MULTIPLE SOLUTIONS WITH SIGN INFORMATION FOR DOUBLE PHASE PROBLEMS WITH UNBALANCED GROWTH

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ABSTRACT. We consider a double phase Dirichlet problem with unbalanced growth and a reaction term which is $(p-1)$-sublinear and has partial interaction with the first eigenvalue of the weighted differential operator Δ_p^a (nonuniform nonresonance). Using the Nehari method, we show that the problem has at least three nontrivial bounded solutions, positive, negative and nodal (sign-changing). This paper extends and complements the main results in the recent paper Papageorgiou-Pudelko-Rădulescu (Nonautonomous (p,q)-equations with unbalanced growth, *Math. Annalen*, 2023).

KEY WORDS: Musielak-Orlicz-Sobolev space, nonuniform nonresonance, Nehari manifold, constant sign solutions, nodal solution.

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1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. We are concerned with the following double phase Dirichlet problem

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ 1 < q < p < N. \end{cases} \tag{1}$$

In this problem, Δ_p^a denotes the weighted p-Laplace operator with weight $a \in L^\infty(\Omega)_+ \backslash \{0\}$, which is defined by

$$\Delta_p^a u = \operatorname{div}(a(z)|Du|^{p-2}Du).$$

Also, Δ_q is the usual q-Laplacian defined by

$$\Delta_q u = \operatorname{div}(|Du|^{q-2}Du).$$

The interest in the study of problems of this type is twofold. From one side, there are motivations from mathematical physics, since the non-autonomous unbalanced operator has been applied to describe steady-state solutions of reaction-diffusion equations arising in biophysics, plasma physics, and chemical reaction analysis. The prototype equation for these models can be written in the form

$$u_t = \Delta_p^a u(z) + \Delta_q u + g(x, u).$$

In this framework, the function u generally stands for a concentration, the term $\Delta_p^a u(z) + \Delta_q u$ corresponds to the diffusion with coefficient $a(z)|Du|^{p-2} + |Du|^{q-2}$, while $g(x,u)$ represents the reaction term related to source and loss processes. On the other hand, such operators provide a valuable framework for explaining the behavior of highly anisotropic materials whose hardening properties, which are linked to the exponent governing the propagation of the gradient variable, differ considerably with the point, where the modulating coefficient $a(z)$ dictates the geometry of a composite made by two different materials.

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Problem (1) is driven by the sum of these two operators. Since $q < p$, the differential operator in (1) is not homogeneous. The differential operator is related to the so-called "double phase" integral functional

$$i(u) = \int_\Omega [a(z)|Du|^p + |Du|^q]\, dz.$$

The integrand of this variational integral is the function

$$\eta(z,t) = a(z)t^p + t^q \text{ for all } z \in \Omega, \text{ all } t \geq 0.$$

A feature of this paper is that we do not assume that the weight function $a(\cdot)$ is bounded away from zero, that is, we do not require $\operatorname{ess\,inf}_\Omega a > 0$. This implies that the integrand $\eta(z,t)$ exhibits unbalanced growth in the t-variable, namely we have

$$t^q \leq \eta(z,t) \leq c_0[1 + t^p] \text{ for all } z \in \Omega, \text{ all } t \geq 0, \text{ some } c_0 > 0.$$

This kind of growth behavior, is often called in the literature as "(p,q)- growth". Such functionals were first investigated by Marcellini [11], [12] and Zhikov [28], in the context of problems of the calculus of variations and of nonlinear elasticity theory. The unbalanced growth of $\eta(z,\cdot)$ has serious implications in the analysis of double phase equations. The functional framework of the problem changes and instead of the classical Sobolev spaces, it is based on generalized Orlicz spaces. In contrast to balanced growth double phase problems ((p,q)-equations, both weighted and unweighted), for which there is a global (that is, up to the boundary) regularity theory (see Lieberman [8]), for unbalanced growth double phase problems no such theory exists. There are only local regularity results for local minimizers. We refer to the recent remarkable works of Baroni-Colombo-Mingione [1], Marcellini [13], [14], Ragusa-Tachikawa [26] and the references therein. We also mention the survey papers of Mingione-Rădulescu [15], Papageorgiou [16], Rădulescu [25].

The lack of a global regularity theory, eliminates many powerful tools which are available in the treatment of balanced growth double phase problems (such as the nonlinear Hopf maximum principle and the comparison of Hölder and Sobolev local minimizers of the energy functionals, see for example, Papageorgiou-Rădulescu-Zhang [21, Proposition A3]). So, we are forced to use a different approach which is based on the Nehari method (ground state solutions).

In problem (1), the reaction function $f(z,x)$ is a measurable function and for a.a. $z \in \Omega$, $f(z,\cdot)$ is continuously differentiable on $\mathbb{R} \setminus \{0\}$ and as $x \to \pm\infty$ exhibits $(p-1)$-sublinear growth and we allow partial interaction with the first eigenvalue of $-\Delta_p^a$ with Dirichlet boundary condition (nonuniform nonresonance). We prove a multiplicity theorem producing three nontrivial bounded solutions and we provide sign information for all them (positive, negative and nodal). In the past, multiplicity theorems for unbalanced growth double phase problems were proved under the assumption that the reaction $f(z,\cdot)$ is $(p-1)$-superlinear as $x \to \pm\infty$. We refer to the works of Deregowska-Gasiński-Papageorgiou [4], Gasiński-Papageorgiou [5], Gasiński-Winkert [6], Liu-Dai [9], Papageorgiou-Rădulescu-Zhang [19] and Papageorgiou-Zhang [23]. Only the very recent work of Papageorgiou-Pudelko-Rădulescu [17], considers problems with $(p-1)$-sublinear reaction and produces two nontrivial solutions with no sign information. Finally, we mention also the recent works on double phase equations with unilateral constraints by Liu-Papageorgiou [10], Papageorgiou-Zhang-Zhang [24], Zeng-Bai-Gasiński-Winkert [27] (existence of nontrivial solutions).

