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Localized concentration of semiclassical solutions for double phase problems with nonlocal reaction

Jian Zhang[1] · Wen Zhang[1] · Vicenţiu D Rădulescu[2,3,4,5,6]

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Abstract

This paper focuses on the study of multiplicity and localized concentration properties of positive solutions for the following singularly perturbed double phase problem with nonlocal Choquard reaction

$$\begin{cases} -\epsilon^p \Delta_p u - \epsilon^q \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) & \\ \quad = \epsilon^{\mu-N}\left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),\ u > 0, & \text{in } \mathbb{R}^N, \end{cases}$$

where $1 < p < q < N$, $0 < \mu < p$, ϵ is a small positive parameter and V is the absorption potential. We assume that the potential V satisfies only a local condition introduced by del Pino and Felmer. Applying suitable variational and topological methods combined with penalization technique, we obtain multiple semiclassical positive solutions for $\epsilon > 0$ sufficiently small as well as related concentration properties, in relationship with the set where the potential V attains its minimum. Moreover, we also investigate the decay property of semiclassical positive solutions. The main

✉ Vicenţiu D Rădulescu
radulescu@agh.edu.pl

Jian Zhang
zhangjian@hutb.edu.cn

Wen Zhang
wenzhang@hutb.edu.cn

1 School of Mathematics and Statistics, Hunan University of Technology and Business, Changsha 410205, Hunan, China

2 Faculty of Applied Mathematics, AGH University of Kraków, Kraków 30-059, Poland

3 Brno University of Technology, Faculty of Electrical Engineering and Communication, Brno 61600, Czech Republic

4 “Simion Stoilow" Institute of Mathematics of the Romanian Academy, Bucharest 014700, Romania

5 Department of Mathematics, University of Craiova, Craiova 200585, Romania

6 Scientific Research Center, Baku Engineering University, Baku AZ0102, Azerbaijan

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results included in this paper complement several recent contributions to the study of concentration phenomena.

Keywords Double phase problem · Semiclassical solution · Localized concentration · Nonlocal reaction

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1 Introduction and main results

In this paper we deal with a class of singularly perturbed double phase problem with nonlocal Choquard reaction

$$\begin{cases} -\epsilon^p \Delta_p u - \epsilon^q \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) & \text{in } \mathbb{R}^N, \\ \quad = \epsilon^{\mu-N} \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),\, u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $1 < p < q < N$, $0 < \mu < p$, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$, with $r \in \{p, q\}$, is the r-Laplacian operator, ϵ is small positive parameter, $*$ represents the convolution between two functions, V is the absorption potential and the nonlinear function G is the primitive function of g.

This paper is inspired by recent fundamental progress in the mathematical analysis of various nonlinear patterns with unbalanced growth and nonlocal reaction. Here we point out that the source term of problem (1.1) is driven by the sum of the (p, q)-Laplacian operator and the nonlocal Choquard reaction term. This problem is also called the double phase problem with unbalanced growth since the behavior of the (p, q)-Laplacian operator switches between two different elliptic situations, which generates an interesting double phase associated energy.

The main interest in the study of problem of this type is twofold. On the one hand, there are motivations from mathematical physics, since the non-autonomous unbalanced operator has been applied to describe steady-state solutions of reaction-diffusion equations arising in biophysics, plasma physics, and chemical reaction analysis. The prototype equation for these models can be written in the form

$$u_t = \operatorname{div}[A(u)\nabla u] + c(x, u) \text{ and } A(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2},$$

where the function u is a state variable and describes density or concentration of multi-component substances, $\operatorname{div}[A(u)\nabla u]$ corresponds to the diffusion with a diffusion coefficient $A(u)$ and $c(x, u)$ represents the reaction term related to source and loss processes, see Cherfils and Il'yasov [19] and Singer [40]. On the other hand, an interesting phenomenon in the paper is that the appearance of Choquard reaction term on the right-hand side generates the nonlocal characteristic, which need more accurate and delicate analyses in mathematical aspects. The nonlocal nonlinearity appears naturally in optical systems with a thermal and it is known to influence the propagation of electromagnetic waves in plasmas, see [16, 30]. This type of nonlocal problem also has attracted considerable interest as a means of eliminating collapse and stabilizing multidimensional solitary waves [12] and it plays an important role in describing the finite-range many-body interactions, see Pekar [39] and Lieb [27].

Furthermore, the double phase problem (1.1) is also motivated by numerous models arising in mathematical physics. For instance, we can refer to the Born-Infeld equation [17] that appears in electromagnetism, electrostatics and electrodynamics as a model based on a modification of Maxwell's Lagrangian density:

$$-\operatorname{div}\left(\frac{\nabla u}{(1-2|\nabla u|^2)^{1/2}}\right)=h(x,u) \text{ in } \Omega.$$

In fact, using the usual Taylor formula, we get

$$(1-x)^{-1/2}=1+\frac{x}{2}+\frac{3}{2\cdot 2^2}x^2+\frac{5!!}{3!\cdot 2^3}x^3+\cdots+\frac{(2n-3)!!}{(n-1)!2^{n-1}}x^{n-1}+\cdots \text{ for } |x|<1.$$

Taking $x=2|\nabla u|^2$ and adopting the first order approximation, we can obtain the double phase problem (1.1) with $p=2$ and $q=4$. Especially, the n-th order approximation problem is driven by the multi-phase differential operator

$$-\Delta u-\Delta_4 u-\frac{3}{2}\Delta_6 u-\cdots-\frac{(2n-3)!!}{(n-1)!}\Delta_{2n} u.$$

We also refer to the following fourth-order relativistic operator

$$u\mapsto \operatorname{div}\left(\frac{|\nabla u|^2}{(1-|\nabla u|^4)^{3/4}}\nabla u\right),$$

which describes large classes of phenomena arising in relativistic quantum mechanics. Similarly, by virtue of Taylor formula again, we have

$$x^2(1-x^4)^{-3/4}=x^2+\frac{3x^6}{4}+\frac{21x^{10}}{32}+\cdots.$$

This illustrates that the fourth-order relativistic operator can be approximated by the following double phase operator

$$u\mapsto \Delta_4 u+\frac{3}{4}\Delta_8 u.$$

For more detailed contents and applications in the physical science and other fields, we refer the readers to Arora-Fiscella-Mukherjee-Winkert [10], Bahrouni-Rădulescu-Repovš [11], Benci-D'Avenia-Fortunato-Pisani [15], and Tao-Li-Winkert [44].

We point out that, when ϵ goes to zero, the singularly perturbed problem (1.1) is known as the semiclassical problem, and the solutions of (1.1) are often referred to as semiclassical solutions which have very plentiful dynamic phenomena, such as concentration, convergence and decay etc. Physically, the concentration phenomenon of semiclassical solutions reflects the transition from quantum mechanics to classical mechanics and it gives rise to significant physical insights. To characterize the concentration phenomenon of semiclassical solutions, we note that if the function v is a solution of (1.1) for $x_0 \in \mathbb{R}^N$, then $u = v(x_0 + \epsilon x)$ verifies

$$-\Delta_p u - \Delta_q u + V(x_0 + \epsilon x)(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), \quad \text{in } \mathbb{R}^N,$$

which means some convergence of the family of solutions to a solution u_0 of the limit problem

$$-\Delta_p u - \Delta_q u + V(x_0)(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), \quad \text{in } \mathbb{R}^N.$$

It is expected that, in the semiclassical limit $\epsilon \to 0$, the concentration phenomenon should be governed by the potential V. In particular, there should be a correspondence between semiclassical solutions of the equation and critical points of the potential. Over the recent years, there have been a great deal of interests in investigating the existence, multiplicity and qualitative properties of semiclassical states as localized solutions for nonlinear mathematical physics equations with various types of concentration phenomena. Let us now briefly recall some related results in this direction.

When $p = q = 2$, problem (1.1) comes back to the following semiclassical Choquard equation

$$-\epsilon^2 \Delta u + V(x)u = \epsilon^{\mu - N} \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

As we know, the question of the existence of semiclassical solutions for the nonlocal problem (1.2) has been posed by Ambrosetti-Malchiodi [7, p.29]. Following the seminal work [7], there have been some works concerning with the study of the semiclassical analysis of problem (1.2) under various assumptions on the potential V, see for example [2, 21, 33, 46] and the references therein. Among them, the authors are mainly interested in the problem how the property of the potential influence the existence, multiplicity and concentration of semiclassical solutions. Applying the Lyapunov-Schmidt reduction method, Wei-Winter [46] constructed families of solutions concentrating to the nondegenerate critical points of V. Cingolani-Secchi-Squassina [21] studied the existence of solutions concentrating around several minimum points of V by using a global penalization method. Moroz-Van Schaftingen [33] used variational methods and developed a novel nonlocal penalization technique to study the localized concentration phenomena: the solutions concentrating around local minima of V. Very recently, based on the global variational argument, Alves-Gao-Squassina-Yang [2] studied the existence and concentration behavior of solutions for the problem (1.2) involving two variable potentials and critical exponential. We also refer the readers to the survey paper [32] and the related works [18, 26, 41] for more details.

When $p = q \neq 2$, problem (1.1) boils down to the following quasi-linear Choquard equation with p-Laplacian operator

$$\begin{cases} -\epsilon^p \Delta_p u + V(x)|u|^{p-2}u = \epsilon^{\mu-N} \left(\frac{1}{|x|^\mu} * G(u) \right) g(u), \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), u > 0, \qquad \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

Regarding the study of semiclassical analysis for problem (1.3), we would like to introduce the recent contributions done by Alves-Yang [4–6]. To be more precise, combining the mountain pass argument with the Ljusternik-Schnirelmann category theory, in [5] they obtained the existence and multiplicity result of positive solutions concentrating at global minimum points of V under the global condition introduced by Rabinowitz [38]. After then, under the local condition on the potential V, they took advantage of a variational method based on the penalized argument, first developed by del Pino-Felmer [22], to obtain a family of solutions concentrating around local minimum points of V [6]. In [4], they also considered the competing potentials and described a new concentration phenomena: the semi-classical solutions concentrating around global minimum points of linear potential and global maximum points of nonlinear potential.

In recent years, the double phase problems with local nonlinear reaction had attracted great attention of many scholars, and some interesting and meaningful results were established by using various variational and topological methods. We refer to the works of [13, 23, 34–37] for the existence and multiplicity results, and the progresses about concentration and qualitative analyses of semiclassical solutions can be found in [3, 8, 9, 48, 50] and the references therein.

Very recently, concerning the semiclassical problem (1.1), there is only a work studying the concentration and multiplicity for semiclassical positive ground state solutions up until now. We refer to the paper by Zhang-Zhang-Rădulescu [49]. Precisely speaking, under the global condition introduced by Rabinowitz [38]

$$0 < \inf_{x \in \mathbb{R}^N} V(x) = V_0 < V_\infty = \liminf_{|x| \to \infty} V(x) < \infty, \tag{1.4}$$

and the nonlinear term g having C^1-smoothness, combining variational and topological arguments from Nehari manifold analysis and Ljusternik-Schnirelmann category theory, the authors proved the existence and multiplicity of semiclassical positive ground state solutions that concentrate around global minimum points of the potential V. Finally, we also mention the recent survey paper by Mingione-Rădulescu [31] where a comprehensive overview of the recent developments concerning elliptic variational problems with unbalanced growth conditions and related to different kinds of nonuniformly elliptic operators.

We would like to emphasize that the global condition (1.4) used in [49] plays a very crucial role in showing the existence and concentration of semiclassical solutions. In fact, the key point is that the limit behavior of V at infinity can help this problem to restore the necessary compactness by using the energy comparison method. So, an interesting question, which motivates the present work, is whether one can seek for localized solutions which concentrate around local minimum points of the potential. As we will see, the answer is affirmative. Motivated by this fact and the above mentioned works, in the present paper we are interested in the qualitative and asymptotic analysis of semiclassical solutions to problem (1.1) under the local condition and we are mainly concerned with the existence, localized concentration, decay estimate and multiplicity properties of solutions. The main features of the present paper are the following:

(1) the problem contains the combined effects of a double phase operator with unbalanced growth and of a Choquard reaction with nonlocal property, so some nontrivial additional technical difficulties arise.

(2) the potential describing the absorption term satisfies a local condition and no information on the behavior of the potential at infinity is available;

(3) the localized concentration phenomenon shows some relationships between the global maximum point of the solution versus the local minimum of the absorption potential;

(4) since the lack of compactness caused by the unboundedness of domain, the Palais-Smale sequences do not have the compactness property.

To state the main results, we need introduce some conditions about the potential V and the nonlinearity g. We first assume that the potential V satisfies the following condition introduced by del Pino and Felmer [22]:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $V(x) \geq \inf_{x\in\mathbb{R}^N} V(x) = V_{\min} > 0$, and there exists an open and bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 := \inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x).$$

We define the local minimum point set $\Pi := \{x \in \Lambda : V(x) = V_0\}$. Without loss of generality, below we may assume $0 \in \Pi$. Meanwhile, the nonlinearity g is assumed to satisfy the following conditions:

(g_1) $g \in C(\mathbb{R}, \mathbb{R})$ and $g(s) = 0$ for all $s < 0$;
(g_2) $g(s) = o(|s|^{p-1})$ as $s \to 0$;
(g_3) there exist $c_0 > 0$ and $\tau \in [q, \frac{(N-\mu)q}{N-q})$ such that

$$|g(s)| \leq c_0(1 + |s|^{\tau-1}) \text{ for all } s \in \mathbb{R};$$

(g_4) there exists $\theta = 2q > q$ such that

$$0 < \theta G(s) = \theta \int_0^s g(t)dt \leq 2g(s)s \text{ for all } s > 0;$$

(g_5) $s \mapsto \frac{g(s)}{s^{\frac{q}{2}-1}}$ is increasing for all $s \in (0, \infty)$.

The first result is the existence and localized concentration of positive solutions, we have the following theorem.

Theorem 1.1 *Let* $0 < \mu < p$ *and assume that conditions* (V) *and* (g_1)-(g_5) *hold, then there exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0)$, *we have the following conclusions:*

(a) *problem (1.1) at least has a positive solution* u_ϵ*;*

(b) *u_ϵ attains its maximum at x_ϵ, moreover, up to a subsequence, there holds*

$$\lim_{\epsilon\to 0} dist(x_\epsilon, \Pi) = 0 \text{ and } \lim_{\epsilon\to 0} V(x_\epsilon) = V_0;$$

(c) *$u_\epsilon(\epsilon x + x_\epsilon) \to u(x)$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ as $\epsilon \to 0$, where u is a positive solution of the limit problem*

$$\begin{cases} -\Delta_p u - \Delta_q u + V_0(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & in\ \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0, & in\ \mathbb{R}^N, \end{cases}$$

(d) *there exist positive constants c, C such that*

$$u_\epsilon(x) \le C \exp\left(-\frac{c}{\epsilon}|x - x_\epsilon|\right).$$

Moreover, we observe that the behavior of V outside Λ is irrelevant from the above results, next we consider the situation where V has multiple isolated local minimum sets, and obtain an immediate consequence which describes some multiple concentration phenomena.

Corollary 1.1 *Let $0 < \mu < p$ and assume that conditions (V) and (g_1)-(g_5) hold. If there exist mutually disjoint bounded domains Λ_j, $j = 1, \cdots, k$ and constants $V_1 < V_2 < \cdots < V_k$ such that*

$$V_j := \inf_{x\in\Lambda_j} V(x) < \min_{x\in\partial\Lambda_j} V(x).$$

Then for sufficiently small $\epsilon > 0$, we have the following conclusions:

(a) *problem (1.1) at least has k positive solution u_ϵ^j for $j = 1, \cdots, k$;*
(b) *u_ϵ^j attains its maximum at x_ϵ^j, moreover, up to a subsequence, there holds*

$$\lim_{\epsilon\to 0} dist(x_\epsilon^j, \Pi_j) = 0 \text{ and } \lim_{\epsilon\to 0} V(x_\epsilon^j) = V_j,$$

where the set $\Pi_j := \{x \in \Lambda_j : V(x) = V_j\}$ for all $j = 1, \cdots, k$;
(c) *$u_\epsilon^j(\epsilon x + x_\epsilon^j) \to u^j(x)$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ as $\epsilon \to 0$, where u^j is ground state solution of the corresponding limit problem*

$$\begin{cases} -\Delta_p u - \Delta_q u + V_j(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & in\ \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0, & in\ \mathbb{R}^N, \end{cases}$$

(d) *there exist positive constants c, C such that*

$$u_\epsilon^j(x) \le C \exp\left(-\frac{c}{\epsilon}|x - x_\epsilon^j|\right).$$

In the second part of the present paper, we shall deal with the multiplicity result of semiclassical solutions to problem (1.1). In this case, we assume the following condition on the potential V:

$(\widehat{V})$ $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $V(x) \geq \inf_{x\in\mathbb{R}^N} V(x) = V_{\min} > 0$, and there exists an open and bounded domain $\Lambda \subset \mathbb{R}^N$ such that

$$V_{\min} < \min_{x\in\partial\Lambda} V(x) \text{ and } 0 \in M := \{x \in \Lambda : V(x) = V_{\min}\} \neq \emptyset.$$

In order to study the multiplicity result, we need apply the Ljusternik-Schnirelman category theory. Now let us remind the definition of the Ljusternik-Schnirelman category. Let X be a topological space and let $Y \neq \emptyset$ be a closed subset of X. The category of Y in X, $\text{cat}_X(Y)$, is the smallest integer n such that

$$Y \subset \bigcup_{k=1}^{n} A_k,$$

where for each $k = 1, \cdot, \cdot, \cdot, n$, A_k is a closed set contractible in X. If such a integer does not exist, then $\text{cat}_X(Y) = +\infty$.

Let us define the following set

$$M_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\} \text{ for } \delta > 0.$$

The multiplicity result of this paper is the following theorem.

Theorem 1.2 *Let $0 < \mu < p$ and assume that conditions $(\widehat{V})$ and (g_1)-(g_5) are satisfied. Then for any $\delta > 0$ there is $\epsilon_\delta > 0$ such that, for any $\epsilon \in (0, \epsilon_\delta)$, problem (1.1) has at least $cat_{M_\delta}(M)$ positive solutions. Furthermore, if u_ϵ denotes one of these solutions and $x_\epsilon \in \mathbb{R}^N$ is its global maximum, then we have*

$$\lim_{\epsilon\to 0} dist(x_\epsilon, M) = 0 \text{ and } \lim_{\epsilon\to 0} V(x_\epsilon) = V_{\min}.$$

Compared with [49], the condition (V) or $(\widehat{V})$ is rather weak, without restriction on the global behavior of V is required, and the behavior of V outside Λ is irrelevant. This fact shows that the limit problem at infinity and its properties are all unknown for the current problem. So, from a variational point of view, one of the major differences between the global condition (1.4) and the local condition (V) is that the associated energy functional, under the local condition (V), does not satisfy the so-called compactness condition in general. In addition, another novelty of this paper is that we remove the C^1-smoothness of g in [49]. Naturally, the argument for C^1-Nehari manifold method used in [49] is not applicable in our situation. Evidently, the problem studied in this paper seems to be more complicated, and the arguments also seem to be more delicate.

Next we sketch the strategies and methods to prove Theorem 1.1 and Theorem 1.2. The proof of Theorem 1.1 will be carried out by using suitable variational techniques. Firstly, since we have no global information on the potential V, we make use of the penalization approach developed by del Pino and Felmer [22] which consists in modifying appropriately the nonlinearity g outside Λ, and thus consider a modified problem whose corresponding energy functional fulfills all the assumptions of the mountain pass theorem. In such a way, the functional of modified problem has the advantage that it satisfies the so-called Palais-Smale compactness condition, which overcome the difficulty caused by the lack of compactness. Secondly, we need to check that the solutions of the modified problem are indeed solutions of the original problem. This goal will be achieved by combing an appropriate De Giorgi iteration argument, the Hölder regularity result and some refined analysis techniques to show the L^∞-estimate and decay property of solutions.

The proof of Theorem 1.2 will be obtained by applying topological techniques. More precisely, in order to obtain multiple solutions of the modified problem, we take advantage of the Ljusternik-Schnirelmann category theory and the topological techniques due to Benci-Cerami [14] based on precise comparisons between the category of some sublevel sets of the modified functional and the category of the set M. Observe that the nonlinearity g is only continuous, one can not apply standard C^1-Nehari manifold arguments due to the lack of differentiability of the corresponding Nehari manifold. To overcome this difficulty, we intend to use some variants of abstract critical point theorems from Szulkin and Weth [42] to deal with our problem. Finally, combining the decay property of solutions we can complete the proof of the multiplicity result to problem (1.1). And then Theorem 1.2 follows naturally.

The paper is organized as follows. In Section 2, we present some preliminary lemmas and obtain the existence result of modified problem. In Section 3, we deal with the autonomous problem associated to problem (1.1). In Section 4, we prove the existence and localized concentration of semiclassical solutions, and complete the proof of Theorem 1.1. In Section 5, we are devoted to the multiplicity result of solutions and we accomplish the proof of Theorem 1.2.

2 Preliminary results and modified problem

Throughout this paper, for the sake of simplicity we will use the following notations.

- $|\cdot|_s$ denotes the usual norm of the space $L^s(\mathbb{R}^N)$, $1 \le s \le \infty$;
- c, C, c_i, C_i denote some different positive constants;
- u^+ and u^- denotes the positive and negative parts of function u, respectively, i.e.,

$$u^+ = \max\{u, 0\} \text{ and } u^- = \min\{u, 0\};$$

- $r^* = \frac{Nr}{N-r}$ denotes the embedding critical exponent of the Sobolev space $W^{1,r}(\mathbb{R}^N)$.

In what follows, we introduce some relevant results about the Sobolev spaces. For $p \in (1, \infty)$ and $N > p$, we define $\mathcal{D}^{1,p}(\mathbb{R}^N)$ as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to

$$|\nabla u|_p^p = \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x.$$

Let $W^{1,p}(\mathbb{R}^N)$ be the usual Sobolev space endowed with the standard norm

$$\|u\|_{W^{1,p}} = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p)\mathrm{d}x\right)^{\frac{1}{p}}.$$

We give the following embedding property for the spaces $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$.

