holds $\langle\xi,\sigma-z\rangle=\langle\xi_1,\sigma_1-z_1\rangle+\langle\xi_2,\sigma_2-z_2\rangle\geq 0$. Since for any $\xi_i\in(\partial I_{K_i})(\sigma_i)$ we have $0=I_{K_i}(z_i)-I_{K_i}(\sigma_i)\geq\langle\xi_i,z_i-\sigma_i\rangle$. This indicates that (4.29) is valid. So, we only need to demonstrate that as $\|\sigma\|_{\mathcal{W}}\to\infty$ with $\sigma\in K$ there holds

$$\inf_{\substack{\tilde{\eta}\in\mathcal{F}_0(\sigma)\\ \tilde{\zeta}\in\mathcal{F}_{0\Gamma}(\sigma)}}\left\langle\hat{\mathcal{A}}\left(\sigma,\tilde{\eta},\tilde{\zeta}\right),\sigma-z\right\rangle\to\infty, \tag{4.30}$$

in which

$$\hat{\mathcal{A}}(\sigma,\tilde{\eta},\tilde{\zeta}):=A\sigma+\tilde{\eta}+\tilde{\zeta}+\mathcal{B}(\sigma)+\lambda\mathcal{D}(\sigma),$$

and $\tilde{\eta}=(\tilde{\eta}_1,\tilde{\eta}_2)(resp.\ \tilde{\zeta}=(\tilde{\zeta}_1,\tilde{\zeta}_2))$, recalling that $\tilde{\eta}_i=\mathcal{I}^*_{\tau_i(\cdot)}\eta_i\in\mathcal{F}^0_i(\sigma)=\left[\mathcal{I}^*_{\tau_i(\cdot)}\tilde{f}^0_i\right](\sigma_1,\sigma_2,\nabla\sigma_1\nabla\sigma_2)(resp.\ \tilde{\zeta}_i=\mathcal{I}^*_{\theta_i(\cdot)}\zeta\mathcal{I}_{\theta_i(\cdot)}\in\mathcal{F}^0_{\Gamma_i}(\sigma)=\left[\mathcal{I}^*_{\theta_i(\cdot)}\tilde{f}^0_{\Gamma_i}\mathcal{I}_{\theta_i(\cdot)}\right](\sigma_1,\sigma_2))$ with $\eta_i\in\tilde{f}^0_i(\sigma_1,\sigma_2)(resp.\ \zeta_i\in\tilde{f}^0_{\Gamma_i}(\sigma_1,\sigma_2))$.

Using (4.13), we infer that for all $\eta_i(x)\in f^0_i(x,\sigma_1,\sigma_2,\nabla\sigma_1,\nabla\sigma_2)$, there holds $|\eta_i|\leq\gamma^\Omega_i(x)+C_{\kappa_i}\left(|\nabla\sigma_1|^{\frac{p_1(x)}{\tau_i'(x)}}+|\nabla\sigma_2|^{\frac{p_2(x)}{\tau_i'(x)}}\right)\in L^{\tau_i'(\cdot)}(\Omega)$. This derives from (4.14) that

$$\begin{aligned}
|\langle\tilde{\eta},\sigma-z\rangle|=&\sum_{i=1}^2\left|\int_\Omega\eta_i(\sigma_i-z_i)dx\right|\\
\leq&\int_\Omega\left(\gamma^\Omega_1(x)+C_{\kappa_1}|\nabla\sigma_1|^{\frac{p_1(x)}{\tau_1'(x)}}+C_{\kappa_1}|\nabla\sigma_2|^{\frac{p_2(x)}{\tau_1'(x)}}\right)(|\sigma_1|+|z_1|)\,dx\\
&+\int_\Omega\left(\gamma^\Omega_2(x)+C_{\kappa_2}|\nabla\sigma_1|^{\frac{p_1(x)}{\tau_2'(x)}}+C_{\kappa_2}|\nabla\sigma_2|^{\frac{p_2(x)}{\tau_2'(x)}}\right)(|\sigma_2|+|z_2|)\,dx\\
\leq&2\sum_{i=1}^2\int_\Omega\varepsilon|\nabla\sigma_i|^{p_i(x)}dx+2\sum_{i=1}^2C(\varepsilon,\kappa_i,\tau_i')\int_\Omega|\sigma_i|^{\tau_i(x)}+|z_i|^{\tau_i(x)}dx\\
&+\sum_{i=1}^2\left(\int_\Omega|\sigma_i|^{\tau_i(x)}dx+C(1,\tau_i)\int_\Omega\left[\gamma^\Omega_i(x)\right]^{\tau_i'(x)}dx+\int_\Omega\gamma^\Omega_i(x)|z_i|dx\right)\\
\leq&\sum_{i=1}^2\left(\int_\Omega 2\varepsilon|\nabla\sigma_i|^{p_i(x)}+(2C(\varepsilon,\kappa_i,\tau_i')+1)|\sigma_i|^{\tau_i(x)}dx+C_i\right)
\end{aligned} \tag{4.31}$$

and

$$
\begin{aligned}
\left|\left\langle\tilde{\zeta}, \sigma-z\right\rangle\right| &= \sum_{i=1}^{2}\left|\int_{\Gamma_i} \zeta_i(\sigma_i - z_i) d\varsigma\right| \\
&\leq \sum_{i=1}^{2}\left(\left\|\gamma_i^{\Gamma_i}\right\|_{\theta_i'(\cdot),\Gamma_i}\left(\|\sigma_i\|_{\theta_i(\cdot),\Gamma_i} + \|z_i\|_{\theta_i(\cdot),\Gamma_i}\right)\right) \\
&\leq \sum_{i=1}^{2}\left(C_i\left(\|\sigma_i\|_{\theta_i(\cdot),\Gamma_i} + 1\right)\right) \\
&\leq C(\|\sigma\|_{\mathcal{W}} + 1).
\end{aligned} \tag{4.32}
$$

Recall that for $i=1,2$, the potential functional with respect to A_i defined by

$$
I_i(\sigma_i) = \int_\Omega \left[|\nabla\sigma_i|^{p_i(x)} + \mu(x)|\nabla\sigma_i|^{q_i(x)}\right]\log(e+\alpha|\nabla\sigma_i|)\mathrm{d}x = \int_\Omega \mathcal{G}_i(x,|\nabla\sigma_i|)dx,
$$

is convex and satisfies

$$
\langle A_i(\sigma_i), \sigma_i - z_i\rangle \geq I_i(\sigma_i) - I_i(z_i) = I_i(\sigma_i) - C. \tag{4.33}
$$

