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On a class of quasilinear eigenvalue problems on unbounded domains

By

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Abstract. We prove several existence and non-existence results for a quasilinear eigenvalue problem with nonlinear boundary condition on unbounded domain. Our paper extends previous results obtained in Chabrowski [1] and Pflüger [4].

1. Preliminaries. Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with smooth boundary Γ. We assume throughout this paper that p, q, r and α_1 are real numbers satisfying

$$1 < p < N, \quad \max\{p, 2\} < q < r < p^\star := \frac{pN}{N-p}, \quad -N < \alpha_1 < q \cdot \frac{N-p}{p} - N. \tag{1}$$

Denote by $C_\delta^\infty(\Omega)$ the space of $C_0^\infty(\mathbb{R}^N)$ – functions restricted to Ω. We define the weighted Sobolev space E as the completion of $C_\delta^\infty(\Omega)$ in the norm

$$\|u\|_E = \left(\int_\Omega |\nabla u(x)|^p + \frac{1}{(1+|x|)^p} |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Denote by $L^q(\Omega;\, w_1)$ and $L^m(\Gamma;\, w_2)$ the weighted Lebesgue spaces with respect to

$$w_i(x) = (1+|x|)^{\alpha_i}, \;\; i = 1, 2, \;\; \alpha_i \in \mathbb{R} \tag{2}$$

and norms

$$\|u\|_{q,w_1}^q = \int_\Omega w_1 |u(x)|^q \, dx \;\text{ and }\; \|u\|_{m,w_2}^m = \int_\Gamma w_2 |u(x)|^m \, d\Gamma.$$

Proposition 1. *Assume* (1) *holds. Then the embedding* $E \subset L^q(\Omega;\, w_1)$ *is compact. If*

$$p \leqq m \leqq p \cdot \frac{N-1}{N-p} \quad \textit{and} \quad -N < \alpha_2 \leqq m \cdot \frac{N-p}{p} - N + 1, \tag{3}$$

then the trace operator $E \to L^m(\Gamma;\, w_2)$ *is continuous. If the upper bounds for* m *in* (3) *are strict, then the trace operator is compact.*

This proposition is a consequence of Theorem 2 and Corollary 6 of [5].

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We assume throughout that $a \in L^{\infty}(\Omega)$ and $b \in L^{\infty}(\Gamma)$ such that

(4) $$a(x) \geqq a_0 > 0 \qquad \text{for a.e. } x \in \Omega$$

(5) $$\frac{c}{(1+|x|)^{p-1}} \leqq b(x) \leqq \frac{C}{(1+|x|)^{p-1}}, \qquad \text{for a.e. } x \in \Gamma, \text{ where } c, C > 0.$$

Lemma 1. *The quantity*

$$\|u\|_b^p = \int_{\Omega} a(x)|\nabla u|^p\, dx + \int_{\Gamma} b(x)|u|^p\, d\Gamma$$

defines an equivalent norm on E.

For the proof of this result we refer to [4], Lemma 2.

Let $h : \Omega \to \mathbb{R}$ be a positive and continuous function satisfying

(6) $$\int_{\Omega} \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\, dx < \infty.$$

We assume that $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies the following conditions:

(g1) $g(\cdot, 0) = 0$, $g(x,s) + g(x,-s) \geqq 0$ for a.e. $x \in \Gamma$ and for any $s \in \mathbb{R}$,

(g2) $|g(x,s)| \leqq g_0(x) + g_1(x)|s|^{m-1}$; $p \leqq m < p \cdot \frac{N-1}{N-p}$, where g_i are nonnegative, measurable functions such that

$$0 \leqq g_i(x) \leqq C_g w_2 \text{ a.e.}, \quad g_0 \in L^{m/(m-1)}\Big(\Gamma;\ w_2^{1/(1-m)}\Big),$$

where $-N < \alpha_2 < m \cdot \frac{N-p}{p} - N + 1$ and w_2 is defined as in (2).

Set $G(x,s) = \int_0^s g(x,t)dt$. We denote by N_g, N_G the corresponding Nemytskii operators.