Lemma 2.1 *Let $N > p$, then there exists a constant $S_* > 0$ such that, for any $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$,*

$$|u|_{p^*}^p \leq S_*^{-1}|\nabla u|_p^p.$$

Furthermore, $W^{1,p}(\mathbb{R}^N)$ is embedded continuously into $L^s(\mathbb{R}^N)$ for any $s \in [p, p^*]$ and compactly into $L^s_{loc}(\mathbb{R}^N)$ for any $s \in [1, p^*)$.

We also have the following Lions concentration compactness lemma due to [28].

Lemma 2.2 *Let $N > p$ and $r \in [p, p^*)$. If $\{u_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^r \mathrm{d}x = 0,$$

where $R > 0$, then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for all $s \in (p, p^*)$.

Observe that, since we shall deal with the double phase problem with (p, q)-Laplacian operator, we introduce the space

$$E = W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$$

endowed with the norm

$$\|u\| = \|u\|_{W^{1,p}} + \|u\|_{W^{1,q}}.$$

Concerning problem (1.1), we use the change of variable $x \mapsto \epsilon x$, then we can see that problem (1.1) is equivalent to the following problem

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\epsilon x)(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \, u > 0, & \text{in } \mathbb{R}^N. \end{cases} \tag{2.1}$$

Evidently, if u is a solution of problem (2.1), then $v(x) := u(x/\epsilon)$ is a solution of problem (1.1). Thus, in the following we will study the equivalent problem (2.1).

For any fixed $\epsilon > 0$, we define the working space

$$E_\epsilon = \left\{u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)(|u|^p + |u|^q)\mathrm{d}x < \infty\right\}$$

endowed with the norm

$$\|u\|_\epsilon = \|u\|_{V_\epsilon,p} + \|u\|_{V_\epsilon,q},$$

where

$$\|u\|_{V_\epsilon,s} = \left(\int_{\mathbb{R}^N}(|\nabla u|^s + V(\epsilon x)|u|^s)\mathrm{d}x\right)^{\frac{1}{s}} \quad \text{for all } s > 1.$$

From the condition on V, it follows that

$$\|u\|^p_{W^{1,p}} + \|u\|^q_{W^{1,q}} \le \left(1 + \frac{1}{V_{\min}}\right)\left(\|u\|^p_{V_\epsilon,p} + \|u\|^q_{V_\epsilon,q}\right).$$

Moreover, according to Alves and Figueiredo [3], we have the following embedding property.

Lemma 2.3 *E_ϵ embeds continuously into $L^s(\mathbb{R}^N)$ for $s \in [p, q^*]$ and compactly into $L^s_{loc}(\mathbb{R}^N)$ for $s \in [1, q^*)$. Furthermore, there exists positive constant ν_s such that*

$$\nu_s|u|_s \le \|u\|_\epsilon, \text{ for all } s \in [p, q^*]. \tag{2.2}$$

Since we will treat the nonlocal problem (2.1) with Choquard reaction, and we need to introduce the classical Hardy-Littlewood-Sobolev inequality (see [29]) which will be frequently used throughout this paper.

Lemma 2.4 (Hardy-Littlewood-Sobolev inequality [29]) *Let $1 < r, s < +\infty$ and $0 < \mu < N$ such that $\frac{1}{r} + \frac{1}{s} + \frac{\mu}{N} = 2$. If $\phi \in L^r(\mathbb{R}^N)$ and $\psi \in L^s(\mathbb{R}^N)$, then there exists a sharp constant $C(N, \mu, r, s) > 0$, independent of ϕ and ψ, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\phi(x)\psi(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \le C(N, \mu, r, s)|\phi|_r|\psi|_s.$$

We observe that if $G(u) = |u|^\tau$ for some $\tau > 0$, then Lemma 2.4 illustrates that the integral

$$\int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * G(u)\right]G(u)\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(u(y))G(u(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x$$

is well defined if $G(u) \in L^s(\mathbb{R}^N)$ for $s > 1$ with $\frac{2}{s} + \frac{\mu}{N} = 2$. Since we will work with $u \in E$, in order to make certain that the above integral is well defined, according to Lemma 2.3 we must require that $s\tau \in [p, q^*]$. Then we can see that τ satisfies the following inequality

$$\frac{(2N-\mu)p}{2N} \le \tau \le \frac{(2N-\mu)q^*}{2N}. \tag{2.3}$$

Here we call that the exponent $(2N - \mu)p/2N$ is the lower critical exponent and $(2N - \mu)q^*/2N$ is the upper critical exponent for the double phase problem with nonlocal Choquard reaction.

Normally, if the growth exponent τ of g lies in the range (2.3), we can obtain the existence result of nontrivial solutions by applying variational approaches. However, in the present paper, since we not only investigate the existence of nontrivial solution, but also want to study some properties of solutions such as positivity, regularity and concentration, Hence, we need to require a stronger condition for the growth exponent, see conditions (g_2) and (g_3).

Moreover, we can infer from conditions (g_2) and (g_3) that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g(s)| \leq \varepsilon|s|^{p-1} + C_\varepsilon|s|^{\tau-1} \text{ and } |G(s)| \leq \varepsilon|s|^p + C_\varepsilon|s|^\tau \text{ for any } s \in \mathbb{R}. \tag{2.4}$$

As we have mentioned in the introduction, generally speaking, the energy functional of problem (2.1) does not satisfy the Palais-Smale compactness condition under local potential condition, we will not deal with problem (2.1) directly. Instead, we need make use of the penalization approach developed by del Pino and Felmer [22] to modify the problem (2.1) such that the modified functional satisfies the Palais-Smale condition. After constructing solutions of the modified problems we will make these solutions localized, so they are solutions of the original problem for small ϵ.

We choose suitable two positive constants ℓ_0 and a_0 such that $g(a_0) = \frac{V_{\min}}{\ell_0} a_0^{q-1}$, and we define the following functions

$$\widetilde{g}(s) = \begin{cases} g(s) & \text{if } s \leq a_0, \\ \frac{V_{\min}}{\ell_0} s^{q-1} & \text{if } s > a_0, \end{cases}$$

and

$$f(x,s) = \begin{cases} \chi_\Lambda(x)g(s) + (1-\chi_\Lambda(x))\widetilde{g}(s) & \text{if } s > 0, \\ 0 & \text{if } s \leq 0, \end{cases} \quad F(x,s) = \int_0^s f(x,t)\mathrm{d}t,$$

where χ_Λ is the characteristic function on Λ. Moreover, we duduce from the conditions (g_1)-(g_5) that f satisfies the following properties:

(f_1) $f(x,s) = o(|s|^{p-1})$ as $s \to 0$ uniformly in $x \in \mathbb{R}^N$;
(f_2) $f(x,s) \leq g(s)$ for all $x \in \mathbb{R}^N$ and $s > 0$;
(f_3) $0 < \frac{\theta}{2}F(x,s) \leq f(x,s)s$ for any $x \in \Lambda$ and $s > 0$, and

$$0 \leq \frac{\theta}{2}F(x,s) \leq f(x,s)s \leq \frac{V_{\min}}{\ell_0}(s^p + s^q) \text{ for any } x \in \Lambda^c \text{ and } s > 0;$$

(f_4) the following monotonicity conditions hold

$$s \mapsto \frac{f(x,s)}{s^{\frac{q}{2}-1}}, \frac{F(x,s)}{s^{\frac{q}{2}}} \text{ are strictly increasing for all } x \in \mathbb{R}^N \text{ and } s > 0.$$

Next we shall consider the following modified problem

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\epsilon x)(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * F(\epsilon x, u)\right) f(\epsilon x, u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0, & \text{in } \mathbb{R}^N. \end{cases} \tag{2.5}$$

We note that if u is a solution of problem (2.5) such that

$$u(x) < a_0 \text{ for all } x \in \Lambda_\epsilon^c, \text{ where } \Lambda_\epsilon := \{x \in \mathbb{R}^N : \epsilon x \in \Lambda\},$$

then $f(\epsilon x, u) = g(u)$ and $F(\epsilon x, u) = G(u)$. So, u is also a solution of problem (2.1).

We define the energy functional associated with the modified problem (2.5)

$$\mathcal{I}_\epsilon(u) = \frac{1}{p}\|u\|_{V_\epsilon,p}^p + \frac{1}{q}\|u\|_{V_\epsilon,q}^q - \frac{1}{2}\int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(\epsilon x, u)\right] F(\epsilon x, u)\mathrm{d}x.$$

Using (2.4), (f_2), Lemma 2.3, Lemma 2.4 and applying some standard arguments, we can easily check that $\mathcal{I}_\epsilon$ is well defined on E_ϵ and belongs to C^1 with its derivative given by

$$\begin{aligned}\langle \mathcal{I}_\epsilon'(u), v\rangle = &\int_{\mathbb{R}^N}\left[|\nabla u|^{p-2}\nabla u \nabla v + |\nabla u|^{q-2}\nabla u\nabla v\right]\mathrm{d}x\\ &+ \int_{\mathbb{R}^N} V(\epsilon x)\left[|u|^{p-2}u + |u|^{q-2}u\right] v\mathrm{d}x - \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(\epsilon x, u)\right] f(\epsilon x, u) v\mathrm{d}x\end{aligned}$$

for all $u, v \in E_\epsilon$. Therefore, it is obvious that the solutions of problem (2.5) correspond to critical points of $\mathcal{I}_\epsilon$.

The following result shows that the energy functional $\mathcal{I}_\epsilon$ satisfies the geometric structure of the mountain pass theorem.

Lemma 2.5 *the energy functional $\mathcal{I}_\epsilon$ has a mountain pass geometry, that is*

(i) *there exist σ_0, $\varrho > 0$ such that $\mathcal{I}_\epsilon(u) \geq \sigma_0$ with $\|u\|_\epsilon = \varrho$;*
(ii) *there exist $u_0 \in E_\epsilon$ and $R > 0$ with $\|u_0\|_\epsilon > R$ such that $\mathcal{I}_\epsilon(u_0) < 0$.*

Proof (i) Let $u \in E_\epsilon$, we first take $\varrho \in (0, 1)$ with $\|u\|_\epsilon = \varrho$, then we have $\|u\|_{V_\epsilon,p}^q \leq \|u\|_{V_\epsilon,p}^p < 1$. Therefore, using Lemma 2.1, Lemma 2.4, (f_2), (2.2) and (2.4) we obtain

$$\begin{aligned}\mathcal{I}_\epsilon(u) &= \frac{1}{p}\|u\|_{V_\epsilon,p}^p + \frac{1}{q}\|u\|_{V_\epsilon,q}^q - \frac{1}{2}\int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(\epsilon x, u)\right] F(\epsilon x, u)\mathrm{d}x\\ &\geq \frac{1}{p}\|u\|_{V_\epsilon,p}^q + \frac{1}{q}\|u\|_{V_\epsilon,q}^q - c_1\left(\varepsilon\|u\|_\epsilon^{2p} + \varepsilon C_\varepsilon\|u\|_\epsilon^{p+\tau} + C_\varepsilon\|u\|_\epsilon^{2\tau}\right)\\ &\geq \frac{c_2}{q}\left(\|u\|_{V_\epsilon,p} + \|u\|_{V_\epsilon,q}\right)^q - c_1\left(\varepsilon\|u\|_\epsilon^{2p} + \varepsilon C_\varepsilon\|u\|_\epsilon^{p+\tau} + C_\varepsilon\|u\|_\epsilon^{2\tau}\right)\\ &= \frac{c_2}{q}\|u\|_\epsilon^q - c_1\left(\varepsilon\|u\|_\epsilon^{2p} + \varepsilon C_\varepsilon\|u\|_\epsilon^{p+\tau} + C_\varepsilon\|u\|_\epsilon^{2\tau}\right).\end{aligned}$$

Since $q < 2\tau$, then there exists $\sigma_0 > 0$ such that $\mathcal{I}_\epsilon(u) \geq \sigma_0 > 0$ when $\|u\|_\epsilon = \varrho$.

(ii) We fix $u_0 \in E_\epsilon \backslash \{0\}$ with $\mathrm{supp}(u_0) \subset \Lambda_\epsilon$. Then we have $F(\epsilon x, u_0) = G(u_0)$, and we set

$$h(t) = \frac{1}{2}\int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * G\left(\frac{tu_0}{\|u_0\|_\epsilon}\right)\right] G\left(\frac{tu_0}{\|u_0\|_\epsilon}\right)\mathrm{d}x \text{ for } t > 0.$$

From (g_4) we can infer that

$$\begin{aligned} h'(t) =& \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G\left(\frac{tu_0}{\|u_0\|_\epsilon} \right) \right] g\left(\frac{tu_0}{\|u_0\|_\epsilon} \right) \frac{u_0}{\|u_0\|_\epsilon} \mathrm{d}x \\ \geq& \frac{\theta}{2t} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G\left(\frac{tu_0}{\|u_0\|_\epsilon} \right) \right] G\left(\frac{tu_0}{\|u_0\|_\epsilon} \right) \mathrm{d}x \\ =& \frac{\theta}{t} h(t). \end{aligned} \tag{2.6}$$

Integrating (2.6) on $[1, s\|u_0\|_\epsilon]$ with $s\|u_0\|_\epsilon > 1$, we have

$$h(s\|u_0\|_\epsilon) \geq h(1)(s\|u_0\|_\epsilon)^\theta.$$

Consequently, we immediately obtain

$$\frac{1}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(su_0) \right] G(su_0) \mathrm{d}x \geq \frac{1}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G\left(\frac{u_0}{\|u_0\|_\epsilon} \right) \right] G\left(\frac{u_0}{\|u_0\|_\epsilon} \right) \mathrm{d}x (s\|u_0\|_\epsilon)^\theta.$$

Combining with the above facts we derive that

$$\begin{aligned} \mathcal{I}_\epsilon(su_0) =& \frac{s^p}{p} \|u_0\|^p_{V_\epsilon,p} + \frac{s^q}{q} \|u_0\|^q_{V_\epsilon,q} - \frac{1}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\epsilon x, su_0) \right] F(\epsilon x, su_0) \mathrm{d}x \\ =& \frac{s^p}{p} \|u_0\|^p_{V_\epsilon,p} + \frac{s^q}{q} \|u_0\|^q_{V_\epsilon,q} - \frac{1}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(su_0) \right] G(su_0) \mathrm{d}x \\ \leq& c_3(s^p + s^q) - c_4 s^\theta \end{aligned}$$

for $s > \frac{1}{\|u_0\|_\epsilon}$. Taking $e = su_0$ with s sufficiently large, we can see that the conclusion (ii) holds since $\theta = 2q > q$. □

According to Lemma 2.5, we can use a version of mountain pass theorem without the Palais-Smale condition [47] to deduce the existence of a Palais-Smale sequence $\{u_n\}$ at level c_ϵ, namely

$$\mathcal{I}_\epsilon(u_n) \to c_\epsilon \text{ and } \mathcal{I}'_\epsilon(u_n) \to 0,$$

where c_ϵ is the mountain pass level of $\mathcal{I}_\epsilon$ defined as

$$c_\epsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{I}_\epsilon(\ell(t)) = \inf_{u \in E_\epsilon \setminus \{0\}} \max_{t \geq 0} \mathcal{I}_\epsilon(tu),$$

and

$$\Gamma = \{\gamma \in C([0,1], E_\epsilon) : \gamma(0) = 0, \mathcal{I}_\epsilon(\gamma(1)) < 0\}.$$

We observe that since $\mathrm{supp}(u_0) \subset \Lambda_\epsilon$, then there exists $\widehat{c} > 0$ independent of ϵ, ℓ_0, a_0 such that

$$c_\epsilon = \inf_{u \in E_\epsilon \setminus \{0\}} \max_{t \geq 0} \mathcal{I}_\epsilon(tu) < \widehat{c}.$$

Lemma 2.6 *Let $\{u_n\}$ be a Palais-Smale sequence of $\mathcal{I}_\epsilon$ at level $0 < c_\epsilon < \widehat{c}$, then $\{u_n\}$ is bounded in E_ϵ and $\|u_n^-\|_\epsilon = o(1)$. Moreover, there holds*

$$\|u_n\|_{V_\epsilon,p}^p + \|u_n\|_{V_\epsilon,q}^q \le 2q(\widehat{c}+1) \text{ for } n \in \mathbb{N} \text{ large enough.} \tag{2.7}$$

Proof Let $\{u_n\}$ be a Palais-Smale sequence of $\mathcal{I}_\epsilon$ at level $0 < c_\epsilon < \widehat{c}$. Observe that $\theta = 2q$, and from (f_3) we can deduce that

$$\begin{aligned}\widehat{c} + o_n(1)\|u_n\|_\epsilon \ge & \mathcal{I}_\epsilon(u_n) - \frac{1}{\theta}\langle \mathcal{I}_\epsilon'(u_n), u_n\rangle \\ = & \left[\frac{1}{p} - \frac{1}{\theta}\right]\|u_n\|_{V_\epsilon,p}^p + \left[\frac{1}{q} - \frac{1}{\theta}\right]\|u_n\|_{V_\epsilon,q}^q \\ & + \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(\epsilon x, u_n)\right]\left[\frac{1}{\theta}f(\epsilon x, u_n(x))u_n(x) - \frac{1}{2}F(\epsilon x, u_n(x))\right]\mathrm{d}x \\ \ge & \left[\frac{1}{p} - \frac{1}{\theta}\right]\|u_n\|_{V_\epsilon,p}^p + \left[\frac{1}{q} - \frac{1}{\theta}\right]\|u_n\|_{V_\epsilon,q}^q \\ \ge & \left[\frac{1}{q} - \frac{1}{\theta}\right](\|u_n\|_{V_\epsilon,p}^p + \|u_n\|_{V_\epsilon,q}^q).\end{aligned} \tag{2.8}$$

We use a contradiction argument to prove this conclusion, and we assume that $\|u\|_\epsilon \to \infty$. In what follows we divide into three cases to finish the proof of the lemma.

Case1. $\|u_n\|_{V_\epsilon,p} \to \infty$ and $\|u_n\|_{V_\epsilon,q} \to \infty$. Since $p < q$, we can see that $\|u_n\|_{V_\epsilon,q}^q \ge \|u_n\|_{V_\epsilon,q}^p > 1$ for n sufficiently large. Therefore, from the above facts we infer that

$$\begin{aligned}\widehat{c} + o_n(1)\|u\|_\epsilon \ge & \left[\frac{1}{q} - \frac{1}{\theta}\right](\|u_n\|_{V_\epsilon,p}^p + \|u_n\|_{V_\epsilon,q}^q) \\ \ge & \left[\frac{1}{q} - \frac{1}{\theta}\right](\|u_n\|_{V_\epsilon,p}^p + \|u_n\|_{V_\epsilon,q}^p) \\ \ge & c_5(\|u_n\|_{V_\epsilon,p} + \|u_n\|_{V_\epsilon,q})^p = c_5\|u_n\|_\epsilon^p.\end{aligned}$$

Evidently, this is impossible, a contradiction.

Case 2. $\|u_n\|_{V_\epsilon,p} \to \infty$ and $\|u_n\|_{V_\epsilon,q}$ is bounded. According to the following fact

$$\widehat{c} + o_n(1)(\|u_n\|_{V_\epsilon,p} + \|u_n\|_{V_\epsilon,q}) \ge \left[\frac{1}{q} - \frac{1}{\theta}\right]\|u_n\|_{V_\epsilon,p}^p,$$

we have

$$\frac{\widehat{c}}{\|u_n\|_{V_\epsilon,p}^p} + \frac{\|u_n\|_{V_\epsilon,p}}{\|u_n\|_{V_\epsilon,p}^p} + o_n(1) \ge \frac{1}{q} - \frac{1}{\theta}.$$

Letting $n \to \infty$, we can see that $0 \ge \frac{1}{q} - \frac{1}{\theta} > 0$, which shows a contradiction.

Case 3. $\|u_n\|_{V_\epsilon,p}$ is bounded and $\|u_n\|_{V_\epsilon,q} \to \infty$. We can proceed similarly as in the Case 2.

From the above three cases we can see that $\{u_n\}$ is bounded, and we have $\langle \mathcal{I}'_\epsilon(u_n), u_n^- \rangle = o(1)$. Employing the definition of f and the following inequality

$$|a-b|^{s-2}(a-b)(a^- - b^-) \geq |a^- - b^-|^s \text{ for all } s > 1, \tag{2.9}$$

we get

$$\begin{aligned}\|u_n^-\|_{V_\epsilon,p}^p + \|u_n^-\|_{V_\epsilon,q}^q &\leq \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla u_n^- \mathrm{d}x + \int_{\mathbb{R}^N} V(\epsilon x)|u_n|^{p-2}u_n u_n^- \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n \nabla u_n^- \mathrm{d}x + \int_{\mathbb{R}^N} V(\epsilon x)|u_n|^{q-2}u_n u_n^- \mathrm{d}x \\ &= \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\epsilon x, u_n)\right] f(\epsilon x, u_n) u_n^- \mathrm{d}x = o_n(1).\end{aligned}$$

which implies that $\|u_n^-\|_\epsilon \to 0$ in E_ϵ. Consequently, we may assume that $u_n \geq 0$ for any $n \in \mathbb{N}$.

Finally, according to (2.8), it is easy to see that

$$\|u_n\|_{V_\epsilon,p}^p + \|u_n\|_{V_\epsilon,q}^q \leq \frac{q\theta}{\theta - q}(\widehat{c}+1) = 2q(\widehat{c}+1) \text{ for } n \in \mathbb{N} \text{ large enough.}$$

The proof is now complete. □

Next we need to fix the notations:

$$\mathscr{B} := \left\{u \in E_\epsilon : \|u\|_{V_\epsilon,p}^p + \|u\|_{V_\epsilon,q}^q \leq 2q(\widehat{c}+1)\right\} \tag{2.10}$$

and

$$K_\epsilon(u)(x) := \frac{1}{|x|^\mu} * F(\epsilon x, u).$$

With the above notations, we are able to prove the following results.