Taking (4.31)–(4.33) as well as inequality (4.23) into account and selecting $\lambda_i > 0$ such that $\lambda_i > a_{2,i}^{-1}\left[2C(\varepsilon,\kappa_i,\tau_i') + 1\right]$, we know that

$$
\begin{aligned}
&\left\langle A\sigma + \tilde{\eta} + \tilde{\zeta} + \mathcal{B}(\sigma) + \lambda\mathcal{D}(\sigma), \sigma - z\right\rangle \\
\geq& \min\{1-2\varepsilon, a_{1,1}, a_{1,2}\}\sum_{i=1}^{2}\int_\Omega \Big[(|\nabla\sigma_i|^{p_i(x)} + \mu(x)|\nabla\sigma_i|^{q_i(x)})\log(e+\alpha|\nabla\sigma_i|) \\
&+ (|\sigma_i|^{p_i(x)} + \mu(x)|\sigma_i|^{q_i(x)})\log(e+\alpha|\sigma_i|)\Big]dx - C(\|\sigma\|_{\mathcal{W}} + 1) \\
=& \min\{1-2\varepsilon, a_{1,1}, a_{1,2}\}\sum_{i=1}^{2}\int_\Omega \mathcal{G}_i(x,|\nabla\sigma_i|) + \mathcal{G}_i(x,|\sigma_i|)dx - C(\|\sigma\|_{\mathcal{W}} + 1)
\end{aligned} \tag{4.34}
$$

for any $\sigma \in K$, $\tilde{\eta} \in \mathcal{F}_0(\sigma)$ and $\tilde{\zeta} \in \mathcal{F}_{0\Gamma}(\sigma)$. Furthermore, for $i=1,2$, we introduce operators $J_i : W^{1,\mathcal{G}_i}(\Omega) \to W^{1,\mathcal{G}_i}(\Omega)^*$:

$$
\langle J_i(\sigma_i), v_i\rangle = \int_\Omega \frac{\mathcal{G}_i'(x,|\nabla\sigma_i|)}{|\nabla\sigma_i|}\nabla\sigma_i\cdot\nabla v_i dx + \int_\Omega \frac{\mathcal{G}_i'(x,|\sigma_i|)}{|\sigma_i|}\sigma_i\cdot v_i dx, \tag{4.35}
$$

for all $\sigma_i, v_i \in W^{1,\mathcal{G}_i}(\Omega)$. Referring to [54, Proposition 3.5], we know that J_i is coercive, that is,

$$
\lim_{\|\sigma_i\|_{1,\mathcal{G}_i}\to\infty} \frac{1}{\|\sigma_i\|_{1,\mathcal{G}_i}}\int_\Omega \mathcal{G}_i(x,|\nabla\sigma_i|) + \mathcal{G}_i(x,|\sigma_i|)dx = \infty.
$$

This associating (4.34) with $\varepsilon < \frac{1}{2}$ derives (4.30). Hence, using Corollary 3.3, we can find σ, η and ζ satisfying (4.26) and (4.27).

Next, we verify that for each $i=1,2$ it holds

$$
\underline{\sigma}_i \leq \sigma \leq \overline{\sigma}_i \quad \text{a.a. in } \Omega. \tag{4.36}
$$

We, first, show that for any $i=1,2$ it holds $\sigma_i \leq \overline{\sigma}_i$. Take $v = \overline{\sigma}\wedge\sigma := \sigma - (\sigma-\overline{\sigma})^+ = \langle \sigma_1 - (\sigma_1 - \overline{\sigma}_1)^+, \sigma_2 - (\sigma_2 - \overline{\sigma}_2)^+\rangle \in K$ into (4.27), it yields

$$
\begin{aligned}
&\sum_{i=1}^{2}\Big[-\left\langle A_i(\sigma_i), (\sigma_i - \overline{\sigma}_i)^+\right\rangle - \int_\Omega \eta_i(\sigma_i - \overline{\sigma}_i)^+\mathrm{d}x - \int_{\Gamma_i}\zeta_i(\sigma_i - \overline{\sigma}_i)^+\mathrm{d}\varsigma \\
&- \int_\Omega [b_i(x,\sigma_i) + \lambda_i d_i(x,\sigma_i)](\sigma_i - \overline{\sigma}_i)^+\mathrm{d}x\Big] \geq 0.
\end{aligned} \tag{4.37}
$$

Then, there exist functions $\overline{\eta}_i \in L^{\tau_i'(\cdot)}(\Omega)$ and $\overline{\zeta}_i \in L^{\theta_i'(\cdot)}(\Gamma_i)$ fulfilling assumptions (i)-(iii) of Definition 2.14. Taking $v = \overline{\sigma} + (\sigma - \overline{\sigma})^+ = \overline{\sigma} \vee \sigma \in \overline{\sigma} \vee K$ into (iii) of Definition 2.14, we obtain

$$\sum_{i=1}^{2} \left[\left\langle A_i\overline{\sigma}_i, (\sigma_i - \overline{\sigma}_i)^+ \right\rangle + \int_\Omega \overline{\eta}_i \,(\sigma_i - \overline{\sigma}_i)^+ \,\mathrm{d}x + \int_{\Gamma_i} \overline{\zeta}_i \,(\sigma_i - \overline{\sigma}_i)^+ \,\mathrm{d}\varsigma \right] \geq 0. \tag{4.38}$$

Adding inequalities (4.37) and (4.38), one has

$$\begin{aligned}\sum_{i=1}^{2} \Bigg[&\left\langle -A_i\sigma_i + A_i\overline{\sigma}_i, (\sigma_i - \overline{\sigma}_i)^+ \right\rangle - \int_\Omega (\eta_i - \overline{\eta}_i)\,(\sigma_i - \overline{\sigma}_i)^+ \,\mathrm{d}x - \int_{\Gamma_i} \left(\zeta_i - \overline{\zeta}_i\right)(\sigma_i - \overline{\sigma}_i)^+ \,\mathrm{d}\varsigma \\ &- \int_\Omega [b_i(x,\sigma_i) + \lambda_i d_i(x,\sigma_i)]\,(\sigma_i - \overline{\sigma}_i)^+ \,\mathrm{d}x \Bigg] \geq 0.\end{aligned}$$

