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Research Article

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Coron Problem for Nonlocal Equations Involving Choquard Nonlinearity

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Abstract: We consider the following Choquard equation:

$$\begin{cases} -\Delta u = \left(\int\limits_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\, dy \right) |u|^{2^*_\mu - 2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $2^*_\mu = \frac{2N-\mu}{N-2}$. This paper is concerned with the existence of a positive high-energy solution of the above problem in an annular-type domain when the inner hole is sufficiently small.

Keywords: Choquard Nonlinearity, Coron Problem, Stationary Nonlinear Schrödinger–Newton Equation, Riesz Potential, Critical Exponent

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1 Introduction

In this paper, we study the existence of a positive solution of the Choquard equation. More precisely, we consider the problem

$$\begin{cases} -\Delta u = \left(\int\limits_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\, dy \right) |u|^{2^*_\mu - 2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 3)$, $2^*_\mu = \frac{2N-\mu}{N-2}$, $0 < \mu < N$.

The work on elliptic equations involving critical Sobolev exponent over non-contractible domains was initiated by J.-M. Coron in 1983. Indeed, Coron [10] proved the existence of a positive solution of the following critical elliptic problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{Q}$$

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where Ω is a smooth bounded domain in $\mathbb{R}^N$ and satisfies the following conditions: there exist constants $0 < R_1 < R_2 < \infty$ such that

$$\{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \subset \Omega, \quad \{x \in \mathbb{R}^N : |x| < R_1\} \nsubseteq \overline{\Omega}. \tag{1.1}$$

Later on, Bahri and Coron [1] proved that if there exists a positive integer d such that $H_d(\Omega, \mathbb{Z}_2) \neq 0$ (where $H_d(\Omega, \mathbb{Z}_2)$ the homology of dimension d of Ω with $\mathbb{Z}_2$ coefficients), then problem (Q) has a positive solution.

Benci and Cerami [4] considered the equation

$$\begin{cases} -\Delta u + \lambda u = u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a smooth bounded domain and $2 < p < 2^*$, $\lambda \in \mathbb{R}_+ \cup \{0\}$. With the help of Ljusternik–Schnirelmann theory, Benci and Cerami showed that there exists a function $\overline{\lambda} : (2, 2^*) \to \mathbb{R}_+ \cup \{0\}$ such that for all $\lambda \geq \overline{\lambda}(p)$, problem (1.2) has at least $\operatorname{cat}(\Omega)$ distinct solutions. We cite [3, 5, 6, 11, 23, 27, 33, 37] and the references therein for the work on the existence of solutions over a non-contractible domain.

We recall that the Choquard equation (1.3) was first introduced in the pioneering work of Fröhlich [13] and Pekar [30] for the modeling of quantum polaron:

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right)u \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

As pointed out by Fröhlich [13] and Pekar, this model corresponds to the study of free electrons in an ionic lattice interact with phonons associated to deformations of the lattice or with the polarization that it creates on the medium (interaction of an electron with its own hole). In the approximation to Hartree–Fock theory of one component plasma, Choquard used equation (1.3) to describe an electron trapped in its own hole.

The Choquard equation is also known as the Schrödinger–Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity. The equation can also be derived from the Einstein–Klein–Gordon and Einstein–Dirac system. Such a model was proposed for boson stars and for the collapse of galaxy fluctuations of scalar field dark matter. We refer for details to Elgart and Schlein [12], Giulini and Großardt [17], Jones [19], and Schunck and Mielke [34]. Penrose [31, 32] proposed equation (1.3) as a model of self-gravitating matter in which quantum state reduction was understood as a gravitational phenomenon.

As pointed out by Lieb [20], Choquard used equation (1.3) to study steady states of the one component plasma approximation in the Hartree–Fock theory. Classification of solutions of (1.3) was first studied by Ma and Zhao [22]. For a broad survey of Choquard equations we refer to Moroz and Van Schaftingen [26] and references therein. We also refer to Battaglia and Van Schaftingen [2], Cassani and Zhang [9], Mingqi, Rădulescu and Zhang [25], and Seok [35] as recent relevant contributions to the study of Choquard-type problems.

Recently, Gao and Yang [16] studied the Brezis–Nirenberg-type result for the following problem:

$$\begin{cases} -\Delta u = \lambda u + \left(\displaystyle\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\right)|u|^{2^*_\mu - 2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $0 < \lambda$, $0 < \mu < N$, $2^*_\mu = \frac{2N-\mu}{N-2}$, Ω is a smooth bounded domain in $\mathbb{R}^N$ and 2^*_μ is the critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality (2.1). They proved the Pohozaev identity for equation (1.4) and used variational methods and the minimizers of the best constant $S_{H,L}$ (defined in (2.3)) to show the existence, non-existence of solution depending on the range of λ. We cite [14, 15] for the Choquard equation with critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality. However, the existence and multiplicity of solutions of nonlocal equations over non-contractible domains is still an open question. Therefore, it is essential to study the existence of a positive solution of elliptic equations involving convolution-type nonlinearity in non-contractible domains.

Inspiring by these results, we study in the present article the Coron problem for problem (P). More precisely, we show the existence of a high-energy positive solution in a non-contractible bounded domain particularly an annulus when the inner hole is sufficiently small. The functional associated with (P) is not C^2 when $\mu > \min\{4, N\}$ and this makes problem (P) more challenging.

In order to achieve the desired aim we first prove the non-existence result using the Pohozaev identity for Choquard equation on $\mathbb{R}^N_+$. We also prove the global compactness lemma for Choquard equation in bounded domains. In case of $\mu = 0$, such a lemma has been proved by Struwe [36] and later generalized to the p-Laplacian case by Mercuri and Willem [24]. In case of $0 < \mu < N$, the method of defining Lévy concentration function is not useful. In the present article we gave the proof of global compactness Lemma 4.5 by introducing the notion of Morrey spaces. Finally, by using the concentration-compactness principle together with the deformation lemma, we prove the existence of high-energy positive solution. To the best of our knowledge, there is no work on Coron's problem for Choquard equation.

We now state the main result of this paper.

Theorem 1.1. *Assume that Ω is a bounded domain in $\mathbb{R}^N$ satisfying condition* (1.1). *If $\frac{R_2}{R_1}$ is sufficiently large, then problem* (P) *admits a positive high-energy solution.*

Turning to the layout of the paper, in Section2 we assemble notations and preliminary results. In Section3, we give the classification of all nonnegative solutions of Choquard equation. In Section4, we analyze the Palais–Smale sequences. In Section5, we prove our main result Theorem 1.1. We refer to the recent monograph by Papageorgiou, Rădulescu and Repovš [29] for some of the basic analytic tools used in this paper.

2 Preliminary Results

This section is devoted to the variational formulation, Pohozaev identity and non-existence result. The outset of the variational framework starts from the following Hardy–Littlewood–Sobolev inequality. We refer to Lieb and Loss [21] for more details.

Proposition 2.1. *Let $t, r > 1$ and $0 < \mu < N$ with $\frac{1}{t} + \frac{\mu}{N} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, r, \mu, N)$ independent of f, h such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu}\,dx\,dy \le C(t, r, \mu, N)\|f\|_{L^t}\|h\|_{L^r}. \tag{2.1}$$

If $t = r = \frac{2N}{2N-\mu}$, then

$$C(t, r, \mu, N) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})} \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{\mu}{2})} \right\}^{-1+\frac{\mu}{N}}.$$

Equality holds in (2.1) *if and only if $\frac{f}{h} \equiv$ constant and*

$$h(x) = A(\gamma^2 + |x-a|^2)^{\frac{2N-\mu}{2}}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

We consider the following functional space:

$$D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\},$$

endowed with the norm defined as

$$\|u\| := \Big(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\Big)^{\frac{1}{2}}.$$

The space $D^{1,2}_0(\Omega)$ is defined as the closure of $C^\infty_c(\Omega)$ in $D^{1,2}(\mathbb{R}^N)$.

Definition 2.2. A function $u \in D^{1,2}_0(\Omega)$ is said to be a solution of (P) if u satisfies

$$\int_\Omega \nabla u \nabla \phi\,dx + \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu-2}u(y)\phi(y)}{|x-y|^\mu}\,dx\,dy \quad \text{for all } \phi \in D^{1,2}_0(\Omega).$$

Notation. We define $u_+ = \max(u, 0)$ and $u_- = \max(-u, 0)$ for all $u \in D^{1,2}(\mathbb{R}^N)$. Moreover, we set

$$\mathbb{R}^N_+ := \{x \in \mathbb{R}^N : x_N > 0\}$$

and we denote by $*$ the standard convolution operator.

Consider functionals $I : D^{1,2}_0(\Omega) \to \mathbb{R}$ and $I_\infty : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2\cdot 2^*_\mu}\int_\Omega\int_\Omega \frac{|u_+(x)|^{2^*_\mu}|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy, \quad u \in D^{1,2}_0(\Omega),$$

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx \frac{1}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_+(x)|^{2^*_\mu}|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy, \quad u \in D^{1,2}(\mathbb{R}^N).$$

By the Hardy–Littlewood–Sobolev inequality, we have

$$\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy\Big)^{\frac{1}{2^*_\mu}} \le C(N,\mu)^{\frac{2N-\mu}{N-2}}\|u\|^2_{L^{2^*}},$$

where $2^* = \frac{2N}{N-2}$. This implies that $I \in C^1(D^{1,2}_0(\Omega), \mathbb{R})$ and $I_\infty \in C^1(D^{1,2}(\mathbb{R}^N), \mathbb{R})$. The best constant for the embedding $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ is defined as

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\Big\{\int_{\mathbb{R}^N}|\nabla u|^2\,dx : \int_{R^N}|u|^{2^*}\,dx = 1\Big\}. \tag{2.2}$$

Consequently, we define

$$S_{H,L} = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\Big\{\int_{\mathbb{R}^N}|\nabla u|^2\,dx : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy = 1\Big\}. \tag{2.3}$$

It was established by Talenti [38] that the best constant S is achieved if and only if u is of the form

$$\Big(\frac{t}{t^2 + |x-(1-t)\sigma|^2}\Big)^{\frac{N-2}{2}} \quad \text{for } \sigma \in \Sigma := \{x \in \mathbb{R}^N : |x| = 1\} \text{ and } t \in (0, 1].$$

Properties of the best constant $S_{H,L}$ were established by Gao and Yang [16]. We recall the following property.

Lemma 2.3. *The constant $S_{H,L}$ defined in* (2.3) *is achieved if and only if*

$$u = C\Big(\frac{b}{b^2 + |x-a|^2}\Big)^{\frac{N-2}{2}},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0,\infty)$ are parameters. Moreover,

$$S_{H,L} = \frac{S}{C(N,\mu)^{\frac{N-2}{2N-\mu}}},$$

where S is defined as in (2.2).

The following property was established in [16].

Lemma 2.4. *If $N \ge 3$ and $0 < \mu < N$, then*

$$\|\cdot\|_{NL} := \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\cdot|^{2^*_\mu}|\cdot|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy\Big)^{\frac{1}{2\cdot 2^*_\mu}}$$

defines a norm on $L^{2^*}(\mathbb{R}^N)$.

Remark 2.5. If we define

$$S_A = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\Big\{\int_{\mathbb{R}^N}|\nabla u|^2\,dx : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_+(x)|^{2^*_\mu}|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy = 1\Big\},$$

then $S_A = S_{H,L}$.