Lemma 2.7 *Let $0 < \mu < p$ and assume that (f_1)-(f_3) are satisfied, then there exists a constant $\ell_0 > 0$ such that*

$$\sup_{u \in \mathscr{B}} |K_\epsilon(u)(x)|_\infty \leq \frac{\ell_0}{2} \text{ for all } \epsilon > 0. \tag{2.11}$$

Proof Firstly, using (2.4) and (f_2) we have

$$|F(\epsilon x, u)| \leq |G(u)| \leq \varepsilon |u|^p + C_\varepsilon |u|^\tau \text{ for any } x \in \mathbb{R}^N \text{ and } u \in \mathbb{R}. \tag{2.12}$$

Therefore, we deduce from Lemma 2.3, (2.10) and (2.12) that

$$
\begin{aligned}
|K_\epsilon(u)(x)| = & \left|\int_{\mathbb{R}^N} \frac{F(\epsilon x, u)}{|x-y|^\mu} \mathrm{d}y\right| = \left|\int_{|x-y|\le 1} \frac{F(\epsilon x, u)}{|x-y|^\mu} \mathrm{d}y\right| + \left|\int_{|x-y|\ge 1} \frac{F(\epsilon x, u)}{|x-y|^\mu} \mathrm{d}y\right| \\
\le & C_1 \int_{|x-y|\le 1} \frac{|u|^p+|u|^\tau}{|x-y|^\mu} \mathrm{d}y + C_1 \int_{|x-y|\ge 1} (|u|^p+|u|^\tau) \mathrm{d}y \\
\le & C_1 \int_{|x-y|\le 1} \frac{|u|^p+|u|^\tau}{|x-y|^\mu} \mathrm{d}y + C_2.
\end{aligned} \tag{2.13}
$$

Since $0 < \mu < p$, we can take $t \in (\frac{N}{N-\mu}, \frac{N}{N-p})$ such that $N - \frac{t\mu}{t-1} > 0$. We infer from the Hölder inequality that

$$
\begin{aligned}
\int_{|x-y|\le 1} \frac{|u|^p}{|x-y|^\mu} \mathrm{d}y \le & \left[\int_{|x-y|\le 1} |u|^{tp} \mathrm{d}y\right]^{\frac{1}{t}} \left[\int_{|x-y|\le 1} |x-y|^{-\frac{t\mu}{t-1}} \mathrm{d}y\right]^{\frac{t-1}{t}} \\
\le & C_3 \left[\int_{|r|\le 1} |r|^{N-1-\frac{t\mu}{t-1}} \mathrm{d}r\right]^{\frac{t-1}{t}} \le C_4.
\end{aligned} \tag{2.14}
$$

Similarly, since $\tau \in [q, \frac{(N-\mu)q}{N-q})$, taking $s \in (\frac{N}{N-\mu}, \frac{Nq}{(N-q)\tau})$ such that $N - \frac{s\mu}{s-1} > 0$, we obtain

$$
\begin{aligned}
\int_{|x-y|\le 1} \frac{|u|^\tau}{|x-y|^\mu} \mathrm{d}y \le & \left[\int_{|x-y|\le 1} |u|^{s\tau} \mathrm{d}y\right]^{\frac{1}{s}} \left[\int_{|x-y|\le 1} |x-y|^{-\frac{s\mu}{s-1}} \mathrm{d}y\right]^{\frac{s-1}{s}} \\
\le & C_5 \left[\int_{|r|\le 1} |r|^{N-1-\frac{s\mu}{s-1}} \mathrm{d}r\right]^{\frac{s-1}{s}} \le C_6.
\end{aligned} \tag{2.15}
$$

Consequently, from (2.13), (2.14) and (2.15) we can conclude that there exists a constant $\ell_0 > 0$ such that (2.11) holds. We complete the proof of lemma. □

Lemma 2.8 *Let $\{u_n\}$ be a Palais-Smale sequence of $\mathcal{I}_\epsilon$ at level $0 < c_\epsilon < \widehat{c}$, then for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that*

$$
\limsup_{n\to\infty} \int_{B_R^c(0)} |\nabla u_n|^p + |\nabla u_n|^q + V(\epsilon x)(|u_n|^q + |u_n|^p) \mathrm{d}x < \varepsilon. \tag{2.16}
$$

Proof For any $R > 0$ and some $c > 0$ independent of R, let $\psi_R \in C^\infty(\mathbb{R}^N)$ be such that

$$
0 \le \psi_R \le 1,\ \psi_R(x) = 0 \text{ in } B_{R/2}(0),\ \psi_R(x) = 1 \text{ in } B_R^c(0) \text{ and } |\nabla \psi_R| \le \frac{c}{R}.
$$

Using Lemma 2.6 we know that $\{u_n\}$ is bounded in E_ϵ, moreover, there exists $n_0 \in \mathbb{N}$ such that

$$
\|u_n\|_{V_\epsilon,p}^p + \|u_n\|_{V_\epsilon,q}^q \le 2q(\widehat{c}+1) \text{ for all } n \ge n_0.
$$

Using the above estimate and Lemma 2.7 we obtain

$$\frac{\sup_{n\geq n_0}|K_\epsilon(u_n)(x)|_\infty}{\ell_0}\leq\frac{1}{2}.$$

On the other hand, according to the boundedness of $\{u_n\}$, we can see that $\langle\mathcal{I}_\epsilon'(u_n),\psi_R u_n\rangle=o_n(1)$. Hence, we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[|\nabla u_n|^p+|\nabla u_n|^q+V(\epsilon x)(|u_n|^q+|u_n|^p)\right]\psi_R\mathrm{d}x\\&=\int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu}*F(\epsilon x,u_n)\right]f(\epsilon x,u_n)u_n\psi_R\mathrm{d}x\\&\quad-\int_{\mathbb{R}^N}\left[|\nabla u_n|^{p-2}+|\nabla u_n|^{q-2}\right]\nabla u_n\nabla\psi_R u_n\mathrm{d}x+o_n(1).\end{aligned}$$

Taking $R>0$ such that $\Lambda_\epsilon\subset B_{R/2}(0)$ and using (f_3) and Hölder inequality, we can derive that

$$\begin{aligned}&\int_{B^c_{R/2}(0)}\left[|\nabla u_n|^p+|\nabla u_n|^q+V(\epsilon x)(|u_n|^q+|u_n|^p)\right]\mathrm{d}x\\&\leq\int_{B^c_{R/2}(0)}\left[\frac{1}{|x|^\mu}*F(\epsilon x,u_n)\right]f(\epsilon x,u_n)u_n\psi_R\mathrm{d}x+\frac{c}{R}(\|u_n\|_\epsilon^p+\|u_n\|_\epsilon^q)+o_n(1)\\&\leq\int_{B^c_{R/2}(0)}\left[\frac{\sup_{n\geq n_0}|K_\epsilon(u_n)(x)|_\infty}{\ell_0}\right]V_{\min}(|u_n|^p+|u_n|^q)\mathrm{d}x+\frac{c}{R}(\|u_n\|_\epsilon^p+\|u_n\|_\epsilon^q)+o_n(1)\\&\leq\frac{1}{2}\int_{B^c_{R/2}(0)}V(\epsilon x)(|u_n|^p+|u_n|^q)\mathrm{d}x+\frac{c}{R}(\|u_n\|_\epsilon^p+\|u_n\|_\epsilon^q)+o_n(1).\end{aligned}$$

Consequently, according to the arbitrariness of R and the boundedness of $\{u_n\}$, we can see that for any $\varepsilon>0$, there exists $R=R(\varepsilon)>0$ such that

$$\limsup_{n\to\infty}\int_{B^c_R(0)}|\nabla u_n|^p+|\nabla u_n|^q+V(\epsilon x)(|u_n|^q+|u_n|^p)\mathrm{d}x<\varepsilon.$$

This finishes the proof of lemma. □

Now, we show that the modified functional $\mathcal{I}_\epsilon$ satisfies the Palais-Smale compactness condition, which will play a crucial role in our arguments.

Lemma 2.9 *$\mathcal{I}_\epsilon$ satisfies the Palais-Smale compactness condition at level $c\in[c_\epsilon,\widehat{c})$.*

Proof Firstly, let $\{u_n\}$ be a Palais-Smale sequence of $\mathcal{I}_\epsilon$ at level $c\in[c_\epsilon,\widehat{c})$. According to Lemma 2.6 we know that $\{u_n\}$ is bounded. Thus, we may assume that, after passing to a subsequence, $u_n\rightharpoonup u$ in E_ϵ, $u_n\to u$ in $L^s_{loc}(\mathbb{R}^N)$ for all $s\in[p,q^*)$ and $u_n(x)\to u(x)$ a.e. in $\mathbb{R}^N$. Moreover, from [49, Lemma 3.1] we can see that u is a critical point of the functional $\mathcal{I}_\epsilon$. It follows that

$$\langle\mathcal{I}_\epsilon'(u),u\rangle=\langle\mathcal{I}_\epsilon'(u),u_n\rangle=0. \tag{2.17}$$

For simplicity, we set

$$
\begin{aligned}
A_n =& \int_{\mathbb{R}^N} \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n - \nabla u\rangle \mathrm{d}x \\
&+ \int_{\mathbb{R}^N} \langle |\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u, \nabla u_n - \nabla u\rangle \mathrm{d}x \\
&+ \int_{\mathbb{R}^N} V(\epsilon x)\Big[(|u_n|^{p-2}u_n - |u|^{p-2}u) + (|u_n|^{q-2}u_n - |u|^{q-2}u)\Big](u_n - u)\mathrm{d}x.
\end{aligned}
$$

Computing directly, we get

$$
\begin{aligned}
A_n =&\langle \mathcal{I}_\epsilon'(u_n), u_n - u\rangle - \langle \mathcal{I}_\epsilon'(u), u_n - u\rangle \\
&+ \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\epsilon x, u_n)\right] f(\epsilon x, u_n)(u_n - u)\mathrm{d}x \\
&- \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\epsilon x, u)\right] f(\epsilon x, u)(u_n - u)\mathrm{d}x.
\end{aligned}
\tag{2.18}
$$

Next we shall prove the following conclusion

$$
\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\epsilon x, u_n)\right] f(\epsilon x, u_n)(u_n - u)\mathrm{d}x = o_n(1). \tag{2.19}
$$

Observe that, from (2.4), (g_3), (f_2) and Lemma 2.3 we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} |F(\epsilon x, u_n)|^{\frac{2N}{2N-\mu}}\mathrm{d}x \le& c_6 \int_{\mathbb{R}^N} ||u_n|^p + |u_n|^\tau|^{\frac{2N}{2N-\mu}}\mathrm{d}x \\
\le& c_7 \int_{\mathbb{R}^N} \left(|u_n|^{\frac{2Np}{2N-\mu}} + |u_n|^{\frac{2N\tau}{2N-\mu}}\right)\mathrm{d}x \\
\le& c_8 \left(\|u_n\|_\epsilon^{\frac{2Np}{2N-\mu}} + \|u_n\|_\epsilon^{\frac{2N\tau}{2N-\mu}}\right) \le C,
\end{aligned}
$$

which implies that $F(\epsilon x, u_n)$ is bounded in $L^{2N/(2N-\mu)}(\mathbb{R}^N)$. Moreover, since $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$, and thus $F(\epsilon x, u_n(x)) \to F(\epsilon x, u(x))$ a.e. in $\mathbb{R}^N$, we also obtain

$$
F(\epsilon x, u_n) \rightharpoonup F(\epsilon x, u) \text{ in } L^{2N/(2N-\mu)}(\mathbb{R}^N).
$$

On account of Lemma 2.4, we know that the convolution term

$$
\frac{1}{|x|^\mu} * w(x) \in L^{2N/\mu}(\mathbb{R}^N) \text{ for all } w \in L^{2N/(2N-\mu)}(\mathbb{R}^N)
$$

is a linear bounded operator from $L^{2N/(2N-\mu)}(\mathbb{R}^N)$ to $L^{2N/\mu}(\mathbb{R}^N)$. So we can see that

$$
\frac{1}{|x|^\mu} * F(\epsilon x, u_n) \rightharpoonup \frac{1}{|x|^\mu} * F(\epsilon x, u) \text{ in } L^{2N/\mu}(\mathbb{R}^N). \tag{2.20}
$$

Since f has a subcritical growth, using Lemma 2.3 and (2.20), we derive that for any fixed $R > 0$,

$$\int_{B_R(0)} \left[\frac{1}{|x|^\mu} * F(\epsilon x, u_n)\right] f(\epsilon x, u_n)(u_n - u)\mathrm{d}x = o_n(1). \tag{2.21}$$

Using the growth condition, the definition of f and the boundedness of $\frac{1}{|x|^\mu} * F(\epsilon x, u_n)$, we obtain

$$\int_{B_R^c(0)} \left[\frac{1}{|x|^\mu} * F(\epsilon x, u_n)\right] |f(\epsilon x, u_n)u_n|\mathrm{d}x \le c_9 \int_{B_R^c(0)} (|u_n|^p + |u_n|^q)\mathrm{d}x,$$

this is because we can choose suitable $R > 0$ such that $\Lambda \subset B_R(0)$. Moreover, from Lemma 2.8, for any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{B_R^c(0)} \left[\frac{1}{|x|^\mu} * F(\epsilon x, u_n)\right] |f(\epsilon x, u_n)u_n|\mathrm{d}x \\
\le &\limsup_{n\to\infty} c_9 \int_{B_R^c(0)} (|u_n|^p + |u_n|^q)\mathrm{d}x \\
\le &\limsup_{n\to\infty} c_{10} \int_{B_R^c(0)} |\nabla u_n|^p + |\nabla u_n|^q + V(\epsilon x)(|u_n|^q + |u_n|^p)\mathrm{d}x \\
\le &c_{10}\varepsilon.
\end{aligned} \tag{2.22}$$

Similarly, using Hölder inequality, we can also show that

$$\limsup_{n\to\infty} \int_{B_R^c(0)} \left[\frac{1}{|x|^\mu} * F(\epsilon x, u_n)\right] |f(\epsilon x, u_n)u|\mathrm{d}x \le c_{11}\varepsilon. \tag{2.23}$$

Consequently, from (2.21), (2.22) and (2.23) we can infer that (2.19) holds.

Following the same argument, we also obtain

$$\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\epsilon x, u)\right] f(\epsilon x, u)(u_n - u)\mathrm{d}x = o_n(1). \tag{2.24}$$

So, we infer from (2.17), (2.18), (2.19) and (2.24) that $A_n = o_n(1)$. Using the following inequality

$$\langle |\xi|^{s-2}\xi - |\eta|^{s-2}\eta, \xi - \eta\rangle \ge \begin{cases} c|\xi - \eta|^s, & \text{if } s \ge 2, \\ c(|\xi| + |\eta|)^{s-2}|\xi - \eta|^2, & \text{if } 1 < s < 2, \end{cases} \forall \xi, \eta \in \mathbb{R}^N \tag{2.25}$$

we conclude that for the case $p > 2$

$$\begin{aligned}\|u_n - u\|^p_{V_\epsilon,p} =& \int_{\mathbb{R}^N} |\nabla(u_n - u)|^p + V(\epsilon x)|u_n - u|^p \mathrm{d}x \\ \leq& c_{11} \int_{\mathbb{R}^N} \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n - \nabla u\rangle \mathrm{d}x \\ &+ c_{12} \int_{\mathbb{R}^N} V(\epsilon x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\mathrm{d}x \\ =& o_n(1).\end{aligned} \tag{2.26}$$

For the case $1 < p < 2$, using the boundedness of $\{u_n\}$, Hölder inequality and (2.25) we have

$$\begin{aligned}\left[\int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^p \mathrm{d}x\right]^{\frac{2}{p}} \leq& \left[\int_{\mathbb{R}^N} \frac{|\nabla u_n - \nabla u|^2}{(|\nabla u_n| + |\nabla u|)^{2-p}} \mathrm{d}x\right]\left[\int_{\mathbb{R}^N} (|\nabla u_n| + |\nabla u|)^p \mathrm{d}x\right]^{\frac{2-p}{p}} \\ \leq& c_{13} \int_{\mathbb{R}^N} \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n - \nabla u\rangle \mathrm{d}x = o_n(1),\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}\left[\int_{\mathbb{R}^N} V(\epsilon x)|u_n - u|^p \mathrm{d}x\right]^{\frac{2}{p}} \leq& \left[\int_{\mathbb{R}^N} \frac{V(\epsilon x)|u_n - u|^2}{(|u_n| + |u|)^{2-p}} \mathrm{d}x\right]\left[\int_{\mathbb{R}^N} V(\epsilon x)(|u_n| + |u|)^p \mathrm{d}x\right]^{\frac{2-p}{p}} \\ \leq& c_{14} \int_{\mathbb{R}^N} V(\epsilon x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\mathrm{d}x = o_n(1).\end{aligned} \tag{2.28}$$

Combining (2.26), (2.27) and (2.28) we can see that $\|u_n - u\|^p_{V_\epsilon,p} = o_n(1)$ for any $p > 1$.

Similarly, we also have $\|u_n - u\|^q_{V_\epsilon,q} = o_n(1)$. Therefore, it is easy to see that

$$\|u_n - u\|_\epsilon = \|u_n - u\|_{V_\epsilon,p} + \|u_n - u\|_{V_\epsilon,q} = o_n(1).$$

We complete the proof of lemma. □

Based on Lemma 2.5 and Lemma 2.9, we can apply the mountain pass theorem [47] to obtain the existence result of solutions for the modified problem (2.5).

Lemma 2.10 *The modified problem (2.5) at least possesses a nontrivial nonnegative solution for all small $\epsilon > 0$.*

Proof Evidently, using Lemma 2.5 and Lemma 2.9 and employing the mountain pass theorem [47], we can infer that for all $\epsilon > 0$ sufficiently small, there exists $u_\epsilon \in E_\epsilon\backslash\{0\}$ such that

$$\mathcal{I}_\epsilon(u_\epsilon) = c_\epsilon \text{ and } \mathcal{I}'_\epsilon(u_\epsilon) = 0. \tag{2.29}$$

Moreover, choosing $u_\epsilon^- = \{u_\epsilon, 0\}$ and recalling that $f(\epsilon x, s) = 0$ for all $s \le 0$ and $\langle \mathcal{I}_\epsilon'(u_\epsilon), u_\epsilon^- \rangle = 0$, we can derive that

$$\begin{aligned}\|u_\epsilon^-\|_{V_\epsilon,p}^p + \|u_\epsilon^-\|_{V_\epsilon,q}^q &\le \int_{\mathbb{R}^N} |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \nabla u_\epsilon^- \mathrm{d}x + \int_{\mathbb{R}^N} V(\epsilon x)|u_\epsilon|^{p-2}u_\epsilon u_\epsilon^- \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} |\nabla u_\epsilon|^{q-2}\nabla u_\epsilon \nabla u_\epsilon^- \mathrm{d}x + \int_{\mathbb{R}^N} V(\epsilon x)|u_\epsilon|^{q-2}u_\epsilon u_\epsilon^- \mathrm{d}x \\ &= \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\epsilon x, u_\epsilon)\right] f(\epsilon x, u_\epsilon)u_\epsilon^- \mathrm{d}x = 0.\end{aligned}$$

which implies that $\|u_\epsilon^-\|_\epsilon = 0$. So, $u_\epsilon \ge 0$ and $u_\epsilon \not\equiv 0$. This ends the proof. □

3 The autonomous problem

For our scope, we shall also investigate the limit problem associated with problem (2.5). To this end, we first discuss in this section the existence of the positive ground state solutions of the autonomous problem.

Let $a > 0$, we consider the following autonomous problem

$$\begin{cases} -\Delta_p u - \Delta_q u + a(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \, u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{3.1}$$

and define the space

$$E_a = \left\{u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(|u|^p + |u|^q)\mathrm{d}x < \infty\right\}$$

with the norm $\|u\|_a = \|u\|_{a,p} + \|u\|_{a,q}$, where

$$\|u\|_{a,s}^s = \int_{\mathbb{R}^N} (|\nabla u|^s + a|u|^s)\mathrm{d}x \text{ for all } s > 1.$$

Clearly, we can see that the norms $\|\cdot\|_a$ and $\|\cdot\|$ are equivalent, accordingly, $E_a = E$.

The corresponding energy functional of problem (3.1) is defined by

$$\mathcal{J}_a(u) = \frac{1}{p}\|u\|_{a,p}^p + \frac{1}{q}\|u\|_{a,q}^q - \frac{1}{2}\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(u)\right] G(u)\mathrm{d}x.$$

Clearly, we can easily see that $\mathcal{J}_a \in C^1(E_a, \mathbb{R})$ and

$$\begin{aligned}\langle \mathcal{J}_a'(u), v\rangle &= \int_{\mathbb{R}^N} \left[|\nabla u|^{p-2}\nabla u \nabla v + |\nabla u|^{q-2}\nabla u \nabla v\right] \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} a[|u|^{p-2}u + |u|^{q-2}u]v\mathrm{d}x - \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(u)\right] g(u)v\mathrm{d}x\end{aligned}$$

for any $u, v \in E_a$. We use $\mathcal{N}_a$ and c_a to denote the corresponding Nehari manifold of $\mathcal{J}_a$

$$\mathcal{N}_a := \{u \in E_a \setminus \{0\} : \langle \mathcal{J}_a'(u), u\rangle = 0\}.$$

Analogous to arguments used in Section 2, it is easy to check that $\mathcal{J}_a$ has a mountain pass geometry. Then, according to the mountain pass theorem, we can find a Palais-Smale sequence $\{u_n\} \subset E_a$ at the level $c_a > 0$ of $\mathcal{J}_a$. Moreover, using (g_5) and a standard argument [47], we can obtain a minimax characterization

$$c_a = \inf_{u \in E_a \setminus \{0\}} \max_{t \geq 0} \mathcal{J}_a(tu) = \inf_{u \in \mathcal{N}_a} \mathcal{J}_a(u).$$

Lemma 3.1 *Let $\{u_n\} \subset E_a$ be a Palais-Smale sequence for $\mathcal{J}_a$ at the level $c \in \mathbb{R}$ and such that $u_n \rightharpoonup 0$ in E_a. Then we have either*

(a) *$u_n \to 0$ in E_a, or*
(b) *there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R, \sigma > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^q \mathrm{d}x > \sigma.$$

Proof Suppose that (b) does not occur. Then, for all $R > 0$ we have

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^q \mathrm{d}x = 0.$$

On account of Lemma 2.2, we can see that $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for all $s \in (p, q^*)$. So, using Lemma 2.4 and (2.4) we obtain

$$\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(u_n)\right] g(u_n)u_n \mathrm{d}x = o_n(1).$$

Finally, according to the fact that $\langle \mathcal{J}_a'(u_n), u_n\rangle = o_n(1)$, we get $\|u_n\|_{a,p}^p + \|u_n\|_{a,q}^q = o_n(1)$, that is $u_n \to 0$ in E_a as $n \to \infty$. □

We now state the main result for the autonomous problem (3.1).