The double-phase problem (1) is motivated by numerous models arising in mathematical physics. For instance, we can refer to the following fourth-order relativistic operator

$$u \mapsto \operatorname{div}\left(\frac{|\nabla u|^2}{(1-|\nabla u|^4)^{3/4}}\,\nabla u\right),$$

which describes large classes of phenomena arising in relativistic quantum mechanics. By Taylor's formula, we have

$$x^2(1-x^4)^{-3/4} = x^2 + \frac{3x^6}{4} + \frac{21x^{10}}{32} + \cdots .$$

This shows that the fourth-order relativistic operator can be approximated by the following autonomous double phase operator

$$u \mapsto \Delta_4 u + \frac{3}{4}\Delta_8 u.$$

The features of the present paper are the following:

(i) The source term of problem (1) is driven by a differential operator with a power-type nonhomogeneous term.

(ii) The corresponding energy functional is a non-autonomous variational integral that satisfies nonstandard growth conditions of (p, q)-type, following the terminology introduced in the basic papers of Marcellini [11, 12, 13, 14].

(iii) The potential that describes the differential operator satisfies general regularity assumptions and it belongs to the p-Muckenhoupt class. Accordingly, the thorough spectral and the qualitative analysis contained in this paper are developed in Musielak-Orlicz-Sobolev spaces.

(iv) The paper covers the nonuniform nonresonance case. The main result establishes the existence of three solutions (all with sign information) and an interesting open problem is whether this result remains valid for the resonance case.

2. Mathematical background and hypotheses

As we already mentioned in the introduction, the functional framework for the analysis of problem (1) is based on the generalized Orlicz spaces. A comprehensive presentation of the theory of these spaces can be found in the book of Harjulehto-Hästö [7].

Let $C^{0,1}(\bar{\Omega})$ be the space of all $\mathbb{R}$-valued Lipschitz continuous function on $\bar{\Omega}$. Let A_p denote the class of p-Muckenhoupt weights (see Cruz Uribe-Fiorenzo [3, p.142] and Harjulehto-Hästö [7, p.114]).

Our hypotheses on the weight function $a(\cdot)$ and the exponents p, q are the following:

H_0 : $a \in C^{0,1}(\bar{\Omega}) \cap A_p$, $a(z) > 0$ for all $z \in \Omega$, $1 < q < p < N$, $\frac{p}{q} < 1 + \frac{1}{N}$.

Remark 1. *The last inequality in the above hypotheses is standard in Dirichlet double phase problems and it says that the two exponents p and q can not be far part. It implies that $p < q^* = \frac{Nq}{N-q}$ and this leads to some useful compact embeddings of certain relevant function spaces. Moreover, this condition together with the Lipschitz continuity of the potential $a(\cdot)$ implies that the Poincaré inequality is valid for the corresponding Musielak-Orlicz-Sobolev space.*

Let $L^0(\Omega)$ denote the space of all measurable functions $u : \Omega \to \mathbb{R}$. As usual, we identify two such functions which differ only on a Lebesgue-null set. Recall that

$$\eta(z,t) = a(z)t^p + t^q \text{ for all } z \in \Omega, \text{ all } t \geq 0.$$

The generalized Orlicz-Lebesgue space, is defined by

$$L^\eta(\Omega) = \{u \in L^0(\Omega) : \rho_\eta(u) < \infty\}$$

with $\rho_\eta(u)$ being the modular function defined by

$$\rho_\eta(u) = \int_\Omega \eta(z, |u|)\mathrm{d}z.$$

We endow $L^\eta(\Omega)$ with the so-called "Luxemburg norm" defined by

$$\|u\|_\eta = \inf\left\{\lambda > 0 : \rho_\eta(\frac{u}{\lambda}) \le 1\right\}.$$

Evidently, this is the Minkowski functional of the set $\{u \in L^\eta(\Omega) : \rho_\eta(u) \le 1\}$. With this norm $L^\eta(\Omega)$ becomes a separable Banach space which is also uniformly convex, hence reflexive (the function $\eta(z,\cdot)$ is uniformly convex).

Using $L^\eta(\Omega)$ we can define the corresponding generalized Musielak-Orlicz-Sobolev space $W^{1,\eta}(\Omega)$ by

$$W^{1,\eta}(\Omega) = \{u \in L^\eta(\Omega) : |Du| \in L^\eta(\Omega)\},$$

where Du denotes the weak gradient of u. This space is endowed with the following norm

$$\|u\|_{1,\eta} = \|u\|_\eta + \|Du\|_\eta \text{ with } \|Du\|_\eta = \||Du|\|_\eta.$$

Also

$$W_0^{1,\eta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\eta}}.$$

For this space the Poincaré inequality holds, namely there exists $\hat{c} = \hat{c}(\Omega) > 0$ such that

$$\|u\|_\eta \le \hat{c}\|Du\|_\eta \text{ for all } u \in W_0^{1,\eta}(\Omega).$$

Therefore on the space $W_0^{1,\eta}(\Omega)$ we can consider the equivalent norm

$$\|u\| = \|Du\|_\eta \text{ for all } u \in W_0^{1,\eta}(\Omega).$$

We have the following useful embeddings among the spaces introduced above.

Proposition 1. *If hypotheses H_0 hold, then*

(a) $L^\eta(\Omega) \hookrightarrow L^r(\Omega)$ *and* $W_0^{1,\eta}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ *continuously for all* $r \in [1,q]$;
(b) $W_0^{1,\eta}(\Omega) \hookrightarrow L^r(\Omega)$ *continuously (resp. compactly) for all* $r \in [1,q^*]$ *(all* $r \in [1,q^*)$*)*;
(c) $L^p(\Omega) \hookrightarrow L^\eta(\Omega)$ *continuously.*

There is a close relation between the norm $\|\cdot\|$ and the modular function $\rho_\eta(\cdot)$.