Lemma 2. *The operators*

$$N_g : L^m(\Gamma;\ w_2) \to L^{m/(m-1)}\Big(\Gamma;\ w_2^{1/(1-m)}\Big), \quad N_G : L^m(\Gamma;\ w_2) \to L^1(\Gamma)$$

are bounded and continuous.

Proof. Let $m' = m/(m-1)$ and $u \in L^m(\Gamma;\ w_2)$. Then, by (g2),

$$\int_{\Gamma} |N_g(u)|^{m'} \cdot w_2^{1/(1-m)}\, d\Gamma \leqq$$
$$2^{m'-1}\bigg(\int_{\Gamma} g_0^{m'} \cdot w_2^{1/(1-m)}\, d\Gamma + \int_{\Gamma} g_1^{m'}|u|^m \cdot w_2^{1/(1-m)}\, d\Gamma\bigg) \leqq$$
$$2^{m'-1}\Big(C + \widetilde{C}_g \cdot \int_{\Gamma} |u|^m \cdot w_2\, d\Gamma\Big),$$

which shows that N_g is bounded. In a similar way we obtain

$$\int_\Gamma |N_G(u)|\, d\Gamma \leqq \int_\Gamma g_0|u|\, d\Gamma + \frac{1}{m}\int_\Gamma g_1|u|^m\, d\Gamma \leqq$$
$$\left(\int_\Gamma g_0^{m'} \cdot w_2^{1/(1-m)}\, d\Gamma\right)^{\frac{1}{m'}} \cdot \left(\int_\Gamma |u|^m \cdot w_2\, d\Gamma\right)^{\frac{1}{m}} + \frac{C_g}{m} \cdot \int_\Gamma |u|^m \cdot w_2\, d\Gamma$$

and the boundedness of N_G follows.

From the usual properties of Nemytskii operators we deduce the continuity of N_g and N_G. $\square$

Define the Banach space

$$X = \left\{u \in E : \int_\Omega h(x)|u|^r\, dx < \infty\right\}$$

endowed with the norm

$$\|u\|_X^p = \|u\|_b^p + \left(\int_\Omega h(x)|u(x)|^r\, dx\right)^{\frac{p}{r}}.$$

For $\lambda > 0$, consider the problem

$$(1_{\lambda,\theta})\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + h(x)|u|^{r-2}u = \lambda(1+|x|)^{\alpha_1}|u|^{q-2}u \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \ \Omega \subset \mathbb{R}^N, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)\cdot |u|^{p-2}u = \theta g(x,u) \quad \text{on } \ \Gamma\,, \\ u \geqq 0,\ u \not\equiv 0 \quad \text{in } \ \Omega\,. \end{cases}$$

The energy functional corresponding to $(1_{\lambda,\theta})$ is given by $\Phi : X \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{p}\int_\Omega a(x)|\nabla u|^p\, dx + \frac{1}{p}\int_\Gamma b(x)|u|^p\, d\Gamma -$$
$$\frac{\lambda}{q}\int_\Omega w_1|u|^q\, dx + \frac{1}{r}\int_\Omega h(x)|u|^r\, dx - \theta\int_\Gamma G(x,u)\, d\Gamma.$$

Proposition 1 implies that Φ is well defined. Solutions to problem $(1_{\lambda,\theta})$ will be found as critical points of Φ. Therefore, a function $u \in X$ is a solution of the problem $(1_{\lambda,\theta})$ provided that, for any $v \in X$,

$$\int_\Omega a|\nabla u|^{p-2}\nabla u \cdot \nabla v + \int_\Gamma b|u|^{p-2}uv = \lambda\int_\Omega w_1|u|^{q-2}uv - \int_\Omega h|u|^{r-2}uv + \theta\int_\Gamma gv\,.$$

Problems of this type are considered in the study of physical phenomena related to equilibrium of anisotropic continuous media which possible are somewhere "perfect" insulators, cf. [2].

2. Main results and proofs.

Theorem 1. *Assume hypotheses* (1), (4), (5), (6), (g1) *and* (g2) *hold. Then there exist real numbers* θ_*, θ^* *and* $\lambda^* > 0$ *such that Problem* $(1_{\lambda,\theta})$ *has no nontrivial solution, provided that* $\theta_* < \theta < \theta^*$ *and* $0 < \lambda < \lambda^*$.