Proposition 2.6. *Let $u \in D_0^{1,2}(\Omega)$ be an arbitrary solution of the problem*

$$\begin{cases} -\Delta u = \Big(\int\limits_{\Omega} \frac{|u_+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dy \Big) |u_+|^{2_\mu^*-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Then

$$I(u) \geq \frac{1}{2}\Big(\frac{N-\mu+2}{2N-\mu}\Big) S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} =: \beta.$$

Moreover, the same conclusion holds for the solution $u \in D^{1,2}(\mathbb{R}^N)$ of

$$-\Delta u = \Big(\int\limits_{\mathbb{R}^N} \frac{|u_+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dy \Big) |u_+|^{2_\mu^*-1} \quad \text{in } \mathbb{R}^N.$$

Proof. If u is a solution of (2.4), then testing (2.4) with u_+ and u_- yields

$$\int\limits_{\Omega} |\nabla u_+|^2\,dx = \int\limits_{\Omega}\int\limits_{\Omega} \frac{|u_+(x)|^{2_\mu^*}|u_+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy \quad \text{and} \quad \int\limits_{\Omega} |\nabla u_-|^2\,dx = 0 \quad \text{a.e. on } \Omega.$$

It follows that

$$(S_A)^{\frac{2_\mu^*}{2_\mu^*-1}} \leq \int\limits_{\Omega}\int\limits_{\Omega} \frac{|u_+(x)|^{2_\mu^*}|u_+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy = \int\limits_{\Omega} |\nabla u_+|^2\,dx = \int\limits_{\Omega} |\nabla u|^2\,dx.$$

We obtain

$$I(u) \geq \Big(\frac{1}{2} - \frac{1}{2\cdot 2_\mu^*}\Big)(S_A)^{\frac{2_\mu^*}{2_\mu^*-1}} = \frac{1}{2}\Big(\frac{N-\mu+2}{2N-\mu}\Big) S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

The proof is now complete. □

Lemma 2.7 (Pohozaev Identity). *Let $N \geq 3$ and assume that $u \in D_0^{1,2}(\mathbb{R}_+^N)$ solves*

$$-\Delta u = \Big(\int\limits_{\mathbb{R}_+^N} \frac{|u_+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dy \Big) |u_+|^{2_\mu^*-1} \quad \text{in } \mathbb{R}_+^N. \tag{2.5}$$

Then the following equality holds:

$$\frac{1}{2}\int\limits_{\partial\mathbb{R}_+^N} (x-x_0)\cdot\nu|\nabla u|^2\,dS + \frac{N-2}{2}\int\limits_{\mathbb{R}_+^N} |\nabla u|^2\,dx = \frac{2N-\mu}{2\cdot 2_\mu^*}\int\limits_{\mathbb{R}_+^N}\int\limits_{\mathbb{R}_+^N} \frac{|u_+(x)|^{2_\mu^*}|u_+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy,$$

where ν is the unit outward normal to $\partial\Omega$ and $x_0 = (0,0,\dots,1)$.

Proof. First observe that any solution of problem (2.5) is nonnegative. This implies

$$\nabla u = \nabla u^+ \quad \text{a.e. on } \mathbb{R}_+^N.$$

Extending $u = 0$ in $\mathbb{R}^N \setminus \mathbb{R}_+^N$, we have $u \in W_{\mathrm{loc}}^{2,2}(\mathbb{R}^N)$ (see Lemma 3.1). Now fix $\varphi \in C_c^1(\mathbb{R}^N)$ such that $\varphi = 1$ on B_1. Let the function $\varphi_\lambda \in D^{1,2}(\mathbb{R}^N)$ be defined for $\lambda \in (0,\infty)$ and $x \in \mathbb{R}^N$ by $\varphi_\lambda(x) = \varphi(\lambda x)$. Multiplying (2.5) with $((x-x_0)\cdot\nabla u)\varphi_\lambda$ and integrating over $\mathbb{R}_+^N$, we obtain

$$\begin{aligned} \int\limits_{\mathbb{R}_+^N} (-\Delta u)((x-x_0)\cdot\nabla u)\varphi_\lambda(x)\,dx &= \int\limits_{\mathbb{R}_+^N}\Big(\int\limits_{\mathbb{R}_+^N} \frac{|u_+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dy \Big)|u_+|^{2_\mu^*-1}((x-x_0)\cdot\nabla u)\varphi_\lambda\,dx \\ &= \int\limits_{\mathbb{R}_+^N} \nabla\Big((x-x_0)\int\limits_{\mathbb{R}_+^N}\Big(\frac{|u_+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dy\Big)|u_+(x)|^{2_\mu^*-1}\varphi_\lambda(x)u(x)\Big)\,dx \\ &\quad - \int\limits_{\mathbb{R}_+^N} u(x)\nabla\Big((x-x_0)\int\limits_{\mathbb{R}_+^N}\Big(\frac{|u_+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dy\Big)|u_+(x)|^{2_\mu^*-1}\varphi_\lambda(x)\Big)\,dx. \end{aligned} \tag{2.6}$$

Using the divergence theorem on the right-hand side of (2.6), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N_+} (-\Delta u)((x-x_0)\cdot\nabla u)\varphi_\lambda(x)\,dx &= \int_{\mathbb{R}^N_+}\bigg(\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\bigg)|u_+|^{2^*_\mu-1}((x-x_0)\cdot\nabla u)\varphi_\lambda\,dx\\ &= -\int_{\mathbb{R}^N_+} u(x)\nabla\bigg((x-x_0)\int_{\mathbb{R}^N_+}\bigg(\frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\bigg)|u_+(x)|^{2^*_\mu-1}\varphi_\lambda(x)\bigg)\,dx. \qquad (2.7)\end{aligned}$$

Now consider the integral

$$\begin{aligned}&\int_{\mathbb{R}^N_+} u(x)\nabla\bigg((x-x_0)\int_{\mathbb{R}^N_+}\bigg(\frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\bigg)|u_+(x)|^{2^*_\mu-1}\varphi_\lambda(x)\bigg)\,dx\\ &= \int_{\mathbb{R}^N_+} Nu(x)\bigg(\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\bigg)|u_+(x)|^{2^*_\mu-1}\varphi_\lambda(x)\,dx\\ &\quad + \int_{\mathbb{R}^N_+}(2^*_\mu-1)u(x)\bigg(\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\bigg)|u_+(x)|^{2^*_\mu-2}\varphi_\lambda(x)(\nabla u\cdot(x-x_0))\,dx\\ &\quad - \mu\int_{\mathbb{R}^N_+} u(x)\varphi_\lambda(x)\bigg(\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}(x-x_0)\cdot(x-y)}{|x-y|^{\mu+2}}\,dy\bigg)|u_+(x)|^{2^*_\mu-1}\,dx\\ &\quad + \lambda\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}|u_+(x)|^{2^*_\mu}}{|x-y|^\mu}(x-x_0)\cdot\nabla\varphi(\lambda x)\,dx\,dy. \qquad (2.8)\end{aligned}$$

Taking into account (2.7) and (2.8), we have

$$\begin{aligned}&2^*_\mu\int_{\mathbb{R}^N_+}(x-x_0)\cdot\nabla u(x)\bigg(\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\bigg)|u_+(x)|^{2^*_\mu-1}\varphi_\lambda(x)\,dx\\ &= -N\int_{\mathbb{R}^N_+} u(x)\bigg(\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\bigg)|u_+(x)|^{2^*_\mu-1}\varphi_\lambda(x)\,dx\\ &\quad + \mu\int_{\mathbb{R}^N_+} u(x)\varphi_\lambda(x)\bigg(\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}(x-x_0).(x-y)}{|x-y|^{\mu+2}}\,dy\bigg)|u_+(x)|^{2^*_\mu-1}\,dx\\ &\quad - \lambda\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}|u_+(x)|^{2^*_\mu}}{|x-y|^\mu}(x-x_0)\cdot\nabla\varphi(\lambda x)\,dx\,dy. \qquad (2.9)\end{aligned}$$

Now, interchanging the role of x and y in (2.9) and combining the resultant equation with (2.9), we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^N_+}(x-x_0)\cdot\nabla u(x)\int_{\mathbb{R}^N_+}\bigg(\frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\bigg)|u_+(x)|^{2^*_\mu-1}\varphi_\lambda(x)\,dx\\ &= \frac{\mu-2N}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}|u_+(x)|^{2^*_\mu}}{|x-y|^\mu}\varphi_\lambda(x)\,dx\,dy\\ &\quad - \frac{\lambda}{2^*_\mu}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}|u_+(x)|^{2^*_\mu}}{|x-y|^\mu}(x-x_0)\cdot\nabla\varphi(\lambda x)\,dx\,dy.\end{aligned}$$

Passing to the limit as $\lambda\to 0$ and using the dominated convergence theorem, we obtain that

$$\int_{\mathbb{R}^N_+}(x-x_0)\cdot\nabla u(x)\bigg(\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\bigg)|u_+(x)|^{2^*_\mu-1}\,dx = \frac{\mu-2N}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}|u_+(x)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy. \qquad (2.10)$$

It is easily seen that

$$\begin{aligned}\Delta u((x-x_0)\cdot\nabla u)\varphi_\lambda &= \operatorname{div}(\nabla u\varphi_\lambda(x-x_0)\cdot\nabla u)-\varphi_\lambda|\nabla u|^2-\varphi_\lambda(x-x_0)\cdot\nabla\Big(\frac{|\nabla u|^2}{2}\Big)\\ &\quad-\lambda((x-x_0)\cdot\nabla u)(\nabla\varphi(\lambda x)\cdot\nabla u)\\ &=\operatorname{div}\Big(\Big(\nabla u(x-x_0)\cdot\nabla u-(x-x_0)\frac{|\nabla u|^2}{2}\Big)\varphi_\lambda\Big)+\frac{N-2}{2}\varphi_\lambda|\nabla u|^2\\ &\quad+\lambda\frac{|\nabla u|^2}{2}((x-x_0)\cdot\nabla\varphi(\lambda x))-\lambda((x-x_0)\cdot\nabla u)(\nabla\varphi(\lambda x)\cdot\nabla u).\end{aligned}$$

Now, integrating by parts we obtain

$$\begin{aligned}\int_{\mathbb{R}^N_+}(\Delta u)((x-x_0)\cdot\nabla u)\varphi_\lambda\,dx &= \int_{\partial\mathbb{R}^N_+}\Big(\nabla u(x-x_0)\cdot\nabla u-(x-x_0)\frac{|\nabla u|^2}{2}\Big)\varphi_\lambda\cdot\nu\,dS\\ &\quad+\frac{N-2}{2}\int_{\mathbb{R}^N_+}\varphi_\lambda|\nabla u|^2\,dx-\int_{\mathbb{R}^N_+}\lambda\frac{|\nabla u|^2}{2}((x-x_0)\cdot\nabla\varphi(\lambda x))\,dx\\ &\quad-\int_{\mathbb{R}^N_+}\lambda((x-x_0)\cdot\nabla u)(\nabla\varphi(\lambda x)\cdot\nabla u)\,dx.\end{aligned}$$

Noticing that $\nabla u=(\nabla u\cdot\nu)\nu$ on $\partial\mathbb{R}^N_+$ and employing dominated convergence theorem for $\lambda\to 0$, we get that

$$\int_{\mathbb{R}^N_+}(\Delta u)((x-x_0)\cdot\nabla u)=\frac{1}{2}\int_{\partial\mathbb{R}^N_+}|\nabla u|^2(x-x_0)\cdot\nu\,dS+\frac{N-2}{2}\int_{\mathbb{R}^N_+}|\nabla u|^2\,dx. \tag{2.11}$$

From equation (2.6), (2.10) and (2.11) we have our desired result. □

We can now deduce the following Liouville-type theorem.