Since for every $i = 1, 2$, A_i is strictly monotone, then

$$\begin{aligned}&\left\langle A_i\sigma_i - A_i\overline{\sigma}_i, (\sigma_i - \overline{\sigma}_i)^+ \right\rangle \\ =& \int_{\{x\in\Omega\,:\,\sigma_i(x)\geq\overline{\sigma}_i(x)\}} \left(\frac{\mathcal{G}_i'(x,|\nabla\sigma_i|)}{|\nabla\sigma_i|}\nabla\sigma_i - \frac{\mathcal{G}_i'(x,|\nabla\overline{\sigma}_i|)}{|\nabla\overline{\sigma}_i|}\nabla\overline{\sigma}_i \right)\cdot\nabla\left(\sigma_i - \overline{\sigma}_i\right)\mathrm{d}x \geq 0.\end{aligned}$$

Note that

$$\begin{aligned}&\int_\Omega (\eta_i - \overline{\eta}_i)\,(\sigma_i - \overline{\sigma}_i)^+ \,\mathrm{d}x \\ =& \int_{\{x\in\Omega\,:\,\sigma_i(x)\geq\overline{\sigma}_i(x)\}} (\eta_i - \overline{\eta}_i)\,(\sigma_i - \overline{\sigma}_i)^+ \,\mathrm{d}x \\ =& \int_{\{x\in\Omega\,:\,\sigma_i(x)\geq\overline{\sigma}_i(x)\}} (\overline{\eta}_i - \overline{\eta}_i)\,(\sigma_i - \overline{\sigma}_i)^+ \,\mathrm{d}x \\ =&0,\end{aligned}$$

we infer

$$\begin{aligned}&\int_{\Gamma_i} \left(\zeta_i - \overline{\zeta}_i\right)(\sigma_i - \overline{\sigma}_i)^+ \,\mathrm{d}\varsigma \\ =& \int_{\{x\in\Omega\,:\,\sigma_i(x)\geq\overline{\sigma}_i(x)\}} \left(\zeta_i - \overline{\zeta}_i\right)(\sigma_i - \overline{\sigma}_i)^+ \,\mathrm{d}\varsigma \\ =& \int_{\{x\in\Omega\,:\,\sigma_i(x)\geq\overline{\sigma}_i(x)\}} \left(\overline{\zeta}_i - \overline{\zeta}_i\right)(\sigma_i - \overline{\sigma}_i)^+ \,\mathrm{d}\varsigma \\ =&0.\end{aligned}$$

The inequalities above say that

$$\begin{aligned}0 \leq& -\sum_{i=1}^{2} \int_\Omega [b_i(x,\sigma_i) + \lambda_i d_i(x,\sigma_i)]\,(\sigma_i - \overline{\sigma}_i)^+ \,\mathrm{d}x \\ =& -\sum_{i=1}^{2} \int_{\{x\in\Omega\,:\,\sigma_i(x)>\overline{\sigma}_i(x)\}} [b_i(x,\sigma_i) + \lambda_i d_i(x,\sigma_i)]\,(\sigma_i - \overline{\sigma}_i)\,\mathrm{d}x.\end{aligned}$$

By (4.15) and (4.16), if $\sigma(x) > \overline{\sigma}_i(x)$, then it derives

$$b_i(x,\sigma_i(x)) = [\sigma_i(x) - \overline{\sigma}_i(x)]^{q_i(x)-1}\log\left(e + \alpha|\sigma_i(x)|\right),$$

and

$$d_i(x,\sigma_i(x)) = [\sigma_i(x) - \overline{\sigma}_i(x)]^{\tau_i(x)-1}.$$

Hence

$$
\begin{aligned}
0 \le &-\sum_{i=1}^{2} \int_{\{x\in\Omega\,:\,\sigma_i(x)>\overline{\sigma}_i(x)\}} |\sigma_i(x)-\overline{\sigma}_i(x)|^{q_i(x)-1}\,[\sigma_i(x)-\overline{\sigma}_i(x)]\log(e+\alpha|\sigma_i(x)|)\,\mathrm{d}x \\
&-\sum_{i=1}^{2} \lambda_i \int_{\{x\in\Omega\,:\,\sigma_i(x)>\overline{\sigma}_i(x)\}} |\sigma_i(x)-\overline{\sigma}_i(x)|^{\tau_i(x)-1}\,[\sigma_i(x)-\overline{\sigma}_i(x)]\,dx.
\end{aligned} \tag{4.39}
$$

But, if $\sigma_i(x) > \overline{\sigma}_i(x)$, we have $\sigma_i(x)-\overline{\sigma}_i(x) > 0$, also, $\log(e+\alpha|\sigma_i(x)|) > 0$. Recalling the fact $\lambda_i > 0$, we can see that inequality (4.39) indicates that the measure of $\{x \in \Omega : \sigma_i(x) > \overline{\sigma}_i(x)\}$ equals 0, hence $\sigma_i(x) \le \overline{\sigma}_i(x)$ for a.a. $x \in \Omega$. Analogously, the left side of inequality (4.36) holds true.

Inequalities (4.36) indicates $\underline{\sigma}_i \le \sigma_i \le \overline{\sigma}_i$ a.a. in $\Omega\cup\Gamma_i$, $b_i(\cdot,\sigma_i) = 0$ in Ω, $f_i^0(x,\sigma_1(x),\sigma_2(x),\nabla\sigma_1,\nabla\sigma_2) = f_i(x,\sigma_1(x),\sigma_2(x),\nabla\sigma_1,\nabla\sigma_2)$ for a.a. $x \in \Omega$, and $f^0_{\Gamma_i}(x,\sigma_1(x),\sigma_2(x)) = f_{\Gamma_i}(x,\sigma_1(x),\sigma_2(x))$ for a.a. $x \in \Gamma_i$. Thus, σ is proved to be a weak solution to (1.1). □

Moreover, invoking the same arguments as in the proof of [12, Theorem 1.9] we get the compactness of solution set of (1.1).

Theorem 4.2. *If (H0), (H1) and (H3) hold. Denote by $\mathcal{S}$ the solution set of (1.1) between $\underline{\sigma}$ and $\overline{\sigma}$, then*

(i) *$\mathcal{S} \subset [\underline{\sigma},\overline{\sigma}]$ is compact in $\mathcal{W}$.*
(ii) *if the lattice conditions hold true*

$$
\mathcal{S} \wedge K \subset K \quad \text{and} \quad \mathcal{S} \vee K \subset K, \tag{4.40}
$$

then each $\sigma \in \mathcal{S}$ is both a subsolution and a supersolution of (1.1).