Lemma 3.2 *Assume that conditions (g_1)-(g_5) hold. Then problem (3.1) has at least one positive ground state solution v such that $\mathcal{J}_a(v) = c_a > 0$.*

Proof Let $\{u_n\}$ be a Palais-Smale sequence at level $c_a > 0$ for $\mathcal{J}_a$, Using (g_4) and the proof of Lemma 2.6 we can show that $\{u_n\}$ is bounded in E_a. Then, there exists $u \in E_a$ such that $u_n \rightharpoonup u$ in E_a, $u_n \to u$ in $L^s_{loc}(\mathbb{R}^N)$ for all $s \in (p, q^*)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. Moreover, using [49, Lemma 3.1] we have $\langle \mathcal{J}_a'(u), \varphi\rangle = 0$ for all $\varphi \in E_a$.

If $u = 0$, then $u_n \nrightarrow 0$ in E_a because $c_a > 0$. Thus, according to Lemma 3.1, we can deduce that there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R, \sigma > 0$ such that

$$\int_{B_R(y_n)} |u_n|^q \mathrm{d}x > \sigma.$$

Let us define $v_n(x) = u_n(x + y_n)$, it follows that

$$\int_{B_R(0)} |v_n|^q \mathrm{d}x \geq \delta. \tag{3.2}$$

Since $\mathcal{J}_a$ and $\mathcal{J}_a'$ are both invariants by translation, we have

$$\mathcal{J}_a(v_n) \to c_a \text{ and } \mathcal{J}_a'(v_n) \to 0. \tag{3.3}$$

After passing to a subsequence, we assume that $v_n \rightharpoonup v$ in E_a, $v_n \to v$ in $L^s_{loc}(\mathbb{R}^N)$ for $s \in (p, q^*)$, and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. Moreover, from (3.2) and (3.3) we infer that $v \neq 0$ and $\mathcal{J}_a'(v) = 0$.

Consequently, we can see that $v \in \mathcal{N}_a$ and $\mathcal{J}_a(v) \geq c_a$. On the other hand, we conclude from Fatou's lemma and (g_4) that

$$\begin{aligned}
c_a &= \lim_{n\to\infty} \left[\mathcal{J}_a(v_n) - \frac{1}{\theta}\langle \mathcal{J}_a'(v_n), v_n\rangle \right] \\
&= \lim_{n\to\infty} \left[\left(\frac{1}{p} - \frac{1}{\theta}\right) \|v_n\|_{a,p}^p + \left(\frac{1}{q} - \frac{1}{\theta}\right) \|v_n\|_{a,q}^q \right. \\
&\quad \left. + \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(v_n)\right]\left[\frac{1}{\theta} g(v_n)v_n - \frac{1}{2}G(v_n)\right] \mathrm{d}x \right] \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right) \|v\|_{a,p}^p + \left(\frac{1}{q} - \frac{1}{\theta}\right) \|v\|_{a,q}^q \\
&\quad + \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(v)\right]\left[\frac{1}{\theta} g(v)v - \frac{1}{2}G(v)\right] \mathrm{d}x \\
&= \mathcal{J}_a(v) - \frac{1}{\theta}\langle \mathcal{J}_a'(v), v\rangle = \mathcal{J}_a(v).
\end{aligned}$$

Therefore, we get $\mathcal{J}_a(v) = c_a$ and v is a ground state solution of problem (3.1). Moreover, by Lemma 2.10 we have $v \geq 0$ in $\mathbb{R}^N$.

Next we show that the function v is positive. In fact, according to the above arguments and Lemma 2.7, we know that v is a solution of problem

$$-\Delta_p v - \Delta_q v + a(|v|^{p-2}v + |v|^{q-2}v) = K(u)(x)g(v) \text{ in } \mathbb{R}^N,$$

where $K \in L^\infty(\mathbb{R}^N)$. Applying the regularity conclusions in [24], we have $v \in L^\infty(\mathbb{R}^N) \cap C^{1,\nu}_{loc}(\mathbb{R}^N)$ for some $\nu \in (0, 1)$. Finally, we take advantage of the Harnack's inequality in [45] to conclude that $v > 0$ in $\mathbb{R}^N$. □

4 Existence and localized concentration

In this section, we are going to investigate the existence and localized concentration of positive solutions to problem (1.1), and we give the proof of Theorem 1.1.

Let $\mathcal{J}_{V_0}$, $\mathcal{N}_{V_0}$ and c_{V_0} denote the corresponding energy functional, Nehari manifold and ground state energy of the following limit problem

$$\begin{cases} -\Delta_p u - \Delta_q u + V_0(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0, & \text{in } \mathbb{R}^N. \end{cases} \tag{4.1}$$

We begin by establishing an important connection between c_ϵ and c_{V_0}, which play a fundamental role in our arguments.

Lemma 4.1 *We have the following estimate*

$$\limsup_{\epsilon\to 0} c_\epsilon \le c_{V_0}.$$

Proof According to Lemma 3.2, we know that there exists a positive ground state solution u to problem (4.1), thus, we have $\mathcal{J}_{V_0}(u) = c_{V_0}$. Let $\psi \in C^\infty(\mathbb{R})$ be such that $0 \le \psi \le 1$, $\psi = 1$ in $[-1, 1]$ and $\psi = 0$ in $\mathbb{R}\backslash(-2, 2)$. Since $0 \in \Pi \subset \Lambda$, we may assume that $B_2 \subset \Lambda$. Define function $u_\epsilon(x) = \psi(2\epsilon|x|)u(x)$ and note that $\text{supp}(u_\epsilon(x)) \subset \Lambda_\epsilon$. By straightforward computation, we can prove

$$\|u - u_\epsilon\| \to 0 \text{ and } \mathcal{J}_{V_0}(u_\epsilon) \to \mathcal{J}_{V_0}(u) \text{ as } \epsilon \to 0. \tag{4.2}$$

On the other hand, by definition of c_ϵ, we immediately obtain

$$\begin{aligned} c_\epsilon &\le \max_{t\ge 0} \mathcal{I}_\epsilon(tu_\epsilon) = \mathcal{I}_\epsilon(t_\epsilon u_\epsilon) \\ &= \frac{t_\epsilon^p}{p}\|u_\varepsilon\|^p_{V_\epsilon,p} + \frac{t_\epsilon^q}{q}\|u_\varepsilon\|^q_{V_\epsilon,q} - \frac{1}{2}\int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * G(t_\epsilon u_\epsilon)\right] G(t_\epsilon u_\epsilon)\mathrm{d}x \end{aligned} \tag{4.3}$$

for some $t_\epsilon > 0$. It follows that

$$t_\epsilon^p\|u_\varepsilon\|^p_{V_\epsilon,p} + t_\epsilon^q\|u_\varepsilon\|^q_{V_\epsilon,q} - \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * G(t_\epsilon u_\epsilon)\right] g(t_\epsilon u_\epsilon)t_\epsilon u_\epsilon \mathrm{d}x = 0 \tag{4.4}$$

Recalling that $u \in \mathcal{N}_{V_0}$, and using (g_5), (4.2) and (4.4), it is easy to verify that $t_\epsilon \to 1$ as $\epsilon \to 0$. We observe that, from (4.3), there holds

$$\mathcal{I}_\epsilon(t_\epsilon u_\epsilon) = \mathcal{J}_{V_0}(t_\epsilon u_\epsilon) + \int_{\mathbb{R}^N}(V(\epsilon x) - V_0)\left[\frac{t_\epsilon^p}{p}|u_\epsilon|^p + \frac{t_\epsilon^q}{q}|u_\epsilon|^q\right]\mathrm{d}x. \tag{4.5}$$

Since $V(\epsilon x)$ is bounded on the support of $u_\epsilon(x)$, and $V(\epsilon x) \to V(0) = V_0$ as $\epsilon \to 0$, we can apply the Lebesgue's dominated convergence theorem and use (4.3) and (4.5) to conclude that

$$\limsup_{\epsilon\to 0} c_\epsilon \le c_{V_0}.$$

The proof is now complete. □

Now we come back to study problem (2.5) and consider the mountain pass solutions u_ϵ obtained in Lemma 2.10.

Lemma 4.2 *Let u_ϵ be a nontrivial solution obtained in Lemma 2.10, then there exist positive constants R, ϵ_0, σ and sequence $\{y_\epsilon\} \subset \mathbb{R}^N$ such that*

$$\int_{B_R(y_\epsilon)} |u_\epsilon|^q \mathrm{d}x \ge \sigma \text{ for all } \epsilon \in (0, \epsilon_0).$$

Proof Since u_ϵ verifies (2.29), we first claim that there exist $\alpha_0 > 0$ such that

$$\|u_\epsilon\|_\epsilon \ge \alpha_0 \text{ for all } \epsilon > 0. \tag{4.6}$$

Indeed, from Lemma 2.3, Lemma 2.4, (2.4) and (f_2) we conclude that

$$\begin{aligned}
\|u_\epsilon\|^p_{V_\epsilon,p} + \|u_\epsilon\|^q_{V_\epsilon,q} &= \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(\epsilon x, u_\epsilon)\right] f(\epsilon x, u_\epsilon)u_\epsilon \mathrm{d}x.\\
&\le c_{15}\left[\varepsilon|u_\epsilon|^p_{\frac{2Np}{2N-\mu}} + C_\varepsilon|u_\epsilon|^\tau_{\frac{2N\tau}{2N-\mu}}\right]^2\\
&\le c_{16}\left(\varepsilon\|u_\epsilon\|^{2p}_\epsilon + C_\varepsilon\|u_\epsilon\|^{p+\tau}_\epsilon + C_\varepsilon\|u_\epsilon\|^{2\tau}_\epsilon\right).
\end{aligned}$$

Without loss of generality, we may assume that $\|u_\epsilon\|_\epsilon < 1$ (because if $\|u_\epsilon\|_\epsilon \ge 1$, the conclusion is obvious). Therefore, using the fact that $\|u_\epsilon\|^q_{V_\epsilon,p} \le \|u_\epsilon\|^p_{V_\epsilon,p} < 1$, and applying the inequality: $a^s + b^s \ge c_s(a+b)^s$ for any $a, b \ge 0$ and $s > 1$, we have

$$c_{16}\left(\varepsilon\|u_\epsilon\|^{2p}_\epsilon + C_\varepsilon\|u_\epsilon\|^{p+\tau}_\epsilon + C_\varepsilon\|u_\epsilon\|^{2\tau}_\epsilon\right) \ge \|u_\epsilon\|^q_{V_\epsilon,p} + \|u_\epsilon\|^q_{V_\epsilon,q} \ge c_{17}\|u_\epsilon\|^q_\epsilon.$$

Evidently, we can easily see that (4.6) holds.

Let $\{\epsilon_n\} \subset (0, \infty)$ be such that $\epsilon_n \to 0$. Arguing by contradiction we assume that there exists $R > 0$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}}\int_{B_R(y)} |u_{\epsilon_n}|^q \mathrm{d}x = 0,$$

then we can use Lemma 2.1 to deduce that $u_{\epsilon_n} \to 0$ in $L^s(\mathbb{R}^N)$ for all $s \in (q, q^*)$. Thus, according to (2.4) and (2.29), we can derive that $\|u_{\epsilon_n}\|_\epsilon \to 0$ as $n \to \infty$. Clearly, this contradicts (4.6). The proof is completed. □

Lemma 4.3 *For any $\epsilon_n \to 0$, consider the sequence $\{y_{\epsilon_n}\} \subset \mathbb{R}^N$ given in Lemma 4.2, and let $v_n(x) = u_{\epsilon_n}(x + y_{\epsilon_n})$. Then there exists a subsequence of $\{v_n\}$, still denoted* *by itself, and some* $v \in E\backslash\{0\}$ *such that* $v_n \to v$ *in* E*. Moreover, there exists* $x_0 \in \Lambda$ *such that*

$$\lim_{n\to\infty} \epsilon_n y_{\epsilon_n} = x_0 \text{ and } V(x_0) = V_0.$$

Proof In what follows, we denote by $\{y_n\}$ and $\{u_n\}$, the sequences $\{y_{\epsilon_n}\}$ and $\{u_{\epsilon_n}\}$, respectively. Since each $\{u_n\}$ satisfies (2.29), we can argue as in the proof of Lemma 2.6 and use Lemma 4.1 to deduce that $\{u_n\}$ is bounded in E, which is also true for $\{v_n\}$. Hence, there exists $v \in E\backslash\{0\}$ such that

$$v_n \rightharpoonup v \text{ in } E \text{ as } n \to \infty, \tag{4.7}$$

and by Lemma 4.2, we have

$$\int_{B_R(0)} |v|^q \mathrm{d}x \geq \sigma > 0. \tag{4.8}$$

Next, we will show that $\{\epsilon_n y_n\}$ is bounded. First of all, we prove that

$$\lim_{n\to\infty} \operatorname{dist}(\epsilon_n y_n, \bar{\Lambda}) = 0. \tag{4.9}$$

Indeed, if (4.9) does not hold, there exists $\delta > 0$ and a subsequence of $\{\epsilon_n y_n\}$, still denoted by itself, such that

$$\operatorname{dist}(\epsilon_n y_n, \bar{\Lambda}) \geq \delta \text{ for all } n \in \mathbb{N}.$$

Consequently, there is some $r > 0$ such that $B_r(\epsilon_n y_n) \subset \Lambda^c$ for all $n \in \mathbb{N}$. Using the fact that $v \geq 0$, we know that there exists $\{\phi_j\} \subset E$ such that $\phi_j \geq 0$, ϕ_j has compact support in $\mathbb{R}^N$ and $\phi_j \to v$ in E as $j \to \infty$. Fixing $j \in \mathbb{N}$ and using ϕ_j as test function, we can obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big[|\nabla v_n|^{p-2}\nabla v_n \nabla\phi_j + |\nabla v_n|^{q-2}\nabla v_n \nabla\phi_j + V(\epsilon_n x + \epsilon_n y_n) \\
&\quad (|v_n|^{p-2} v_n \phi_j + |v_n|^{q-2} v_n \phi_j) \Big] \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\epsilon_n x + \epsilon_n y_n, v_n) \right] f(\epsilon_n x + \epsilon_n y_n, v_n)\phi_j \mathrm{d}x.
\end{aligned}$$

We observe that since $x \in B_{r/\epsilon_n}(0)$, then $\epsilon_n x + \epsilon_n y_n \in B_r(\epsilon_n y_n) \subset \Lambda^c$. So, for n large enough, we deduce from (f_2), (f_3) and Lemma 2.7 that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(\epsilon_n x + \epsilon_n y_n, v_n)\right] f(\epsilon_n x + \epsilon_n y_n, v_n)\phi_j \mathrm{d}x\\ &\le \frac{\ell_0}{2}\int_{\mathbb{R}^N} f(\epsilon_n x + \epsilon_n y_n, v_n)\phi_j \mathrm{d}x\\ &= \frac{\ell_0}{2}\int_{B_{r/\epsilon_n}(0)} f(\epsilon_n x + \epsilon_n y_n, v_n)\phi_j \mathrm{d}x + \frac{\ell_0}{2}\int_{B^c_{r/\epsilon_n}(0)} f(\epsilon_n x + \epsilon_n y_n, v_n)\phi_j \mathrm{d}x\\ &\le \frac{1}{2}\int_{\mathbb{R}^N} V_{\min}(|v_n|^{p-1} + |v_n|^{q-1})\phi_j \mathrm{d}x + c_{18}\int_{B^c_{r/\epsilon_n}(0)} g(v_n)\phi_j \mathrm{d}x.\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[|\nabla v_n|^{p-2}\nabla v_n \nabla\phi_j + |\nabla v_n|^{q-2}\nabla v_n\nabla\phi_j + V_{\min}(|v_n|^{p-2}v_n\phi_j + |v_n|^{q-2}v_n\phi_j)\right]\mathrm{d}x\\ &\le c_{18}\int_{B^c_{r/\epsilon_n}(0)} g(v_n)\phi_j \mathrm{d}x.\end{aligned}$$

Taking into account ϕ_j has compact support, and using (2.4) and (4.7) we infer that

$$\int_{B^c_{r/\epsilon_n}(0)} g(v_n)\phi_j \mathrm{d}x \to 0,$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[|\nabla v_n|^{p-2}\nabla v_n \nabla\phi_j + |\nabla v_n|^{q-2}\nabla v_n\nabla\phi_j + V_{\min}(|v_n|^{p-2}v_n\phi_j + |v_n|^{q-2}v_n\phi_j)\right]\mathrm{d}x\\ &\to \int_{\mathbb{R}^N}\left[|\nabla v|^{p-2}\nabla v \nabla\phi_j + |\nabla v|^{q-2}\nabla v\nabla\phi_j + V_{\min}(|v|^{p-2}v\phi_j + |v|^{q-2}v\phi_j)\right]\mathrm{d}x\end{aligned}$$

as $n \to \infty$. Combining the above facts, we immediately obtain that

$$\int_{\mathbb{R}^N}\left[|\nabla v|^{p-2}\nabla v \nabla\phi_j + |\nabla v|^{q-2}\nabla v\nabla\phi_j + V_{\min}(|v|^{p-2}v\phi_j + |v|^{q-2}v\phi_j)\right]\mathrm{d}x \le 0.$$

Since j is arbitrary, passing to the limit as $j \to \infty$, it is easy to see that

$$\int_{\mathbb{R}^N}\left[|\nabla v|^p + |\nabla v|^q + V_{\min}(|v|^p + |v|^q)\right]\mathrm{d}x = 0.$$

This shows that $v \equiv 0$ in $\mathbb{R}^N$ and contradicts (4.8). As a result, we have proved (4.9).

From (4.9), we get that there exists a subsequence of $\{\epsilon_n y_n\}$, and $x_0 \in \bar{\Lambda}$ such that $\epsilon_n y_n \to x_0$. Next we claim that $x_0 \in \Lambda$.

Indeed, we note that v_n is a solution of the following problem

$$\begin{cases} -\Delta_p u - \Delta_q u + V_n(x)(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * F_n(x,u)\right) f_n(x,u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \ u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{4.10}$$

where $V_n(x) = V(\epsilon_n x + \epsilon_n y_n)$, $f_n(x,u) = f(\epsilon_n x + \epsilon_n y_n, u)$ and $F_n(x,u) = F(\epsilon_n x + \epsilon_n y_n, u)$. Then, by taking $n \to \infty$, the weak limit v must be the critical point of the energy functional

$$\begin{aligned} \widehat{\mathcal{I}}_0(u) =& \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p + V(x_0)|u|^p \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q + V(x_0)|u|^q \mathrm{d}x \\ & - \frac{1}{2}\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(x_0,u)\right] F(x_0,u)\mathrm{d}x, \end{aligned}$$

where

$$F(x_0,u) = \chi(x_0)G(u) + (1-\chi(x_0))\widetilde{G}(u).$$

Let $\widehat{c_0}$ denote the mountain pass level of $\widehat{\mathcal{I}}_0$. Similar argument as discussed in Section 2 can be used to obtain $F(x_0,s) \le G(s)$ for all $s > 0$, which gives then $c_0 \le \widehat{c_0}$. Here we use c_0 to denote the mountain pass level associated with $\mathcal{I}_0$, where $\mathcal{I}_0 : E \to \mathbb{R}$ is given by

$$\begin{aligned} \mathcal{I}_0(u) = & \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p + V(x_0)|u|^p \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q + V(x_0)|u|^q \mathrm{d}x \\ & - \frac{1}{2}\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(u)\right] G(u)\mathrm{d}x. \end{aligned}$$

Moreover, using Fatou's lemma and Lemma 4.1 we can derive that

$$\begin{aligned} c_0 \le & \widehat{c_0} \le \widehat{\mathcal{I}}_0(v) = \widehat{\mathcal{I}}_0(v) - \frac{1}{\theta}\langle \widehat{\mathcal{I}}_0'(v), v\rangle \\ =& \left(\frac{1}{p} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v|^p + V(x_0)|v|^p \mathrm{d}x + \left(\frac{1}{q} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v|^q + V(x_0)|v|^q \mathrm{d}x \\ & + \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(x_0,v)\right]\left[\frac{1}{\theta} f(x_0,v)v - \frac{1}{2}F(x_0,v)\right]\mathrm{d}x \\ =& \liminf_{n\to\infty}\left[\left(\frac{1}{p} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v_n|^p + V_n(x)|v|^p \mathrm{d}x + \left(\frac{1}{q} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v_n|^q + V_n(x)|v_n|^q \mathrm{d}x\right] \\ & + \liminf_{n\to\infty}\left[\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F_n(x,v_n)\right]\left[\frac{1}{\theta} f_n(x,v_n)v_n - \frac{1}{2}F_n(x,v_n)\right]\mathrm{d}x\right] \\ =& \liminf_{n\to\infty}\left[\mathcal{I}_{\epsilon_n}(u_n) - \frac{1}{\theta}\langle \mathcal{I}_{\epsilon_n}'(u_n), u_n\rangle\right] \\ =& \liminf_{n\to\infty} c_{\epsilon_n} \le c_{V_0}. \end{aligned}$$

Consequently, $c_0 \le c_{V_0}$, which implies that $V(x_0) \le V_0 = V(0)$. Then, from condition (V), we can see that $x_0 \notin \partial\Lambda$, and so $x_0 \in \Lambda$. Moreover, we also have $V(x_0) = V_0$ and $x_0 \in \Pi$.