Theorem 2.8. *Let $N\geq 3$ and let $u\in D^{1,2}_0(\mathbb{R}^N_+)$ be any solution of*

$$-\Delta u=\Big(\int_{\mathbb{R}^N_+}\frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\Big)|u_+|^{2^*_\mu-1}\quad\text{in }\mathbb{R}^N_+. \tag{2.12}$$

Then $u\equiv 0$ on $\mathbb{R}^N_+$.

Proof. If u is a solution of (2.12), then

$$\int_{\mathbb{R}^N_+}\nabla u\cdot\nabla\phi\,dx-\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{|u_+(x)|^{2^*_\mu}|u_+(y)|^{2^*_\mu-1}\phi(y)}{|x-y|^\mu}\,dx\,dy\quad\text{for all }\phi\in D^{1,2}_0(\mathbb{R}^N_+).$$

Taking $\phi=u_-$ we obtain $u_-=0$ a.e. on $\mathbb{R}^N$. This implies that u is a nonnegative solution of (2.12). Now, by Lemma 2.7 we have

$$\int_{\partial\mathbb{R}^N_+}|\nabla u|^2(x-x_0)\cdot\nu\,dS=0.$$

But $(x-x_0)\cdot\nu>0$ for $x\in\partial\mathbb{R}^N_+$. Since u is a nontrivial solution, we get a contradiction from the Hopf boundary point lemma. Hence, $u\equiv 0$ on $\mathbb{R}^N_+$. □

3 Classification of Solutions

In this section we will discuss the regularity and classification of nonnegative solutions of the following equation:

$$-\Delta u=(|x|^{\mu-N}*|u|^p)|u|^{p-2}u\quad\text{in }\mathbb{R}^N, \tag{3.1}$$

where $p := \frac{N+\mu}{N-2}$ and $0 < \mu < N$. Consider the following integral system of equations:

$$\begin{cases} u(x) = \int\limits_{\mathbb{R}^N} \frac{u^{p-1}(y)v(y)}{|x-y|^{N-2}}\,dy, & u \geq 0 \quad \text{in } \mathbb{R}^N, \\ v(x) = \int\limits_{\mathbb{R}^N} \frac{u^{p}(y)}{|x-y|^{N-\mu}}\,dy, & v \geq 0 \quad \text{in } \mathbb{R}^N. \end{cases} \tag{3.2}$$

We note that if $u \in D^{1,2}(\mathbb{R}^N)$, then $(u,v) \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) \times L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$.

First we will discuss the regularity of nonnegative solutions of (3.1). In this regard, we will prove the following auxiliary result.

Lemma 3.1. *Let $u \in D^{1,2}(\mathbb{R}^N)$ be a nonnegative solutions of* (3.1). *Then $u \in W^{2,s}_{\mathrm{loc}}(\mathbb{R}^N)$ for all $1 \leq s < \infty$.*

Proof. Let $u \in D^{1,2}(\mathbb{R}^N)$ be a nonnegative solution of (3.1). Now following the same approach as in proof of [18, Lemma 3.1], we have $(u,v) \in L^r(\mathbb{R}^N) \times L^s(\mathbb{R}^N)$ for all $1 < r, s < \infty$. In particular, $u^p \in L^{\frac{N}{\mu}}(\mathbb{R}^N)$, and now using the boundedness of Riesz potential operator, we have $|x|^{\mu-N} * u^p \in L^\infty(\mathbb{R}^N)$. Thus, from (3.1), we have

$$|-\Delta u| \leq C|u|^{p-1}.$$

By the classical elliptic regularity theory for subcritical problems in local bounded domains, we have $u \in W^{2,s}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $1 \leq s < \infty$. ☐

Next, we will discuss the classification of all positive solutions of the following system:

$$\begin{cases} u(x) = \int\limits_{\mathbb{R}^N} \frac{u^{a}(y)v^{b}(y)}{|x-y|^{N-\alpha}}\,dy, & u > 0 \quad \text{in } \mathbb{R}^N, \\ v(x) = \int\limits_{\mathbb{R}^N} \frac{u^{c}(y)v^{d}(y)}{|x-y|^{N-\beta}}\,dy, & v > 0 \quad \text{in } \mathbb{R}^N, \end{cases} \tag{3.3}$$

where $a \geq 0$, $b, c, d \in \{0\} \cup [1, \infty)$, $0 < \alpha, \beta < N$.

Let $(u,v) \in L^{q_1}(\mathbb{R}^N) \times L^{q_2}(\mathbb{R}^N)$ be a solution of (3.3). Now for all $\lambda \in \mathbb{R}$, we define

$$T_\lambda := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^N : x_1 = \lambda\}$$

as the moving plane. Let

$$x^\lambda := (2\lambda - x_1, x_2, \dots, x_n),$$

let

$$\Sigma_\lambda := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^N : x_1 < \lambda\}$$

and let

$$\Sigma'_\lambda := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^N : x_1 \geq \lambda\}$$

be the reflection of Σ_λ about the plane T_λ. Moreover, define $u_\lambda(y) := u(y^\lambda)$ and $v_\lambda(y) = v(y^\lambda)$. Immediately, we have the following property whose proof is just an elementary computation.

Lemma 3.2. *Assume that (u,v) is a positive pair of solution of* (3.3). *Then*

$$\begin{cases} u(y^\lambda) - u(y) = \int\limits_{\Sigma_\lambda} \Big(\frac{1}{|y-x|^{N-\alpha}} - \frac{1}{|y^\alpha - x|^{N-\alpha}}\Big)[u^a(x^\lambda)v^b(x^\lambda) - u^a(x)v^b(x)]\,dx, \\ v(y^\lambda) - v(y) = \int\limits_{\Sigma_\lambda} \Big(\frac{1}{|y-x|^{N-\beta}} - \frac{1}{|y^\alpha - x|^{N-\beta}}\Big)[u^c(x^\lambda)v^d(x^\lambda) - u^c(x)v^d(x)]\,dx. \end{cases}$$

Lemma 3.3. *There exists $\eta > 0$ such that for all $\lambda < -\eta$,*

$$u(y^\lambda) \geq u(y), \quad v(y^\lambda) \geq v(y) \quad \textit{for all } y \in \Sigma_\lambda.$$

Proof. Define $\Sigma_\lambda^u := \{y \in \Sigma_\lambda : u(y) > u_\lambda(y)\}$, $\Sigma_\lambda^v := \{y \in \Sigma_\lambda : v(y) > v_\lambda(y)\}$. By Lemma 3.2, we obtain

$$\begin{aligned} u(y^\lambda) - u(y) &= \int\limits_{\Sigma_\lambda} \Big(\frac{1}{|y-x|^{N-\alpha}} - \frac{1}{|y^\lambda - x|^{N-\alpha}}\Big)[u^a(x^\lambda)v^b(x^\lambda) - u^a(x)v^b(x)]\,dx \\ &\leq \int\limits_{\Sigma_\lambda} \Big(\frac{1}{|y-x|^{N-\alpha}} - \frac{1}{|y^\lambda - x|^{N-\alpha}}\Big)[u_\lambda^a(v^b - v_\lambda^b)^+ + v^b(u^a - u_\lambda^a)^+]\,dx \\ &\leq \int\limits_{\Sigma_\lambda} \frac{1}{|y-x|^{N-\alpha}}[u_\lambda^a(v^b - v_\lambda^b)^+ + v^b(u^a - u_\lambda^a)^+]\,dx. \end{aligned}$$

By the Hardy–Littlewood–Sobolev inequality, we obtain

$$\begin{aligned} \|u - u_\lambda\|_{L^{q_1}(\Sigma_\lambda^u)} &\leq \|u - u_\lambda\|_{L^{q_1}(\Sigma_\lambda)} \\ &\leq C\|u_\lambda^a(v^b - v_\lambda^b)^+ + v^b(u^a - u_\lambda^a)^+\|_{L^r(\Sigma_\lambda)} \\ &\leq C\|u_\lambda^a(v^b - v_\lambda^b)\|_{L^r(\Sigma_\lambda^v)} + \|v^b(u^a - u_\lambda^a)\|_{L^r(\Sigma_\lambda^u)}, \end{aligned}$$

where $r = \frac{Nq_1}{N+\alpha q_1}$. Now if $a, b > 1$, then by Hölder's inequality, we get

$$\begin{aligned} \|u - u_\lambda\|_{L^{q_1}(\Sigma_\lambda^u)} &\leq C\|u_\lambda^a v^{b-1}(v - v_\lambda)\|_{L^r(\Sigma_\lambda^v)} + C\|v^b u^{a-1}(u - u_\lambda)\|_{L^r(\Sigma_\lambda^u)} \\ &\leq C\|u_\lambda\|_{L^{q_1}(\Sigma_\lambda^v)}^a \|v^{b-1}(v - v_\lambda)\|_{L^s(\Sigma_\lambda^v)} + C\|v\|_{L^{q_2}(\Sigma_\lambda^u)}^b \|u^{a-1}(u - u_\lambda)\|_{L^t(\Sigma_\lambda^u)} \\ &\leq C\|u_\lambda\|_{L^{q_1}(\Sigma_\lambda')}^a \|v\|_{L^{q_2}(\Sigma_\lambda^v)}^{b-1} \|v - v_\lambda\|_{L^{q_2}(\Sigma_\lambda^v)} + C\|v\|_{L^{q_2}(\Sigma_\lambda)}^b \|u\|_{L^{q_1}(\Sigma_\lambda^u)}^{a-1} \|u - u_\lambda\|_{L^{q_1}(\Sigma_\lambda^u)}, \end{aligned} \tag{3.4}$$

and if $0 < a < 1$, $b > 1$, then we have

$$\begin{aligned} \|u - u_\lambda\|_{L^{q_1}(\Sigma_\lambda^u)} &\leq C\|u_\lambda^a v^{b-1}(v - v_\lambda)\|_{L^r(\Sigma_\lambda^v)} + C\|v^b(u - u_\lambda)^a\|_{L^r(\Sigma_\lambda^u)} \\ &\leq C\|u_\lambda\|_{L^{q_1}(\Sigma_\lambda^v)}^a \|v^{b-1}(v - v_\lambda)\|_{L^s(\Sigma_\lambda^v)} + C\|v\|_{L^{q_2}(\Sigma_\lambda^u)}^b \|u - u_\lambda\|_{L^{q_1}(\Sigma_\lambda^u)}^a \\ &\leq C\|u_\lambda\|_{L^{q_1}(\Sigma_\lambda')}^a \|v\|_{L^{q_2}(\Sigma_\lambda^v)}^{b-1} \|v - v_\lambda\|_{L^{q_2}(\Sigma_\lambda^v)} + C\|v\|_{L^{q_2}(\Sigma_\lambda)}^b \|u - u_\lambda\|_{L^{q_1}(\Sigma_\lambda^u)}, \end{aligned} \tag{3.5}$$

where

$$s = \frac{rq_1}{q_1 - ar}, \quad t = \frac{rq_2}{q_2 - br} = \frac{q_1}{r} \quad \text{and} \quad \frac{b}{q_2} + \frac{a-1}{q_1} = \frac{\alpha}{N}.$$