5. A further discussion for Special case 1.1

This section is aimed at establishing the existence of a weak solution for Special cases 1.1 to 1.4 by constructing suitable sub-supersolutions. We only show the existence result for Special case 1.1, since the corresponding results for Special cases 1.2 to 1.4 could be proved by the similar ways.

If (H1) and (H2) hold true, we can reformulate multivalued functions f_i and f_{Γ_i} as

$$
f_i(x,y_1,y_2,\varphi_1,\varphi_2) = [f_{ia}(x,y_1,y_2,\varphi_1,\varphi_2), f_{ib}(x,y_1,y_2,\varphi_1,\varphi_2)] \tag{5.1}
$$

for all $(x,y_1,y_2,\varphi_1,\varphi_2) \in \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$, and

$$
f_{\Gamma_i}(x,y_1,y_2) = [f_{\Gamma_i,a}(x,y_1,y_2), f_{\Gamma_i,b}(x,y_1,y_2)] \tag{5.2}
$$

for all $(x,y_1,y_2) \in \Gamma_i\times\mathbb{R}\times\mathbb{R}$, where for each $i = 1,2$ functions $f_{ia}(x,y_1,y_2,\varphi_1,\varphi_2)$, $f_{ib}(x,y_1,y_2,\varphi_1,\varphi_2)$, $f_{\Gamma_i,a}(x,y_1,y_2)$, $f_{\Gamma_i,b}(x,y_1,y_2)$ are single-valued functions. Thanks for (H1), if $x \mapsto \sigma(x)$ and $x \mapsto \nabla\sigma(x)$ are measurable, we see that for each $i = 1,2$, $x \mapsto f_{ia}(x,\sigma_1(x),\sigma_2(x),\nabla\sigma_1(x),\nabla\sigma_2(x))$, $x \mapsto f_{ib}(x,\sigma_1(x),\sigma_2(x),\nabla\sigma_1(x),\nabla\sigma_2(x))$, $x \mapsto f_{\Gamma_i,a}(x,\sigma_1(x),\sigma_2(x))$, $x \mapsto f_{\Gamma_i,b}(x,\sigma_1(x),\sigma_2(x))$, $i = 1,2$ are measurable. Also, $(y_1,y_2,\varphi_1,\varphi_2) \mapsto f_{ia}(y_1,y_2,\varphi_1,\varphi_2)$, $(y_1,y_2) \mapsto f_{\Gamma_i,a}(x,y_1,y_2)$ are lower semicontinuous, $(y_1,y_2,\varphi_1,\varphi_2) \mapsto f_{ib}(x,y_1,y_2,\varphi_1,\varphi_2)$, $(y_1,y_2) \mapsto f_{\Gamma_i,b}(x,y_1,y_2)$ are upper semicontinuous.

We suppose that the obstacle functions $\phi = (\phi_1,\phi_2)$, multivalued functions $f_i : \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$ as well as $f_{\Gamma_i} : \Gamma_i\times\mathbb{R}\times\mathbb{R}$ enjoy the following condition.

(Hϕ) Assume that $\phi = (\phi_1,\phi_2) \in \mathcal{W}$ and one can find $c_\phi > 0$ satisfying

$$
\phi_i(x) \le c_\phi,
$$

for a.a. $x \in \Omega$.

(Hf) There exist $h_i, j_i \in \mathbb{R}$ such that $h_i \le j_i, j_i \ge c_\phi$, and

$$
\begin{aligned}
&f_{1a}(x,h_1,y_2,0,\varphi_2) \le 0, \quad f_{1b}(x,j_1,y_2,0,\varphi_2) \ge 0, \quad \text{for a.a. } x \in \Omega,\\
&f_{2a}(x,y_1,h_2,\varphi_1,0) \le 0, \quad f_{2b}(x,y_1,j_2,\varphi_1,0) \ge 0, \quad \text{for a.a. } x \in \Omega,\\
&f_{\Gamma_1,a}(x,h_1,y_2) \le 0, \quad f_{\Gamma_1,b}(x,j_1,y_2) \ge 0, \quad \text{for a.a. } x \in \Gamma_1,\\
&f_{\Gamma_2,a}(x,y_1,h_2) \le 0, \quad f_{\Gamma_2,b}(x,y_1,j_2) \ge 0, \quad \text{for a.a. } x \in \Gamma_2,
\end{aligned} \tag{5.3}
$$

for all $(y_1, y_2) \in [h_1, j_1] \times [h_2, j_2]$, all $\varphi_i \in \mathbb{R}^N$.

In Special case (1.12), K_i is given by

$$K_i = \left\{\sigma_i \in W^{1,\mathcal{G}_i}(\Omega) \ : \ \sigma_i(x) \geq \phi_i(x) \text{ a.a. in } \Omega\right\}.$$

We are ready to show the following existence result to Special case (1.12).

Theorem 5.1. *If (H0), (H1) and (Hf) hold. Assume that (H3) holds with $\underline{\sigma}_i := h_i$ and $\overline{\sigma}_i := j_i$ for $i = 1, 2$. Then, for $i = 1, 2$, there exists at least a weak solution $\sigma = (\sigma_1, \sigma_2)$ of problem* (1.12) *fulfilling $h_i \leq \sigma_i \leq j_i$ in Ω.*

Proof. This theorem will be proved via Theorem 4.1, so we first construct a pair of sub-supersolutions to (1.12). Set $\underline{\sigma}_i := h_i$ and $\overline{\sigma}_i := j_i$, we claim that $\underline{\sigma}$ is a subsolution of (1.12), in the same time, $\overline{\sigma}$ is a supersolution of (1.12). Since hypotheses (Hf) are satisfied, it is easy to see that $\underline{\sigma}_i \leq \overline{\sigma}_i$. Firstly, we are going to check conditions (i)-(iii) of Definition 2.13 to show that $\underline{\sigma} := (h_1, h_2)$ is a subsolution for (1.12).