Proposition 2. *If hypotheses H_0 hold, then*

(a) $\|u\| = \lambda \Leftrightarrow \rho_\eta(\frac{Du}{\lambda}) = 1$;
(b) $\|u\| < 1$ *(resp* $= 1, > 1$*)* $\Leftrightarrow \rho_\eta(u) < 1 (= 1, > 1)$;
(c) $\|u\| < 1 \Rightarrow \|u\|^p \le \rho_\eta(Du) \le \|u\|^q$;
(d) $\|u\| > 1 \Rightarrow \|u\|^q \le \rho_\eta(Du) \le \|u\|^p$;
(e) $\|u\| \to 0$ *(resp* $\to \infty$*)* $\Leftrightarrow \rho_\eta(Du) \to 0$ *(resp* $\to \infty$*)*.

Let $V : W_0^{1,\eta}(\Omega) \to W_0^{1,\eta}(\Omega)^*$ be defined by

$$\langle V(u), h\rangle = \int_\Omega \left[a(z)|Du|^{p-2}(Du,Dh)_{\mathbb{R}^N} + |Du|^{q-2}(Du,Dh)_{\mathbb{R}^N}\right] dz \text{ for all } u,h \in W_0^{1,\eta}(\Omega).$$

This operator is bounded (maps bounded sets to bounded sets), continuous, strictly monotone (thus, maximal monotone too) and coercive.

Let $\eta_0(z,t)$ be the following integrand

$$\eta_0(z,t) = a(z)t^p \text{ for all } z \in \Omega, \text{ all } t \ge 0.$$

We consider the corresponding generalized Orlicz spaces $L^{\eta_0}(\Omega)$ and $W_0^{1,\eta_0}(\Omega)$. These are separable Banach spaces which are uniformly convex (hence reflexive too). Moreover, from Papageorgiou-Rădulescu-Zhang [20, Lemma 2.1], we have that

$$W_0^{1,\eta_0}(\Omega) \hookrightarrow L^{\eta_0}(\Omega) \text{ compactly.} \tag{2}$$

We consider the following nonlinear eigenvalue problem

$$-\Delta_p^a u(z) = \widehat{\lambda} a(z)|u(z)|^{p-2}u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{3}$$

Problem (3) was examined recently by Papageorgiou-Pudelko-Rădulescu [17]. Using (2) they proved among other things, that there is a smallest eigenvalue $\widehat{\lambda}_1 > 0$ which is simple and isolated in the spectrum of (3). Moreover, $\widehat{\lambda}_1 > 0$ has the following variational characterization

$$\widehat{\lambda}_1 = \inf\left\{\frac{\rho_{\eta_0}(Du)}{\rho_{\eta_0}(u)} : u \in W_0^{1,\eta_0}(\Omega), u \neq 0\right\}. \tag{4}$$

As we already mentioned, the corresponding eigenspace is one-dimensional with elements in $W_0^{1,\eta_0}(\Omega) \cap L^\infty(\Omega)$ which have fixed sign, more precisely, if $\widehat{u}$ is an eigenfunction corresponding to $\widehat{\lambda}_1 > 0$, then $\widehat{u}(z) > 0$ or $\widehat{u}(z) < 0$ for a.a. $z \in \Omega$. In fact, $\widehat{\lambda}_1$ is the only eigenvalue with eigenfunctions of constant sign. All other eigenvalues have eigenfunctions which are nodal (sign-changing).

We will also use some tools from nonsmooth analysis and in particular the subdifferential theory of Clarke [2]. So, let X be a Banach space and $\varphi : X \to \mathbb{R}$ a locally Lipschitz function. The generalized directional derivative of $\varphi(\cdot)$ at $x \in X$ in the direction $h \in X$ is defined by

$$\varphi^0(x;h) = \limsup_{x' \to +x, \lambda \downarrow 0} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

Then $\varphi^0(x;\cdot)$ is sublinear, continuous and by the Hahn-Banach theorem, we can define the nonempty, convex, w^*-compact set

$$\partial\varphi(x) = \left\{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x;h) \text{ for all } h \in X\right\}.$$

The multifunction $x \mapsto \partial\varphi(x)$ is known as the generalized (or Clarke) subdifferential of $\varphi(\cdot)$. It coincides with the convex subdifferential if $\varphi(\cdot)$ is continuous, convex. The Clarke subdifferential has a rich calculus that parallels the one for the convex subdifferential and for smooth functions. For details we refer to the book of Clarke [2].

Given $u \in L^0(\Omega)$, we introduce the positive and negative parts of u defined by

$$u^+(z) = \max\{u(z), 0\}, \quad u^-(z) = \max\{-u(z), 0\}, \ z \in \Omega.$$

We have $u = u^+ - u^-$, $|u| = u^+ + u^-$. Moreover, if $u \in W_0^{1,\eta}(\Omega)$, then $u^\pm \in W_0^{1,\eta}(\Omega)$. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Finally, if X is a Banach space and $\varphi \in C^1(X)$, then we define

$$K_\varphi = \left\{u \in X : \varphi'(u) = 0\right\},$$

the critical set of $\varphi(\cdot)$.

Our hypotheses on the reaction $f(z,x)$ are the following:

H : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R}\backslash\{0\})$ and

(i) $|f(z,x)| \leq \hat{a}(z)[1 + |x|^{p-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\hat{a}(z) \in L^\infty(\Omega)$;

(ii) there exists a function $\theta \in L^\infty(\Omega)$ such that

$$\theta(z) \leq \widehat{\lambda}_1 a(z) \text{ for } a.a.\ z \in \Omega, \ \theta \not\equiv \widehat{\lambda}_1 a,$$

$$\limsup_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \leq \theta(z) \text{ uniformly for } a.a. z \in \Omega;$$

(iii) $\lim_{x\to 0} \frac{f(z,x)}{|x|^{q-2}x} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iv) for a.a. $z \in \Omega$ and all $x \neq 0$, we have

$$(p-1)f(z,x)x \leq f'_x(z,x)x^2.$$

Remark 2. *Hypothesis H(iv) implies that for a.a. $x \in \Omega$, the function*

$$x \mapsto \frac{f(z,x)x}{|x|^p}$$

is nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, +\infty)$. If $1 < p < 2$ and $a \in (0, 2-p)$, then a differential function which for small $|x| \le 1$ has the form xe^{-ax} and for $|x| > 1$ large is of the form $\theta |x|^{p-2}x$ with $\theta < \widehat{\lambda}_1 a$, satisfies hypotheses H.