Similarly, for $c, d > 1$ we have

$$\|v - v_\lambda\|_{L^{q_2}(\Sigma_\lambda^v)} \leq C\|v\|_{L^{q_2}(\Sigma_\lambda')}^d \|u\|_{L^{q_1}(\Sigma_\lambda^u)}^{c-1} \|u - u_\lambda\|_{L^{q_1}(\Sigma_\lambda^u)} + C\|u\|_{L^{q_1}(\Sigma_\lambda)}^c \|v\|_{L^{q_2}(\Sigma_\lambda^v)}^{d-1} \|v - v_\lambda\|_{L^{q_2}(\Sigma_\lambda^v)}, \tag{3.6}$$

where q_1 and q_2 are positive constant such that $\frac{d-1}{q_2} + \frac{c}{q_1} = \frac{\beta}{N}$. Taking into account (3.4), (3.5) and (3.6), for all $\lambda \in \mathbb{R}$ we have

$$\|u - u_\lambda\|_{L^{q_1}(\Sigma_\lambda^u)} \leq \Bigg\{ \frac{C\|v\|_{L^{q_2}(\Sigma_\lambda')}^d \|u\|_{L^{q_1}(\Sigma_\lambda^u)}^{c-1}}{1 - C\|u\|_{L^{q_1}(\Sigma_\lambda)}^c \|v\|_{L^{q_2}(\Sigma_\lambda^v)}^{d-1}} \|u_\lambda\|_{L^{q_1}(\Sigma_\lambda')}^a \|v\|_{L^{q_2}(\Sigma_\lambda^v)}^{b-1} + C\|v\|_{L^{q_2}(\Sigma_\lambda)}^b \|u\|_{L^{q_1}(\Sigma_\lambda^u)}^{a-1} \Bigg\} \|u - u_\lambda\|_{L^{q_1}(\Sigma_\lambda^u)}.$$

Using the fact that $(u, v) \in L^{q_1}(\mathbb{R}^N) \times L^{q_2}(\mathbb{R}^N)$, we can choose $\eta > 0$ sufficiently large such that for all $\lambda < -\eta$.

$$\frac{C\|v\|_{L^{q_2}(\Sigma_\lambda')}^d \|u\|_{L^{q_1}(\Sigma_\lambda^u)}^{c-1}}{1 - C\|u\|_{L^{q_1}(\Sigma_\lambda)}^c \|v\|_{L^{q_2}(\Sigma_\lambda^v)}^{d-1}} \|u_\lambda\|_{L^{q_1}(\Sigma_\lambda')}^a \|v\|_{L^{q_2}(\Sigma_\lambda^v)}^{b-1} + C\|v\|_{L^{q_2}(\Sigma_\lambda)}^b \|u\|_{L^{q_1}(\Sigma_\lambda^u)}^{a-1} \leq \frac{1}{2}.$$

It follows that $\|u - u_\lambda\|_{L^{q_1}(\Sigma_\lambda^u)} = 0$ and hence Σ_λ^u must be measure zero and empty when $\lambda < -\eta$. In the similar manner, Σ_λ^v must be of measure zero and empty when $\lambda < -\eta$. For all other cases, the proof follows analogously. This concludes the proof of the lemma. □

Now using the same assertions and arguments as in Huang, Li and Wang [18] in combination with Lemma 3.3, we have the following theorem.

Theorem 3.4. *Assume that* $a \geq 0$, $b, c, d \in \{0\} \cup [1, \infty)$, $0 < \alpha, \beta < N$ *and* $(u, v) \in L^{q_1}(\mathbb{R}^N) \times L^{q_2}(\mathbb{R}^N)$ *is a pair of positive solutions of* (3.3) *with* q_1 *and* q_2 *satisfying*

$$q_1, q_2 > 1, \quad \frac{b}{q_2} + \frac{a-1}{q_1} = \frac{\alpha}{N}, \quad \frac{c}{q_1} + \frac{d-1}{q_2} = \frac{\beta}{N}.$$

Then (u, v) is radially symmetric and monotone decreasing about some point in $\mathbb{R}^N$. Moreover, if

$$b = \frac{1}{N-\beta}[(N+\alpha) - a(N-\alpha)], \quad c = \frac{1}{N-\alpha}[(N+\beta) - d(N-\beta)],$$

then (u, v) must be of the form

$$u(x) = \Big(\frac{d_1}{e_1 + |x - x_1|^2}\Big)^{\frac{N-\alpha}{2}}, \quad v(x) = \Big(\frac{d_2}{e_2 + |x - x_2|^2}\Big)^{\frac{N-\beta}{2}}$$

for some constants $d_1, d_2, e_1, e_2 > 0$ and some $x_1, x_2 \in \mathbb{R}^N$.

As an immediate corollary, we have the following result on radial symmetry of nonnegative solutions of (3.1).

Corollary 3.5. *Every nonnegative solution $u \in D^{1,2}(\mathbb{R}^N)$ of equation (3.1) is radially symmetric, monotone decreasing and of the form*

$$u(x) = \Big(\frac{c_1}{c_2 + |x - x_0|^2}\Big)^{\frac{N-2}{2}}$$

for some constants $c_1, c_2 > 0$ and some $x_0 \in \mathbb{R}^N$.

Proof. Let u be any nonnegative solution of equation (3.1). Then by Lemma 3.1, we have $u \in W^{2,s}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $1 \le s < \infty$. Hence, by the strong maximum principle, we have that u is a positive function in $\mathbb{R}^N$. It implies that $(u, v) \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) \times L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$ is a positive solution of the integral system (3.2). Now employing Theorem 3.4 for $\alpha = 2$, $a = p - 1$, $b = 1$, $\beta = \mu$, $c = p$, $d = 0$ and using the fact $u \in D^{1,2}(\mathbb{R}^N)$, that is, $u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ and $v \in L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$, we have the desired result. □

4 Palais–Smale Analysis

Lemma 4.1. *Let $u_n \rightharpoonup u$ be weakly convergent in $D^{1,2}(\mathbb{R}^N)$ and $u_n \to u$ a.e. on $\mathbb{R}^N$. Then*

$$\begin{aligned}(|x|^{-\mu} * |(u_n)_+|^{2^*_\mu})|(u_n)_+|^{2^*_\mu-2}(u_n)_+ - (|x|^{-\mu} * |(u_n - u)_+|^{2^*_\mu})|(u_n - u)_+|^{2^*_\mu-2}(u_n - u)_+ \\ \to (|x|^{-\mu} * |u_+|^{2^*_\mu})|u_+|^{2^*_\mu-2}u_+ \quad \textit{in } (D^{1,2}(\mathbb{R}^N))'.\end{aligned} \tag{4.1}$$

Proof. Since $u_n \rightharpoonup u$ weakly in $D^{1,2}(\mathbb{R}^N)$, there exists $M > 0$ such that $\|u_n\| < M$ for all $n \in \mathbb{N}$. Let $\phi \in D^{1,2}(\mathbb{R}^N)$ and

$$I = \int_{\mathbb{R}^N} [(|x|^{-\mu} * |(u_n)_+|^{2^*_\mu})|(u_n)_+|^{2^*_\mu-2}(u_n)_+(|x|^{-\mu} * |(u_n - u)_+|^{2^*_\mu})|(u_n - u)_+|^{2^*_\mu-2}(u_n - u)_+]\phi\, dx.$$

Then $I = I_1 + I_2 + I_3 - 2I_4$, where

$$I_1 = \int_{\mathbb{R}^N} (|x|^{-\mu} * (|(u_n)_+|^{2^*_\mu} - |(u_n - u)_+|^{2^*_\mu}))(|(u_n)_+|^{2^*_\mu-2}(u_n)_+ - |(u_n - u)_+|^{2^*_\mu-2}(u_n - u)_+)\phi\, dx,$$

$$I_2 = \int_{\mathbb{R}^N} (|x|^{-\mu} * |(u_n)_+|^{2^*_\mu})|(u_n - u)_+|^{2^*_\mu-2}(u_n - u)_+\phi\, dx,$$

$$I_3 = \int_{\mathbb{R}^N} (|x|^{-\mu} * |(u_n - u)_+|^{2^*_\mu})|(u_n)_+|^{2^*_\mu-2}(u_n)_+\phi\, dx,$$

$$I_4 = \int_{\mathbb{R}^N} (|x|^{-\mu} * |(u_n - u)_+|^{2^*_\mu})|(u_n - u)_+|^{2^*_\mu-2}(u_n - u)_+\phi\, dx.$$

Claim 1. *We have*

$$\lim_{n\to\infty} I_1 = \int_{\mathbb{R}^N} (|x|^{-\mu} * |u_+|^{2^*_\mu})|u_+|^{2^*_\mu-2}u_+\phi\, dx.$$

Similar to the proof of the Brezis–Lieb lemma [8] we have

$$|(u_n)_+|^{2^*_\mu} - |(u_n - u)_+|^{2^*_\mu} \to |u_+|^{2^*_\mu} \quad \text{in } L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N) \quad \text{as } n \to \infty.$$

Since the Hardy–Littlewood–Sobolev inequality implies that the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ to $L^{\frac{2N}{\mu}}(\mathbb{R}^N)$, we get

$$|x|^{-\mu} * (|(u_n)_+|^{2^*_\mu} - |(u_n - u)_+|^{2^*_\mu}) \to |x|^{-\mu} * |u_+|^{2^*_\mu} \quad \text{strongly in } L^{\frac{2N}{\mu}}(\mathbb{R}^N) \quad \text{as } n \to \infty. \tag{4.2}$$

Since both $|(u_n)_+|^{2^*_\mu-2}(u_n)_+\phi \rightharpoonup |u_+|^{2^*_\mu-2}u_+\phi$ and $|(u_n-u)_+|^{2^*_\mu-2}(u_n-u)_+\phi \rightharpoonup 0$ converge weakly in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$, we obtain

$$|(u_n)_+|^{2^*_\mu-2}(u_n)_+\phi - |(u_n - u)_+|^{2^*_\mu-2}(u_n - u)_+\phi \rightharpoonup |u_+|^{2^*_\mu-2}u_+\phi \tag{4.3}$$

weakly in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$. Thus, Claim 1 follows from (4.2) and (4.3).

Claim 2. *We have* $\lim_{n\to\infty} I_2 = 0$.

Since $|(u_n)_+|^{2^*_\mu} \rightharpoonup |(u)_+|^{2^*_\mu}$ weakly in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$, by the Hardy–Littlewood–Sobolev inequality (2.1) we have

$$|x|^{-\mu} * |(u_n)_+|^{2^*_\mu} \rightharpoonup |x|^{-\mu} * |u_+|^{2^*_\mu} \quad \text{weakly in } L^{\frac{2N}{\mu}}(\mathbb{R}^N). \tag{4.4}$$

We observe that

$$|(u_n - u)_+|^{2^*_\mu-2}(u_n - u)_+\phi \to 0 \quad \text{a.e. in } \mathbb{R}^N$$

and for any open subset $U \subset \mathbb{R}^N$, we have

$$\begin{aligned} \int_U \big||(u_n - u)_+|^{2^*_\mu-2}(u_n - u)_+\phi\big|^{\frac{2N}{2N-\mu}}\,dx &\le \Big(\int_U |(u_n - u)_+|^{2^*}\,dx\Big)^{\frac{N-\mu+2}{2N-\mu}}\Big(\int_U |\phi|^{2^*}\,dx\Big)^{\frac{N-2}{2N-\mu}} \\ &\le \|u_n\|^{2^*(2^*_\mu-1)}\Big(\int_U |\phi|^{2^*}\,dx\Big)^{\frac{N-2}{2N-\mu}} \\ &\le M\Big(\int_U |\phi|^{2^*}\,dx\Big)^{\frac{N-2}{2N-\mu}}. \end{aligned}$$

This implies that $\{\big||(u_n - u)_+|^{2^*_\mu-2}(u_n - u)_+\phi\big|^{\frac{2N}{2N-\mu}}\}_n$ is equi-integrable in $L^1(\mathbb{R}^N)$. Hence, by the Vitali convergence theorem we get that

$$|(u_n - u)_+|^{2^*_\mu-2}(u_n - u)_+\phi \to 0 \quad \text{strongly in } L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N).$$

This fact together with (4.4) completes the proof of Claim 2.