- For (i), by the definition of K_i we have $h_i \vee K_i \subset K_i$ for $i = 1, 2$.
- With respect to (ii), let $\underline{\eta}_1(x) := f_{1a}(x, h_1, y_2, 0, \varphi_2)$, $\underline{\eta}_2(x) := f_{2a}(x, y_1, h_2, \varphi_1, 0)$ and $\underline{\zeta}_1(x) := f_{\Gamma_1,a}(x, h_1, y_2)$, $\underline{\zeta}_2(x) := f_{\Gamma_2,a}(x, y_1, h_2)$, we see that $\underline{\eta}_1(x) \in f_1(x, h_1, y_2, 0, \varphi_2)$, $\underline{\eta}_2(x) \in f_2(x, y_1, h_2, \varphi_1, 0)$, $\underline{\zeta}_1(x) \in f_{\Gamma_1}(x, h_1, y_2)$, $\underline{\zeta}_2(x) \in f_{\Gamma_2}(x, y_1, h_2)$ and $\underline{\eta}_i \in L^{r_i'(\cdot)}(\Omega)$, $\underline{\zeta}_i \in L^{\iota_i'(\cdot)}(\Gamma_i)$ with $1 < r_i(\cdot) < p_i^*(\cdot)$ and $1 < \iota_i(\cdot) < (p_i)_*(\cdot)$.
- As (iii), we need to verify that for each $i = 1, 2$ fixed, the following inequality

$$\sum_{i=1}^{2} \left[\langle A_i(h_i), v_i - h_i\rangle + \int_\Omega \underline{\eta}_i\,(v_i - h_i)\,\mathrm{d}x + \int_{\Gamma_i} \underline{\zeta}_i\,(v_i - h_i)\,\mathrm{d}\varsigma\right] \geq 0 \quad \text{for all } v_i \in h_i \wedge K_i, \tag{5.4}$$

where

$$\langle A_i\sigma_i, v_i - \sigma_i\rangle = \int_\Omega \frac{\mathcal{G}_i'(x, |\nabla\sigma_i|)}{|\nabla\sigma_i|}\nabla\sigma_i \cdot \nabla\left(v_i - \sigma_i\right)\mathrm{d}x \text{ for all } \sigma_i \in W^{1,\mathcal{G}_i}(\Omega).$$

Because $v_i \in h_i \wedge K_i$ means $v_i = h_i \wedge \psi_i = h_i - (h_i - \psi_i)^+$ for any $\psi_i \in K_i$, we use the fact $\nabla h_i = 0$ to rewrite inequality (5.4) to the following one

$$\sum_{i=1}^{2} \left[\int_\Omega \underline{\eta}_i\,(h_i - \psi_i)^+\,\mathrm{d}x + \int_{\Gamma_i} \underline{\zeta}_i\,(h_i - \psi_i)^+\,\mathrm{d}\varsigma\right] \leq 0 \quad \text{for all } \psi_i \in K_i. \tag{5.5}$$

Owing to $(h_i - \psi_i)^+ \in \left\{v_i \in W^{1,\mathcal{G}_i}(\Omega) : v_i \geq 0\right\}$, $\underline{\eta}_1 = f_{1a}\left(\cdot, h_1, y_2, 0, \varphi_2\right)$, $\underline{\eta}_2 = f_{2a}\left(\cdot, y_1, h_2, \varphi_1, 0\right)$, and $\underline{\zeta}_1 = f_{\Gamma_1,a}\left(\cdot, h_1, y_2\right)$, $\underline{\zeta}_2 = f_{\Gamma_2,a}\left(\cdot, y_1, h_2\right)$, it could utilize (Hf) to find

$$\begin{aligned}&\int_\Omega f_{1a}\left(x, h_1, y_2, 0, \varphi_2\right) v_1 \ \mathrm{d}x + \int_\Omega f_{2a}\left(x, y_1, h_2, \varphi_1, 0\right) v_2 \ \mathrm{d}x \\ &+ \int_{\Gamma_1} f_{\Gamma_1,a}\left(x, h_1, y_2\right) v_1 \ \mathrm{d}\varsigma + \int_{\Gamma_2} f_{\Gamma_2,a}\left(x, y_1, h_2\right) v_2 \ \mathrm{d}\varsigma \leq 0,\end{aligned}$$

for all $v = (v_1, v_2) \in \mathcal{W}$ with $v \geq 0$, thus, (5.5) is verified.

This indicates that $\underline{\sigma} := (h_1, h_2)$ is a subsolution of (1.12).

Secondly, by checking the conditions of Definition 2.13 we verify that $\overline{\sigma} = (j_1, j_2)$ is a supersolution of (1.12).

- For (i), thanks to $j_i \geq c_\phi \geq \phi_i(x)$ (that is, $j_i \wedge K_i \subset K_i$).
- With respect to (ii), let $\overline{\eta}_1(x) := f_{1b}(x, h_1, y_2, 0, \varphi_2)$, $\overline{\eta}_2(x) := f_{2b}(x, y_1, h_2, \varphi_1, 0)$ and $\overline{\zeta}_1(x) := f_{\Gamma_1,b}(x, h_1, y_2)$, $\overline{\zeta}_2(x) := f_{\Gamma_2,b}(x, y_1, h_2)$, then $\overline{\eta}_1(x) \in f_1(x, h_1, y_2, 0, \varphi_2)$, $\overline{\eta}_2(x) \in f_2(x, y_1, h_2, \varphi_1, 0)$, $\overline{\zeta}_1(x) \in f_{\Gamma_1}(x, h_1, y_2)$, $\overline{\zeta}_2(x) \in f_{\Gamma_2}(x, y_1, h_2)$ and $\overline{\eta}_i \in L^{r_i'(\cdot)}(\Omega)$, $\overline{\zeta}_i \in L^{\iota_i'(\cdot)}(\Gamma_i)$ with $1 < r_i(\cdot) < p_i^*(\cdot)$ and $1 < \iota_i(\cdot) < (p_i)_*(\cdot)$.