Finally, we are able to show the strong convergence $v_n \to v$ in E as $n \to \infty$. We first define

$$\Lambda_n = \frac{\Lambda - \epsilon_n y_n}{\epsilon_n}$$

and

$$\chi_n^1(x) = \begin{cases} 1 & \text{if } x \in \Lambda_n, \\ 0 & \text{if } x \in \Lambda_n^c, \end{cases} \text{ and } \chi_n^2(x) = 1 - \chi_n^1(x).$$

Let us also consider the following functions:

$$\begin{aligned}
h_n^1(x) &:= \left[\left(\frac{1}{p} - \frac{1}{\theta}\right) V_n(x)|v_n|^p + \left(\frac{1}{q} - \frac{1}{\theta}\right) V_n(x)|v_n|^q\right] \chi_n^1(x), \\
h^1(x) &:= \left(\frac{1}{p} - \frac{1}{\theta}\right) V(x_0)|v|^p + \left(\frac{1}{q} - \frac{1}{\theta}\right) V(x_0)|v|^q, \\
h_n^2(x) &:= \Bigg[\left(\frac{1}{p} - \frac{1}{\theta}\right) V_n(x)|v_n|^p + \left(\frac{1}{q} - \frac{1}{\theta}\right) V_n(x)|v_n|^q \\
&\quad + \left(\frac{1}{|x|^\mu} * F_n(x, v_n)\right)\left(\frac{1}{\theta} f_n(x, v_n)v_n - \frac{1}{2} F_n(x, v_n)\right)\Bigg]\chi_n^2(x), \\
h_n^3(x) &:= \left[\left(\frac{1}{|x|^\mu} * F_n(x, v_n)\right)\left(\frac{1}{\theta} f_n(x, v_n)v_n - \frac{1}{2} F_n(x, v_n)\right)\right] \chi_n^1(x) \\
&= \left[\left(\frac{1}{|x|^\mu} * G(v_n)\right)\left(\frac{1}{\theta} g(v_n)v_n - \frac{1}{2} G(v_n)\right)\right] \chi_n^1(x), \\
h^3(x) &:= \left(\frac{1}{|x|^\mu} * G(v)\right)\left(\frac{1}{\theta} g(v)v - \frac{1}{2} G(v)\right).
\end{aligned}$$

From (g_4) and (f_3) we see that the above functions are nonnegative in $\mathbb{R}^N$. Moreover, since

$$v_n(x) \to v(x) \text{ a.e in } \mathbb{R}^N \text{ and } \epsilon_n y_n \to x_0 \in \Lambda$$

as $n \to \infty$, we have

$$\chi_n^1(x) \to 1, h_n^1(x) \to h^1(x), h_n^2(x) \to 0 \text{ and } h_n^3(x) \to h^3(x) \text{ a.e in } \mathbb{R}^N.$$

Then, using Fatou's lemma and Lemma 4.1 we conclude that

$$\begin{aligned}
c_{V_0} &\geq \limsup_{n\to\infty} c_{\epsilon_n} = \limsup_{n\to\infty} \left[\mathcal{I}_{\epsilon_n}(u_n) - \frac{1}{\theta}\langle \mathcal{I}'_{\epsilon_n}(u_n), u_n\rangle\right] \\
&\geq \limsup_{n\to\infty} \Bigg[\left(\frac{1}{p} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v_n|^p \mathrm{d}x + \left(\frac{1}{q} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} (h_n^1 + h_n^2 + h_n^2)\mathrm{d}x\Bigg] \\
&\geq \liminf_{n\to\infty} \Bigg[\left(\frac{1}{p} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v_n|^p \mathrm{d}x + \left(\frac{1}{q} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} (h_n^1 + h_n^2 + h_n^2)\mathrm{d}x\Bigg] \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v|^p \mathrm{d}x + \left(\frac{1}{q} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v|^q \mathrm{d}x + \int_{\mathbb{R}^N} (h^1 + h^3)\mathrm{d}x \geq c_{V_0}.
\end{aligned}$$

The above inequalities show that

$$\int_{\mathbb{R}^N} |\nabla v_n|^p + |\nabla v_n|^q \mathrm{d}x \to \int_{\mathbb{R}^N} |\nabla v|^p + |\nabla v|^q \mathrm{d}x \tag{4.11}$$

and

$$h_n^1 \to h^1, h_n^2 \to 0 \text{ and } h_n^3 \to h^3 \text{ in } L^1(\mathbb{R}^N).$$

Consequently, we have

$$\int_{\mathbb{R}^N} V_n(x)(|v_n|^p + |v_n|^q)\mathrm{d}x \to \int_{\mathbb{R}^N} V(x_0)(|v|^p + |v|^q)\mathrm{d}x. \tag{4.12}$$

From (4.11) and (4.12) we can get immediately $v_n \to v$ in E, finishing the proof. □

To characterize the concentration of solutions, next we use an appropriate De Giorgi iteration argument and some refined analysis techniques to show the L^∞-estimate and decay property of solutions. Furthermore, these properties are contributed to determine the concentration location of solutions.

Lemma 4.4 *Let v_n be the sequence defined as in Lemma 4.3. Then we have $v_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $|v_n|_\infty \le C$ for all $n \in \mathbb{N}$. Moreover,*

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N}.$$

Proof We note that v_n is a solution of the following problem

$$\begin{cases} -\Delta_p v_n - \Delta_q v_n + V_n(|v_n|^{p-2}v_n + |v_n|^{q-2}v_n) \text{ in } \mathbb{R}^N, \\ \quad = \left(\frac{1}{|x|^\mu} * F_n(x, v_n)\right) f_n(x, v_n), \\ v_n \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), v_n > 0, \qquad \text{in } \mathbb{R}^N, \end{cases}$$

where $V_n(x)$, $f_n(x, v_n)$ and $F_n(x, v_n)$ are given in (4.10). From Lemma 4.3 we know that $\{v_n\}$ is bounded and $v_n \to v$ in E. Let us define

$$K_n(x) = \frac{1}{|x|^\mu} * F_n(x, v_n).$$

Then, according to the boundedness of $\{v_n\}$ and following the proof of Lemma 2.7, we can get

$$|K_n(x)| \le C \text{ for some } C > 0 \text{ and any } n \in \mathbb{N}. \tag{4.13}$$

We adopt some ideas found in [1, 25] (see also [49] for more details) to prove this lemma. Let $x_0 \in \mathbb{R}^N$, $R_0 > 1$ and $0 < t < s < 1$, and let smooth function $\psi \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$0 \le \psi(x) \le 1, \ \mathrm{supp}\psi \subset B_s(x_0), \psi(x) = 1, \forall x \in B_t(x_0) \text{ and } |\nabla\psi| \le \frac{2}{s-t}.$$

For $l \geq 1$ and $\rho > 0$, we denote $\Omega_{n,l,\rho} = \{x \in B_\rho(x_0) : v_n(x) > l\}$ and

$$Q_n = \int_{\Omega_{n,l,s}} (|\nabla v_n|^p + |\nabla v_n|^q)\psi^q \mathrm{d}x.$$

Since v_n is a solution of the above problem, then, for any $\varphi \in E_\epsilon$ we have the following relation

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla\varphi \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v_n|^{q-2}\nabla v_n \cdot \nabla\varphi \mathrm{d}x\\&+ \int_{\mathbb{R}^N} V_n(x)(v_n^{p-1} + v_n^{q-1})\varphi \mathrm{d}x = \int_{\mathbb{R}^N} K_n(x) f_n(x, v_n)\varphi \mathrm{d}x.\end{aligned}$$

Taking $\varphi_n = \psi^q(v_n - l)_+$ as test function, we can deduce that

$$\begin{aligned}&q\int_{\Omega_{n,l,s}} \psi^{q-1}(v_n - l)_+|\nabla v_n|^{p-2}\nabla v_n \cdot \nabla\psi \mathrm{d}x + \int_{\Omega_{n,l,s}} \psi^q|\nabla v_n|^p \mathrm{d}x\\&+q\int_{\Omega_{n,l,s}} \psi^{q-1}(v_n - l)_+|\nabla v_n|^{q-2}\nabla v_n \cdot \nabla\psi \mathrm{d}x + \int_{\Omega_{n,l,s}} \psi^q|\nabla v_n|^q \mathrm{d}x\\&+\int_{\Omega_{n,l,s}} V_n(x)(v_n^{p-1} + v_n^{q-1})\psi^q(v_n - l)_+ \mathrm{d}x = \int_{\Omega_{n,l,s}} K_n(x) f_n(x, v_n)\psi^q(v_n - l)_+ \mathrm{d}x,\end{aligned}$$

and combining (V) and (f_2) we obtain

$$\begin{aligned}Q_n \leq &C_7\int_{\Omega_{n,l,s}} \psi^{q-1}(v_n - l)_+|\nabla\psi|(|\nabla v_n|^{p-1} + |\nabla v_n|^{q-1})\mathrm{d}x\\&-\int_{\Omega_{n,l,s}} V_{\min}\psi^{q-1}(v_n - l)_+(v_n^{p-1} + v_n^{q-1})\mathrm{d}x + \int_{\Omega_{n,l,s}} K_n(x)g(v_n)\psi^q(v_n - l)_+ \mathrm{d}x.\end{aligned}$$

Moreover, using (2.4) and (4.13) we get

$$\begin{aligned}Q_n \leq C_8 \Bigg[&\int_{\Omega_{n,l,s}} \psi^{q-1}(v_n - l)_+|\nabla\psi|(|\nabla v_n|^{p-1} + |\nabla v_n|^{q-1})\mathrm{d}x\\&+\int_{\Omega_{n,l,s}} v_n^{q^*-1}\psi^q(v_n - l)_+ \mathrm{d}x\Bigg].\end{aligned}$$

Since $0 < s-t < 1$, applying Young inequality and Hölder inequality, we conclude that

$$\begin{aligned}&\int_{\Omega_{n,l,s}} v_n^{q^*-1}\psi^q(v_n - l)_+ \mathrm{d}x \leq \int_{\Omega_{n,l,s}} (|v_n - l| + l)^{q^*-1}\psi^q(v_n - l)_+ \mathrm{d}x\\&\leq C_9\left[\int_{\Omega_{n,l,s}} |v_n - l|^{q^*} \mathrm{d}x + l^{q^*-1}\int_{\Omega_{n,l,s}} |v_n - l| \mathrm{d}x\right]\end{aligned}$$

$$\leq C_9\left[\int_{\Omega_{n,l,s}}\left|\frac{v_n-l}{s-t}\right|^{q^*}\mathrm{d}x+l^{q^*}\left[\int_{\Omega_{n,l,s}}\left|\frac{v_n-l}{s-t}\right|^{q^*}\mathrm{d}x\right]^{\frac{1}{q^*}}|\Omega_{n,l,s}|^{\frac{q^*-1}{q^*}}\right]$$
$$\leq C_{10}\left[\int_{\Omega_{n,l,s}}\left|\frac{v_n-l}{s-t}\right|^{q^*}\mathrm{d}x+l^{q^*}|\Omega_{n,l,s}|\right].$$

Using the properties of Young functions and following the proof of Theorem 3.1 in [43], for some $\varepsilon_1 \in (0, 1)$ we obtain

$$\int_{\Omega_{n,l,s}}(|\nabla v_n|^{p-1}+|\nabla v_n|^{q-1})\psi^{q-1}|\nabla\psi|(v_n-l)_+\mathrm{d}x$$
$$\leq \varepsilon_1\int_{\Omega_{n,l,s}}(|\nabla v_n|^{p}+|\nabla v_n|^{q})\psi^{q}\mathrm{d}x+C_{\varepsilon_1}\left[\int_{\Omega_{n,l,s}}\left|\frac{v_n-l}{s-t}\right|^{q^*}\mathrm{d}x+|\Omega_{n,l,s}|\right].$$

Therefore, we can infer from the above facts that

$$Q_n\leq C_{11}\left[\int_{\Omega_{n,l,s}}\left|\frac{v_n-l}{s-t}\right|^{q^*}\mathrm{d}x+(l^{q^*}+1)|\Omega_{n,l,s}|\right].$$

Exploiting the definition of ψ, we conclude that

$$\int_{\Omega_{n,l,t}}|\nabla v_n|^q\mathrm{d}x\leq C_{11}\left[\int_{\Omega_{n,l,s}}\left|\frac{v_n-l}{s-t}\right|^{q^*}\mathrm{d}x+(l^{q^*}+1)|\Omega_{n,l,s}|\right], \tag{4.14}$$

where C_5 does not depend on l and $l \geq l_0 \geq 1$ for some constant l_0.

We fix $R_1 \in (0, 1)$ and define

$$\sigma_j=\frac{R_1}{2}\left(1+\frac{1}{2^j}\right),\bar{\sigma}_j=\frac{1}{2}(\sigma_j+\sigma_{j+1}),l_j=\frac{l_0}{2}\left(1-\frac{1}{2^{j+1}}\right)$$

and

$$Q_{j,n}=\int_{\Omega_{n,l_j,\sigma_j}}(v_n-l_j)_+^{q^*}\mathrm{d}x \text{ and } \xi_j=\xi\left(\frac{2^{j+1}}{R_1}\left(|x-x_0|-\frac{R_1}{2}\right)\right),$$

where $\xi \in C^1(\mathbb{R})$ satisfies

$$0\leq\xi\leq 1,\xi(s)=1 \text{ for } s\leq\frac{1}{2},\xi(s)=0 \text{ for } s\geq\frac{3}{4} \text{ and } |\xi'|\leq c_0.$$

Clearly, we can see that the following conclusions hold

$$\sigma_j\to\frac{R_1}{2}\text{ (decreasing)},\ l_j\to\frac{l_0}{2}\text{ (increasing) and }\sigma_{j+1}<\bar{\sigma}_j<\sigma_j<1.$$

Since $\xi_j = 1$ in $B_{\sigma_{j+1}}(x_0)$ and $\xi_j = 0$ outside $B_{\bar{\sigma}_j}(x_0)$, then from Lemma 2.1 we get

$$\begin{aligned} Q_{j+1,n} &= \int_{\Omega_{n,l_{j+1},\sigma_{j+1}}} (v_n - l_{j+1})_+^{q^*} \mathrm{d}x \le \int_{B_{R_1(x_0)}} \left[(v_n - l_{j+1})_+ \xi_j\right]^{q^*} \mathrm{d}x \\ &\le C_{12} \left[\int_{B_{R_1(x_0)}} |\nabla \left[(v_n - l_{j+1})_+ \xi_j\right]|^q \mathrm{d}x\right]^{\frac{q^*}{q}} \\ &\le C_{13} \left[\int_{\Omega_{n,l_{j+1},\bar{\sigma}_j}} |\nabla v_n|^q \mathrm{d}x + 2^{jp} \int_{\Omega_{n,l_{j+1},\bar{\sigma}_j}} (v_n - l_{j+1})_+^q \mathrm{d}x\right]^{\frac{q^*}{q}}. \end{aligned} \tag{4.15}$$

From (4.14) we further deduce that

$$\begin{aligned} \int_{\Omega_{n,l_{j+1},\bar{\sigma}_j}} |\nabla v_n|^q \mathrm{d}x &\le C_{14} \left[\int_{\Omega_{n,l_{j+1},\sigma_j}} \left|\frac{v_n - l_{j+1}}{\sigma_j - \bar{\sigma}_j}\right|^{q^*} \mathrm{d}x + (l_{j+1}^{q^*} + 1)|\Omega_{n,l_{j+1},\sigma_j}|\right] \\ &\le C_{15} \left[2^{jq^*} \int_{\Omega_{n,l_{j+1},\sigma_j}} (v_n - l_{j+1})_+^{q^*} \mathrm{d}x + (l_{j+1}^{q^*} + 1)|\Omega_{n,l_{j+1},\sigma_j}|\right]. \end{aligned} \tag{4.16}$$

Using Hölder inequality and Young's inequality, it follows that

$$\begin{aligned} \int_{\Omega_{n,l_{j+1},\bar{\sigma}_j}} (v_n - l_{j+1})_+^q \mathrm{d}x &\le \left[\int_{\Omega_{n,l_{j+1},\bar{\sigma}_j}} (v_n - l_{j+1})_+^{q^*} \mathrm{d}x\right]^{\frac{q}{q^*}} |\Omega_{n,l_{j+1},\bar{\sigma}_j}|^{\frac{q^*-q}{q^*}} \\ &\le C_{16} \left[\int_{\Omega_{n,l_{j+1},\bar{\sigma}_j}} (v_n - l_{j+1})_+^{q^*} \mathrm{d}x + |\Omega_{n,l_{j+1},\bar{\sigma}_j}|\right]. \end{aligned} \tag{4.17}$$

From (4.15), (4.16) and (4.17) we can obtain the following estimate

$$\begin{aligned} Q_{j+1,n}^{\frac{q}{q^*}} &\le C_{17} \left[(2^{jq^*} + 2^{jp}) \int_{\Omega_{n,l_{j+1},\sigma_j}} (v_n - l_{j+1})_+^{q^*} \mathrm{d}x + (l_{j+1}^{q^*} + 1 + 2^{jq})|\Omega_{n,l_{j+1},\sigma_j}|\right] \\ &\le C_{18} \left[(2^{jq^*} + 2^{jp}) \int_{\Omega_{n,l_{j+1},\sigma_j}} (v_n - l_{j+1})_+^{q^*} \mathrm{d}x + 2^{jq}|\Omega_{n,l_{j+1},\sigma_j}|\right]. \end{aligned} \tag{4.18}$$

We note that

$$Q_{j,n} \ge \int_{\Omega_{n,l_{j+1},\sigma_j}} (v_n - l_j)_+^{q^*} \mathrm{d}x \ge (l_{j+1} - l_j)^{q^*} |\Omega_{n,l_{j+1},\sigma_j}|,$$

which implies that

$$|\Omega_{n,l_{j+1},\sigma_j}| \le \left(\frac{1}{l_{j+1} - l_j}\right)^{q^*} Q_{j,n} = \left(\frac{2^{j+3}}{l_0}\right)^{q^*} Q_{j,n}. \tag{4.19}$$

Combining (4.18) and (4.19), we immediately obtain

$$Q_{j+1,n}^{\frac{q}{q^*}} \le C_{19}\left[(2^{jq^*}+2^{jp})Q_{j,n}+2^{j(q^*+q)}Q_{j,n}\right] \le C_{20}2^{j(q^*+q)}Q_{j,n}.$$

Therefore, we can get the following iteration formula

$$Q_{j+1,n} \le C_{21}B^j Q_{j,n}^{1+\beta},$$

where C_{21} depends on N, q, R_1, l_0, $B = 2^{(q^*+q)q/q^*} > 1$ and $\beta = q^*/q - 1$.

According to the fact $v_n \to v$ in E, we get

$$\limsup_{l_0\to\infty}\left[\limsup_{n\to\infty} Q_{0,n}\right] = \limsup_{l_0\to\infty}\left[\limsup_{n\to\infty}\int_{\Omega_{n,l_0,\sigma_0}}(v_n-\frac{l_0}{4})_+^{q^*}\mathrm{d}x\right] = 0.$$

Thus, there exists N_0 and $L_0 > 0$ such that

$$Q_{0,n} \le C^{-\frac{1}{\beta}}B^{-\frac{1}{\beta^2}} \text{ for } n \ge N_0 \text{ and } l_0 \ge L_0.$$

Exploiting Lemma 4.7 of [25], we have that

$$\lim_{j\to\infty} Q_{j,n} = 0 \text{ for } n \ge N_0.$$

On the other hand, there holds

$$\lim_{j\to\infty} Q_{j,n} = \lim_{j\to\infty}\int_{\Omega_{n,l_j,\sigma_j}}(v_n-l_j)_+^{q^*}\mathrm{d}x = \int_{\Omega_{n,l_0/2,R_1/2}}(v_n-\frac{l_0}{2})_+^{q^*}\mathrm{d}x.$$

Then, we obtain

$$\int_{\Omega_{n,l_0/2,R_1/2}}(v_n-\frac{l_0}{2})_+^{q^*}\mathrm{d}x = 0 \text{ for all } n \ge N_0,$$

and consequently,

$$v_n(x) \le \frac{l_0}{2} \text{ for a.e. } x \in B_{\frac{R_1}{2}}(x_0) \text{ and for all } n \ge N_0.$$

From the arbitrariness of $x_0 \in \mathbb{R}^N$, we can see that

$$v_n(x) \le \frac{l_0}{2} \text{ for a.e. } x \in \mathbb{R}^N \text{ and for all } n \ge N_0,$$

that is,

$$|v_n|_\infty \le \frac{l_0}{2} \text{ for all } n \ge N_0.$$

Setting $C = \max\{\frac{l_0}{2}, |v_1|_\infty, \ldots, |v_{N_0-1}|_\infty\}$, we have $|v_n|_\infty \le C$ for all $n \in \mathbb{N}$. Moreover, using the regularity conclusion found in [24] (see Theorem 1 and Theorem 2), we can see that $v_n \in C^{1,\nu}_{loc}(\mathbb{R}^N)$ for some $\nu \in (0, 1)$.