Claim 3. *We have* $\lim_{n\to\infty} I_3 = 0$.

Similar to the proof of Claim 2, we have

$$|x|^{-\mu} * |(u_n - u)_+|^{2^*_\mu} \rightharpoonup 0 \quad \text{weakly in } L^{\frac{2N}{\mu}}(\mathbb{R}^N)$$

and

$$|(u_n)_+|^{2^*_\mu-2}(u_n)_+\phi \to |u_+|^{2^*_\mu-2}u_+\phi \quad \text{strongly in } L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N).$$

Thus, Claim 3 follows.

Claim 4. *We have* $\lim_{n\to\infty} I_4 = 0$.

Similar to the proof of Claim 2, we have

$$|x|^{-\mu} * |(u_n - u)_+|^{2^*_\mu} \rightharpoonup 0 \quad \text{weakly in } L^{\frac{2N}{\mu}}(\mathbb{R}^N)$$

and

$$|(u_n - u)_+|^{2^*_\mu-2}(u_n - u)_+\phi \to 0 \quad \text{strongly in } L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N).$$

Thus, Claim 4 follows. Hence

$$I \to \int_{\mathbb{R}^N} (|x|^{-\mu} * |u_+|^{2^*_\mu})|u_+|^{2^*_\mu-2}u_+\phi\,dx,$$

that is, (4.1) holds. □

Lemma 4.2. *If $u_n \rightharpoonup u$ weakly in $D_0^{1,2}(\Omega)$, $u_n \to u$ a.e. on Ω, $I(u_n) \to c$, $I'(u_n) \to 0$ in $(D_0^{1,2}(\Omega))'$, then $I'(u) = 0$ and $v_n := u_n - u$ satisfies*

$$\|v_n\|^2 = \|u_n\|^2 - \|u\|^2 + o(1), \quad I_\infty(v_n) \to c - I(u) \quad \text{and} \quad I'_\infty(v_n) \to 0 \quad \text{in } (D_0^{1,2}(\Omega))'.$$

Proof. Let us prove the following:

Claim. *We have $I'(u) = 0$.*

Note that

$$u_n \rightharpoonup u \text{ weakly in } D_0^{1,2}(\Omega) \implies |(u_n)_+|^{2^*_\mu} \rightharpoonup |u_+|^{2^*_\mu} \text{ weakly in } L^{\frac{2N}{2N-\mu}}(\Omega).$$

Since Riesz potential is a linear continuous map from $L^{\frac{2N}{2N-\mu}}(\Omega)$ to $L^{\frac{2N}{\mu}}(\Omega)$, we obtain that

$$\int_\Omega \frac{|(u_n)_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy \rightharpoonup \int_\Omega \frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy \quad \text{weakly in } L^{\frac{2N}{\mu}}(\Omega).$$

Also, $|(u_n)_+|^{2^*_\mu-2}(u_n)_+ \rightharpoonup |u_+|^{2^*_\mu-2}u_+$ weakly in $L^{\frac{2N}{N-\mu+2}}(\Omega)$. Combining these facts, we have

$$\Big(\int_\Omega \frac{|(u_n)_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\Big)|(u_n)_+|^{2^*_\mu-2}(u_n)_+ \rightharpoonup \Big(\int_\Omega \frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\Big)|u_+|^{2^*_\mu-2}u_+ \quad \text{weakly in } L^{\frac{2N}{N+2}}(\Omega).$$

This implies for any $\phi \in D_0^{1,2}(\Omega)$, we have

$$\int_\Omega\int_\Omega \frac{|(u_n)_+(x)|^{2^*_\mu}|(u_n)_+(y)|^{2^*_\mu-2}(u_n)_+(y)\phi(y)}{|x-y|^\mu}\,dx\,dy \to \int_\Omega\int_\Omega \frac{|u_+(x)|^{2^*_\mu}|u_+(y)|^{2^*_\mu-2}u_+(y)\phi(y)}{|x-y|^\mu}\,dx\,dy. \tag{4.5}$$

Now, for $\phi \in D_0^{1,2}(\Omega)$ consider

$$\begin{aligned}\langle I'(u_n) - I'(u), \phi\rangle &= \int_\Omega \nabla u_n.\nabla\phi\,dx - \int_\Omega\int_\Omega \frac{|(u_n)_+(x)|^{2^*_\mu}|(u_n)_+(y)|^{2^*_\mu-2}(u_n)_+\phi(y)}{|x-y|^\mu}\,dx\,dy \\ &\quad - \int_\Omega \nabla u.\nabla\phi\,dx + \int_\Omega\int_\Omega \frac{|u_+(x)|^{2^*_\mu}|u_+(y)|^{2^*_\mu-2}u_+\phi(y)}{|x-y|^\mu}\,dx\,dy.\end{aligned}$$

By using (4.5) and the fact that $u_n \rightharpoonup u$ weakly in $D_0^{1,2}(\Omega)$, the claim follows. By the Brezis–Lieb lemma (see [8, 16]) we have

$$\begin{aligned}I_\infty(v_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2 - \frac{1}{2\cdot 2^*_\mu}\int_\Omega\int_\Omega \frac{|(u_n-u)_+(x)|^{2^*_\mu}|(u_n-u)_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy + o(1) \\ &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2\cdot 2^*_\mu}\int_\Omega\int_\Omega \frac{|(u_n)_+(x)|^{2^*_\mu}|(u_n)_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy \\ &\quad - \frac{1}{2}\|u\|^2 + \frac{1}{2\cdot 2^*_\mu}\int_\Omega\int_\Omega \frac{|u_+(x)|^{2^*_\mu}|u_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy + o(1) \\ &= I(u_n) - I(u) + o(1) \to c - I(u).\end{aligned}$$

Now we will show that $I'_\infty(v_n) \to 0$ in $(D_0^{1,2}(\Omega))'$. By Lemma 4.1, for any $\phi \in D_0^{1,2}(\Omega)$,

$$\langle I'_\infty(v_n), \phi\rangle = \langle I'(v_n), \phi\rangle = \langle I'(u_n), \phi\rangle - \langle I'(u), \phi\rangle + o(1) \to 0.$$

This implies $I'_\infty(v_n) \to 0$ in $(D_0^{1,2}(\Omega))'$. □

Lemma 4.3. *Let $\{y_n\} \subset \Omega$ and $\{\lambda_n\} \subset (0,\infty)$ be such that $\frac{1}{\lambda_n}\operatorname{dist}(y_n, \partial\Omega) \to \infty$. Assume the sequence $\{u_n\}$ and the rescaled sequence*

$$f_n(x) = \lambda_n^{\frac{N-2}{2}} u_n(\lambda_n x + y_n)$$

is such that $f_n \rightharpoonup f$ weakly in $D^{1,2}(\mathbb{R}^N)$, $f_n \to f$ a.e. on $\mathbb{R}^N$, $I_\infty(u_n) \to c$, $I'_\infty(u_n) \to 0$ in $(D_0^{1,2}(\Omega))'$. Then $I'_\infty(f) = 0$. Also, the sequence

$$z_n(x) = u_n(x) - \lambda_n^{\frac{2-N}{2}} f\Big(\frac{x-y_n}{\lambda_n}\Big)$$

satisfies $\|z_n\|^2 = \|u_n\|^2 - \|f\|^2 + o(1)$, $I_\infty(z_n) \to c - I_\infty(f)$ and $I'_\infty(z_n) \to 0$ in $(D_0^{1,2}(\Omega))'$.

Proof. For $\phi \in C_c^\infty(\mathbb{R}^N)$ define $\phi_n(x) := \lambda_n^{\frac{2-N}{2}} \phi(\frac{x-y_n}{\lambda_n})$. If $\phi \in C_c^\infty(B_k)$, then for large n, $\phi_n \in C_c^\infty(\Omega)$. It implies

$$\langle I'_\infty(f_n), \phi\rangle = \langle I'_\infty(u_n), \phi_n\rangle \leq \|I'_\infty(u_n)\|\|\phi_n\| = \|I'_\infty(u_n)\|\|\phi\| \to 0.$$

Hence, $I'_\infty(f_n) \to 0$ as $n \to \infty$ in $(D_0^{1,2}(B_k))'$ for each k.

Claim. *We have $I'_\infty(f) = 0$.*

Since $\phi \in C_c^\infty(\mathbb{R}^N)$, we obtain $\phi \in C_c^\infty(B_k)$ for some k. Now, using the fact $\frac{1}{\lambda_n}\operatorname{dist}(y_n, \partial\Omega) \to \infty$, $I'_\infty(f_n) \to 0$ in $(D_0^{1,2}(B_k))'$ and following the steps of Claim of Lemma 4.2, we have $\langle I'_\infty(f_n) - I'_\infty(f), \phi\rangle \to 0$, that is, the claim holds. By the Brezis–Lieb lemma (see [8, 16]),

$$I_\infty(z_n) = I_\infty(f_n - f) = I_\infty(u_n) - I_\infty(f) + o(1) \to c - I_\infty(f).$$

As $f_n \rightharpoonup f$ weakly in $D^{1,2}(\mathbb{R}^N)$, we obtain

$$\|z_n\|^2 = \int_{\mathbb{R}^N} \left|\nabla u_n(x) - \lambda_n^{\frac{-N}{2}} \nabla f\left(\frac{x-y_n}{\lambda_n}\right)\right|^2 dx = \|u_n\|^2 - \|f\|^2 + o(1).$$

By Lemma 4.1, for any $\phi \in D_0^{1,2}(\Omega)$, we have

$$\langle I'_\infty(z_n), \phi\rangle = \left\langle I'_\infty(u_n) - I'_\infty\left(\lambda_n^{\frac{2-N}{2}} f\left(\frac{\cdot - y_n}{\lambda_n}\right)\right), \phi\right\rangle + o(1) = \langle I'_\infty(u_n), \phi\rangle + o(1) = o(1).$$

This implies $I'_\infty(z_n) \to 0$ in $(D_0^{1,2}(\Omega))'$. □

Before proving the global compactness lemma for the Choquard equation, we will define the well-known Morrey spaces.

Definition 4.4. A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to Morrey space $\mathcal{L}^{r,\gamma}(\mathbb{R}^N)$, with $r \in [1, \infty)$ and $\gamma \in [0, N]$, if and only if

$$\|u\|^r_{\mathcal{L}^{r,\gamma}(\mathbb{R}^N)} := \sup_{R>0,\, x\in\mathbb{R}^N} R^{\gamma-N} \int_{B(x,R)} |u|^r \, dy < \infty.$$

By Hölder's inequality, we have $L^{2^*}(\mathbb{R}^N) \hookrightarrow \mathcal{L}^{2,N-2}(\mathbb{R}^N)$.