- As (iii), we need to show that for $i=1,2$ fixed the inequality is available

$$\sum_{i=1}^{2}\left[\langle A_i(j_i), v_i - j_i\rangle + \int_\Omega \overline{\eta}_i(v_i-j_i)\mathrm{d}x + \int_{\Gamma_i}\overline{\zeta}_i(v_i-j_i)\mathrm{d}\varsigma\right] \geq 0 \quad \text{for all } v_i \in \overline{\sigma}_i \vee K_i. \tag{5.6}$$

Indeed, for any $v_i \in j_i \vee K_i$ it has $v_i = j_i \vee \psi_i = j_i + (\psi_i - j_i)^+, \psi_i \in K_i$, and $\nabla j_i = 0$, then (5.6) is equivalent to

$$\sum_{i=1}^{2}\left[\int_\Omega \overline{\eta}_i(\psi_i-j_i)^+\mathrm{d}x + \int_{\Gamma_i}\overline{\zeta}_i(\psi_i-j_i)^+\mathrm{d}\varsigma\right] \geq 0 \quad \text{for all } \psi_i \in K_i. \tag{5.7}$$

Due to $(\psi_i - j_i)^+ \in \left\{v_i \in W^{1,\mathcal{G}_i}(\Omega) : v_i \geq 0\right\}$, it uses (Hf) to get

$$\begin{aligned}&\int_\Omega f_{1b}\left(x,h_1,y_2,0,\varphi_2\right)v_1 \ \mathrm{d}x + \int_\Omega f_{2b}\left(x,y_1,h_2,\varphi_1,0\right)v_2\ \mathrm{d}x\\ &+\int_{\Gamma_1} f_{\Gamma_1,b}\left(x,h_1,y_2\right)v_1\ \mathrm{d}\varsigma + \int_{\Gamma_2} f_{\Gamma_2,b}\left(x,y_1,h_2\right)v_2\ \mathrm{d}\varsigma \geq 0,\end{aligned}$$

for all $v=(v_1,v_2)\in\mathcal{W}$ with $v\geq 0$, which implies (5.7).

This leads to that $\overline{\sigma} = (j_1, j_2)$ is a supersolution of (1.12). Invoking Theorem 4.1, we show the solvability of (1.12). □

Recalling the definition of K, we see that K satisfies the lattice condition (4.40). Therefore, it could apply Theorem 4.2 to get the following results.

Corollary 5.2. *Assume that the the same hypotheses of Theorem 5.1 are satisfied, then the solution set $\mathcal{S}$ of* (1.12) *is compact in $\mathcal{W}$.*

Motivated by [25, Theorem 4.2], indeed, we can find more solutions to problem (1.12) by strengthening the hypotheses (Hf), that is, we make the following assumptions.

(Hf)' For all $n\in\mathbb{N}$ and $i=1,2$, let $h_i^{(n)}, j_i^{(n)} \in \mathbb{R}$ be such that $j_i^{(n)} \geq c_\phi$,

$$\text{either}\quad h_i^{(n)} \leq j_i^{(n)} < h_i^{(n+1)} \quad \text{or} \quad j_i^{(n+1)} < h_i^{(n)} \leq j_i^{(n)} \tag{5.8}$$

and

$$\begin{aligned}
f_{1a}(x,h_1^{(n)},y_2,0,\varphi_2)\leq 0,\quad & f_{1b}(x,j_1^{(n)},y_2,0,\varphi_2)\geq 0,\quad && \text{for a.a. } x\in\Omega,\\
f_{2a}(x,y_1,h_2^{(n)},\varphi_1,0)\leq 0,\quad & f_{2b}(x,y_1,j_2^{(n)},\varphi_1,0)\geq 0,\quad && \text{for a.a. } x\in\Omega,\\
f_{\Gamma_1,a}(x,h_1^{(n)},y_2)\leq 0,\quad & f_{\Gamma_1,b}(x,j_1^{(n)},y_2)\geq 0,\quad && \text{for a.a. } x\in\Gamma_1,\\
f_{\Gamma_2,a}(x,y_1,h_2^{(n)})\leq 0,\quad & f_{\Gamma_2,b}(x,y_1,j_2^{(n)})\geq 0,\quad && \text{for a.a. } x\in\Gamma_2,
\end{aligned} \tag{5.9}$$

for all $(y_1,y_2)\in[h_1^{(n)},j_1^{(n)}]\times[h_2^{(n)},j_2^{(n)}]$, all $\varphi_i\in\mathbb{R}^N$ and all $n\in\mathbb{N}$.

From [25, Theorem 4.2], we can get the following theorem.

Theorem 5.3. *If (H0), (H1) and (Hf)' hold true. Let (H3) hold for a.a. $x\in\Omega$, for all $y_i\in[h_i^{(n)},j_i^{(n)}]$ and for all $\varphi_i\in\mathbb{R}^N$ with $i=1,2$. Then there exists a sequence of weak solutions $\sigma^{(n)}=(\sigma_1^{(n)},\sigma_2^{(n)})$ to problem* (1.1) *satisfying $\sigma_i^{(n)}<\sigma_i^{(n+1)}$(resp. $\sigma_i^{(n+1)}<\sigma_i^{(n)}$) if $j_i^{(n)}<h_i^{(n+1)}$(resp. $j_i^{(n+1)}<h_i^{(n)}$) for all $n\in\mathbb{N}$ with $i=1,2$.*

Proof. Utilizing Theorem 5.1 with $h_i = h_i^{(n)}$ and $j_i = j_i^{(n)}$ for all $n\in\mathbb{N}$, we can find solutions $\sigma^{(n)}=(\sigma_1^{(n)},\sigma_2^{(n)})$ of (1.12) fulfilling $h_i^{(n)}\leq\sigma_i^{(n)}\leq j_i^{(n)}$ with $i=1,2$ and $n\in\mathbb{N}$. Moreover, by (Hf)', there hold $\sigma_i^{(n)}\leq j_i^{(n)}<h_i^{(n+1)}\leq\sigma_i^{(n+1)}$ for a.a. $x\in\Omega$ if $j_i^{(n)}<h_i^{(n+1)}$. Similarly, we can proof the results under the case that $j_i^{(n+1)}<h_i^{(n)}$. □

In fact, there are a large of functions that satisfy the corresponding hypotheses in this paper. We end this section to provide the following example for function f_1, f_2, f_{Γ_1} and f_{Γ_2}, more precisely, we have the following example.