Finally, we show that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$. In fact, following the above arguments, for each $\varepsilon > 0$, we have that

$$\limsup_{|x_0|\to\infty} \left[\limsup_{n\to\infty} Q_{0,n} \right] = \limsup_{|x_0|\to\infty} \left[\limsup_{n\to\infty} \int_{\Lambda_{n,l_0,\sigma_0}} (v_n - \frac{\varepsilon}{4})_+^{q^*} \mathrm{d}x \right] = 0.$$

Thereby, employing Lemma 4.7 of [25], there exist $R_* > 0$ and $N_0 \in \mathbb{N}$ such that

$$\lim_{j\to\infty} Q_{j,n} = 0 \text{ if } |x_0| > R_* \text{ and } n \ge N_0,$$

this shows that

$$v_n(x) \le \frac{\varepsilon}{4} \text{ for } x \in B_{\frac{R_1}{2}}(x_0) \text{ and } |x_0| > R_*, n \ge N_0.$$

Now, increasing R_* if necessary, it follows that

$$v_n(x) \le \frac{\varepsilon}{4} \text{ for } |x_0| > R_* \text{ and for all } n \in \mathbb{N}.$$

According to the arbitrariness of ε, we can see that

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N},$$

and we complete the proof of this lemma. □

Lemma 4.5 *There exist c, $C > 0$ such that for all small $\epsilon > 0$, there holds*

$$v_n(x) \le C \exp(-c|x|) \text{ for all } x \in \mathbb{R}^N.$$

Proof According to Lemma 4.4, (g_2), (f_2) and (4.13), we find that there exists $R > 0$ such that

$$K_n(x) f_n(x, v_n) \le K_n(x) g(v_n) \le \frac{V_{\min}}{2}(v_n^{p-1} + v_n^{q-1}) \text{ for all } |x| \ge R.$$

Then, for $|x| \ge R$ we get

$$\begin{aligned} &-\Delta_p v_n - \Delta_q v_n + \frac{V_{\min}}{2}(v_n^{p-1} + v_n^{q-1}) \\ &= K_n(x) f_n(x, v_\epsilon) - \left[V_n(x) - \frac{V_{\min}}{2} \right] (v_n^{p-1} + v_n^{q-1}) \\ &\le K_n(x) g(v_n) - \frac{V_{\min}}{2}(v_n^{p-1} + v_n^{q-1}) \le 0. \end{aligned} \tag{4.20}$$

Let $\psi(x) = C_0 \exp(-c_0|x|)$ with $c_0, C_0 > 0$ such that

$$c_0^p(p-1) < \frac{V_{\min}}{2} \text{ and } c_0^q(q-1) < \frac{V_{\min}}{2}$$

and $v_n(x) \le C_0 \exp(-c_0 R)$ for all $|x| = R$. Then, computing directly, we obtain

$$\begin{aligned} &-\Delta_p\psi - \Delta_q\psi + \frac{V_{\min}}{2}(\psi^{p-1} + \psi^{q-1}) \\ =&\psi^{p-1}\left[\frac{V_{\min}}{2} - c_0^p(p-1) + \frac{N-1}{|x|}c_0^{p-1}\right] \\ &+ \psi^{q-1}\left[\frac{V_{\min}}{2} - c_0^q(q-1) + \frac{N-1}{|x|}c_0^{q-1}\right] \\ >&0 \end{aligned} \tag{4.21}$$

for all $|x| \ge R$. Let $\Omega = \{|x| \ge R\} \cap \{v_n > \psi\}$. Using the following inequality

$$\langle |\xi|^{s-2}\xi - |\eta|^{s-2}\eta, \xi - \eta\rangle \ge 0 \text{ for all } s > 1 \text{ and } \xi, \eta \in \mathbb{R}^N$$

and choosing $\phi = \max\{v_n - \psi, 0\} \in W_0^{1,p}(\mathbb{R}^N \backslash B_R) \cap W_0^{1,q}(\mathbb{R}^N \backslash B_R)$ as a test function in (4.20) and (4.21), we can infer that

$$\begin{aligned} 0 \ge &\int_\Omega \left[(|\nabla v_n|^{p-2}\nabla v_n - |\nabla\psi|^{p-2}\nabla\psi)\nabla\phi + (|\nabla v_n|^{q-2}\nabla v_n - |\nabla\psi|^{q-2}\nabla\psi)\nabla\phi\right]\mathrm{d}x \\ &+ \frac{V_{\min}}{2}\int_\Omega \left[(v_n^{p-1} - \psi^{p-1}) + (v_n^{q-1} - \psi^{q-1})\right]\phi\mathrm{d}x \\ \ge&0. \end{aligned}$$

Evidently, we know that Ω is empty, and we can easily conclude that $v_n(x) \le \psi(x)$ for all $|x| \ge R$, and

$$v_n(x) \le \psi(x) = C_0 \exp(-c_0|x|) \text{ for all } |x| \ge R.$$

Thus, there exist $c, C > 0$, we have

$$v_n(x) \le C \exp(-c|x|) \text{ for all } x \in \mathbb{R}^N.$$

The proof is completed. □

Lemma 4.6 *There exists* $\nu_0 > 0$ *such that* $|v_n|_\infty \ge \nu_0$ *for all* $n \in \mathbb{N}$.

Proof According to Lemma 4.2, we can get

$$\int_{B_R(0)} |v_n(x)|^q \mathrm{d}x \ge \delta > 0$$

for some $\delta > 0$, $R > 0$ and $n \geq N_0$. Assume by contradiction that $|v_n|_\infty \to 0$ as $n \to +\infty$, then it follows from Lemma 4.4 that

$$0 < \delta \leq \int_{B_R(0)} |v_n(x)|^q \mathrm{d}x \leq |B_R(0)||v_n(x)|_\infty^q \to 0 \text{ as } n \to \infty,$$

which implies a contradiction. This completes the proof. □

Now we are in a position to finish the proof of Theorem 1.1.

Proof of Theorem 1.1. As we have introduced in the Section 2, in order to show the solution of the modified problem (2.5) is indeed the solution of the original problem (2.1), we need to prove that there exists $\epsilon_0 > 0$ such that, for any $\epsilon \in (0, \epsilon_0)$ and any mountain pass solution u_ϵ of problem (2.5), it holds

$$|u_\epsilon(x)|_{L^\infty(\Lambda_\epsilon^c)} < a_0. \tag{4.22}$$

Assume by contradiction that for some subsequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$, we can find $u_n := u_{\epsilon_n}$ such that $\mathcal{J}_{\epsilon_n}(u_n) = c_{\epsilon_n}$, $\mathcal{J}'_{\epsilon_n}(u_n) = 0$ and

$$|u_n(x)|_{L^\infty(\Lambda_{\epsilon_n}^c)} \geq a_0. \tag{4.23}$$

According to Lemma 4.2 and Lemma 4.3, we can see that $\{y_n\} \subset \mathbb{R}^N$ such that $v_n = u_n(x + y_n) \to v$ in E and $\epsilon_n y_n \to x_0$ for some $x_0 \in \Pi \subset \Lambda$.

Since $x_0 \in \Pi \subset \Lambda$, we can choose a suitable $r > 0$ such that $B_r(x_0) \subset B_{2r}(x_0) \subset \Lambda$. Then, we see that $B_{2r/\epsilon_n}(\frac{x_0}{\epsilon_n}) \subset \Lambda_{\epsilon_n}$. Moreover, for any $x \in B_{r/\epsilon_n}(y_n)$, there holds

$$\left|x - \frac{x_0}{\epsilon_n}\right| \leq |x - y_n| + \left|y_n - \frac{x_0}{\epsilon_n}\right| \leq \frac{1}{\epsilon_n}(r + o_n(1)) < \frac{2r}{\epsilon_n}$$

for n sufficiently large, which implies $x \in B_{2r/\epsilon_n}(\frac{x_0}{\epsilon_n})$ for n enough large. Therefore, we obtain

$$\Lambda_{\epsilon_n}^c \subset B_{r/\epsilon_n}^c(y_n) \text{ for } n \text{ enough large.} \tag{4.24}$$

Using Lemma 4.4, we know that

$$v_n(x) \to 0 \text{ as } |x| \to \infty \text{ uniformly in } n \in \mathbb{N}.$$

Then, there exists $R > 0$ such that

$$v_n(x) < a_0 \text{ for any } x \in B_R^c \text{ and } n \in \mathbb{N}.$$

This shows that

$$u_n(x) = v_n(x - y_n) < a_0 \text{ for any } x \in B_R^c(y_n) \text{ and } n \in \mathbb{N}.$$

On the other hand, on account of (4.24), there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we get

$$\Lambda^c_{\epsilon_n} \subset B^c_{r/\epsilon_n}(y_n) \subset B^c_R(y_n).$$

which implies that

$$u_n(x) < a_0 \text{ for any } \Lambda^c_{\epsilon_n} \text{ and } n \geq n_0.$$

Evidently, this is impossible according to (4.23). Consequently, by the definition of f, we know that u_ϵ is a solution of problem (2.1) for all $\epsilon \in (0, \epsilon_0)$. Moreover, by repeating the arguments explored in Lemma 3.2, we deduce that $u_\epsilon \in C^{1,\nu}_{loc}$ for some $\nu \in (0,1)$. Applying the Harnack's inequality, we can conclude that $u_\epsilon(x) > 0$ in $\mathbb{R}^N$.

Next we prove the concentration behavior and decay property of solution. From Lemma 4.3 we can know that

$$v_n(x) = u_n(x + y_n) \to v \text{ in } E \text{ and } \epsilon_n y_n \to x_0 \in \Pi \subset \Lambda.$$

If p_n is a global maximum point of $v_n(x)$, then, by Lemma 4.6 we know that there exists $R_0 > 0$ such that $p_n \in B_{R_0}(0)$. Therefore, $z_n = p_n + y_n$ is a global maximum point of $u_n(x)$. We deduce from the boundedness of $\{p_n\}$ and the continuity of V that

$$\lim_{n\to\infty} \epsilon_n z_n = x_0 \in \Pi \text{ and } \lim_{n\to\infty} V(\epsilon_n z_n) = V(x_0) = V_0.$$

Moreover, according to the fact that $V(x_0) = V_0$ and the proof of Lemma 4.3, we also know that v is a positive solution of the following limit problem

$$\begin{cases} -\Delta_p u - \Delta_q u + V_0(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),\ u > 0, & \text{in } \mathbb{R}^N. \end{cases} \tag{4.25}$$

From Lemma 4.5 and the boundedness of $\{p_n\}$, we derive that

$$\begin{aligned} |u_n(x)| =& |v_n(x - y_n)| \leq C \exp(-c|x - y_n|) = C \exp(-c|x - z_n + p_n|) \\ \leq& C \exp(-c|x - z_n| + c|p_n|) \leq \widetilde{C} \exp(-\tilde{c}|x - z_n|) \end{aligned}$$

for some $\tilde{c}, \widetilde{C} > 0$ and all $x \in \mathbb{R}^N$.

Finally, we observe that if $u_\epsilon(x)$ is a positive solution of problem (2.1), then $\widehat{u}_\epsilon(x) = u_\epsilon(\frac{x}{\epsilon})$ is a positive solution of problem (1.1). So, the conclusion (a) of Theorem 1.1 holds. Moreover, the maximum points x_ϵ and z_ϵ of $\widehat{u}_\epsilon$ and u_ϵ, respectively, satisfy $x_\epsilon = \epsilon z_\epsilon$. Consequently, according to the above discussions, we have the following conclusions

(b) $\lim_{\epsilon\to 0} \operatorname{dist}(x_\epsilon, \Pi) = 0$ and $\lim_{\epsilon\to 0} V(x_\epsilon) = V_0$;
(c) $\widehat{u}_\epsilon(\epsilon x + x_\epsilon) \to \widehat{u}(x)$ in E, where $\widehat{u}$ is a positive solution of the limit problem (4.25).
(d) there exist positive constants $\tilde{c}, \widetilde{C}$ such that

$$\widehat{u}_\epsilon(x) \leq \widetilde{C} \exp\left(-\frac{\tilde{c}}{\epsilon}|x - x_\epsilon|\right).$$

We finish the proof of all conclusions of Theorem 1.1. □

5 Multiplicity of positive solutions

In this section we are going to investigate the multiplicity of positive solutions. In what follows we always assume that condition $(\widehat{V})$ holds.

Let us denote by $\mathcal{N}_\epsilon$ the Nehari manifold associated with $\mathcal{I}_\epsilon$, given by

$$\mathcal{N}_\epsilon = \{u \in E_\epsilon : u \neq 0, \langle \mathcal{I}'_\epsilon(u), u\rangle = 0\},$$

and let S_ϵ be the unit sphere of E_ϵ. Similarly to the proof of (4.6), we can see that for all $u \in \mathcal{N}_\epsilon$, there exists $\alpha_0 > 0$ such that

$$\|u\|_\epsilon \geq \alpha_0 \text{ for all } \epsilon > 0. \tag{5.1}$$

Since the nonlinearity g is only continuous, $\mathcal{N}_\epsilon$ is not differentiable and some well-known arguments for C^1-Nehari manifold are not applicable in our situation. So, to conquer this difficulty created by the non-differentiability, we will adapt some variants of critical point theorems due to Szulkin-Weth [42]. Let us start with the following results which will play a crucial role in our analysis.

Lemma 5.1 *The following conclusions hold:*

(a) *For each $u \in E_\epsilon\backslash\{0\}$ there exists an unique t_u such that if $h_u(t) := \mathcal{I}_\epsilon(tu)$, then $h'_u(t) > 0$ for $0 < t < t_u$ and $h'_u(t) < 0$ for $t > t_u$.*
(b) *There exists $\delta > 0$ such that $t_u \geq \delta$ for all $u \in S_\epsilon$, and for each compact subset $\mathcal{W} \subset S_\epsilon$ there exists a constant $C_{\mathcal{W}}$ such that $t_u \leq C_{\mathcal{W}}$ for all $u \in \mathcal{W}$.*

Proof (a) Let $u \in E_\epsilon\backslash\{0\}$, from the proof of Lemma 2.5, we can know that $h(0) = 0$, $h(t) > 0$ for t sufficiently small and $h(t) < 0$ for t sufficiently large. So, there is $t = t_u$ such that $\max_{t>0} h(t)$ is achieved at t_u, so $h'(t_u) = 0$ and $t_u u \in \mathcal{N}_\epsilon$.

Next, we claim that t_u is the unique critical point of h. Assume by contradiction that there exist t_1 and t_2 with $0 < t_1 < t_2$ such that $t_1 u, t_2 u \in \mathcal{N}_\epsilon$, then it follows that

$$t_1^{p-q}\|u\|^p_{V_\epsilon,p} + \|u\|^q_{V_\epsilon,q} = \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * \frac{F(\epsilon x, t_1 u)}{t_1^{\frac{q}{2}}}\right]\frac{f(\epsilon x, t_1 u)}{t_1^{\frac{q}{2}-1}} u\mathrm{d}x$$

and

$$t_2^{p-q}\|u\|^p_{V_\epsilon,p} + \|u\|^q_{V_\epsilon,q} = \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * \frac{F(\epsilon x, t_2 u)}{t_2^{\frac{q}{2}}}\right]\frac{f(\epsilon x, t_2 u)}{t_2^{\frac{q}{2}-1}} u\mathrm{d}x.$$

Subtracting term by term in the above equalities and using (f_4) we can infer that

$$\begin{aligned}0 <&(t_1^{p-q} - t_2^{p-q})\|u\|_{V_\epsilon,p}^p\\ =&\int_{\mathbb{R}^N}\left(\left[\frac{1}{|x|^\mu} * \frac{F(\epsilon x, t_1 u)}{t_1^{\frac{q}{2}}}\right]\frac{f(\epsilon x, t_1 u)}{t_1^{\frac{q}{2}-1}} - \left[\frac{1}{|x|^\mu} * \frac{F(\epsilon x, t_2 u)}{t_2^{\frac{q}{2}}}\right]\frac{f(\epsilon x, t_2 u)}{t_2^{\frac{q}{2}-1}}\right) u \mathrm{d}x < 0.\end{aligned}$$

which implies a contradiction, and we get the uniqueness of t_u.

(b) For each $u \in S_\epsilon$, according to (5.1) and conclusion (a), we can see that there exists $t_u > 0$ such that $t_u u \in \mathcal{N}_\epsilon$, and $t_u = \|t_u u\|_\epsilon \geq \delta$ for some $\delta > 0$. It remains we prove that $t_u \leq C_{\mathcal{W}}$ for all $u \in \mathcal{W} \subset S_\epsilon$. Arguing by contradiction we assume that there exist a sequence $\{u_n\} \subset \mathcal{W} \subset S_\epsilon$ and $\{t_n\}$ such that $t_n \to \infty$. Note that, since $\mathcal{W}$ is compact, there exists $u \in \mathcal{W}$ such that $u_n \to u$ in E_ϵ. Similarly to the proof of Lemma 2.5, we can know that $\mathcal{I}_\epsilon(t_n u_n) \to -\infty$. However, for any $u \in \mathcal{N}_\epsilon$, using (f_2) we have

$$\begin{aligned}\mathcal{I}_\epsilon(u) =&\mathcal{I}_\epsilon(u) - \frac{1}{\theta}\langle \mathcal{I}_\epsilon'(u), u\rangle\\ =&\left(\frac{1}{p} - \frac{1}{\theta}\right)\|u\|_{V_\epsilon,p}^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u\|_{V_\epsilon,q}^q\\ &+ \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(\epsilon x, u)\right]\left(\frac{1}{\theta} f(\epsilon x, u)u - \frac{1}{2}F(\epsilon x, u)\right)\mathrm{d}x > 0.\end{aligned}$$

So, the conclusion $\mathcal{I}_\epsilon(t_n u_n) \to -\infty$ is impossible, a contradiction. This completes the proof. □

Lemma 5.2 *The map $\widehat{m}_\epsilon : E_\epsilon\backslash\{0\} \to \mathcal{N}_\epsilon$ is continuous, and the map $m_\epsilon := \widehat{m}_\epsilon|_{S_\epsilon} : S_\epsilon \to \mathcal{N}_\epsilon$ is a homeomorphism between S_ϵ and $\mathcal{N}_\epsilon$ with inverse given by*

$$m_\epsilon^{-1} : \mathcal{N}_\epsilon \to S_\epsilon,\ m_\epsilon^{-1}(u) = u/\|u\|_\epsilon.$$

Proof Assume that $u_n \to u \neq 0$. Since $\widehat{m}_\epsilon(tu) = \widehat{m}_\epsilon(u)$ for each $t > 0$, we may assume $u_n \in S_\epsilon$ for all n and it is sufficient to verify that $\widehat{m}_\epsilon(u_n) \to \widehat{m}_\epsilon(u)$ after passing to a subsequence. From Lemma 5.1, it is easy to see that $\hat{m}_\epsilon(u_n) = t_{u_n}u_n \in \mathcal{N}_\epsilon$, and $\{t_{u_n}\}$ is bounded and bounded away from 0, Therefore, up to a subsequence, $t_{u_n} \to t_0 > 0$. Since $\mathcal{N}_\epsilon$ is closed and $\widehat{m}_\epsilon(u_n) \to t_0 u$ and $t_0 u \in \mathcal{N}_\epsilon$. Hence $t_0 u = t_u u = \widehat{m}_\epsilon(u)$. From the above proof, the second conclusion is an immediate consequence. □

Let us define the functional $\widehat{\Psi}_\epsilon : E_\epsilon\backslash\{0\} \to \mathbb{R}$ and the restriction $\Psi_\epsilon : S_\epsilon \to \mathbb{R}$ as follows

$$\widehat{\Psi}_\epsilon(u) = \mathcal{I}_\epsilon(\widehat{m}_\epsilon(u)) \text{ and } \Psi_\epsilon = \widehat{\Psi}_\epsilon|_{S_\epsilon}.$$

Moreover, from Lemma 5.1 and Lemma 5.2 and arguing as in the proofs of Proposition 9 and Corollary 10 in [42], we can obtain the following important results.

Lemma 5.3 *We have the following conclusions:*

(a) $\widehat{\Psi}_\epsilon \in C^1(E_\epsilon\backslash\{0\}, \mathbb{R})$ *and for* $u, v \in E_\epsilon$ *and* $u \neq 0$,

$$\langle \widehat{\Psi}'_\epsilon(u), v\rangle = \frac{\|\widehat{m}_\epsilon(u)\|_\epsilon}{\|u\|_\epsilon}\langle \mathcal{I}'_\epsilon(\widehat{m}_\epsilon(u)), v\rangle.$$

(b) $\Psi_\epsilon \in C^1(S_\epsilon, \mathbb{R})$ *and* $\langle \Psi'_\epsilon(u), v\rangle = \|\widehat{m}_\epsilon(u)\|_\epsilon \langle \mathcal{I}'_\epsilon(\widehat{m}_\epsilon(u)), v\rangle$ *for* $v \in T_u(S_\epsilon)$, *where* $T_u(S_\epsilon)$ *is the tangent space of* S_ϵ *at* u.
(c) $\{u_n\}$ *is a Palais-Smale sequence for* Ψ_ϵ *if and only if* $\{\widehat{m}_\epsilon(u_n)\}$ *is a Palais-Smale sequence for* $\mathcal{I}_\epsilon$.
(d) $u \in S_\epsilon$ *is a critical point of* Ψ_ϵ *if and only if* $\widehat{m}_\epsilon(u) \in \mathcal{N}_\epsilon$ *is a critical point of* $\mathcal{I}_\epsilon$, *the corresponding values of* Ψ_ϵ *and* $\mathcal{I}_\epsilon$ *coincide and*

$$\inf_{S_\epsilon} \Psi_\epsilon = \inf_{\mathcal{N}_\epsilon} \mathcal{I}_\epsilon.$$

As in [42], we have the following variational characterization of the infimum of $\mathcal{I}_\epsilon$ on $\mathcal{N}_\epsilon$:

$$c_\epsilon = \inf_{u\in\mathcal{N}_\epsilon} \mathcal{I}_\epsilon(u) = \inf_{u\in E_\epsilon\backslash\{0\}} \max_{t\geq 0} \mathcal{I}_\epsilon(tu) = \inf_{u\in S_\epsilon} \max_{t\geq 0} \mathcal{I}_\epsilon(tu).$$

Lemma 5.4 *The functional* Ψ_ϵ *satisfies the Palais-Smale compactness condition on* S_ϵ *at any level* $c \in \mathbb{R}$.

Proof Let $\{w_n\}$ be a Palais-Smale sequence for Ψ_ϵ at the level c, then

$$\Psi_\epsilon(w_n) \to c \text{ and } \|\Psi'_\epsilon(w_n)\|_* \to 0,$$

where $\|\cdot\|_*$ is the norm in the dual space $(T_{w_n}S_\epsilon)^*$. From Lemma 5.3-(c), we can know that $u_n := m_\epsilon(w_n)$ is a Palais-Smale sequence for $\mathcal{I}_\epsilon$ at the level c. So, according to Lemma 2.9, we find that there exists $w \in S_\epsilon$ such that, up to a subsequence, $m_\epsilon(w_n) \to m_\epsilon(w)$. Moreover, using Lemma 5.2 we can derive that $w_n \to w$ in S_ϵ. □

It is clear that the previous results can be easily extended for the autonomous problem (3.1). Arguing as in the proofs of Lemma 5.2 and Lemma 5.3 we have the following result.

Lemma 5.5 *The following conclusions hold:*

(a) *For each* $u \in E_a\backslash\{0\}$ *there exists an unique* t_u *such that if* $h_u(t) := \mathcal{J}_a(tu)$, *then* $h'_u(t) > 0$ *for* $0 < t < t_u$ *and* $h'_u(t) < 0$ *for* $t > t_u$.
(b) *There exists* $\delta > 0$ *such that* $t_u \geq \delta$ *for all* $u \in S_a$, *and for each compact subset* $\mathcal{W} \subset S_a$ *there exists a constant* $C_{\mathcal{W}}$ *such that* $t_u \leq C_{\mathcal{W}}$ *for all* $u \in \mathcal{W}$, *where* S_a *is the unit sphere of* E_a.
(c) *The map* $\widehat{m}_a : E_a\backslash\{0\} \to \mathcal{N}_a$ *given by* $\widehat{m}_a(u) := t_u u$ *is continuous, and the map* $m_a := \widehat{m}_a|_{S_a}$ *is a homeomorphism between* S_a *and* $\mathcal{N}_a$, *moreover,* $m_a^{-1}(u) = u/\|u\|_a$.