P r o o f. Suppose that u is a solution in X of $(1_{\lambda,\theta})$. Then u satisfies

$$\begin{aligned}\int_\Omega a(x)|\nabla u|^p\,dx+\int_\Gamma b(x)|u|^p\,d\Gamma-\theta\int_\Gamma g(x,u)u\,d\Gamma+\int_\Omega h(x)|u|^r\,dx\\ =\lambda\int_\Omega w_1|u|^q\,dx.\end{aligned}\tag{7}$$

It follows from the Young inequality that

$$\begin{aligned}\lambda\int_\Omega w_1|u|^q\,dx&=\int_\Omega \frac{\lambda w_1}{h^{q/r}}\cdot h^{q/r}|u|^q\,dx\\ &\leqq \frac{r-q}{r}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx+\frac{q}{r}\int_\Omega h|u|^r\,dx.\end{aligned}$$

This combined with (7) gives

$$\begin{aligned}\|u\|_b^p-\theta\int_\Gamma g(x,u)u\,d\Gamma&\leqq \frac{r-q}{r}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx+\frac{q-r}{r}\int_\Omega h|u|^r\,dx\\ &\leqq \frac{r-q}{r}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx.\end{aligned}\tag{8}$$

Set

$$A=\left\{u\in X:\int_\Gamma g(x,u)u\,d\Gamma<0\right\},\qquad B=\left\{u\in X:\int_\Gamma g(x,u)u\,d\Gamma>0\right\}$$

$$\theta_*=\sup_{u\in A}\frac{\|u\|_b^p}{\int_\Gamma g(x,u)u\,d\Gamma},\qquad \theta^*=\inf_{u\in B}\frac{\|u\|_b^p}{\int_\Gamma g(x,u)u\,d\Gamma}\tag{9}$$

We introduce the convention that if $A=\emptyset$ then $\theta_*=-\infty$ and if $B=\emptyset$ then $\theta^*=+\infty$.

We show that if we take $\theta_*<\theta<\theta^*$ then there exists $C_0>0$ such that

$$C_0\|u\|_b^p\leqq \|u\|_b^p-\theta\int_\Gamma g(x,u)u\,d\Gamma\quad\text{for all } u\in X.\tag{10}$$

If $\theta<\theta^*$ then there exists a constant $C_1\in(0,1)$ such that

$$\theta\leqq(1-C_1)\theta^*\leqq(1-C_1)\frac{\|u\|_b^p}{\int_\Gamma g(x,u)u\,d\Gamma}\quad\text{for all } u\in B$$

which implies

$$\|u\|_b^p-\theta\int_\Gamma g(x,u)u\,d\Gamma\geqq C_1\|u\|_b^p\quad\text{for all } u\in B.\tag{11}$$

If $\theta_*<\theta$ then there exists a constant $C_2\in(0,1)$ such that

$$(1-C_2)\frac{\|u\|_b^p}{\int_\Gamma g(x,u)u\,d\Gamma}\leqq(1-C_2)\theta_*\leqq\theta\quad\text{for all } u\in A$$

which yields

$$\|u\|_b^p-\theta\int_\Gamma g(x,u)u\,d\Gamma\geqq C_2\|u\|_b^p\quad\text{for all } u\in A.\tag{12}$$

From (11) and (12) we conclude that

$$\|u\|_b^p - \theta\int_\Gamma g(x,u)u\,d\Gamma \geqq \min\{C_1, C_2\}\|u\|_b^p \quad \text{for all } u \in X$$

and taking $C_0 = \min\{C_1, C_2\}$ we obtain (10).