Example 5.4. Consider the following mutilvalued functions:

$$\begin{aligned} f_1(x, y_1, y_2, \varphi_1, \varphi_2) &= P_1(y_1) + \frac{1}{2}P_2(y_2) + c_1|\varphi_1|^{p_1(x)-1} + \frac{1}{2}P_2(|\varphi_2|), \\ f_2(x, y_1, y_2, \varphi_1, \varphi_2) &= \frac{1}{2}P_1(y_1) + P_2(y_2) + \frac{1}{2}P_1(|\varphi_1|) + c_2|\varphi_2|^{p_2(x)-1}, \\ f_{\Gamma_1}(x, y_1, y_2) &= P_1(y_1) + \frac{1}{2}P_2(y_2) + u_1(x), \\ f_{\Gamma_2}(x, y_1, y_2) &= \frac{1}{2}P_1(y_1) + P_2(y_2) + u_2(x), \end{aligned}$$

where $P_1, P_2 : \mathbb{R} \to \mathbb{R}$ are given by

$$P_1(y) = \begin{cases} [-2, -1] & y = 4m - 1 \\ 2(y - 4m) & y \in (4m - 1, 4m + 1) \\ [1, 2] & y = 4m + 1 \\ -y + 4m + 2 & y \in (4m + 1, 4m + 3) \quad m \in \mathbb{Z}, \end{cases} \tag{5.10}$$

and

$$P_2(y) = \begin{cases} [-2, -1] & y = 4m - 2 \\ 2(y - 4m + 1) & y \in (4m - 2, 4m) \\ [1, 2] & y = 4m \\ -y + 4m + 1 & y \in (4m, 4m + 2) \quad m \in \mathbb{Z}, \end{cases} \tag{5.11}$$

$c_i > 0$ and $u_i \in L^\infty(\Gamma_i)$ fulfilling $|u_i(x)| \le \frac{1}{2}$ for a.e. $x \in \Gamma_i$ for $i = 1, 2$.

Note that for $i = 1, 2$, the multivalued functions f_i and f_{Γ_i} fulfill hypotheses (H1) and (H3). Moreover, if we take $h_1^{(n)} = -1 + 4n$, $j_1^{(n)} = 1 + 4n$, $h_2^{(n)} = -2 + 4n$ and $j_2^{(n)} = 4n$ then the above functions satisfy hypotheses (Hf)' (so, they fulfill (Hf) as well).

Declarations

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References

[1] E. Amoroso, Á. Crespo-Blanco, P. Pucci, P. Winkert, *Superlinear elliptic equations with unbalanced growth and nonlinear boundary condition,* Bull. Sci. Math. **197** (2024), 103534.

[2] R. Arora, Á. Crespo-Blanco, P. Winkert, *Logarithmic double phase problems with generalized critical growth,* https://arxiv.org/abs/2501.17985.

[3] R. Arora, Á. Crespo-Blanco, P. Winkert, *On logarithmic double phase problems,* arXiv preprint arXiv: 2309.09174 (2023).

[4] P. Baroni, M. Colombo, G. Mingione, *Non-autonomous functionals, borderline cases and related function classes,* St. Petersb. Math. J. **27** (2016), no 3, 347–379.

[5] P. Baroni, M. Colombo, G. Mingione, *Regularity for general functionals with double phase,* Calc. Var. Partial Differ. Equ. **57** (2018), 1–48.

[6] A. Bahrouni, V.D. Rădulescu, D.D. Repovš, *A weighted anisotropic variant of the Caffarelli-KohnNirenberg inequality and applications,* Nonlinearity **30** (2018), 1516.

[7] A. Bahrouni, V.D. Rădulescu, D.D. Repovš, *Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves,* Nonlinearity **32** (2019), no 7, 2481.

[8] L. Beck, G. Mingione, *Lipschitz bounds and nonuniform ellipticity,* Commun. Pure Appl. Math. **73** (2020), no. 5, 944–1034.

[9] V. Benci, P. D'Avenia, D. Fortunato, L. Pisani, *Solitons in several space dimensions: Derrick's problem and infinitely many solutions,* Arch. Ration. Mech. Anal. **154** (2000), 297–324.

[10] S.-S. Byun, J. Ok, K. Song, *Hölder regularity for weak solutions to nonlocal double phase problems,* J. Math. Pures Appl. **168** (2022), 110–142.

[11] S. Carl, V.K. Le, "Multi-valued Variational Inequalities and Inclusions", Cham: Springer, 2021.

[12] S. Carl, V.K. Le, P. Winkert, *Multi-valued variational inequalities for variable exponent double phase problems: comparison and extremality results,* J. Elliptic Parabol. Equat. (2025), 1–42.

[13] Á. Crespo-Blanco, L. Gasiński, P. Harjulehto, P. Winkert, *A new class of double phase variable exponent problems: Existence and uniqueness,* J. Differ. Equ. **323** (2022), 182–228.

[14] C. De Filippis, G. Mingione, *Regularity for double phase problems at nearly linear growth,* Arch. Ration. Mech. Anal **247** (2023), no 5, 85.

[15] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, "Lebesgue and Sobolev Spaces with Variable Exponents", Springer, 2011.

[16] L. Dupaigne, M. Ghergu, V. D. Rădulescu, *Lane–Emden–Fowler equations with convection and singular potential,* J. Math. Pures Appl. **87** (2007), no. 6, 563–581.

[17] G. Duvaut, J.-L. Lions, "Inequalities in mechanics and physics", Springer-Verlag, Berlin-New York, 1976.

[18] S. El Manouni, G. Marino, P. Winkert, *Existence results for double phase problems depending on Robin and Steklov eigenvalues for the p-Laplacian,* Adv. Nonlinear Anal. **11** (2022), no. 1, 304–320.

[19] X. Fan, D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$,* J. Math. Anal. Appl. **263** (2001), no 2, 424–446.

[20] F. Faraci, D. Motreanu, D. Puglisi, *Positive solutions of quasi-linear elliptic equations with dependence on the gradient,* Calc. Var. Partial Differ. Equ. **54** (2015), no. 1, 525–538.

[21] G.M. Figueiredo, G.F. Madeira, *Positive maximal and minimal solutions for non-homogeneous elliptic equations depending on the gradient,* J. Differ. Equ. **274** (2021), 857–875.

[22] M. Fuchs, G. Mingione, *Full $C^{1,\alpha}$-regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth,* Manuscr. Math. **102** (2000), no 2, 227–250.

[23] L. Gasiński, N.S. Papageorgiou, *Positive solutions for nonlinear elliptic problems with dependence on the gradient,* J. Differ. Equ. **263** (2017), no. 2, 1451–1476.

[24] L. Gasiński, P. Winkert, *Existence and uniqueness results for double phase problems with convection term,* J. Differ. Equ. **268** (2020), 4183–4193.