Lemma 4.5 (Global Compactness Lemma). *Let $\{u_n\}_{n\in\mathbb{N}} \subset D_0^{1,2}(\Omega)$ be such that $I(u_n) \to c$, $I'(u_n) \to 0$. Then passing if necessary to a subsequence, there exists a solution $v_0 \in D_0^{1,2}(\Omega)$ of*

$$-\Delta u = \left(\int_\Omega \frac{|u_+(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right)|u_+|^{2^*_\mu - 1} \quad \text{in } \Omega \tag{4.6}$$

and (possibly) $k \in \mathbb{N} \cup \{0\}$, nontrivial solutions $\{v_1, v_2, \ldots, v_k\}$ of

$$-\Delta u = (|x|^{-\mu} * |u_+|^{2^*_\mu})|u_+|^{2^*_\mu - 1} \quad \text{in } \mathbb{R}^N \tag{4.7}$$

with $v_i \in D^{1,2}(\mathbb{R}^N)$ and k sequences $\{y_n^i\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $\{\lambda_n^i\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, $i = 1, 2, \ldots, k$, satisfying

$$\frac{1}{\lambda_n^i}\operatorname{dist}(y_n^i, \partial\Omega) \to \infty \quad \text{and} \quad \left\|u_n - v_0 - \sum_{i=1}^k (\lambda_n^i)^{\frac{2-N}{2}} v_i\left(\frac{\cdot - y_n^i}{\lambda_n^i}\right)\right\| \to 0, \quad n \to \infty,$$

$$\|u_n\|^2 \to \sum_{i=0}^k \|v_i\|^2,\ n \to \infty, \quad I(v_0) + \sum_{i=1}^k I_\infty(v_i) = c. \tag{4.8}$$

Proof. We divide the proof into several steps.

Step 1. By coercivity of the functional I, we get $\{u_n\}$ is a bounded sequence in $D_0^{1,2}(\Omega)$. It implies that there exists a $v_0 \in D_0^{1,2}(\Omega)$ such that $u_n \rightharpoonup v_0$ weakly in $D_0^{1,2}(\Omega)$, $u_n \to v_0$ a.e. on Ω. By Lemma 4.2, $I'(v_0) = 0$. Set $u_n^1 = u_n - v_0$. Then

$$\|u_n^1\|^2 = \|u_n\|^2 - \|v_0\|^2 + o(1), \quad I_\infty(u_n^1) \to c - I(v_0) \quad \text{and} \quad I'_\infty(u_n^1) \to 0 \quad \text{in } (D_0^{1,2}(\Omega))'. \tag{4.9}$$

Moreover, there exists a constant $M_1 > 0$ such that $\|u_n^1\| < M_1$ for all $n \in \mathbb{N}$.

Step 2. If

$$\int_\Omega\int_\Omega \frac{|(u_n^1)_+(x)|^{2_\mu^*}|(u_n^1)_+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy \to 0,$$

then using the fact that $I'(u_n) \to 0$, it follows that $u_n^1 \to 0$ in $D_0^{1,2}(\Omega)$ and we are done. If

$$\int_\Omega\int_\Omega \frac{|(u_n^1)_+(x)|^{2_\mu^*}|(u_n^1)_+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy \nrightarrow 0,$$

then we may assume that

$$\int_\Omega\int_\Omega \frac{|(u_n^1)_+(x)|^{2_\mu^*}|(u_n^1)_+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy > \delta \quad \text{for some } \delta > 0.$$

This together with the Hardy–Littlewood–Sobolev inequality gives $\|u_n^1\|_{L^{2^*}} > \delta_1$ for all n and for an appropriate constant $\delta_1 > 0$. Taking into account that u_n^1 is a bounded sequence in $L^{2^*}(\mathbb{R}^N)$, $L^{2^*}(\mathbb{R}^N) \hookrightarrow \mathcal{L}^{2,N-2}(\mathbb{R}^N)$, and [28, Theorem 2], we obtain

$$c_2 < \|u_n^1\|_{\mathcal{L}^{2,N-2}(\mathbb{R}^N)} < c_1 \quad \text{for all } n.$$

Thus, there exists a positive constant C_0 such that for all n, we have

$$C_0 < \|u_n^1\|_{\mathcal{L}^{2,N-2}(\mathbb{R}^N)} < C_0^{-1}. \tag{4.10}$$

Now employing the definition of Morrey spaces and (4.10), for each $n \in \mathbb{N}$ there exists $\{y_n^1, \lambda_n^1\} \in \mathbb{R}^N \times \mathbb{R}^+$ such that

$$0 < \widehat{C_0} < \|u_n^1\|^2_{\mathcal{L}^{r,\gamma}(\mathbb{R}^N)} - \frac{C_0^2}{2n} < (\lambda_n^1)^{-2} \int_{B(y_n^1,\lambda_n^1)} |u_n^1|^2\,dy$$

for some suitable positive constant $\widehat{C_0}$. Now, define

$$f_n^1(x) := (\lambda_n^1)^{\frac{N-2}{2}} u_n^1(\lambda_n^1 x + y_n^1).$$

Since $\|f_n^1\| = \|u_n^1\|$, we have $\|f_n^1\| < M_1$ for all $n \in \mathbb{N}$ and we can assume that $f_n^1 \rightharpoonup v_1$ weakly in $D^{1,2}(\mathbb{R}^N)$ and $f_n^1 \to v_1$ a.e. on $\mathbb{R}^N$. Moreover,

$$\int_{B(0,1)} |f_n^1|^2\,dx = (\lambda_n^1)^{N-2} \int_{B(0,1)} |u_n^1(\lambda_n^1 x + y_n^1)|^2\,dx = (\lambda_n^1)^{-2} \int_{B(y_n^1,\lambda_n^1)} |u_n^1(y)|^2\,dy > \widehat{C_0} > 0.$$

Since, $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2_{\mathrm{loc}}(\mathbb{R}^N)$ is compact, we have $\int_{B(0,1)} |v_1|^2\,dx > \widehat{C_0} > 0$. It implies that $v_1 \neq 0$.

Step 3. We claim that $\lambda_n \to 0$ and $y_n^1 \to y_0 \in \overline{\Omega}$. Let if possible $\lambda_n \to \infty$. As $\{u_n^1\}$ is a bounded sequence in $D_0^{1,2}(\Omega)$, it implies $\{u_n^1\}$ is a bounded sequence in $L^2(\Omega)$. Thus, if we define $\Omega_n = \frac{\Omega - y_n^1}{\lambda_n^1}$, then

$$\int_{\Omega_n} |f_n^1|^2\,dx = \frac{1}{(\lambda_n^1)^2} \int_\Omega |u_n^1|^2\,dx \le \frac{C}{\lambda_n^2} \to 0.$$

Contrary to this, using Fatou's lemma, we have

$$0 = \liminf_{n\to\infty} \int_{\Omega_n} |f_n^1|^2\,dx \ge \int_{\Omega_n} |v_1|^2\,dx.$$

This means that $v \equiv 0$, which is not possible by Step 2. Hence $\{\lambda_n^1\}$ is bounded in $\mathbb{R}$, that is, there exists $0 \le \lambda_0^1 \in \mathbb{R}$ such that $\lambda_n^1 \to \lambda_0^1$ as $n \to \infty$. If $|y_n^1| \to \infty$, then for any $x \in \Omega$ and large n, $\lambda_n x + y_n \notin \overline{\Omega}$. Since $u_n \in D_0^{1,2}(\Omega)$, it follows that $u_n^1(\lambda_n x + y_n) = 0$ for all $x \in \Omega$, which yields a contradiction to the assumption

$$\|u_n\|_{NL}^{2\cdot 2_\mu^*} > \delta > 0.$$

Therefore, y_n^1 is bounded, it implies that $y_n^1 \to y_0^1 \in \mathbb{R}^N$. Now let if possible $\lambda_n^1 \to \lambda_0^1 > 0$. Then

$$\Omega_n \to \frac{\Omega - y_0^1}{\lambda_0^1} = \Omega_0 \neq \mathbb{R}^N.$$

Hence using the fact that $u_n^1 \rightharpoonup 0$ weakly in $D_0^{1,2}(\Omega)$, we have $f_n^1 \rightharpoonup 0$ weakly in $D^{1,2}(\mathbb{R}^N)$ which is not possible since by Step 2, $v_1 \neq 0$. This implies $\lambda_n^1 \to 0$. Arguing by contradiction, we assume that

$$y_0^1 \notin \overline{\Omega}. \tag{4.11}$$

In view of the fact that $\lambda_n^1 x + y_n^1 \to y_0^1$ for all $x \in \Omega$ as $n \to \infty$. Now using (4.11), we have $\lambda_n^1 x + y_n^1 \notin \overline{\Omega}$ for all $x \in \Omega$ and n large enough. It implies that $u_n^1(\lambda_n^1 x + y_n^1) = 0$ for n large enough, which is not possible. Therefore, $y_0^1 \in \overline{\Omega}$. This completes the proof of claim and Step 3.

Step 4: Assume that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^1} \operatorname{dist}(y_n^1, \partial\Omega) \to \alpha < \infty.$$

Then v_1 is a solution of (2.12) and by Theorem 2.8 we have $v_1 \equiv 0$, which is not possible. Therefore,

$$\frac{1}{\lambda_n^1} \operatorname{dist}(y_n^1, \partial\Omega) \to \infty \quad \text{as } n \to \infty.$$

Thus by (4.9) and Lemma 4.3, we have $I'_\infty(v_1) = 0$ and the sequence

$$u_n^2(x) = u_n^1(x) - \lambda_n^{\frac{2-N}{2}} v_1\Big(\frac{x - y_n}{\lambda_n}\Big)$$

satisfies

$$I_\infty(u_n^2) \to c - I_\infty(v_0) - I_\infty(v_1) \quad \text{and} \quad I'_\infty(u_n^2) \to 0 \quad \text{in } (D_0^{1,2}(\Omega))'.$$

By Proposition 2.6, we have $I_\infty(v_1) \geq \beta$. So, iterating the above procedure, we can construct sequences $\{v_i\}$, $\{\lambda_n^i\}$, $\{f_n^i\}$ and after k iterations we obtain

$$I_\infty(u_n^{k+1}) < I(u_n) - I(v_0) - \sum_{i=1}^{k} I_\infty(v_i) \leq I(u_n) - I(v_0) - k\beta.$$

As the later will be negative for large k, the induction process terminates after some index $k \geq 0$. Consequently, we get k sequences $\{y_n^i\}_n \subset \Omega$ and $\{\lambda_n^i\}_n \subset \mathbb{R}_+$, satisfying (4.8). □

Definition 4.6. We say that I satisfies the Palais–Smale condition at c if for any sequence $u_k \in D_0^{1,2}(\Omega)$ such that $I(u_k) \to c$ and $I'(u_k) \to 0$ there exists a subsequence that converges strongly in $D_0^{1,2}(\Omega)$.

Lemma 4.7. *The functional I satisfies the Palais–Smale condition for any $c \in (\beta, 2\beta)$, where*

$$\beta = \frac{1}{2}\Big(\frac{N - \mu + 2}{2N - \mu}\Big) S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

Proof. For some $c \in (\beta, 2\beta)$, we assume that there exists $\{u_n\} \in D_0^{1,2}(\Omega)$ such that

$$I(u_n) \to c, I'(u_n) \to 0 \quad \text{in } (D_0^{1,2}(\Omega))'.$$

By Lemma 4.5, passing to a subsequence (if necessary), there exists a solution $v_0 \in D_0^{1,2}(\Omega)$ of (4.6) and $k \in \mathbb{N} \cup \{0\}$, nontrivial solutions $\{v_1, v_2, \dots, v_k\}$ of (4.7) with $v_i \in D^{1,2}(\mathbb{R}^N)$ and k sequences $\{y_n^i\}_n \subset \mathbb{R}^N$ and $\{\lambda_n^i\}_n \subset \mathbb{R}_+$ satisfying (4.8). Now, by equation (4.8) and Proposition 2.6 we have $k\beta \leq c < 2\beta$. This implies $k \leq 1$.

