OPEN ACCESS
Bulletin of Mathematical Sciences
(2026) 2650006 (33 pages)

DOI: 10.1142/S1664360726500062

Normalized solutions of quasilinear problems with nonlocal reaction

Haidong Liu [*,‖] and Vicenţiu D. Rădulescu [†,‡,§,¶,**]

[*] *Institute of Mathematics*
Jiaxing University, Zhejiang 314001, P. R. China

[†] *Faculty of Applied Mathematics*
AGH University of Kraków
al. Mickiewicza 30, 30-059 Kraków, Poland

[‡] *Brno University of Technology*
Faculty of Electrical Engineering and Communication
Technická 3058/10, Brno 61600, Czech Republic

[§] *Scientific Research Center*
Baku Engineering University, AZ0102 Baku, Azerbaijan

[¶] *Simion Stoilow Institute of Mathematics of the Romanian Academy*
21 Calea Griviţei, 010702 Bucharest, Romania
[‖] *liuhaidong@zjxu.edu.cn*
[**] *radulescu@agh.edu.pl*

Received 8 January 2026
Revised 12 February 2026
Accepted 12 February 2026
Published 21 February 2026

Communicated by Ahmed Alsaedi

In this paper, we investigate the normalized solutions of a class of quasilinear elliptic equations characterized by convolution nonlinearity. We establish the existence and nonexistence of global and local minimizers across various ranges of the exponent, thereby extending many existing results in the literature.

Keywords: Quasilinear elliptic equation; convolution nonlinearity; normalized solution.

Mathematics Subject Classification 2020: 35A15, 35J62

[**] Corresponding author.

1. Introduction

We look for solutions $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ of the quasilinear problem

$$-\Delta u - \Delta(u^2)u + \lambda u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N \tag{1.1}$$

satisfying the additional condition

$$\int_{\mathbb{R}^N} u^2\,dx = c > 0. \tag{1.2}$$

Such solutions are commonly referred to as normalized solutions since (1.2) imposes a normalization on the L^2-mass of the solution u. Therefore, $\lambda \in \mathbb{R}$ in (1.1) cannot be prescribed a priori but rather emerges as part of the unknown. Here and throughout the subsequent discussion, we consistently assume that the spatial dimension $N \geq 3$ and the exponent p satisfies $\frac{N+\alpha}{N} \leq p \leq \frac{2(N+\alpha)}{N-2}$ unless otherwise specified. In addition, I_α represents the Riesz potential of order $\alpha \in (0, N)$, which is defined as

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{\frac{N}{2}} \Gamma(\frac{\alpha}{2})|x|^{N-\alpha}}, \quad \text{for } x \in \mathbb{R}^N \backslash \{0\}.$$

1.1. *Motivation*

Clearly, problem (1.1) is a special case of the quasilinear elliptic equation

$$-\Delta u - \Delta(u^2)u + \lambda u = f(x, u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

which originates from the study of solitary waves of nonlinear Schrödinger equation

$$i\partial_t \psi + \Delta\psi + \Delta(|\psi|^2)\psi + g(x, |\psi|^2)\psi = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N. \tag{1.4}$$

It is known that (1.4) appears in many different physical contexts, including condensed matter theory, dissipative quantum mechanics, fluid mechanics and plasma physics, and the theory of Heisenberg ferromagnets and magnons (see, e.g. [3, 22, 23, 28, 37, 41]).

The study of (1.3) admits two different approaches. One can either consider the chemical potential $\lambda \in \mathbb{R}$ to be given, or regard it as part of the unknown. In the former case, (1.3) has been widely investigated over the past two decades, and the situation is now well understood. We do not even attempt to review the available results here, instead we refer the interested reader to [2, 7, 9, 19–21, 30–36, 40, 42, 51]. See also the references therein. In the latter case, the L^2-mass normalization of the solution u becomes essential. This constraint naturally introduces $\lambda \in \mathbb{R}$ in (1.3) as a Lagrange multiplier. The classical variational approach to finding normalized solutions of (1.3) is to seek critical points of the energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx - \int_{\mathbb{R}^N} F(x, u)dx$$