[25] U. Guarnotta, R. Livrea, P. Winkert, *The sub-supersolution method for variable exponent double phase systems with nonlinear boundary conditions,* Rend. Lincei-Mat. Appl. **34** (2023), no 3, 617–639.

[26] U. Guarnotta, S.A. Marano, *Infinitely many solutions to singular convective Neumann systems with arbitrarily growing reactions,* J. Differ. Equ. **271** (2021), 849–863.

[27] U. Guarnotta, S.A. Marano, *Corrigendum to "Infinitely many solutions to singular convective Neumann systems with arbitrarily growing reactions" [J. Differential Equations 271 (2021) 849–863],* J. Differ. Equ. **274** (2021), 1209–1213.

[28] U. Guarnotta, S.A. Marano, A. Moussaoui, *Singular quasilinear convective elliptic systems in $\mathbb{R}^N$,* Adv. Nonlinear Anal. **11** (2022), no. 1, 741–756.

[29] P. Harjulehto, P. Hästö, "Orlicz Spaces and Generalized Orlicz Spaces", Springer International Publishing, 2019.

[30] P. Hästö, J. Ok, *Maximal regularity for local minimizers of non-autonomous functionals,* J. Eur. Math. Soc. **24** (2021), no. 4, 1285–1334.

[31] K. Ho, Y.H. Kim, C. Zhang, *Double phase anisotropic variational problems involving critical growth*, Adv. Nonlinear Anal. **13** (2024), no. 1, Paper No. 20240010, 38 pp.

[32] O. Kováčik, J. Rákosník, *On spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$,* Czech. Math. J. **41** (1991), no 4, 592–618.
[33] V.K. Le, *A range and existence theorem for pseudomonotone perturbations of maximal monotone operators,* Proc. Amer. Math. Soc. **139** (2011), 1645–1658.
[34] J.-L. Lions, "Quelques Méthodes de Résolution des Problémes aux Limites Non Linéaires", Dunod (1969).
[35] W. Liu, G. Dai, *Existence and multiplicity results for double phase problem,* J. Differ. Equ. **265** (2018), no 9, 4311–4334.
[36] Y. Liu, Y. Lu, C. Vetro, *A new kind of double phase elliptic inclusions with logarithmic perturbation terms I: Existence and extremality results,* Commun. Nonlinear Sci. Numer. Simul. **129** (2024), 107683.
[37] Z. Liu, D. Motreanu, S. Zeng, *Positive solutions for nonlinear singular elliptic equations of p-Laplacian type with dependence on the gradient,* Calc. Var. Partial Differ. Equ. **58** (2019), no. 1, 28.
[38] P. Marcellini, G. Papi, *Nonlinear elliptic systems with general growth,* J. Differ. Equ. **221** (2006), no 2, 412–443.
[39] D. Motreanu, E. Tornatore, *Nonhomogeneous degenerate quasilinear problems with convection,* Nonlinear Anal.-Real World Appl. **71** (2023), 103800.
[40] D. Motreanu, C. Vetro, F. Vetro, *Systems of quasilinear elliptic equations with dependence on the gradient via subsolution-supersolution method,* Discrete Contin. Dyn. Syst. Ser. S **11** (2018), no 2, 309–321.
[41] A. Moussaoui, D. Nabab, J. Vélin, *Singular quasilinear convective systems involving variable exponents*, Opuscula Math. **44** (2024), no. 1, 105–134.
[42] P.D. Panagiotopoulos, *Nonconvex problems of semipermeable media and related topics,* ZAMM J. Appl. Math. Mech./Z. Angew. Math. Mech. **65** (1985), 29–36.
[43] P.D. Panagiotopoulos, "Hemivariational inequalities", Springer Berlin Heidelberg, 1993.
[44] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Positive solutions for nonlinear Neumann problems with singular terms and convection,* J. Math. Pures Appl. **136** (2020), 1–21.
[45] M.A. Ragusa, A. Tachikawa, *Regularity of minimizers for double phase functionals of borderline case with variable exponents*, Adv. Nonlinear Anal. 13 (2024), no. 1, Paper No. 20240017, 27 pp.
[46] V.D. Rădulescu, C. Vetro, *Anisotropic Navier Kirchhoff problems with convection and Laplacian dependence,* Math. Meth. Appl. Sci. **46** (2023), no. 1, 461–478.
[47] J.F. Rodrigues, "Obstacle problems in mathematical physics", Elsevier, 1987.
[48] J. Stefan, *Über einige Probleme der Theorie der Wärmeleitung,* Wien. Ber. **98** (1889), 473–484.
[49] F. Vetro, P. Winkert, *Logarithmic double phase problems with convection: existence and uniqueness results,* Commun. Pure Appl. Anal. **23** (2024), no. 9, 1325–1339.
[50] C. Vetro, S. Zeng, *Regularity and Dirichlet problem for double phase energy functionals of different power growth,* J. Geometric Anal., **34** (2024), no. 4, 1–27.
[51] M. Xiang, Y. Ma, M. Yang, *Normalized homoclinic solutions of discrete nonlocal double phase problems*, Bull. Math. Sci. **14** (2024), no. 2, Paper No. 2450003, 18 pp.
[52] S. Zeng, Y. Bai, L. Gasiński, P. Winkert, *Existence results for double phase implicit obstacle problems involving multivalued operators,* Calc. Var. Partial Differ. Equ. **59** (2020), 1–18.
[53] S. Zeng, L. Gasiński, P. Winkert, Y. Bai, *Existence of solutions for double phase obstacle problems with multivalued convection term,* J. Math. Anal. Appl. **501** (2021), 123997.
[54] S. Zeng, Y. Lu, N.S. Papageorgiou, *A class of double phase variable exponent energy functionals with different power growth and logarithmic perturbation,* submitted.
[55] W. Zhang, J. Zhang, V.D. Rădulescu, *Concentrating solutions for singularly perturbed double phase problems with nonlocal reaction,* J. Differ. Equ. **347** (2023): 56–103.
[56] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory,* Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 4, 675–710.
[57] V.V. Zhikov, *On variational problems and nonlinear elliptic equations with nonstandard growth conditions,* J. Math. Sci. **173** (2011), 463–570.
[58] V.V. Zhikov, *On Lavrentiev's phenomenon,* Russian J. Math. Phys. **3** (1995), 249–269.

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