If $k = 0$, compactness holds and we are done. If $k = 1$, then we have two possibilities: either $v_0 \not\equiv 0$ or $v_0 \equiv 0$. If $v_0 \not\equiv 0$, since $I(v_0) \geq \beta$ and by [16, Lemma 1.3], β is never achieved on bounded domain, we have $I(v_0) > \beta$ and this is not possible. If $v_0 \equiv 0$, then by Theorem 2.8, $I_\infty(v_1) = c$ and v_1 is a nonnegative solution of (4.7).

Next, by Corollary 3.5, we deduce that v_1 is radially symmetric, monotonically deceasing and of the form $v_1(x) = (\frac{a}{b+|x-x_0|^2})^{\frac{N-2}{2}}$, for some constants $a, b > 0$ and some $x_0 \in \mathbb{R}^N$. Therefore by Lemma 2.3, we conclude that $S_{H,L}$ is achieved by v_1. It follows that $I_\infty(v_1) = \beta$, which is a contradiction since $I_\infty(v_1) = c > \beta$. □

5 Proof of Theorem 1.1

To prove Theorem 1.1, we shall first establish some auxiliary results.

Let R_1, R_2 be the radii of the annulus as in Theorem 1.1. Without loss of generality, we can assume $x_0 = 0$, $R_1 = \frac{1}{4R}$, $R_2 = 4R$, where $R > 0$ will be chosen sufficiently large. Consider the family of functions

$$u_t^\sigma(x) := S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} C(N,\mu)^{\frac{2-N}{2(N-\mu+2)}} \left(\frac{1-t}{(1-t)^2 + |x - t\sigma|^2}\right)^{\frac{N-2}{2}} \in D^{1,2}(\mathbb{R}^N),$$

where $\sigma \in \Sigma := \{x \in \mathbb{R}^N : |x| = 1\}$, $t \in [0,1)$. Note that if $t \to 1$, then u_t^σ concentrates at σ. Also, if $t \to 0$ then

$$u_t^\sigma \to u_0 := S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} C(N,\mu)^{\frac{2-N}{2(N-\mu+2)}} \left(\frac{1}{1+|x|^2}\right)^{\frac{N-2}{2}}.$$

Now, define $\upsilon \in C_c^\infty(\Omega)$ such that $0 \le \upsilon \le 1$ on Ω and

$$\upsilon(x) = \begin{cases} 1, & \frac{1}{2} < |x| < 2, \\ 0, & |x| > 4,\ |x| < \frac{1}{4}. \end{cases}$$

Subsequently, we can define

$$\upsilon_R(x) = \begin{cases} \upsilon(Rx), & 0 < |x| < \frac{1}{2R}, \\ 1, & \frac{1}{2R} \le |x| \le R, \\ \upsilon(\frac{x}{R}), & |x| \ge R. \end{cases}$$

We now define

$$g_t^\sigma(x) = u_t^\sigma(x)\upsilon_R(x) \in D_0^{1,2}(\Omega), \quad g_0(x) = u_0(x)\upsilon_R(x).$$

We establish the following auxiliary result.

Lemma 5.1. *Let $\sigma \in \Sigma$ and $t \in (0,1]$. Then the following holds:*
(i) $\|u_t^\sigma\| = \|u_0\|$,
(ii) $\|(u_t^\sigma)_+\|_{NL} = \|(u_0)_+\|_{NL}$,
(iii) $\|u_t^\sigma\|^2 = S_{H,L}\|(u_t^\sigma)_+\|_{NL}^2$,
(iv) $\lim_{R\to\infty} \sup_{\sigma\in\Sigma, t\in[0,1)} \|g_t^\sigma - u_t^\sigma\| = 0$,
(v) $\lim_{R\to\infty} \sup_{\sigma\in\Sigma, t\in[0,1)} \|g_t^\sigma\|_{NL}^{2\cdot 2_\mu^*} = \|u_t^\sigma\|_{NL}^{2\cdot 2_\mu^*}$.

Proof. By trivial transformations, we can get first two properties u_t^σ and since u_t^σ is a minimizer of $S_{H,L}$ therefore, third ones holds.

We have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla g_t^\sigma - \nabla u_t^\sigma|^2\,dx &\le 2\int_{\mathbb{R}^N} |u_t^\sigma(x)\nabla\upsilon_R(x)|^2\,dx + 2\int_{\mathbb{R}^N} |\nabla u_t^\sigma(x)\upsilon_R(x) - \nabla u_t^\sigma(x)|^2\,dx \\ &\le C\Big(R^2 \int_{B_{\frac{1}{2R}}} |u_t^\sigma(x)|^2\,dx + \int_{B_{\frac{1}{2R}}} |\nabla u_t^\sigma(x)|^2\,dx\Big) \\ &\quad + C\Big(\frac{1}{R^2}\int_{B_{4R}\setminus B_{2R}} |u_t^\sigma(x)|^2\,dx + \int_{\mathbb{R}^N\setminus B_{2R}} |\nabla u_t^\sigma(x)|^2\,dx\Big), \end{aligned} \tag{5.1}$$

where B_α is a ball of radius α and center 0.

From the definition of u_t^σ, we have

$$R^2 \int_{B_{\frac{1}{2R}}} |u_t^\sigma(x)|^2\,dx \le CR^2 \int_{B_{\frac{1}{2R}}} dx \le \frac{C}{R^{N-2}},$$

$$\int_{B_{\frac{1}{2R}}} |\nabla u_t^\sigma(x)|^2\,dx \le C\int_{B_{\frac{1}{2R}}} |x - t\sigma|\,dx \le C\int_{B_{\frac{1}{2R}}} dx \le \frac{C}{R^N},$$

and

$$\frac{1}{R^2}\int_{B_{4R}\setminus B_{2R}} |u_t^\sigma(x)|^2\,dx \le \frac{C}{R^2}\int_{B_{4R}\setminus B_{2R}} \frac{1}{|x|^{2N-4}}\,dx \le \frac{C}{R^{N-2}},$$
$$\int_{\mathbb{R}^N\setminus B_{2R}} |\nabla u_t^\sigma(x)|^2\,dx \le C\int_{\mathbb{R}^N\setminus B_{2R}} \frac{1}{|x|^{2N-2}}\,dx \le \frac{C}{R^{N-2}}.$$

Therefore, from (5.1) if $R\to\infty$, we get $\sup_{\sigma\in\Sigma,t\in(0,1]}\|g_t^\sigma - u_t^\sigma\|\to 0$.

Next, we shall prove that

$$\lim_{R\to\infty}\sup_{\sigma\in\Sigma,t\in(0,1]}\|g_t^\sigma\|_{NL}^{2\cdot 2_\mu^*} = \|u_t^\sigma\|_{NL}^{2\cdot 2_\mu^*}.$$

Consider

$$\|g_t^\sigma\|_{NL}^{2\cdot 2_\mu^*} - \|u_t^\sigma\|_{NL}^{2\cdot 2_\mu^*} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v_R^{2_\mu^*}(x)v_R^{2_\mu^*}(y)-1)|u_t^\sigma(x)|^{2_\mu^*}|u_t^\sigma(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy \le C\sum_{i=1}^{5} J_i,$$

where

$$J_1 = \int_{B_{2R}\setminus B_{\frac{1}{2R}}}\int_{B_{\frac{1}{2R}}} \frac{|u_t^\sigma(x)|^{2_\mu^*}|u_t^\sigma(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy, \qquad J_2 = \int_{B_{2R}\setminus B_{\frac{1}{2R}}}\int_{\mathbb{R}^N\setminus B_{2R}} \frac{|u_t^\sigma(x)|^{2_\mu^*}|u_t^\sigma(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy,$$
$$J_3 = \int_{B_{\frac{1}{2R}}}\int_{B_{\frac{1}{2R}}} \frac{|u_t^\sigma(x)|^{2_\mu^*}|u_t^\sigma(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy, \qquad J_4 = \int_{B_{\frac{1}{2R}}}\int_{\mathbb{R}^N\setminus B_{2R}} \frac{|u_t^\sigma(x)|^{2_\mu^*}|u_t^\sigma(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy,$$
$$J_5 = \int_{\mathbb{R}^N\setminus B_{2R}}\int_{\mathbb{R}^N\setminus B_{2R}} \frac{|u_t^\sigma(x)|^{2_\mu^*}|u_t^\sigma(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy.$$

By the Hardy–Littlewood–Sobolev inequality, we have the following estimates:

$$\begin{aligned}
J_1 &\le C(N,\mu)\bigg(\int_{B_{\frac{1}{2R}}} \frac{(1-t)^N\,dx}{((1-t)^2+|x-t\sigma|^2)^N}\bigg)^{\frac{2N-\mu}{2N}}\bigg(\int_{B_{2R}\setminus B_{\frac{1}{2R}}} \frac{(1-t)^N\,dx}{((1-t)^2+|x-t\sigma|^2)^N}\bigg)^{\frac{2N-\mu}{2N}}\\
&\le C\bigg(\int_{B_{\frac{1}{2R}}}(1-t)^{N-2}\,dx\bigg)^{\frac{2N-\mu}{2N}} \le C\Big(\frac{1}{2R}\Big)^{\frac{2N-\mu}{2}},\\
J_2 &\le C(N,\mu)\bigg(\int_{B_{2R}\setminus B_{\frac{1}{2R}}} \frac{(1-t)^N\,dx}{((1-t)^2+|x-t\sigma|^2)^N}\bigg)^{\frac{2N-\mu}{2N}}\bigg(\int_{\mathbb{R}^N\setminus B_{2R}} \frac{(1-t)^N\,dx}{((1-t)^2+|x-t\sigma|^2)^N}\bigg)^{\frac{2N-\mu}{2N}}\\
&\le C\bigg(\int_{\mathbb{R}^N\setminus B_{2R}} \frac{dx}{|x-t\sigma|^{2N}}\bigg)^{\frac{2N-\mu}{2N}} \le C\bigg(\int_{|y+t\sigma|\ge 2R} \frac{dy}{|y|^{2N}}\bigg)^{\frac{2N-\mu}{2N}} \le C\bigg(\int_{|y|\ge 2R-1} \frac{dy}{|y|^{2N}}\bigg)^{\frac{2N-\mu}{2N}}\\
&\le C\Big(\frac{1}{2R-1}\Big)^{\frac{2N-\mu}{2}},\\
J_3 &\le C(N,\mu)\bigg(\int_{B_{\frac{1}{2R}}} \frac{(1-t)^N\,dx}{((1-t)^2+|x-t\sigma|^2)^N}\bigg)^{\frac{2N-\mu}{N}} \le C\bigg(\int_{B_{\frac{1}{2R}}}(1-t)^{N-2}\,dx\bigg)^{\frac{2N-\mu}{N}} \le C\Big(\frac{1}{2R}\Big)^{2N-\mu}.
\end{aligned}$$

Using the same estimates as above, we can easily obtain

$$J_4 \le C\Big(\frac{1}{2R}\Big)^{\frac{2N-\mu}{2}} \quad\text{and}\quad J_5 \le C\Big(\frac{1}{2R-1}\Big)^{2N-\mu}.$$