Let $F(z,x) = \int_0^x f(z,s)\mathrm{d}s$. We introduce the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega F(z,u)\mathrm{d}z \text{ for all } u \in W_0^{1,\eta}(\Omega).$$

Evidently $\varphi \in C^1(W_0^{1,\eta}(\Omega))$ and the Nehari manifold corresponding to $\varphi(\cdot)$ is the following set

$$N = \left\{ u \in W_0^{1,\eta}(\Omega) : \langle \varphi'(u), u\rangle = 0, u \neq 0 \right\}.$$

Since we want to have solutions of constant sign, we introduce the following positive and negative truncations of $\varphi(\cdot)$:

$$\varphi_+(u) = \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega F(z,u^+)\mathrm{d}z$$
$$\varphi_-(u) = \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega F(z,-u^-)\mathrm{d}z \text{ for all } u \in W_0^{1,\eta}(\Omega).$$

Note that $\varphi_\pm \in C^1(W_0^{1,\eta}(\Omega))$. We introduce the Nehari manifolds for these two functions

$$N_+ = \left\{ u \in W_0^{1,\eta}(\Omega) : \langle \varphi_+'(u), u\rangle = 0, u \neq 0 \right\},$$
$$N_- = \left\{ u \in W_0^{1,\eta}(\Omega) : \langle \varphi_-'(u), u\rangle = 0, u \neq 0 \right\}.$$

Finally for the nodal solutions, we consider the set

$$N_0 = \left\{ u \in W_0^{1,\eta}(\Omega) : u^+ \in N,\ u^- \in N \right\}.$$

3. Three solutions theorem

First we prove a property of the Nehari manifold N which implies that $N \neq \emptyset$.

Proposition 3. *If hypotheses H_0, H hold and $u \in W_0^{1,\eta}(\Omega), u \neq 0$, then there exists unique $t_u > 0$ such that $t_u u \in N$.*

Proof. Consider the fibering function $\mu_u(\cdot)$ defined by

$$\mu_u(t) = \varphi(tu) \text{ for all } t \geq 0.$$

Evidently, $\mu_u \in C^1(0,\infty)$ and for $t > 0$, we have

$$\mu_u'(t) = \langle \varphi'(tu), u\rangle \text{ (by the chain rule)},$$
$$\Rightarrow \mu_u'(t) = \langle V(tu), u\rangle - \int_\Omega f(z,tu)u\mathrm{d}z$$
$$= t^{p-1}\rho_{\eta_0}(Du) + t^{q-1}\|Du\|_q^q - \int_\Omega f(z,tu)u\mathrm{d}z.$$

Therefore we see that

$$\mu_u'(t) = 0 \Leftrightarrow tu \in N. \tag{5}$$

On account of (5), we consider the equation

$$\mu_u'(t) = 0,$$
$$\Rightarrow \rho_{\eta_0}(Du) + \frac{1}{t^{p-q}}\|Du\|_q^q = \int_\Omega \frac{f(z,tu)u}{t^{p-1}}\mathrm{d}z. \tag{6}$$

Equation (6) suggests to consider the function

$$\gamma_u(t) = \rho_{\eta_0}(Du) - \int_\Omega \frac{f(z,tu)u}{t^{p-1}}\mathrm{d}z + \frac{1}{t^{p-q}}\|Du\|_q^q, \ t > 0.$$

We have

$$t^{p-q}\gamma_u(t) = t^{p-q}\rho_{\eta_0}(Du) - \int_\Omega \frac{f(z,tu)u}{t^{q-1}}\mathrm{d}z + \|Du\|_q^q.$$

Passing to the limit as $t \to 0^+$ and using hypothesis H(iii), we have that

$$\gamma_u(t) \to -\infty \text{ as } t \to 0^+. \tag{7}$$

On the other hand, on account of hypothesis H(ii), we see that

$$\gamma_u(t) \to \rho_{\eta_0}(Du) - \int_\Omega \theta(z)|u|^p\mathrm{d}z > 0 \text{ as } t \to +\infty, \tag{8}$$

(recall that $p > q$ and see Proposition 5 of [20]).

From (7), (8) and the Bolzano-Weiestrass' theorem, we infer that there exists $t_u > 0$ such that

$$\gamma_u(t_u) = 0.$$

Therefore $t_u > 0$ is a solution of (6). Note that in (6) the left hand side is strictly decreasing as a function of $t > 0$. For the right hand side, let

$$\ell(t) = \int_\Omega \frac{f(z,tu)u}{t^{p-1}}\mathrm{d}z = \int_\Omega \frac{f(z,tu)tu}{t^p}\mathrm{d}z, \quad t > 0.$$

Then, from H(iv) we can infer that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\ell(t) &= \int_\Omega \frac{f'_x(z,tu)t^{p+1}u^2 + f(z,tu)t^p u - pt^{p-1}f(z,tu)tu}{t^{2p}}\mathrm{d}z \\ &= \int_\Omega \frac{f'_x(z,tu)tu^2 + f(z,tu)u - pf(z,tu)u}{t^p}\mathrm{d}z \\ &= \int_\Omega \frac{f'_x(z,tu)(tu)^2 - (p-1)f(z,tu)(tu)}{t^{p+1}}\mathrm{d}z \geq 0.\end{aligned}$$

Hence the right hand side is nondecreasing. It follows that the solution $t_u > 0$ is unique and we have

$$\begin{aligned}&\mu'_u(t_u) = 0, \\ \Rightarrow &t_u u \in N \text{ (see (5))}.\end{aligned}$$

The proof is now complete. □

Corollary 4. *If hypotheses H_0, H hold, then $N \neq \emptyset$.*

Another important consequence of Proposition 3 is the nonemptiness of N_0, the set where nodal solutions are located.