constrained on the L^2-sphere

$$\mathcal{S}_c = \left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx < \infty, \ \int_{\mathbb{R}^N} u^2 \, dx = c \right\},$$

where $F(x,u) = \int_0^u f(x,t)\,dt$. A first strategy is to find minimizers of J restricted to $\mathcal{S}_c$. It is known that the number $4+\frac{4}{N}$ plays a pivotal role and serves as a watershed for the minimization method. To elucidate this, we consider the prototypical nonlinearity $f(x,u) = |u|^{p-2}u$, where $2 < p < \frac{4N}{N-2}$. Utilizing the Gagliardo–Nirenberg inequality together with L^2-invariant rescaling argument, we see that J is bounded from below on $\mathcal{S}_c$ if $p < 4 + \frac{4}{N}$, whereas it becomes unbounded from below on $\mathcal{S}_c$ if $p > 4 + \frac{4}{N}$. Consequently, the threshold number $4 + \frac{4}{N}$ is designated as the mass critical exponent for (1.3).

We summarize some known results on normalized solutions of (1.3), which primarily focus on the power nonlinearity $f(x,u) = |u|^{p-2}u$. In the mass subcritical and critical case $2 < p \le 4 + \frac{4}{N}$, existence and nonexistence of global minimizers for J on $\mathcal{S}_c$ were established in [8, 15, 49]. When $2 + \frac{4}{N} < p < 4 + \frac{4}{N}$, Jeanjean *et al.*, [16] obtained both local minimizer and mountain pass normalized solution by the perturbation method from [36]. When $p = 4 + \frac{4}{N}$, existence and concentration behavior of normalized solutions were studied in [24, 49, 52, 53] provided that the mass c is suitably large. Later, Zhang *et al.*, [54] adopted the dual approach from [7, 33] to find infinitely many normalized solutions when $2 < p < 2 + \frac{4}{N}$ and as many normalized solutions as prescribed when $2 + \frac{4}{N} \le p < 4 + \frac{4}{N}$ and the mass c is suitably large. See [48] for analogous results of (1.3) with general nonlinearities in the mass subcritical regime.

The mass supercritical case $4 + \frac{4}{N} < p < \frac{4N}{N-2}$ has been less studied due to inherent technical difficulties. Li and Zou [24] employed the perturbation method to establish the existence of a positive ground state normalized solution and applied the index theory to obtain infinitely many normalized solutions for (1.3). They required that the spatial dimension $N \le 3$ and the exponent p is also less than H^1-critical exponent when $N = 3$, stemming from delicate control on the sign property of the Lagrange multiplier. Zhang *et al.*, [53] also established the existence of a positive ground state normalized solution via the Pohožaev manifold technique, successfully handling the case of H^1-critical exponent. Recently, inspired by [4], Jeanjean *et al.*, [17] studied a relaxed minimization problem

$$m_{c,\mathcal{D}} = \inf_{u \in \mathcal{D}_c} J(u),$$

where

$$\mathcal{D}_c = \left\{ u \in H^1(\mathbb{R}^N)\backslash\{0\} \,\middle|\, \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx < \infty, \ \int_{\mathbb{R}^N} u^2 \, dx \le c, \ P(u) = 0 \right\}$$

with

$$P(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + (N+2) \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx - \frac{N(p-2)}{2p} \int_{\mathbb{R}^N} |u|^p \, dx.$$

They proved that $m_{c,\mathcal{D}}$ has a minimizer if either $N \leq 4$ or $N \geq 5$ and $c > 0$ is suitably small. Such a minimizer is indeed a solution of (1.3) for some $\lambda \in \mathbb{R}$ and belongs to $\mathcal{S}_c$. Asymptotic behavior of the minimizers was further analyzed in [17] as the mass c tends to some limit. Gao and Zhang [13] combined the perturbation method with the Pohožaev manifold technique to obtain infinitely many normalized solutions for (1.3) in the dimensions $N = 3, 4$.