By (7), (10) and Proposition 1 we have

$$C_0\overline{C}\left(\int_\Omega w_1|u|^q\,dx\right)^{\frac{p}{q}} \leqq C_0\|u\|_b^p \leqq \lambda\int_\Omega w_1|u|^q\,dx, \tag{13}$$

for some constant $\overline{C} > 0$. This inequality implies

$$(\overline{C}\lambda^{-1}C_0)^{q/(q-p)} \leqq \int_\Omega w_1|u|^q\,dx$$

which combined with (13) yields

$$C_0\overline{C}(\overline{C}\lambda^{-1}C_0)^{p/(q-p)} \leqq C_0\|u\|_b^p.$$

Combining this with (8) and (10) we obtain

$$C_0\overline{C}(\overline{C}\lambda^{-1}C_0)^{p/(q-p)} \leqq \frac{r-q}{r}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx.$$

If we take

$$\lambda^* = \left((C_0\overline{C})^{q/(q-p)}\frac{r}{r-q}\left(\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx\right)^{-1}\right)^{\frac{(r-q)(q-p)}{q(r-p)}}$$

the result follows. □

Set

$$U = \left\{u \in X : \int_\Gamma G(x,u)\,d\Gamma < 0\right\}, \qquad V = \left\{u \in X : \int_\Gamma G(x,u)\,d\Gamma > 0\right\}$$

$$\theta_- = \sup_{u\in U}\frac{\|u\|_b^p}{p\int_\Gamma G(x,u)\,d\Gamma}, \qquad \theta^+ = \inf_{u\in V}\frac{\|u\|_b^p}{p\int_\Gamma G(x,u)\,d\Gamma} \tag{14}$$

If $U = \emptyset$ (resp. $V = \emptyset$) then we set $\theta_- = -\infty$ (resp. $\theta^+ = +\infty$). Proceeding in the same manner as we did for proving (10) we can show that if we take $\theta_- < \theta < \theta^+$ then there exists $c > 0$ such that

$$\frac{1}{p}\|u\|_b^p - \theta\int_\Gamma G(x,u)\,d\Gamma \geqq c\|u\|_b^p \qquad \text{for all } u \in X. \tag{15}$$

We shall employ in what follows the following elementary inequality

$$k|u|^\beta - h|u|^\gamma \leqq C_{\beta,\gamma}k\left(\frac{k}{h}\right)^{\beta/(\gamma-\beta)} \qquad \forall u \in \mathbb{R},\ \forall h, k \in (0,\infty),\ \forall 0 < \beta < \gamma\,. \tag{16}$$

Proposition 2. *If $\theta_- < \theta < \theta^+$ then the functional Φ is coercive.*

Proof. By virtue of (16) we have

$$\int_\Omega \left(\frac{\lambda}{q}|u|^q w_1 - \frac{h}{2r}|u|^r\right) dx \leqq C_{r,q}\int_\Omega \lambda w_1 \left(\frac{\lambda w_1}{h}\right)^{q/(r-q)} dx$$
$$= C_{r,q}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}dx.$$

Using (15) it follows that

$$\Phi(u) = \frac{1}{p}\|u\|_b^p - \theta\int_\Gamma G(x,u)\,d\Gamma - \int_\Omega \left(\frac{\lambda}{q}|u|^q w_1 - \frac{h}{2r}|u|^r\right) dx + \frac{1}{2r}\int_\Omega h|u|^r\,dx$$
$$\geqq c\|u\|_b^p + \frac{1}{2r}\int_\Omega h|u|^r\,dx - C_1$$

and the coercivity of Φ follows. □

Proposition 3. *Suppose $\theta_- < \theta < \theta^+$ and let $\{u_n\}$ be a sequence in X such that $\Phi(u_n)$ is bounded. Then there exists a subsequence of $\{u_n\}$, relabelled again by $\{u_n\}$, such that $u_n \rightharpoonup u_0$ in X and*

$$\Phi(u_0) \leqq \liminf_{n\to\infty} \Phi(u_n).$$

Proof. Since Φ is coercive in X we see that the boundedness of $\Phi(u_n)$ implies that $\|u_n\|_b$ and $\int_\Omega h|u_n|^r\,dx$ are bounded. From Proposition 1 we know that the embedding $E \subset L^q(\Omega;\ w_1)$ is compact and using the fact that $\{u_n\}$ is bounded in E we may assume that $u_n \rightharpoonup u_0$ in E and $u_n \to u_0$ in $L^q(\Omega;\ w_1)$.