Corollary 5. *If hypotheses H_0, H hold, then $N_0 \neq \emptyset$.*

Proof. Let $u \in W_0^{1,\eta}(\Omega)$ with $u^+ \neq 0$, $u^- \neq 0$. Proposition 3 says that there exist unique positive numbers t_+ and $t_- > 0$ such that

$$t_+u^+ \in N \text{ and } t_-(-u^-) \in N.$$

We set $y = t_+u^+ - t_-u^-$. Then $y \in W_0^{1,\eta}(\Omega)$ and

$$\begin{aligned}&y^+ = t_+u^+, \ y^- = t_-u^-, \\ \Rightarrow &y \in N_0 \text{ and so } N_0 \neq \emptyset.\end{aligned}$$

The proof is complete. □

Proposition 6. *If hypotheses H_0, H hold and $u \in N$, then $\varphi(tu) \leq \varphi(u)$ for all $t > 0$.*

Proof. On account of hypotheses H(i), H(iii), given $M > 0$ we can find $c_1 = c_1(M) > 0$ such that

$$F(z,x) \geq \frac{M}{q}|x|^q - c_1|x|^p \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{9}$$

Then for $t > 0$, we have

$$\begin{aligned} \varphi(tu) &= \frac{t^p}{p}\rho_{\eta_0}(Du) + \frac{t^q}{q}\|Du\|_q^q - \int_\Omega F(z,tu)\mathrm{d}z \\ &\leq \frac{t^p}{p}\rho_{\eta_0}(Du) + \frac{t^q}{q}\left[\|Du\|_q^q - M\|u\|_q^q\right] + c_1 t^p\|u\|_p^p. \qquad \text{(see (9))}. \end{aligned} \tag{10}$$

Choosing $M > 0$ large from (10), we see that

$$\varphi(tu) \leq c_2 t^p - c_3 t^q \text{ for some } c_2, c_3 > 0.$$

Since $q < p$, we infer that

$$\varphi(tu) < 0 \text{ for } t \in (0,1) \text{ small}. \tag{11}$$

On the other hand, hypotheses H(i), (ii) imply that given $\epsilon > 0$ and $\tau \in (1,q]$, we can find $c_4 = c_4(\epsilon,\tau) > 0$ such that

$$F(z,x) \geq \frac{1}{p}\left[\theta(z) + \epsilon\right]|x|^p + c_4|x|^\tau \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Then for $t > 1$, we have

$$\varphi(tu) \geq \frac{t^p}{p}\left[\rho_{\eta_0}(Du) - \int_\Omega \theta(z)|u|^p\mathrm{d}z - \epsilon\|u\|_p^p\right] - c_4 t^\tau\|u\|_\tau^\tau.$$

Using Proposition 5 of [20] and choosing $\epsilon > 0$ small, we obtain

$$\varphi(tu) \geq c_5 t^p - c_3 t^\tau \text{ for some } c_5, c_6 > 0 \text{ all } t > 0.$$

Since $p > \tau$, we have that

$$\varphi(tu) > 0 \text{ for } t > 1 \text{ large}. \tag{12}$$

We consider the fibering function

$$\mu_u(t) = \varphi(tu) \text{ for all } t > 0.$$

By hypothesis $u \in N$. So, $t = 1$ is the unique critical point of $\mu_u(\cdot)$ (see Proposition 3). Then from (11) and (12), it follows that $t = 1$ is the unique maximizer of $\mu_u(\cdot)$. Therefore we conclude that

$$\varphi(tu) \leq \varphi(u) \text{ for all } t > 0.$$

This completes the proof. □

Let $m = \inf_N \varphi$.

Proposition 7. *If hypotheses H_0 and H hold, then $m > 0$.*

Proof. Recall that

$$F(z,x) \leq \frac{1}{p}\left[\theta(z) + \epsilon\right]|x|^p + c_7|x|^\tau \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}$$
$$\text{(with } \tau \in (1,q], c_7 > 0).$$

Then for $u \in W_0^{1,\eta}(\Omega)$, we have

$$\varphi(u) \geq \frac{1}{p}\left[\rho_{\eta_0}(Du) - \int_\Omega \theta(z)|u|^p \mathrm{d}z - \epsilon\|u\|_p^p\right] - c_7\|u\|_\tau^\tau,$$
$$\geq c_8\|u\|^p - c_9\|u\|^\tau \text{ for some } c_8, c_9 > 0 \text{ (see [20] and Proposition 1).}$$

So, if $\rho > 1$ is large, we see that

$$\varphi(u) \geq \widehat{\mu} > 0 \text{ for all } \|u\| = \rho.$$

If $u \in N$, then let $\widehat{t}_u > 0$ such that $\|\widehat{t}_u u\| = \rho$. From Proposition 6, we have

$$\varphi(u) \geq \varphi(\widehat{t}_u u) \geq \widehat{\mu} > 0.$$

Since $u \in N$ is arbitrary, we deduce that

$$m \geq \widehat{\mu} > 0,$$

which completes the proof. □

Proposition 8. *If hypotheses H_0 and H hold, then the functionals $\varphi(\cdot)$, $\varphi_\pm(\cdot)$ are all coercive.*

Proof. Hypotheses H(i), (ii) imply that given $\epsilon > 0$, we can find $c_{10} = c_{10}(\epsilon) > 0$ such that

$$F(z,x) \leq \frac{1}{p}\left[\theta(z) + \epsilon\right]|x|^p + c_{10} \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{13}$$

Let $u \in W_0^{1,\eta}(\Omega)$ with $\|u\|_{1,\eta_0} \geq 1$. We have

$$\varphi(u) \geq \frac{1}{p}\left[\rho_{\eta_0}(Du) - \int_\Omega \theta(z)|u|^p \mathrm{d}z\right] + \left(\frac{1}{q} - \epsilon c_{11}\right)\|u\|_q^q - c_{10}|\Omega|_N$$
$$\text{for some } c_{11} > 0 \text{ (recall that } W_0^{1,q}(\Omega) \hookrightarrow L^p(\Omega) \text{ since } p < q^*).$$