This implies that

$$\sup_{\sigma\in\Sigma,\,t\in[0,1)}\Big(\|g_t^\sigma\|_{NL}^{2\cdot 2_\mu^*} - \|u_t^\sigma\|_{NL}^{2\cdot 2_\mu^*}\Big)\to 0 \quad\text{as } R\to\infty$$

and completes the proof. □

In order to proceed further we define the manifold $\mathcal{M}$ and the functions $G : \mathcal{M} \to \mathbb{R}^N$ as follows:

$$\mathcal{M} = \left\{ u \in D_0^{1,2}(\Omega) : \int_\Omega \int_\Omega \frac{|u_+(x)|^{2_\mu^*} |u_+(y)|^{2_\mu^*}}{|x-y|^\mu} \, dx \, dy = 1 \right\} \quad \text{and} \quad G(u) = \int_\Omega x |\nabla u|^2 \, dx.$$

We also define $S_{H,L}(u, \Omega) : D_0^{1,2}(\Omega) \setminus \{0\} \to \mathbb{R}$, $S_{H,L} : D^{1,2}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}$ and $\tau : D_0^{1,2}(\Omega) \to \mathbb{R}$ as

$$S_{H,L}(u, \Omega) = \frac{\int_\Omega |\nabla u|^2 \, dx}{\left(\int_\Omega \int_\Omega \frac{|u_+(x)|^{2_\mu^*} |u_+(y)|^{2_\mu^*}}{|x-y|^\mu} \, dx \, dy\right)^{\frac{1}{2_\mu^*}}},$$

$$S_{H,L}(u) = \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\|u_+\|_{NL}^2},$$

$$\tau(u) = \left(\int_\Omega \int_\Omega \frac{|u_+(x)|^{2_\mu^*} |u_+(y)|^{2_\mu^*}}{|x-y|^\mu} \, dx \, dy \right)^{\frac{1}{2_\mu^*}}.$$

Proposition 5.2. *If $S_{H,L}(\,\cdot\,, \Omega) \in C^1(D_0^{1,2}(\Omega) \setminus \{0\})$ and $S'_{H,L}(u, \Omega) = 0$ for $u \in D_0^{1,2}(\Omega)$, then one has $I'(\lambda u) = 0$ for some $\lambda > 0$.*

Proof. Let $w \in D_0^{1,2}(\Omega)$. Then

$$\langle S'_{H,L}(u, \Omega), w \rangle = \frac{2\tau(u) \int_\Omega \nabla u . \nabla w \, dx - 2\|u\|^2 \tau(u)^{1-2_\mu^*} \int_\Omega \int_\Omega \frac{|u_+(x)|^{2_\mu^*} |u_+(y)|^{2_\mu^*-2} u_+(y) w(y)}{|x-y|^\mu} \, dx \, dy}{\tau(u)^2}.$$

As $S'_{H,L}(u, \Omega)(w) = 0$, it implies

$$\tau(u) \int_\Omega \nabla u . \nabla w \, dx = \|u\|^2 \tau(u)^{1-2_\mu^*} \int_\Omega \int_\Omega \frac{|u_+(x)|^{2_\mu^*} |u_+(y)|^{2_\mu^*-2} u_+(y) w(y)}{|x-y|^\mu} \, dx \, dy,$$

that is,

$$\int_\Omega \nabla u . \nabla w \, dx = \frac{\|u\|^2 \int_\Omega \int_\Omega \frac{|u_+(x)|^{2_\mu^*} |u_+(y)|^{2_\mu^*-2} u_+(y) w(y)}{|x-y|^\mu} \, dx \, dy}{\int_\Omega \int_\Omega \frac{|u_+(x)|^{2_\mu^*} |u_+(y)|^{2_\mu^*}}{|x-y|^\mu} \, dx \, dy}.$$

Therefore, if we choose

$$\lambda^{2(2_\mu^*-1)} = \frac{\|u\|^2}{\int_\Omega \int_\Omega \frac{|u_+(x)|^{2_\mu^*} |u_+(y)|^{2_\mu^*}}{|x-y|^\mu} \, dx \, dy},$$

then we get $I'(\lambda u) = 0$. ◻

Proposition 5.3. *Let $\{v_n\} \subset \mathcal{M}$ be a Palais–Smale sequence for $S_{H,L}(\,\cdot\,, \Omega)$ at level c. Then the sequence $\{u_n\}$ given by*

$$u_n = \lambda_n v_n, \quad \lambda_n = (S_{H,L}(v_n, \Omega))^{\frac{N-2}{2(N-\mu+2)}}$$

is a Palais–Smale sequence for I at level $\frac{N-\mu+2}{2(2N-\mu)} c^{\frac{2N-\mu}{N-\mu+2}}$.

Proof. By the calculations of Proposition 5.2 for any $w \in D_0^{1,2}(\Omega)$, we have

$$\frac{1}{2} \langle S'_{H,L}(v_n, \Omega), w \rangle = \int_\Omega \nabla v_n . \nabla w \, dx - \lambda_n^{2(2_\mu^*-1)} \int_\Omega \int_\Omega \frac{|(v_n)_+(x)|^{2_\mu^*} |(v_n)_+(y)|^{2_\mu^*-2} (v_n)_+(y) w(y)}{|x-y|^\mu} \, dx \, dy.$$

Now by multiplying the above equation by λ_n for any $w \in D_0^{1,2}(\Omega)$, we obtain

$$\langle I'(u_n), w \rangle = \int_\Omega \nabla u_n . \nabla w \, dx - \int_\Omega \int_\Omega \frac{|(u_n)_+(x)|^{2_\mu^*} |(u_n)_+(y)|^{2_\mu^*-2} (u_n)_+(y) w(y)}{|x-y|^\mu} \, dx \, dy.$$

Since $v_n \in \mathcal{M}$, it follows that $\lambda^{2(2_\mu^*-1)} = \|v_n\|^2 = S_{H,L}(v_n, \Omega)$, that is,

$$\lambda_n = S_{H,L}(v_n, \Omega)^{\frac{N-2}{2(N-\mu+2)}}.$$

From $S_{H,L}(v_n, \Omega) = c + o(1)$ we get λ_n is bounded. In particular, it follows that $\langle I'(\lambda_n v_n), w\rangle \to 0$ as $n \to \infty$. Also, we have u_n is bounded and this yields

$$o(1) = \langle I'(u_n), u_n\rangle = \|u_n\|^2 - \int_\Omega\int_\Omega \frac{|(u_n)_+(x)|^{2^*_\mu}|(u_n)_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy.$$

All the above facts imply that

$$\lim_{n\to\infty} I(u_n) = \frac{N-\mu+2}{2(2N-\mu)} \lim_{n\to\infty} \lambda_n^{2\cdot 2^*_\mu} = \frac{N-\mu+2}{2(2N-\mu)} c^{\frac{2N-\mu}{N-\mu+2}}. \qquad \square$$

Remark 5.4. Since we proved that I satisfies the Palais–Smale condition in $(\beta, 2\beta)$, it follows that $S_{H,L}(\cdot\,,\Omega)$ satisfies the Palais–Smale condition in $(S_{H,L}, 2^{\frac{N-\mu+2}{2N-\mu}} S_{H,L})$ by using Proposition 5.2.

Lemma 5.5. *If* $f_t^\sigma(x) := \frac{g_t^\sigma(x)}{\|g_t^\sigma\|_{NL}}$ *and* $f_0(x) := \frac{g_0(x)}{\|g_0\|_{NL}}$, *then*

$$\lim_{R\to\infty} S_{H,L}(f_t^\sigma, \Omega) = S_{H,L}(u_t^\sigma) = S_{H,L}$$

uniformly with respect to $\sigma \in \Sigma$ *and* $t \in [0, 1)$.

Proof. This is a trivial consequence of Lemma 5.1. $\square$

In particular, if $R > 1$ sufficiently large, then we can achieve that

$$\sup_{\sigma,t}(f_t^\sigma, \Omega) < S_1 < 2^{\frac{N-\mu+2}{2N-\mu}} S_{H,L} \quad \text{for some } S_1 \in \mathbb{R}.$$

Proof of Theorem 1.1 completed. As we have established, $S_{H,L}(\cdot\,,\Omega)$ satisfies Palais–Smale at level α on $\mathcal{M}$ for $\alpha \in (S_{H,L}, 2^{\frac{N-\mu+2}{2N-\mu}} S_{H,L})$. We will argue by contradiction. If $S_{H,L}(\cdot\,,\Omega)$ does not admit a critical value in this range, by the deformation lemma (see Bonnet [7, Theorem 2.5]) for any $\alpha \in (S_{H,L}, 2^{\frac{N-\mu+2}{2N-\mu}} S_{H,L})$ there exist $\delta > 0$ and an onto homeomorphism function $\psi : \mathcal{M} \to \mathcal{M}$ such that

$$\psi(\mathcal{M}_{\alpha+\delta}) \subset \mathcal{M}_{\alpha-\delta},$$

where $\mathcal{M}_\alpha = \{u \in \mathcal{M} : S_{H,L}(u, \Omega) < \alpha\}$. For a given fixed $\varepsilon > 0$ we can cover the interval $[S_{H,L} + \varepsilon, S_1]$ by finitely many such δ-intervals and composing the deformation maps, we get an onto-homeomorphism function $\psi : \mathcal{M} \to \mathcal{M}$ such that

$$\psi(\mathcal{M}_{S_1}) \subset \mathcal{M}_{S_{H,L}+\varepsilon}.$$

Also, we can assume $\psi(u) = u$ for all u whenever $S_{H,L}(u, \Omega) \le S_{H,L} + \frac{\varepsilon}{2}$.

By the concentration-compactness lemma (see [14]) and [16, Lemma 1.2], we have that for any sequence $\{u_m\} \in \mathcal{M}_{S_{H,L}+\frac{1}{m}}$ there exists a subsequence and $x^{(0)} \in \overline{\Omega}$ such that

$$\Big(\int_\Omega \frac{|(u_m)_+(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\Big)|(u_m)_+|^{2^*_\mu}\,dx \rightharpoonup \delta_{x^{(0)}}, \quad |\nabla u_m|^2\,dx \rightharpoonup S_{H,L}\delta_{x^{(0)}}$$

weakly in the sense of measure. This implies given any neighborhood V of $\overline{\Omega}$, there exists an $\varepsilon > 0$ such that $G(\mathcal{M}_{S_{H,L}}) \subset V$.

Since Ω is a smooth bounded domain, we can find a neighborhood V of $\overline{\Omega}$ such that for any $q \in V$ there exists a unique nearest neighbor $r = \pi(q) \in \overline{\Omega}$ such that the projection π is continuous. Let ε be chosen for such a neighborhood V, and let $\psi : \mathcal{M} \to \mathcal{M}$ be the corresponding onto homeomorphism. Define the map $D : \Sigma \times [0, 1] \to \overline{\Omega}$ given by

$$D(\sigma, t) = \pi(G(\psi(f_t^\sigma))).$$

It is easy to see that D is well-defined, continuous and satisfies

$$D(\sigma, 0) = \pi(G(\psi(f_0))) =: y_0 \in \overline{\Omega} \qquad \text{and} \qquad D(\sigma, 1) = \sigma \quad \text{for all } \sigma \in \Sigma.$$

This implies that D is a contraction of Σ in $\overline{\Omega}$ contradicting the hypothesis of Ω. Hence, our assumption is wrong implies that $S_{H,L}(\cdot\,,\Omega)$ has a critical value, that is, there exists a $u \in D_0^{1,2}(\Omega)$ such that u is a solution to problem (P). Now, using [15, Lemma 4.4], we have $u \in L^\infty(\Omega) \cap C^2(\overline{\Omega})$. Thus, by the maximum principle, u is a positive solution of problem (P). Hence the proof of Theorem 1.1 is complete. $\square$

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