Set $F(x,u) = \frac{\lambda}{q}|u|^q w_1 - \frac{1}{r}h|u|^r$ and $f(x,u) = F_u(x,u)$. A simple computation yields

$$f_u(x,u) = (q-1)\lambda|u|^{q-2}w_1 - (r-1)h|u|^{r-2} \leqq C_{r,q}\lambda w_1\left(\frac{\lambda w_1}{h}\right)^{(q-2)/(r-q)} \tag{17}$$

where the last inequality follows from (16). We obtain

$$\Phi(u_0)-\Phi(u_n) = \tfrac{1}{p}\int_\Omega a(x)|\nabla u_0|^p\,dx + \tfrac{1}{p}\int_\Gamma b(x)|u_0|^p\,d\Gamma - \tfrac{1}{p}\int_\Omega a(x)|\nabla u_n|^p\,dx -$$
$$\tfrac{1}{p}\int_\Gamma b(x)|u_n|^p\,d\Gamma - \theta\int_\Gamma G(x,u_0)\,d\Gamma + \theta\int_\Gamma G(x,u_n)\,d\Gamma + \int_\Omega (F(x,u_n) -$$
$$F(x,u_0))\,dx = \tfrac{1}{p}\left(\|u_0\|_b^p - \|u_n\|_b^p\right) + \theta\left(\int_\Gamma G(x,u_n)\,d\Gamma - \int_\Gamma G(x,u_0)\,d\Gamma\right) +$$
$$\int_\Omega\left(\int_0^1\int_0^s f_u(x,u_0+t(u_n-u_0))\,dt\,ds\right)(u_n-u_0)^2\,dx \leqq \tfrac{1}{p}\left(\|u_0\|_b^p - \|u_n\|_b^p\right) +$$
$$\theta\left(\int_\Gamma G(x,u_n)\,d\Gamma - \int_\Gamma G(x,u_0)\,d\Gamma\right) + C_2\int_\Omega (u_n-u_0)^2\frac{w_1^{(r-2)/(r-q)}}{h^{(q-2)/(r-q)}}dx,$$

where $C_2 = \frac{1}{2}C_{r,q}\lambda^{(r-2)/(r-q)}$. We show that the last integral tends to 0 as $n\to\infty$. Indeed, applying Hölder's inequality we obtain

$$\int_\Omega (u_n-u_0)^2\frac{w_1^{(r-2)/(r-q)}}{h^{(q-2)/(r-q)}}\,dx \leqq \left(\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx\right)^{(q-2)/q}\cdot\left(\int_\Omega w_1|u_n-u_0|^q\,dx\right)^{\frac{2}{q}}.$$

Since $u_n \to u_0$ in $L^q(\Omega;\ w_1)$ we have

$$\lim_{n\to\infty} \int_\Omega (u_n - u_0)^2 \frac{w_1^{(r-2)/(r-q)}}{h^{(q-2)/(r-q)}}\, dx = 0. \tag{18}$$

The compactness of the trace operator $E \to L^m(\Gamma;\ w_2)$ and the continuity of the Nemytskii operator $N_G : L^m(\Gamma;\ w_2) \to L^1(\Gamma)$ imply $N_G(u_n) \to N_G(u_0)$ in $L^1(\Gamma)$ i.e. $\int_\Gamma |N_G(u_n) - N_G(u_0)|\, d\Gamma \to 0$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \int_\Gamma G(x, u_n)\, d\Gamma = \int_\Gamma G(x, u_0)\, d\Gamma. \tag{19}$$

Since the norm in E is lower semicontinuous with respect to the weak topology our conclusion follows from (18) and (19). $\square$

Proposition 4. *If $\theta_* < \theta < \theta^*$ and u is a solution of Problem $(1_{\lambda,\theta})$, then*

$$C_0 \|u\|_b^p + \frac{r-q}{r} \int_\Omega h|u|^r\, dx \leqq \frac{r-q}{r} \lambda^{r/(r-q)} \int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\, dx$$

and

$$\|u\|_b \geqq K\lambda^{-1/(q-p)},$$

where $K > 0$ is a constant independent of u.