Similarly, we define the map $\widehat{\Psi}_a : E_a\backslash\{0\} \to \mathbb{R}$ and the restriction $\Psi_a : S_a \to \mathbb{R}$ as follows

$$\widehat{\Psi}_a(u) = \mathcal{J}_a(\widehat{m}_a(u)) \text{ and } \Psi_a = \widehat{\Psi}_a|_{S_a}.$$

Lemma 5.6 *The following results hold:*

(a) $\widehat{\Psi}_a \in C^1(E_a\backslash\{0\}, \mathbb{R})$ *and for* $u, v \in E_a$ *and* $u \neq 0$,

$$\langle \widehat{\Psi}_a'(u), v\rangle = \frac{\|\widehat{m}_a(u)\|_a}{\|u\|_a}\langle \mathcal{J}_a'(\widehat{m}_a(u)), v\rangle.$$

(b) $\Psi_a \in C^1(S_a, \mathbb{R})$ *and* $\langle \Psi_a'(u), v\rangle = \|\widehat{m}_a(u)\|_a\langle \mathcal{J}_a'(\widehat{m}_a(u)), v\rangle$ *for* $v \in T_u(S_a)$, *where* $T_u(S_a)$ *is the tangent space of* S_a *at* u.
(c) $\{u_n\}$ *is a Palais-Smale sequence for* Ψ_a *if and only if* $\{\widehat{m}_a(u_n)\}$ *is a Palais-Smale sequence for* $\mathcal{J}_a$.
(d) $u \in S_a$ *is a critical point of* Ψ_a *if and only if* $\widehat{m}_a(u) \in \mathcal{N}_a$ *is a critical point of* $\mathcal{J}_a$, *the corresponding values of* Ψ_a *and* $\mathcal{J}_a$ *coincide and*

$$\inf_{S_a} \Psi_a = \inf_{\mathcal{N}_a} \mathcal{J}_a.$$

Moreover, we also have the following variational characterization of the infimum of $\mathcal{J}_a$ on $\mathcal{N}_a$:

$$c_a = \inf_{u\in\mathcal{N}_a} \mathcal{J}_a(u) = \inf_{u\in E_a\backslash\{0\}} \max_{t\geq 0} \mathcal{J}_a(tu) = \inf_{u\in S_a} \max_{t\geq 0} \mathcal{J}_a(tu).$$

Next we establish a important compactness result for the autonomous problem which will be used later.

Lemma 5.7 *Let* $\{u_n\} \subset \mathcal{N}_a$ *be a sequence such that* $\mathcal{J}_a(u_n) \to c_a$, *then* $\{u_n\}$ *has a convergent subsequence in* E_a.

Proof Since $\{u_n\} \subset \mathcal{N}_a$ and $\mathcal{J}_a(u_n) \to c_a$, applying Lemma 5.6-(d) we can deduce that

$$v_n := m_a^{-1}(u_n) = u_n/\|u_n\|_a \in S_a \text{ and } \Psi_a(v_n) = \mathcal{J}_a(u_n) \to c_a.$$

Hence, using the Ekeland's variational principle we can find that there is a Palais-Smale sequence $\{\tilde{v}_n\} \subset S_a$ of Ψ_a at the level c_a such that $\|\tilde{v}_n - v_n\|_a = o_n(1)$ as $n \to \infty$. Let $\tilde{u}_n = m_a(\tilde{v}_n)$, then we have $\|\tilde{u}_n - u_n\|_a = o_n(1)$. It follows from Lemma 5.6-(c) that $\{\tilde{u}_n\} \subset \mathcal{N}_a$ is a Palais-Smale sequence of $\mathcal{J}_a$ at the level c_a. Then, it is clear that $\{\tilde{u}_n\}$ is bounded. According to the proof of Lemma 3.2, we can see that there exists $\tilde{u} \in E_a\backslash\{0\}$, after passing to a subsequence, such that $\tilde{u}_n \rightharpoonup \tilde{u}$, $\mathcal{J}_a(\tilde{u}) = c_a$ and $\mathcal{J}_a'(\tilde{u}) = 0$. Applying Brezis-Lieb lemma and a standard argument, we derive that

$$\mathcal{J}_a(\tilde{u}_n - \tilde{u}) = o_n(1) \text{ and } \mathcal{J}_a'(\tilde{u}_n - \tilde{u}) = o_n(1).$$

This fact, together with (g_4), yields that

$$\begin{aligned} o_n(1) =&\mathcal{J}_a(\tilde{u}_n - \tilde{u}) - \frac{1}{\theta}\langle \mathcal{J}_a'(\tilde{u}_n - \tilde{u}), \tilde{u}_n - \tilde{u}\rangle \\ =&\left(\frac{1}{p} - \frac{1}{\theta}\right)\|\tilde{u}_n - \tilde{u}\|_{a,p}^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|\tilde{u}_n - \tilde{u}\|_{a,q}^q \\ &+ \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * G(\tilde{u}_n - \tilde{u})\right]\left[\frac{1}{\theta}g(\tilde{u}_n - \tilde{u})(\tilde{u}_n - \tilde{u}) - \frac{1}{2}G(\tilde{u}_n - \tilde{u})\right]\mathrm{d}x \\ \geq&\left(\frac{1}{p} - \frac{1}{\theta}\right)\|\tilde{u}_n - \tilde{u}\|_{a,p}^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|\tilde{u}_n - \tilde{u}\|_{a,q}^q, \end{aligned}$$

which implies that $\tilde{u}_n \to \tilde{u}$ in E_a. So, we can easily conclude that $u_n \to \tilde{u}$ in E_a. This completes the proof. □

We introduce some technical results. Let u be a positive ground state solution of the problem

$$\begin{cases} -\Delta_p u - \Delta_q u + V_{\min}(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right)g(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{5.2}$$

and let ζ be a smooth nonincreasing cut-off function in $[0, +\infty)$ such that $\zeta(s) = 1$ if $0 \leq s \leq \frac{\delta}{2}$ and $\zeta(s) = 0$ if $s \geq \delta$, $0 \leq \zeta \leq 1$ and $|\zeta'(s)| \leq c$ for some $c > 0$. For any $z \in M$, we define the function

$$\Psi_{\epsilon,z}(x) = \zeta(|\epsilon x - z|)u(\frac{\epsilon x - z}{\epsilon}).$$

It follows from Lemma 5.1 that there exists $t_\epsilon > 0$ such that

$$\max_{t\geq 0}\mathcal{I}_\epsilon(t\Psi_{\epsilon,z}) = \mathcal{I}_\epsilon(t_\epsilon\Psi_{\epsilon,z}).$$

So, we define $\Phi_\epsilon : M \to \mathcal{N}_\epsilon$ by $\Phi_\epsilon(z) = t_\epsilon\Psi_{\epsilon,z}$. According to the construction of $\Psi_{\epsilon,z}$, we can see that $\Phi_\epsilon(z)$ has compact support for any $z \in M$. The following lemma describes an important relationship between Φ_ϵ and the set M.

Lemma 5.8 *We have the limit*

$$\lim_{\epsilon\to 0}\mathcal{I}_\epsilon(\Phi_\epsilon(z)) = c_{V_{\min}} \text{ uniformly in } z \in M.$$

Proof Assume by contradiction that there exist $\varepsilon_0 > 0$, $\{z_n\} \subset M$ and $\epsilon_n \to 0$ such that

$$|\mathcal{I}_{\epsilon_n}(\Phi_{\epsilon_n}(z)) - c_{V_{\min}}| \geq \varepsilon_0. \tag{5.3}$$

Considering the change of variable $\widehat{x} = (\epsilon_n x - z_n)/\epsilon_n$ and $\widehat{y} = (\epsilon_n y - z_n)/\epsilon_n$, if $\widehat{x} \in B_{\delta/\epsilon_n}(0)$, it follows that $\epsilon_n\widehat{x} \in B_\delta(0)$ and $\epsilon_n\widehat{x} + z_n \in B_\delta(z_n) \subset M_\delta \subset \Lambda$. Since

$F(\epsilon x, s) = G(s)$ in Λ, we can deduce that

$$\begin{aligned}\mathcal{I}_{\epsilon_n}(\Phi_{\epsilon_n}(z_n)) =&\frac{t_{\epsilon_n}^p}{p}|\nabla\Psi_{\epsilon_n,z_n}|_p^p + \frac{t_{\epsilon_n}^q}{q}|\nabla\Psi_{\epsilon_n,z_n}|_q^q \\ &+ \int_{\mathbb{R}^N} V(\epsilon_n x)\left[\frac{t_{\epsilon_n}^p}{p}|\Psi_{\epsilon_n,z_n}|^p + \frac{t_{\epsilon_n}^q}{q}|\Psi_{\epsilon_n,z_n}|^q\right]\mathrm{d}x \\ &- \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * F(\epsilon_n x, \Psi_{\epsilon_n,z_n})\right]F(\epsilon_n x, \Psi_{\epsilon_n,z_n})\mathrm{d}x \\ =&\frac{t_{\epsilon_n}^p}{p}|\nabla(\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x}))|_p^p + \frac{t_{\epsilon_n}^q}{q}|\nabla(\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x}))|_q^q \\ &+ \int_{\mathbb{R}^N} V(\epsilon_n\widehat{x} + z_n)\left[\frac{t_{\epsilon_n}^p}{p}|\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x})|^p + \frac{t_{\epsilon_n}^q}{q}|\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x})|^q\right]\mathrm{d}\widehat{x} \\ &- \int_{\mathbb{R}^N}\left[\frac{1}{|\widehat{x}|^\mu} * G(t_{\epsilon_n}\zeta(|\epsilon_n\widehat{y}|)u(\widehat{y}))\right]G(t_{\epsilon_n}\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x}))\mathrm{d}\widehat{x}.\end{aligned} \tag{5.4}$$

Applying the Lebesgue's dominated convergence theorem, we can easily check that

$$|\nabla\Psi_{\epsilon_n,z_n}|_p^p + \int_{\mathbb{R}^N} V(\epsilon_n x)|\Psi_{\epsilon_n,z_n}|^p\mathrm{d}x \to |\nabla u|_p^p + \int_{\mathbb{R}^N} V_{\min}|u|^p\mathrm{d}x, \tag{5.5}$$

$$|\nabla\Psi_{\epsilon_n,z_n}|_q^q + \int_{\mathbb{R}^N} V(\epsilon_n x)|\Psi_{\epsilon_n,z_n}|^q\mathrm{d}x \to |\nabla u|_q^q + \int_{\mathbb{R}^N} V_{\min}|u|^q\mathrm{d}x, \tag{5.6}$$

and

$$\int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * G(\Psi_{\epsilon_n,z_n})\right]G(\Psi_{\epsilon_n,z_n})\mathrm{d}x \to \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * G(u)\right]G(u)\mathrm{d}x. \tag{5.7}$$

Moreover, using the fact $\langle\mathcal{I}_{\epsilon_n}'(t_{\epsilon_n}\Psi_{\epsilon_n,z_n}), t_{\epsilon_n}\Psi_{\epsilon_n,z_n}\rangle = 0$ and the arguments used in [49], it is easy to prove that $t_{\epsilon_n} \to t_0 > 0$. So, from (5.5), (5.6) and (5.7) we can derive that

$$t_0^p|\nabla u|_p^p + t_0^q|\nabla u|_q^q + V_{\min}\int_{\mathbb{R}^N}(|u|^p + |u|^q)\mathrm{d}x = \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * G(t_0 u)\right]g(t_0 u)t_0 u\mathrm{d}x. \tag{5.8}$$

Since u is a positive ground state solution of problem (5.2), it follows that

$$|\nabla u|_p^p + |\nabla u|_q^q + V_{\min}\int_{\mathbb{R}^N}(|u|^p + |u|^q)\mathrm{d}x = \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * G(u)\right]g(u)u\mathrm{d}x. \tag{5.9}$$

Then, according to (g_5), (5.8) and (5.9), we can get $t_0 = 1$. Therefore, we infer from (5.4), (5.5), (5.6) and (5.7) that

$$\begin{aligned}\mathcal{I}_{\epsilon_n}(\Phi_{\epsilon_n}(z_n)) \to &\frac{1}{p}\|u\|_{V_{\min},p}^p + \frac{1}{q}\|u\|_{V_{\min},q}^q - \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * G(u)\right]G(u)\mathrm{d}x \\ =&\mathcal{J}_{V_{\min}}(u) = c_{V_{\min}}.\end{aligned}$$

From (5.3) we can know that this is impossible. Thus, we complete the proof of the lemma. □

Next we are in the position to introduce the barycenter map. For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. We define $\eta : \mathbb{R}^N \to \mathbb{R}^N$ as follows

$$\eta(x) = x \text{ for } |x| \le \rho \ \text{ and } \ \eta(x) = \frac{\rho x}{|x|} \text{ for } |x| \ge \rho.$$

The barycenter map $\beta_\epsilon : \mathcal{N}_\epsilon \to \mathbb{R}^N$ is defined by

$$\beta_\epsilon(u) = \frac{\int_{\mathbb{R}^N} \eta(\epsilon x)(|u|^p + |u|^q)\mathrm{d}x}{\int_{\mathbb{R}^N}(|u|^p + |u|^q)\mathrm{d}x}.$$

Combining the above definitions, we can prove the following result.

Lemma 5.9 *We have the limit*

$$\lim_{\epsilon \to 0} \beta_\epsilon(\Phi_\epsilon(z)) = z \textit{ uniformly in } z \in M.$$

Proof Arguing by contradiction, we assume that there exist $\sigma_0 > 0$, $\{z_n\} \subset M$ and $\epsilon_n \to 0$ such that

$$|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(z_n)) - z_n| \ge \sigma_0 > 0. \tag{5.10}$$

According to the definitions of Φ_{ϵ_n} and β_{ϵ_n}, and making the change of variable $\widehat{x} = (\epsilon_n x - z_n)/\epsilon_n$ we immediately obtain

$$\beta_{\epsilon_n}(\Phi_{\epsilon_n}(z_n)) = z_n + \frac{\int_{\mathbb{R}^N}[\eta(\epsilon_n\widehat{x} + z_n) - z_n](|\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x})|^p + |\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x})|^q)\mathrm{d}\widehat{x}}{\int_{\mathbb{R}^N}(|\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x})|^p + |\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x})|^q)\mathrm{d}\widehat{x}}.$$

Since $\{z_n\} \subset M \subset B_\rho(0)$, employing the Lebesgue's dominates convergence theorem, we have

$$|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(z_n)) - z_n| \to 0,$$

and we also get a contradiction according to (5.10). □

Now we give the following useful compactness result, which plays a fundamental role to prove that solutions of the modified problem are solutions of the original problem.

Lemma 5.10 *Let $\epsilon_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\epsilon_n}$ be a sequence satisfying $\mathcal{I}_{\epsilon_n}(u_n) \to c_{V_{\min}}$. Then there exists $\{\tilde{z}_n\} \subset \mathbb{R}^N$ such that $v_n = u_n(x + \tilde{z}_n)$ has a convergent subsequence. Moreover, up to a subsequence, $z_n \to z \in M$, where $z_n = \epsilon_n \tilde{z}_n$.*

Proof Since $\{u_n\} \subset \mathcal{N}_{\epsilon_n}$ and $\mathcal{I}_{\epsilon_n}(u_n) \to c_{V_{\min}}$, then, a standard argument shows that $\{u_n\}$ is bounded. We claim that there are $R_0, \delta > 0$ and $\tilde{z}_n \in \mathbb{R}^N$ such that

$$\liminf_{n \to \infty} \int_{B_{R_0}(\tilde{z}_n)} |u_n|^q \mathrm{d}x \ge \delta. \tag{5.11}$$

Otherwise, Lemma 2.2 implies that $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $s \in (q, q^*)$. According to (2.4) and Lemma 2.4, we can easily prove that $u_n \to 0$ in E_ϵ, this is impossible because $\mathcal{I}_{\epsilon_n}(u_n) \to c_{V_{\min}} > 0$. Consequently, (5.11) holds.

Let us define $v_n(x) = u_n(x + \tilde{z}_n)$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v \neq 0$. From Lemma 5.5 we can see that there exists $t_n > 0$ such that $\tilde{v}_n = t_n v_n \in \mathcal{N}_{V_{\min}}$. Then we have

$$c_{V_{\min}} \leq \mathcal{J}_{V_{\min}}(\tilde{v}_n) = \mathcal{J}_{V_{\min}}(t_n u_n) \leq \mathcal{I}_{\epsilon_n}(t_n u_n) \leq \mathcal{I}_{\epsilon_n}(u_n) \to c_{V_{\min}},$$

which shows $\mathcal{J}_{V_{\min}}(\tilde{v}_n) \to c_{V_{\min}}$. In particular, we get $\tilde{v}_n \rightharpoonup \tilde{v}$ and $t_n \to t_0 > 0$. Then, from the uniqueness of the weak limit, we have $\tilde{v} = t_0 v \not\equiv 0$. By using Lemma 5.7, we can see that

$$\tilde{v}_n \to \tilde{v} \text{ in } E_{V_{\min}},$$

which implies that $v_n \to v$ in $E_{V_{\min}}$ and $\mathcal{J}_{V_{\min}}(\tilde{v}) = c_{V_{\min}}$ and $\langle \mathcal{J}'_{V_{\min}}(\tilde{v}), \tilde{v}\rangle = 0$.

In order to finish the proof of the lemma, we consider $z_n = \epsilon_n \tilde{z}_n$. Our claim is to show that $\{z_n\}$ admits a subsequence, still denoted by z_n, such that $z_n \to z_0$ for some $z_0 \in M$. Indeed, we first prove that $\{z_n\}$ is bounded. We argue by contradiction, and we assume that, passing to a subsequence, $|z_n| \to \infty$ as $n \to \infty$. Since

$$\|u_n\|^p_{V_\epsilon,p} + \|u_n\|^q_{V_\epsilon,q} = \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\epsilon x, u_n)\right] f(\epsilon x, u_n) u_n \mathrm{d}x$$

and $\mathcal{I}_\epsilon(u_n) \to c_{V_{\min}}$, by Lemma 2.6 we can see that there exists $n_0 \in \mathbb{N}$ such that $u_n \in \mathscr{B}$ for all $n \geq n_0$. Then, using Lemma 2.7 we get

$$\sup_{n \geq n_0} \left| \frac{1}{|x|^\mu} * F(\epsilon x, u_n) \right|_\infty < \frac{\ell_0}{2}.$$

Fixed $R > 0$ such that $\Lambda \subset B_R(0)$, we assume that $|z_n| > 2R$. Then, for all $x \in B_{R/\epsilon_n}(0)$, we can know that $|\epsilon_n x + z_n| \geq |z_n| - |\epsilon_n x| > R$ for all n large enough. Hence, we deduce from $(\widehat{V})$, (f_3) and the definition of f that

$$\begin{aligned}
&\|v_n\|^p_{V_{\min},p} + \|v_n\|^q_{V_{\min},q} \leq \frac{\ell_0}{2} \int_{\mathbb{R}^N} f(\epsilon_n x + z_n, v_n) v_n \mathrm{d}x \\
&\leq \frac{\ell_0}{2} \left[\int_{B_{R/\epsilon_n}(0)} \widetilde{g}(\epsilon_n x + z_n, v_n) v_n \mathrm{d}x + \int_{B^c_{R/\epsilon_n}(0)} g(\epsilon_n x + z_n, v_n) v_n \mathrm{d}x \right] \\
&\leq \frac{\ell_0}{2} \left[\int_{B_{R/\epsilon_n}(0)} \frac{V_{\min}}{\ell_0} (|v_n|^p + |v_n|^q) \mathrm{d}x + \int_{B^c_{R/\epsilon_n}(0)} g(\epsilon_n x + z_n, v_n) v_n \mathrm{d}x \right].
\end{aligned}$$

Observe that, since $v_n \to v$ in $E_{V_{\min}}$, we obtain

$$\int_{B^c_{R/\epsilon_n}(0)} g(\epsilon_n x + z_n, v_n) v_n \mathrm{d}x = o_n(1).$$

Combining the above estimates, we can conclude that

$$\|v_n\|^p_{V_{\min},p} + \|v_n\|^q_{V_{\min},q} = o_n(1),$$

that is $v_n \to 0$ in $E_{V_{\min}}$, which is a contradiction. Thus, $\{z_n\}$ is bounded, and we may assume that $z_n \to z_0 \in \mathbb{R}^N$.

It remains to check that $z_0 \in M$. Arguing by contradiction again, we assume that $V(z_0) > V_{\min}$, Then, recalling that $\tilde{v}_n \to \tilde{v}$ in $E_{V_{\min}}$, we can use Fatou's lemma to get

$$\begin{aligned} c_{V_{\min}} = \mathcal{J}_{V_{\min}}(\tilde{v}) <& \frac{1}{p}\|\tilde{v}\|^p_{V(z_0),p} + \frac{1}{q}\|\tilde{v}\|^q_{V(z_0),q} - \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * G(\tilde{v})\right]G(\tilde{v})\mathrm{d}x \\ \leq& \liminf_{n\to\infty}\left[\frac{1}{p}\int_{\mathbb{R}^N}|\nabla\tilde{v}_n|^p\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}|\nabla\tilde{v}_n|^q\mathrm{d}x\right. \\ &+ \int_{\mathbb{R}^N} V(\epsilon_n x + z_n)\left[\frac{1}{p}|\tilde{v}_n|^p + \frac{1}{q}|\tilde{v}_n|^q\right]\mathrm{d}x \\ &\left. - \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * G(\tilde{v}_n)\right]G(\tilde{v}_n)\mathrm{d}x\right] \\ \leq& \liminf_{n\to\infty}\mathcal{I}_{\epsilon_n}(t_n u_n) \\ \leq& \liminf_{n\to\infty}\mathcal{I}_{\epsilon_n}(u_n) = c_{V_{\min}}. \end{aligned}$$

Evidently, this is a contradiction. So, $z_0 \in M$, and this ends the proof of the lemma. □

Let $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ be a positive function given by

$$\vartheta(\epsilon) = \max_{z\in M}|\mathcal{I}_\epsilon(\Phi_\epsilon(z)) - c_{V_{\min}}|.$$

It follows from Lemma 5.8 that $\vartheta(\epsilon) \to 0$ as $\epsilon \to 0$. We introduce a subset $\widetilde{\mathcal{N}_\epsilon}$ of $\mathcal{N}_\epsilon$, and set

$$\widetilde{\mathcal{N}_\epsilon} := \{u \in \mathcal{N}_\epsilon : \mathcal{I}_\epsilon(u) \leq c_{V_{\min}} + \vartheta(\epsilon)\},$$

Since $\Phi_\epsilon(z) \in \widetilde{\mathcal{N}_\epsilon}$ for all $z \in M$, then we deduce that $\widetilde{\mathcal{N}_\epsilon} \neq \emptyset$. Moreover, we have the following result.