1.2. *Main results*

It should be noted that quasilinear problems involving potentials or general nonlinearities remain largely unexplored in the literature. Some partial results can be found in [10–12, 14, 18, 25, 29, 38, 43, 44, 48, 50]. In particular, we refer to Chen and Tang [6] for a comprehensive review on the existence of normalized solutions for several classes of nonlinear elliptic equations.

In this paper, we are interested in solutions of (1.1) satisfying the mass constraint (1.2). The corresponding energy functional is defined by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \int_{\mathbb{R}^N} u^2|\nabla u|^2\, dx - \frac{1}{2p}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p\, dx,$$

which is not of class C^1 in $H^1(\mathbb{R}^N)$ due to the second integral term. To avoid this difficulty, we shall adopt a direct approach inspired by [8] and investigate the following minimization problem

$$m_c = \inf_{u \in \mathcal{S}_c} \Phi(u),$$

where again

$$\mathcal{S}_c = \left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} u^2|\nabla u|^2\, dx < \infty,\ \int_{\mathbb{R}^N} u^2\, dx = c \right\}.$$

In order to present our main results, we introduce the function

$$Q_{\frac{N+\alpha}{N}}(x) = C_0 \left(\frac{\sigma}{\sigma^2 + |x - z|^2} \right)^{\frac{N}{2}}, \quad \text{for } x \in \mathbb{R}^N,$$

where $\sigma > 0$ and $z \in \mathbb{R}^N$ are parameters, and $C_0 > 0$ is a fixed constant such that

$$\int_{\mathbb{R}^N} \Big(I_\alpha * \big|Q_{\frac{N+\alpha}{N}}\big|^{\frac{N+\alpha}{N}} \Big) \big|Q_{\frac{N+\alpha}{N}}\big|^{\frac{N+\alpha}{N}}\, dx = 1.$$

Theorem 1.1. *If* $p = \frac{N+\alpha}{N}$, *then*

$$m_c = -\frac{N c^{\frac{N+\alpha}{N}}}{2(N+\alpha)\big\|Q_{\frac{N+\alpha}{N}}\big\|_{L^2(\mathbb{R}^N)}^{\frac{2(N+\alpha)}{N}}} < 0 \tag{1.5}$$

and m_c *is not achieved for any* $c > 0$.

Theorem 1.2. *Let $\frac{N+\alpha}{N} < p < \frac{2N+\alpha+2}{N}$. We have the following conclusions*:

(1) *If $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$, then $m_c < 0$ and m_c is achieved for any $c > 0$.*
(2) *If $p = \frac{N+\alpha+2}{N}$, then there exists a threshold mass $c_1^* > 0$ such that*

 (2.1) *$m_c = 0$ and m_c is not achieved for any $0 < c \le c_1^*$.*
 (2.2) *$m_c < 0$ and m_c is achieved for any $c > c_1^*$.*

(3) *If $\frac{N+\alpha+2}{N} < p < \frac{2N+\alpha+2}{N}$ and, in addition,*

$$\max\left\{N-4, \frac{2(N+1)(N-2)}{3N+2}\right\} \le \alpha < N \text{ when } N \ge 4, \tag{1.6}$$

then there exists a threshold mass $c_2^* > 0$ such that

 (3.1) *$m_c = 0$ and m_c is not achieved for any $0 < c < c_2^*$.*
 (3.2) *$m_c = 0$ and m_c is achieved for any $c = c_2^*$.*
 (3.3) *$m_c < 0$ and m_c is achieved for any $c > c_2^*$.*

Theorem 1.3. *Let $\frac{2N+\alpha+2}{N} \le p < \frac{2(N+\alpha)}{N-2}$. We have the following conclusions*:

(1) *If $p = \frac{2N+\alpha+2}{N}$, then there exists a threshold mass $c_3^* > 0$ such that*

$$m_c = \begin{cases} 0, & \text{for any } 0 < c \le c_3^*, \\ -\infty, & \text{for any } c > c_3^*. \end{cases}$$

(2) *If $\frac{2N+\alpha+2}{N} < p < \frac{2(N+\alpha)}{N-2}$, then $m_c = -\infty$ for any $c > 0$.*
(3) *The infimum m_c fails to be attained for any $c > 0$.*