Proof. If u is a solution of $(1_{\lambda,\theta})$ then

$$\|u\|_b^p - \theta \int_\Gamma g(x,u)u\, d\Gamma + \int_\Omega h|u|^r\, dx = \lambda \int_\Omega w_1 |u|^q\, dx \leqq$$

$$\frac{r-q}{r} \lambda^{r/(r-q)} \int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\, dx + \frac{q}{r} \int_\Omega h|u|^r\, dx.$$

Using (10) we obtain the first part of the assertion.

From Proposition 1 we have that there exists $C_q > 0$ such that

$$\|u\|^q_{L^q(\Omega;\, w_1)} \leqq C_q \|u\|_b^q, \quad \text{for all } u \in E.$$

This inequality and (10) imply

$$\|u\|_b \geqq C_0^{1/(q-p)} C_q^{-1/(q-p)} \lambda^{-1/(q-p)}$$

and taking $K = C_0^{1/(q-p)} C_q^{-1/(q-p)}$ the second part follows. $\square$

Theorem 2. *Assume hypotheses* (1), (4), (5), (6), (g1) *and* (g2) *hold. Set $\underline{\theta} = \max\{\theta_*, \theta_-\}$, if $g(x,\cdot)$ is odd, and $\underline{\theta} = 0$ elsewhere, $\overline{\theta} = \min\{\theta^*, \theta^+\}$. Suppose that $J = (\underline{\theta}, \overline{\theta}) \neq \Phi$. There exists $\lambda_0 > 0$ such that the following hold:*

(i) *Problem $(1_{\lambda,\theta})$ admits a nontrivial solution, for any $\lambda \geqq \lambda_0$ and every $\theta \in J$;*
(ii) *Problem $(1_{\lambda,\theta})$ does not have any nontrivial solution, provided that $0 < \lambda < \lambda_0$ and $\theta \in J$.*

Proof. According to Propositions 2 and 3, Φ is coercive and lower semicontinuous. Therefore there exists $\tilde{u} \in X$ such that $\Phi(\tilde{u}) = \inf_X \Phi(u)$. To ensure that $\tilde{u} \not\equiv 0$ we shall prove that $\inf_X \Phi < 0$. Set

$$\tilde{\lambda} := \inf\left\{\frac{q}{p}\|u\|_b^p - q\theta\int_\Gamma G(x,u)\,d\Gamma + \frac{q}{r}\int_\Omega h|u|^r\,dx : u \in X, \int_\Omega w_1|u|^q\,dx = 1\right\}.$$

First we check that $\tilde{\lambda} > 0$. For this aim we consider the constrained minimization problem

$$M := \inf\left\{\int_\Omega a(x)|\nabla u|^p\,dx + \int_\Gamma b(x)|u|^p\,d\Gamma : u \in E, \int_\Omega w_1|u|^q\,dx = 1\right\}.$$

Clearly, $M > 0$. Since X is embedded in E, we have

$$\int_\Omega a(x)|\nabla u|^p\,dx + \int_\Gamma b(x)|u|^p\,d\Gamma \geqq M$$

for all $u \in X$ with $\int_\Omega w_1|u|^q\,dx = 1$. Now, applying the Hölder inequality we find

$$1 = \int_\Omega w_1|u|^q\,dx = \int_\Omega \frac{w_1}{h^{q/r}}h^{q/r}|u|^q\,dx \leqq \left(\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx\right)^{(r-q)/r} \cdot \left(\int_\Omega h|u|^r\,dx\right)^{\frac{q}{r}}. \tag{20}$$

Relation (15) implies

$$\frac{q}{p}\|u\|_b^p - q\,\theta\int_\Gamma G(x,u)\,d\Gamma \geqq q\,c\|u\|_b^p.$$

By virtue of (20) we have

$$\frac{q}{p}\|u\|_b^p - q\,\theta\int_\Gamma G(x,u)\,d\Gamma + \frac{q}{r}\int_\Omega h|u|^r\,dx \geqq qc\|u\|_b^p + \frac{q}{r}\int_\Omega h|u|^r\,dx \geqq$$

$$qcM + \frac{q}{r}\left(\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx\right)^{-(r-q)/q}$$

for all $u \in X$ with $\int_\Omega w_1|u|^q\,dx = 1$. It follows that

$$\tilde{\lambda} \geqq qcM + \frac{q}{r}\left(\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx\right)^{-(r-q)/q}$$

and our claim follows.