Using once again Proposition 5 of [20], we have

$$\rho_{\eta_0}(Du) - \int_\Omega \theta(z)|u|^p \mathrm{d}z \geq c_{12}\|u\|_{1,\eta_0}^p \text{ for some } c_{12} > 0.$$

Choosing $\epsilon \in (0,1)$ small, we have

$$\begin{aligned}
\varphi(u) &\geq c_{12}\left[\|u\|_{1,\eta_0}^p + \|Du\|_q^q\right] - c_{10}|\Omega|_N \\
&\geq c_{12}\left[\rho_{\eta_0}(Du) + \|Du\|_q^q\right] - c_{10}|\Omega|_N \text{ (recall } \|u\|_{1,\eta_0} \geq 1 \text{ and see Proposition 2)} \\
&= c_{12}\rho_\eta(Du) - c_{10}|\Omega|_N, \\
\Rightarrow &\varphi(u) \text{ is coercive (see Proposition 2).}
\end{aligned}$$

Similarly we argue for the functionals $\varphi_\pm(\cdot)$. □

Let

$$m_+ = \inf_{N_+} \varphi_+, \quad m_- = \inf_{N_-} \varphi_-, \quad m_0 = \inf_{N_0} \varphi.$$

By producing solutions of these minimization problems, we will be able to have the multiplicity theorem for problem (1) (three solutions theorem).

Reasoning as we did for the functional $\varphi(\cdot)$ on N, we can obtain the following properties for the pairs (φ_+, N_+) and (φ_-, N_-).

Proposition 9. *If hypotheses H_0 and H hold, then*

(*a*) *for every $u \in W_0^{1,\eta}(\Omega)$, $u \neq 0$, there exists unique $t_u^+ > 0$ such that $t_u^+ u \in N_+$, hence $N_+ \neq \emptyset$;*
(*b*) *if $u \in N_+$, then $\varphi_+(tu) \leq \varphi_+(u)$ for all $t > 0$;*
(*c*) $m_+ > 0$.

Proposition 10. *If hypotheses H_0 and H hold, then*

(a) *for every $u \in W_0^{1,\eta}(\Omega)$, $u \neq 0$, there exists unique $t_u^- > 0$ such that $t_u^- u \in N_-$, hence $N_- \neq \emptyset$;*
(b) *if $u \in N_-$, then $\varphi_-(tu) \leq \varphi_-(u)$ for all $t > 0$;*
(c) *$m_- > 0$.*

Now we will produce solutions for the three minimization problems.

Proposition 11. *If hypotheses H_0, H hold, then there exists $u_0 \in N_0$ such that $\varphi(u_0) = m_0$.*

Proof. Consider a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq N_0$, $\varphi(u_n) \downarrow m_0$. We have

$$\begin{aligned}\varphi(u_n) &= \varphi(u_n^+) + \varphi(-u_n^-)\\ &\geq 2m \quad (\text{since } u_n^+ \in N, -u_n^- \in N),\\ \Rightarrow & m_0 \geq 2m > 0.\end{aligned}$$

From Proposition 8, we see that

$$\{u_n^+\}_{n\in N}, \{u_n^-\}_{n\in N} \subseteq W_0^{1,\eta}(\Omega) \text{ are bounded.}$$

So, we may assume that

$$\left\{\begin{array}{l} u_n^+ \rightharpoonup y_1, u_n^- \rightharpoonup y_2 \text{ in } W_0^{1,\eta}(\Omega),\\ y_1, y_2 \geq 0 \text{ and } \{y_1 > 0\} \cap \{-y_2 < 0\} = \emptyset. \end{array}\right. \tag{14}$$

Since $u_n^+, -u_n^- \in N$, we have

$$\rho_\eta(Du_n^+) = \int_\Omega f(z, u_n^+)u_n^+ \mathrm{d}z,$$

$$\rho_\eta(D(-u_n^-)) = \int_\Omega f(z, -u_n^-)(-u_n^-)\mathrm{d}z \text{ for all } n \in \mathbb{N}.$$

If $y_1 = 0$, then from (14), we see that $\rho_\eta(Du_n^+) \to 0$ and so $\|u_n^+\| \to 0$ (see Proposition 2). Hence

$$0 < m \leq \varphi(u_n^+) \text{ for all } n \in N, \varphi(u_n^+) \to \varphi(0) = 0,$$

a contradiction. Therefore $y_1 \neq 0$. In a similar fashion, we show that $y_2 \neq 0$.

Using Proposition 3, we can find $t_1, t_2 > 0$ such that

$$t_1 y_1 \in N \text{ and } t_2(-y_2) \in N.$$

Let $u_0 = t_1 y_1 - t_2 y_2$. Evidently $u_0^+ = t_1 y_1, u_0^- = t_2 y_2$. Hence $u_0 \in N$. We have

$$\begin{aligned} m_0 &= \lim_{n\to\infty} \varphi(u_n)\\ &= \lim_{n\to\infty} \left[\varphi(u_n^+) + \varphi(-u_n^-)\right]\\ &\geq \liminf_{n\to\infty} \left[\varphi(t_1 u_n^+) + \varphi(-t_2 u_n^-)\right] \ \ (\text{see Proposition 6}),\\ &\geq \varphi(t_1 y_1) + \varphi(-t_2 y_2) \ (\text{see}(14))\\ &= \varphi(u_0) \geq m_0 \ (\text{since } u_0 \in N_0),\\ \Rightarrow & \varphi(u_0) = m_0 \text{ with } u_0 \in N_0. \end{aligned}$$

The proof is now complete. ☐

Evidently, $u_0 \in N_0$ is a candidate for a nodal solution of (1). Now by solving the other two minimization problems we will generate candidates for positive and negative solutions.