Lemma 5.11 *For any $\delta > 0$, then the following limit holds*

$$\lim_{\epsilon\to 0}\sup_{u\in\widetilde{\mathcal{N}_\epsilon}}\inf_{z\in M_\delta}|\beta_\epsilon(u) - z| = 0.$$

Proof Let $\epsilon_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, there exists $\{u_n\} \subset \widetilde{\mathcal{N}_{\epsilon_n}}$, such that

$$\inf_{z\in M_\delta}|\beta_{\epsilon_n}(u_n) - z| = \sup_{u\in\widetilde{\mathcal{N}_{\epsilon_n}}}\inf_{z\in M_\delta}|\beta_{\epsilon_n}(u) - z| + o_n(1).$$

Hence, it is sufficient to prove that there exists $\{z_n\} \subset M_\delta$ such that

$$\lim_{n\to\infty}|\beta_{\epsilon_n}(u_n) - z_n| = 0.$$

In fact, since $\{u_n\} \subset \widetilde{\mathcal{N}_{\epsilon_n}}$, then we have

$$c_{V_{\min}} \leq c_{\epsilon_n} \leq \mathcal{I}_{\epsilon_n}(u_n) \leq c_{V_{\min}} + \vartheta(\epsilon_n),$$

which implies that

$$\mathcal{I}_{\epsilon_n}(u_n) \to c_{V_{\min}} \text{ and } \{u_n\} \subset \mathcal{N}_{\epsilon_n}.$$

According to Lemma 5.10, there exists $\{\tilde{z}_n\} \subset \mathbb{R}^N$ such that $v_n(x) = u_n(x + \tilde{z}_n)$ has a convergent subsequence. Moreover, up to a subsequence, $z_n = \epsilon_n \tilde{z}_n \to z \in M$, and we can conclude that

$$\begin{aligned}
\beta_{\epsilon_n}(u_n) =& \frac{\int_{\mathbb{R}^N} \eta(\epsilon_n x)(|u_n|^p + |u_n|^q)\mathrm{d}x}{\int_{\mathbb{R}^N} (|u_n|^p + |u_n|^q)\mathrm{d}x} \\
=& \frac{\int_{\mathbb{R}^N} \eta(\epsilon_n y + z_n)(|u_n(y + \tilde{z}_n)|^p + |u_n(y + \tilde{z}_n)|^q)\mathrm{d}y}{\int_{\mathbb{R}^N} (|u_n(y + \tilde{z}_n)|^p + |u_n(y + \tilde{z}_n)|^q)\mathrm{d}y} \\
=& z_n + \frac{\int_{\mathbb{R}^N} [\eta(\epsilon_n y + z_n) - z_n](|v_n(y)|^p + |v_n(y)|^q)\mathrm{d}y}{\int_{\mathbb{R}^N} (|v_n(y)|^p + |v_n(y)|^q)\mathrm{d}y} \\
\to& z \in M.
\end{aligned}$$

Therefore, there exists $\{z_n\} \subset M_\delta$ such that

$$\lim_{n\to\infty} |\beta_{\epsilon_n}(u_n) - z_n| = 0.$$

The proof is now complete. □

In order to achieve our aim, we recall the following result for critical points involving Ljusternik-Schnirelmann category. We refer to [14, 20] for more details.

Proposition 5.1 *Let U be a complete C^1 Riemannian manifold. Assume that $\varphi \in C^1(U, \mathbb{R})$ is bounded from below and satisfies $-\infty < \inf_U \varphi < d < k < \infty$. Moreover, suppose that φ satisfies the Palais-Smale condition on the sublevel $\{u \in U : \varphi(u) \leq k\}$ and that d is not a critical level for φ. Then*

$$card\{u \in \varphi^d : \varphi'(u) = 0\} \geq cat_{\varphi^d}(\varphi^d),$$

where $\varphi^d = \{u \in U : \varphi(u) \leq d\}$.

With a view to utilize Proposition 5.1, the following abstract lemma serves as a particularly useful tool in that it relates the topology of some sublevel of a functional to the topology of some subset of the space $\mathbb{R}^N$, For detailed proof, we refer to [14].

Proposition 5.2 *Let Ω, Ω_1 and Ω_2 be closed sets with $\Omega_1 \subset \Omega_2$ and let $\pi : \Omega \to \Omega_2$, $\phi : \Omega_1 \to \Omega$ be continuous maps such that $\pi \circ \phi$ is homotopically equivalent to the inclusion mapping $id : \Omega_1 \to \Omega_2$. Then $cat_\Omega(\Omega) \geq cat_{\Omega_2}(\Omega_1)$.*

We shall make use of the Ljusternik-Schnirelmann category theory to prove the multiplicity result of positive solutions for problem (2.5). Since $\mathcal{N}_\epsilon$ is not a C^1-submanifold of E_ϵ, we cannot directly apply the Ljusternik-Schnirelmann category theory. Fortunately, on account of Lemma 5.2, we can know that the mapping m_ϵ is a homeomorphism between $\mathcal{N}_\epsilon$ and S_ϵ, and S_ϵ is a C^1-submanifold of E_ϵ. So we can apply the Ljusternik-Schnirelmann category theoty to the functional $\Psi_\epsilon(u) = \mathcal{I}_\epsilon(\widehat{m}_\epsilon(u))|_{S_\epsilon} = \mathcal{I}_\epsilon(m_\epsilon(u))$. Based on the above facts, we can obtain the following multiplicity result for problem (2.5).

Lemma 5.12 *For any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\widehat{\epsilon}_\delta > 0$ such that, for any $\epsilon \in (0, \widehat{\epsilon}_\delta)$, problems (2.5) has at least $cat_{M_\delta}(M)$ positive solutions.*

Proof For any $\epsilon > 0$, we define $\pi_\epsilon : M \to S_\epsilon$ as follows

$$\pi_\epsilon(z) = m_\epsilon^{-1}(t_\epsilon \Psi_{\epsilon,z}) = m_\epsilon^{-1}(\Phi_\epsilon(z)) \text{ for all } z \in M.$$

Using Lemma 5.8, we see that

$$\lim_{\epsilon\to 0} \Psi_\epsilon(\pi_\epsilon(z)) = \lim_{\epsilon\to 0} \mathcal{I}_\epsilon(\Phi_\epsilon(z)) = c_{V_{\min}} \text{ uniformly in } z \in M.$$

Moreover, we define

$$\widetilde{S}_\epsilon = \{u \in S_\epsilon : \Psi_\epsilon(u) \le c_{V_{\min}} + \vartheta(\epsilon)\},$$

with $\vartheta(\epsilon) = \sup_{z\in M} |\Psi_\epsilon(\pi_\epsilon(z)) - c_{V_{\min}}| \to 0$ as $\epsilon \to 0$. Therefore, $\pi_\epsilon(z) \in \widetilde{S}_\epsilon$ for all $z \in M$, and this shows that $\widetilde{S}_\epsilon \neq \emptyset$ for all $\epsilon > 0$.

From Lemma 5.2, Lemma 5.3, Lemma 5.8 and Lemma 5.11, we can find $\widehat{\epsilon}_\delta > 0$ such that the diagram

$$M \xrightarrow{\Phi_\epsilon} \widetilde{\mathcal{N}_\epsilon} \xrightarrow{m_\epsilon^{-1}} \widetilde{S}_\epsilon \xrightarrow{m_\epsilon} \widetilde{\mathcal{N}_\epsilon} \xrightarrow{\beta_\epsilon} M_\delta$$

is well defined for any $\epsilon \in (0, \widehat{\epsilon}_\delta)$. By Lemma 5.9, there exists a function $\gamma(\epsilon, z)$ with $|\gamma(\epsilon, z)| < \frac{\delta}{2}$ uniformly in $z \in M$ for all $\epsilon \in (0, \epsilon_\delta)$, such that $\beta_\epsilon(\Phi_\epsilon(z)) = z + \gamma(\epsilon, z)$ for all $z \in M$. We define the function $H(t, z) = z + (1 - t)\gamma(\epsilon, z)$. Then, $H : [0, 1] \times M \to M_\delta$ is continuous. Evidently, $H(0, z) = \beta_\epsilon(\Phi_\epsilon(z))$ and $H(1, z) = z$ for all $z \in M$, and $\beta_\epsilon \circ \Phi_\epsilon = (\beta_\epsilon \circ m_\epsilon) \circ \pi_\epsilon$ is homotopic to the inclusion mapping $id : M \to M_\delta$. So, applying Proposition 5.2, we have

$$\text{cat}_{\widetilde{S}_\epsilon}(\widetilde{S}_\epsilon) \ge \text{cat}_{M_\delta}(M).$$

On the other hand, let us choose a function $\vartheta(\epsilon) > 0$ such that $\vartheta(\epsilon) \to 0$ as $\epsilon \to 0$ and such that $c_{V_{\min}} + \vartheta(\epsilon)$ is not a critical level for $\mathcal{I}_\epsilon$. For $\epsilon > 0$ small enough, we deduce from Lemma 5.4 that Ψ_ϵ satisfies the Palais-Smale condition in $\widetilde{S}_\epsilon$. Consequently, exploiting Proposition 5.1, we find that Ψ_ϵ has at least $\text{cat}_{\widetilde{S}_\epsilon}(\widetilde{S}_\epsilon)$ critical points on $\widetilde{S}_\epsilon$. Then, using Lemma 5.3, we can deduce that $\mathcal{I}_\epsilon$ has at least $\text{cat}_{M_\delta}(M)$ critical points. We finish the proof of the lemma. □

Lemma 5.13 *Let $\epsilon_n \to 0$ and $\{u_n\}$ be a sequence of solution to (2.5). Then, $v_n(x) = u_n(x+\tilde{z}_n)$ satisfies $v_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $|v_n|_\infty \leq C$ for all $n \in \mathbb{N}$, where $\{\tilde{z}_n\}$ is given by Lemma 5.10. Moreover, we also have*

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N}.$$

Proof Observe that, since $\mathcal{I}_{\epsilon_n}(u_n) \leq c_{V_{\min}} + \vartheta(\epsilon_n)$ with $\vartheta(\epsilon_n) \to 0$ as $n \to \infty$, according to the proof of Lemma 5.10 we can infer that $\mathcal{I}_{\epsilon_n}(u_n) \to c_{V_{\min}}$, and there exists a sequence $\{\tilde{z}_n\} \subset \mathbb{R}^N$ such that $\epsilon_n \tilde{z}_n \to z_0 \in M$ and $v_n(x) = u_n(x+\tilde{z}_n)$ strongly converges in $E_{V_{\min}}$. Using Lemma 4.4 we can deduce that the conclusions of Lemma 5.13 hold. This completes the proof. □

Finally, we are ready to present the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\delta > 0$ be such that $M_\delta \subset \Lambda$. Firstly, we show that there exists $\widetilde{\epsilon}_\delta > 0$ such that for any $\epsilon \in (0, \widetilde{\epsilon}_\delta)$ and any solution $u_\epsilon \in \widetilde{\mathcal{N}_\epsilon}$ of (2.5), the following estimate holds

$$|u_\epsilon(x)|_{L^\infty(\Lambda_\epsilon^c)} < a_0. \tag{5.12}$$

Assume by contradiction that for some subsequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$, we can find $u_n := u_{\epsilon_n}$ such that $\mathcal{I}'_{\epsilon_n}(u_n) = 0$ and

$$|u_n(x)|_{L^\infty(\Lambda_{\epsilon_n}^c)} \geq a_0.$$

Since $\mathcal{I}_{\epsilon_n}(u_n) \leq c_{V_{\min}} + \vartheta(\epsilon_n)$, from the proof of Lemma 5.10 we can show that $\mathcal{I}_{\epsilon_n}(u_n) \to c_{V_{\min}}$. According to Lemma 5.13, there exists a sequence $\{\tilde{z}_n\} \subset \mathbb{R}^N$ such that $\epsilon_n \tilde{z}_n \to z_0 \in M$ and $v_n(x) = u_n(x+\tilde{z}_n)$ strongly converges in $E_{V_{\min}}$. Moreover, using Lemma 5.13 and combining the proof of Theorem 1.1 we get a contradiction.

Let $\widehat{\epsilon}_\delta > 0$ be given by Lemma 5.12 and we set $\epsilon_\delta := \min\{\widetilde{\epsilon}_\delta, \widehat{\epsilon}_\delta\}$. Fix $\epsilon \in (0, \epsilon_\delta)$, employing Lemma 5.12 we can see that problem (2.5) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions. If $u_\epsilon \in E_\epsilon$ is one of these solutions, then $u_\epsilon \in \widetilde{\mathcal{N}_\epsilon}$, and we can deduce from (5.12) and the definition of f that u_ϵ is also a solution to (2.1). Therefore, $\widehat{u}_\epsilon(x) = u_\epsilon(\frac{x}{\epsilon})$ is a positive solution of problem (1.1), and we can infer that problem (1.1) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions. Finally, to study the concentration behavior of the solutions, we argue as in the proof of Theorem 1.1. We finish the proof of Theorem 1.2. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Alves, C.O., da Silva, A.: Multiplicity and concentration of positive solutions for a class of quasilinear problems through Orlicz-Sobolev space. J. Math. Phys. **57**, 111502 (2016)
2. Alves, C.O., Gao, F., Squassina, M., Yang, M.: Singularly perturbed critical Choquard equations. J. Differ. Equ **263**, 3943–3988 (2017)
3. Alves, C.O., Figueiredo, G.: Multiplicity and concentration of positive solutions for a class of quasilinear problems. Adv. Nonlinear Stud. **11**, 265–294 (2011)
4. Alves, C.O., Yang, M.: Existence of semiclassical ground state solutions for a generalized Choquard equation. J. Differ. Equ **257**, 4133–4164 (2014)
5. Alves, C.O., Yang, M.: Multiplicity and concentration of solutions for a quasilinear Choquard equation. J. Math. Phys. **55**, 061502 (2014)
6. Alves, C.O., Yang, M.: Investigating the multiplicity and concentration behaviour of solutions for a quasi-linear Choquard equation via the penalization method. Proc. Roy. Soc. Edinburgh Sect. A **146**, 23–58 (2016)
7. Ambrosetti, A., Malchiodi, A.: Concentration phenomena for nonlinear Schrödinger equations: recent results and new perspectives. Perspectives in nonlinear partial differential equations, 19-30, Contemp. Math., 446, Amer. Math. Soc., Providence, RI, (2007)
8. Ambrosio, V., Repovš, D.: Multiplicity and concentration results for a (p, q)-Laplacian problem in $\mathbb{R}^N$. Z. Angew. Math. Phys. **72**, 33 (2021)
9. Ambrosio, V., Rădulescu, V.D.: Fractional double-phase patterns: concentration and multiplicity of solutions. J. Math. Pures Appl. **142**, 101–145 (2020)
10. Arora, R., Fiscella, A., Mukherjee, T., Winkert, P.: Existence of ground state solutions for a Choquard double phase problem. Nonlinear Anal. Real World Appl. **73**, 103914 (2023)
11. Bahrouni, A., Rădulescu, V.D., Repovš, D.: Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves. Nonlinearity **32**, 2481–2495 (2019)
12. Bang, O., Krolikowski, W., Wyller, J., Rasmussen, J.: Collapse arrest and soliton stabilization in nonlocal nonlinear media. Phys. Rev. E **66**, 046619 (2002)
13. Bartolo, R., Candela, A.M., Salvatore, A.: On a class of superlinear (p, q)-Laplacian type equations on $\mathbb{R}^N$. J. Math. Anal. Appl. **438**, 29–41 (2016)
14. Benci, V., Cerami, G.: Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology. Calc. Var. Partial. Differ. Equ. **2**, 29–48 (1994)
15. Benci, V., d'Avenia, P., Fortunato, D., Pisani, L.: Solitonsin severalspace dimensions: Derrick's problem and infinitely many solutions. Arch. Ration. Mech. Anal. **154**, 297–324 (2000)
16. Bergé, L., Couairon, A.: Nonlinear propagation of self-guided ultra-short pulses in ionized gases. Phys. Plasmas **7**, 210–230 (2000)
17. Bonheure, D., d'Avenia, P., Pomponio, A.: On the electrostatic Born-Infeld equation with extended charges. Comm. Math. Phys. **346**, 877–906 (2016)

18. Chen, S., Tang, X.: A comprehensive review on the existence of normalized solutions for four classes of nonlinear elliptic equations. Opuscula Math. **45**, 739–763 (2025). https://doi.org/10.7494/OpMath.2025.45.6.739
19. Cherfils, L., Il'yasov, Y.: On the stationary solutions of generalized reaction diffusion equations with $p\&q$-Laplacian, Commun. Pure. Appl. Anal. **4**, 9–22 (2005)
20. Cingolani, S., Lazzo, M.: Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions. J. Differ. Equ **160**, 118–138 (2000)
21. Cingolani, S., Secchi, S., Squassina, M.: Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities. Proc. R. Soc. Edinb. Sect. A **140**, 973–1009 (2010)
22. del Pino, M., Felmer, P.: Local mountain pass for semilinear elliptic problems in unbounded domains. Calc. Var. Partial. Differ. Equ. **4**, 121–137 (1996)
23. Figueiredo, G.M.: Existence of positive solutions for a class of p&q elliptic problems with critical growth on $\mathbb{R}^N$. J. Math. Anal. Appl. **378**, 507–518 (2011)
24. He, C., Li, G.: The regularity of weak solutions to nonlinear scalar field elliptic equations containing p&q-Laplacians. Ann. Acad. Sci. Fenn. Math. **33**, 337–371 (2006)
25. Ladyzhenskaya, O.A., Ural'tseva, N.N.: Linear and Quasilinear Elliptic Equations, Academic Press, (1968)
26. Li, Y., Van Nguyen, T., Zhang, B.: Existence, concentration and multiplicity of solutions for (p, N)-Laplacian equations with convolution term. Bull. Math. Sci. **15**(1), 2450009 (2025). https://doi.org/10.1142/S1664360724500097
27. Lieb, E.H.: Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. Stud. Appl. Math. **57**, 93–105 (1976/77)
28. Lions, P.L.: The concentration compactness principle in the calculus of variations. The locally compact case. Part II, Ann. Inst. H. Poincaré. Anal. Non Linéaire **1**, 223–283 (1984)
29. Lieb, E., Loss, M.: Analysis, Graduate Studies in Mathematics, American Mathematical Society, (2001)
30. Litvak, A.G.: Self-focusing of powerful light beams by thermal effects. JETP Lett. **4**, 230–232 (1966)
31. Mingione, G., Rădulescu, V.D.: Recent developments in problems with nonstandard growth and nonuniform ellipticity. J. Math. Anal. Appl. **501**, 125197 (2021)
32. Moroz, V., Van Schaftingen, J.: A guide to the Choquard equation. J. fixed point theory appl. **19**, 773–813 (2017)
33. Moroz, V., Van Schaftingen, J.: Semi-classical states for the Choquard equation. Calc. Var. Partial. Differ. Equ. **52**, 199–235 (2015)
34. Papageorgiou, N.S., Pudełko, A., Rădulescu, V.D.: Non-autonomous (p, q)-equations with unbalanced growth. Math. Ann. **385**, 1707–1745 (2023)
35. Papageorgiou, N.S., Qin, D., Rădulescu, V.D.: Nonlinear eigenvalue problems for the (p, q)–Laplacian. Bull. Sci. Math. **172**, 103038 (2021)
36. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.: Existence and multiplicity of solutions for double-phase Robin problems. Bull. London Math. Soc. **52**, 546–560 (2020)
37. Papageorgiou, N.S., Rădulescu, V.D., Zhang, J.: Ambrosetti-Prodi problems for the Robin (p, q)-Laplacian. Nonlinear Anal. Real World Appl. **67**, 103640 (2022)
38. Rabinowitz, P.H.: On a class of nonlinear Schrödinger equations. Z. Angew. Math. Phys. **43**, 270–291 (1992)
39. Pekar, S.: Untersuchungen über die Elektronentheorie der Kristalle. Akademie-Verlag, Berlin (1954)
40. Singer, T.: Existence of weak solutions of parabolic systems with (p, q)-growth. Manuscr. Math. **151**, 87–112 (2016)
41. Sun, X., Fu, Y., Liang, S.: Multiplicity and concentration of solutions for Kirchhoff equations with exponential growth. Bull. Math. Sci. **14**(3), 2450004 (2024). https://doi.org/10.1142/S1664360724500048
42. Szulkin, A., Weth, T.: The method of Nehari manifold. In: Gao, D.Y., Motreanu, D. (eds.) Handbook of Nonconvex Analysis and Applications, pp. 597–632. International Press, Boston (2010)
43. Tan, Z., Fang, F.: Orlicz-Sobolev versus Hölder local minimizer and multiplicity results for quasilinear elliptic equations. J. Math. Anal. Appl. **402**, 348–370 (2013)
44. Tao, H., Li, L., Winkert, P.: Existence and concentration of solutions for a 1-biharmonic Choquard equation with steep potential well in $\mathbb{R}^N$. J. Geom. Anal. **33**(9), 276 (2023)
45. Trudinger, N.S.: On Harnack type inequalities and their application to quasilinear elliptic equations. Commun. Pure Appl. Math. **20**, 721–747 (1967)
46. Wei, J., Winter, M.: Strongly interacting bumps for the Schrödinger-Newton equations. J. Math. Phys. **50**, 012905 (2009)

47. Willem, M.: Minimax Theorems, Progress in Nonlinear Differential Equations and their Applications, vol. 24. Birkhäuser, Boston (1996)
48. Zhang, W., Zhang, J.: Multiplicity and concentration of positive solutions for fractional unbalanced double phase problems. J. Geom. Anal. **32**, 235 (2022)
49. Zhang, W., Zhang, J., Rădulescu, V.D.: Concentrating solutions for singularly perturbed double phase problems with nonlocal reaction. J. Differ. Equ **347**, 56–103 (2023)
50. Zhang, J., Zhang, W., Rădulescu, V.D.: Double phase problems with competing potentials: concentration and multiplication of ground states. Math. Z. **301**, 4037–4078 (2022)