Let $\lambda > \tilde{\lambda}$. Then there exists a function $u \in X$ with $\int_\Omega w_1|u|^q\,dx = 1$ such that

$$\lambda > \frac{q}{p}\|u\|_b^p - q\theta\int_\Gamma G(x,u)\,d\Gamma + \frac{q}{r}\int_\Omega h|u|^r\,dx.$$

This can be rewritten as

$$\Phi(u) = \frac{1}{p}\|u\|_b^p - \theta\int_\Gamma G(x,u)\,d\Gamma + \frac{1}{r}\int_\Omega h|u|^r\,dx - \frac{\lambda}{q}\int_\Omega w_1|u|^q\,dx < 0$$

and consequently $\inf_{u\in X}\Phi(u) < 0$. By Propositions 2 and 3 it follows that the problem $(1_{\lambda,\theta})$ has a solution.

We set

$$\lambda_0 = \inf\{\lambda > 0 : (1_{\lambda,\theta}) \text{ admits a solution}\}.$$

Suppose $\lambda_0 = 0$. Then taking $\lambda_1 \in (0, \lambda^*)$ (where λ^* is given by Theorem 1) we have that there is $\bar{\lambda}$ such that the problem $(1_{\bar{\lambda},\theta})$ admits a solution. But this is a contradiction, according to Theorem 1. Consequently, $\lambda_0 > 0$.

We now show that for each $\lambda > \lambda_0$ problem $(1_{\lambda,\theta})$ admits a solution. Indeed, for every $\lambda > \lambda_0$ there exists $\rho \in (\lambda_0, \lambda)$ such that problem $(1_{\rho,\theta})$ has a solution u_ρ which is a subsolution of problem $(1_{\lambda,\theta})$. We consider the variational problem

$$\inf\{\Phi(u) : u \in X \text{ and } u \geqq u_\rho\}.$$

By Propositions 2 and 3 this problem admits a solution $\bar{u}$. This minimizer $\bar{u}$ is a solution of problem $(1_{\lambda,\theta})$. Since the hypothesis $g(x,s) + g(x,-s) \geqq 0$ for a.e. $x \in \Gamma$ and for all $s \in \mathbb{R}$ implies that $G(x, |\bar{u}|) \geqq G(x, \bar{u})$ (that is, $\Phi(|\bar{u}|) \leqq \Phi(\bar{u})$) we may assume that $\bar{u} \geqq 0$ on Ω. It remains to show that problem $(1_{\lambda_0,\theta})$ has also a solution. Let $\lambda_n \to \lambda_0$ and $\lambda_n > \lambda_0$ for each n. Problem $(1_{\lambda_n,\theta})$ has a solution u_n for each n. By Proposition 4 the sequence $\{u_n\}$ is bounded in X. Therefore we may assume that $u_n \rightharpoonup u_0$ in X and $u_n \to u_0$ in $L^q(\Omega;\, w_1)$. We have that u_0 is a solution of $(1_{\lambda_0}, \theta)$. Since u_n and u_0 are solutions of $(1_{\lambda_n,\theta})$ and $(1_{\lambda_0,\theta})$, respectively, we have

$$\begin{aligned}
&\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0)(\nabla u_n - \nabla u_0)\,dx + \\
&\int_\Gamma b(x)(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0)(u_n - u_0)\,d\Gamma + \\
&\int_\Omega h(|u_n|^{r-2}u_n - |u_0|^{r-2}u_0)(u_n - u_0)\,dx = \\
&\lambda_n \int_\Omega w_1(|u_n|^{q-2}u_n - |u_0|^{q-2}u_0)(u_n - u_0)\,dx + \\
&(\lambda_n - \lambda_0)\int_\Omega w_1|u_0|^{q-2}u_0(u_n - u_0)\,dx + \\
&\theta\int_\Gamma (g(x,u_n) - g(x,u_0))(u_n - u_0)\,d\Gamma = J_{1,n} + J_{2,n} + J_{3,n},
\end{aligned}$$