Proposition 12. *If hypotheses H_0, H hold, then there exist $\widehat{u} \in N_+$ and $\widehat{v} \in N_-$ such that*

$$\varphi_+(\widehat{u}) = m_+ \text{ and } m_- = \varphi_-(\widehat{v}).$$

Proof. Let $\{u_n\}_{n\in N} \subseteq N_+$ be a minimizing sequence for $\varphi_+(\cdot)$, that is,

$$\varphi_+(u_n) \downarrow m_+.$$

Proposition 8 implies that

$$\{u_n\}_{n\in N} \subseteq W_0^{1,\eta}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \rightharpoonup \overline{u} \text{ in } W_0^{1,\eta}(\Omega). \tag{15}$$

Suppose, by contradiction, that $\overline{u} = 0$. Since $u_n \in N_+, n \in \mathbb{N}$, we have

$$\rho_{\eta_0}(Du_n) + \|Du\|_q^q = \int_\Omega f(z, u_n^+)u_n^+ dz \text{ for all } n \in \mathbb{N}. \tag{16}$$

From (15) and since $W_0^{1,\eta}(\Omega) \hookrightarrow L^p(\Omega)$ compactly (recall $p < q^*$ and see Proposition 1), we have

$$\begin{aligned} &u_n \to \overline{u} = 0 \text{ in } L^p(\Omega), \\ \Rightarrow\ &u_n^+ \to \overline{u}^+ = 0 \text{ in } L^p(\Omega). \end{aligned} \tag{17}$$

So, if in (16), we pass to the limit as $n \to \infty$ and use (17), we obtain

$$\begin{aligned} &\rho_\eta(Du_n) \to 0, \\ \Rightarrow &u_n \to 0 \text{ in } W_0^{1,\eta}(\Omega) \text{ (see Proposition 2)}, \\ \Rightarrow &\varphi_+(u_n) \to \varphi_+(0) = 0 = m_+ > 0 \text{ (see Proposition 9)}, \end{aligned}$$

a contradiction. Therefore $\overline{u} \neq 0$ and so by Proposition 9, we can find unique $t_+ > 0$ such that

$$\widehat{u} = t_+\overline{u} \in N_+.$$

We have

$$\begin{aligned} m_+ &= \lim_{n\to\infty} \varphi_+(u_n) \\ &\geq \liminf_{n\to\infty} \varphi_+(t_+u_n) \text{ (see Proposition 9 (6))} \\ &\geq \varphi_+(t_+\overline{u}) \text{ (see (15))} \\ &= \varphi_+(\widehat{u}) \geq m_+ \text{ (see } \widehat{u} \in N_+), \\ \Rightarrow &\varphi_+(\widehat{u}) = m_+ \text{ with } \widehat{u} \in N_+. \end{aligned}$$

Similarly for $\varphi_-(\cdot)$ using now Proposition 10. ☐

Next, we show that $\widehat{u}, \widehat{v}$ and u_0 are critical points of $\varphi_+(\cdot)$, $\varphi_-(\cdot)$ and $\varphi(\cdot)$ respectively. Therefore we conclude that N_+, N_- and N_0 are natural constraints for $\varphi_+(\cdot)$, $\varphi_-(\cdot)$ and $\varphi(\cdot)$ respectively (see Papageorgiou-Rădulescu-Repovš [18, p.425]).

Proposition 13. *If hypotheses H_0, H hold, then $\widehat{u} \in K_{\varphi_+}$, $\widehat{v} \in K_{\varphi_-}$, $u_0 \in K_\varphi$.*

Proof. Consider the locally Lipschitz functional $\psi_+ : W_0^{1,\eta}(\Omega) \hookrightarrow \mathbb{R}$ defined by

$$\psi_+(u) = \rho_{\eta_0}(Du_n) + \|Du\|_q^q - \int_\Omega f(z, u^+)u^+ dz \text{ for all } u \in W_0^{1,\eta}(\Omega).$$

Then we have

$$m_+ = \varphi_+(\widehat{u}) = \inf\left[\varphi_+(u) : \psi_+(u) = 0, u \neq 0\right].$$

By the nonsmooth Lagrange multiplier rule of Clarke [2, p.228], we can find $(\lambda, \beta) \in \mathbb{R}^2 \backslash \{0\}$ such that

$$\begin{aligned} &0 \in \lambda\varphi_+'(\widehat{u}) + \beta\partial\psi_+'(\widehat{u}), \\ \Rightarrow &\lambda\varphi_+'(\widehat{u}) + \beta h^* = 0 \text{ for some } h^* \in \partial\psi_+'(0). \end{aligned} \tag{18}$$

If $\beta=0$, then $\lambda\neq 0$ and we have

$$\begin{aligned}&\lambda\varphi_+'(\widehat{u})=0,\\ &\Rightarrow\varphi_+'(\widehat{u})=0,\\ &\Rightarrow u\in K_{\varphi_+}.\end{aligned}\tag{19}$$

On (18) we act with $\widehat{u}\in N_+$. We obtain

$$\begin{aligned}&\beta\langle h^*,\widehat{u}\rangle=0 \text{ (recall } \langle\varphi_+'(\widehat{u}),\widehat{u}\rangle=0),\\
&\Rightarrow\beta\left[p\rho_{\eta_0}(D\widehat{u})+q\|D\widehat{u}\|_q^q-\int_\Omega f_x'(z,\widehat{u}^+)(\widehat{u}^+)^2\mathrm{d}z-\int_\Omega f(z,\widehat{u}^+)\widehat{u}^+\mathrm{d}z\right]=0\\
&\quad\text{(recall } f(z,\cdot)\in C^1(\mathbb{R}\backslash\{0\}) \text{ and see Clarke [2, pp.48, 80]),}\\
&\Rightarrow\beta\left[p\left(\rho_{\eta_0}(D\widehat{u})+\|D\widehat{u}\|_q^q-\int_\Omega f(z,\widehat{u}^+)\widehat{u}^+\mathrm{d}z\right)\right]\\
&\quad+\beta\left[-(p-q)\|D\widehat{u}\|_q^q-\int_\Omega\left[f_x'(z,\widehat{u}^+)(\widehat{u}^+)^2-(p-1)f(z,\widehat{u}^+)\widehat{u}^+\right]\mathrm{d}z\right]=0,\\
&\Rightarrow\beta\left[-(p-q)\|D\widehat{u}\|_q^q-\int_\Omega\left[f_x'(z,\widehat{u}^+)(\widehat{u}^+)^2-(p-1)f(z,\widehat{u}^+)\widehat{u}^+\right]\mathrm{d}z\right]=0,\ \text{(since } \widehat{u}\in N_+),\end{aligned}$$

a contradiction, since $p>q$, $\widehat{u}\neq 0$ and see H(iv). Therefore $\beta=0$ and so from (19) we have $\widehat{u}\in K_{\varphi_+}$.

Similarly for $\widehat{v}\in N_-$ using this time $\varphi_-(\cdot)$ and the locally Lipschitz function

$$\psi_-(u)=\rho_{\eta_0}(Du)+\|Du\|_q^q-\int_\Omega f(z,-u^-)(-u^-)\mathrm{d}z \text{ for all } u\in W_0^{1,\eta}(\Omega).$$

We conclude that

$$\widehat{v}\in K_{\varphi_-}.$$

Consider the C^1-functional

$$\psi(u)=\rho_{\eta_0}(Du)+\|Du\|_q^q-\int_\Omega f(z,u)u\mathrm{d}z \text{ for all } u\in W_0^{1,\eta}(\Omega).$$

Also let $\xi_\pm: W_0^{1,\eta}(\Omega)\hookrightarrow W_0^{1,\eta}(\Omega)$ be the Lipschitz maps defined by

$$\xi_+(u)=u^+,\ \xi_-(u)=-u^-.$$

We set $\widehat{\psi}_+=\psi\circ\xi_+$ and $\widehat{\psi}_-=\psi\circ\xi_-$, we have

$$m_0=\varphi(u_0)=\inf\left[\varphi(u):\widehat{\psi}_+(u)=0,\widehat{\psi}_-(u)=0,u^\pm\neq 0\right].$$

By the nonsmooth Lagrange multiplier rule, we can find $\lambda\in\mathbb{R}$, $(\theta_1,\theta_2)\in\mathbb{R}^2$ not both zero such that

$$0\in\lambda\varphi'(u_0)+\theta_1\partial\widehat{\psi}_+(u_0)+\theta_2\partial\widehat{\psi}_-(u_0).\tag{20}$$

Suppose that $(\beta_1,\beta_2)\neq 0$, then at least one of the components is nonzero, say $\beta_1\neq 0$. We act with $u_0^+\in N$ and obtain as before

$$\begin{aligned}&0=\beta_1\left[p\rho_{\eta_0}(Du_0^+)+q\|Du_0^+\|_q^q-\int_\Omega\left[f_x'(z,u_0^+)(u_0^+)^2-(p-1)f(z,u_0^+)u_0^+\right]\mathrm{d}z\right]\\
&\Rightarrow\beta_1=0 \text{ (see H(iv) and recall } u_0^+\neq 0).\end{aligned}$$

Similarly we show that $\beta_2\neq 0$. Therefore $\lambda\neq 0$ and we have

$$\begin{aligned}&\lambda\varphi'(u_0)=0,\\ &\Rightarrow\varphi'(u_0)=0 \text{ and so } u_0\in K_\varphi.\end{aligned}$$

This completes the proof. □

Now we are ready for the multiplicity theorem.

Theorem 14. *If hypotheses H_0, H hold, then problem has at least three nontrivial solutions*

$$\widehat{u} \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega), \widehat{u}(z) > 0 \text{ for a.a. } z \in \Omega,$$
$$\widehat{v} \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega), \widehat{v}(z) < 0 \text{ for a.a. } z \in \Omega,$$
$$u_0 \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega) \text{ nodal solution.}$$

Proof. We know that there exists $\widehat{u} \in W_0^{1,\eta}(\Omega)$ such that

$$\varphi_+(\widehat{u}) = m_+, \ \widehat{u} \in K_{\varphi_+}.$$

So, $\widehat{u}$ is a ground state solution of (1). We have

$$\langle \varphi'_+(\widehat{u}), h\rangle = 0 \text{ for all } h \in W_0^{1,\eta}(\Omega).$$

Let $h = -\widehat{u}^- \in W_0^{1,\eta}(\Omega)$. We obtain

$$\begin{aligned}&\rho_{\eta_0}(D\widehat{u}^-) + \|D\widehat{u}^-\|_q^q = 0,\\ \Rightarrow &\rho_\eta(D\widehat{u}^-) = 0,\\ \Rightarrow &\widehat{u} \geq 0, \widehat{u} \neq 0 \text{ (recall } \widehat{u} \in N_+).\end{aligned}$$

Invoking Theorem 3.1 of Gasiński-Winkert [6], we have that $\widehat{u} \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$. Finally Proposition 2.4 of Papageorgiou-Vetro-Vetro [22] implies that $\widehat{u} > 0$ for a.a. $z \in \Omega$. Similarly for $\widehat{v} \in N_-$ with $\varphi_-(\widehat{v}) = m_-, \widehat{v} \in K_{\varphi_-}$. Finally from Proposition 11, we obtain $u_0 \in N_0$ such that

$$\varphi(u_0) = \inf_{N_0} \varphi$$

and also we have

$$u_0 \in K_\varphi \text{ (see Proposition 13).}$$

Therefore $u_0 \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$ is a nodal solution of (1). ☐

Remark 3. *We see that all three functions are ground state solutions of problem* (1). *An interesting open problem is whether Theorem 14 remains valid if we have resonance with respect to $\widehat{\lambda}_1 > 0$ as $x \to \pm\infty$, that is,*

$$\limsup_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \leq \widehat{\lambda}_1 a(z) \text{ uniformly for a.a. } z \in \Omega.$$

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