where

$$\begin{aligned}
J_{1,n} &= \lambda_n \int_\Omega w_1(|u_n|^{q-2}u_n - |u_0|^{q-2}u_0)(u_n - u_0)\,dx,\\
J_{2,n} &= (\lambda_n - \lambda_0)\int_\Omega w_1|u_0|^{q-2}u_0(u_n - u_0)\,dx,\\
J_{3,n} &= \theta\int_\Gamma (g(x,u_n) - g(x,u_0))(u_n - u_0)\,d\Gamma.
\end{aligned}$$

We have

$$|J_{1,n}| \leqq \sup_{n \geqq 1} \lambda_n \left(\int_\Omega w_1|u_n|^{q-1}|u_n - u_0|\,dx + \int_\Omega w_1|u_0|^{q-1}|u_n - u_0|\,dx \right)$$

and it follows from the Hölder inequality that

$$\begin{aligned}
|J_{1,n}| \leqq \sup_{n \geqq 1} \lambda_n \Bigg[&\left(\int_\Omega w_1|u_n|^q\,dx \right)^{(q-1)/q} \cdot \left(\int_\Omega w_1|u_n - u_0|^q\,dx \right)^{\frac{1}{q}} + \\
&\left(\int_\Omega w_1|u_0|^q\,dx \right)^{(q-1)/q} \cdot \left(\int_\Omega w_1|u_n - u_0|^q\,dx \right)^{\frac{1}{q}} \Bigg].
\end{aligned}$$

We easily observe that $J_{1,n} \to 0$ as $n \to \infty$.

From the estimate

$$|J_{2,n}| \leqq |\lambda_n - \lambda_0| \left(\int_\Omega w_1|u_0|^q\,dx \right)^{(q-1)/q} \cdot \left(\int_\Omega w_1|u_n - u_0|^q\,dx \right)^{\frac{1}{q}}$$

we obtain that $J_{2,n} \to 0$ as $n \to \infty$.

Using the compactness of the trace operator $E \to L^m(\Gamma;\ w_2)$, the continuity of Nemytskii operator $N_g : L^m(\Gamma;\ w_2) \to L^{m/(m-1)}(\Gamma;\ w_2^{1/(1-m)})$ and the estimate

$$\int_\Gamma |g(x,u_n) - g(x,u_0)| \cdot |u_n - u_0|\, d\Gamma \leqq$$

$$\left(\int_\Gamma |g(x,u_n) - g(x,u_0)|^{m/(m-1)} w_2^{1/(1-m)}\, d\Gamma\right)^{(m-1)/m} \cdot \left(\int_\Gamma w_2 |u_n - u_0|^m\, d\Gamma\right)^{\frac{1}{m}}$$

we see that $J_{3,n} \to 0$ as $n \to \infty$.

We have so proved that

$$\lim_{n\to\infty} \left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0)(\nabla u_n - \nabla u_0)\, dx + \right.$$

$$\left.\int_\Gamma b(x)(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0)(u_n - u_0)\, d\Gamma\right) = 0.$$

Applying the inequality (see [3], Lemma 4.10)

$$|\xi - \zeta|^p \leqq C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta), \qquad \forall \xi, \zeta \in \mathbb{R}^N \ \ \forall p \geqq 2$$

we find

$$\|u_n - u_0\|_b^p = \int_\Omega a(x)|\nabla u_n - \nabla u_0|^p\, dx + \int_\Gamma b(x)|u_n - u_0|^p\, dx \leqq$$

$$C\left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0)(\nabla u_n - \nabla u_0)\, dx + \right.$$

$$\left.\int_\Gamma b(x)(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0)(u_n - u_0)\, d\Gamma\right) \to 0 \text{ as } n \to \infty$$

which shows that $\|u_n\|_b \to \|u_0\|_b$. If $1 < p < 2$ we use a similar argument based on the inequality $|\xi - \zeta|^2 \leqq C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p}$. By Proposition 4 we have $u_0 \not\equiv 0$. This concludes our proof. $\square$

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