In Theorems 1.1–1.3, we require that $\frac{N+\alpha}{N} \le p < \frac{2(N+\alpha)}{N-2}$, which is equivalent to $2 \le \frac{2Np}{N+\alpha} < \frac{4N}{N-2}$. Indeed, using the Hardy–Littlewood–Sobolev inequality (see Lemma 2.2), the Hölder inequality and the Sobolev inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\,dx &\le C\left(\int_{\mathbb{R}^N} |u|^{\frac{2Np}{N+\alpha}}\,dx\right)^{\frac{N+\alpha}{N}} \\ &\le C\left(\int_{\mathbb{R}^N} u^2\,dx\right)^{\frac{2(N+\alpha)-p(N-2)}{N+2}} \left(\int_{\mathbb{R}^N} |u|^{\frac{4N}{N-2}}\,dx\right)^{\frac{(N-2)(Np-N-\alpha)}{N(N+2)}} \\ &\le C\left(\int_{\mathbb{R}^N} u^2\,dx\right)^{\frac{2(N+\alpha)-p(N-2)}{N+2}} \left(\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx\right)^{\frac{Np-N-\alpha}{N+2}}\end{aligned} \tag{1.7}$$

for any $u \in X := \left\{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx < \infty\right\}$, where C is a positive constant. This inequality ensures that the energy functional Φ is well defined in X under the condition $\frac{N+\alpha}{N} \le p < \frac{2(N+\alpha)}{N-2}$. Consequently, the bounds $\frac{N+\alpha}{N}$ and $\frac{2(N+\alpha)}{N-2}$ are termed lower and upper critical exponents for (1.1), respectively.

Let $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and Q_p be a ground state solution of

$$-\frac{Np-(N+\alpha)}{2}\Delta u + \frac{N+\alpha-(N-2)p}{2}u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

Then any $u \in H^1(\mathbb{R}^N)$ satisfies

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\,dx \le \frac{p}{\|Q_p\|_{L^2(\mathbb{R}^N)}^{2p-2}} \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\frac{Np-N-\alpha}{2}} \left(\int_{\mathbb{R}^N} u^2\,dx\right)^{\frac{N+\alpha-(N-2)p}{2}}, \tag{1.8}$$

with equality attained when $u = Q_p$. Theorems 1.2 and 1.3 reveal that the threshold mass emerges when the exponent p lies within the interval $[\frac{N+\alpha+2}{N}, \frac{2N+\alpha+2}{N}]$. The constants c_1^*, c_2^* and c_3^* will be rigorously constructed in Secs. 4 and 5, defined as follows:

$$c_1^* = \|Q_{\frac{N+\alpha+2}{N}}\|_{L^2(\mathbb{R}^N)}^2,$$
$$c_2^* = \inf\{c>0 \mid m_c < 0\},$$
$$c_3^* = \left(\frac{4N+2\alpha+4}{N} \inf_{u\in X\setminus\{0\}} \frac{\left(\int_{\mathbb{R}^N} u^2\,dx\right)^{\frac{\alpha+2}{N}} \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx}{\int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{2N+\alpha+2}{N}}\right)|u|^{\frac{2N+\alpha+2}{N}}\,dx}\right)^{\frac{N}{\alpha+2}}.$$

Remark 1.4. Theorem 1.3 states that Φ has no global minimizer on $\mathcal{S}_c$ for $p = \frac{2N+\alpha+2}{N}$ and $0 < c \le c_3^*$. In this special case, we can further establish the nonexistence of normalized solutions of (1.1) subject to the mass constraint (1.2). Assume by contradiction that $(u,\lambda) \in H^1(\mathbb{R}^N)\times\mathbb{R}$ solves (1.1) and (1.2). Then

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx + 4\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx + \lambda\int_{\mathbb{R}^N} u^2\,dx - \int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{2N+\alpha+2}{N}}\right)|u|^{\frac{2N+\alpha+2}{N}}\,dx = 0. \tag{1.9}$$

In addition, using standard arguments, we see that u also satisfies

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + (N-2)\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx + \frac{N\lambda}{2}\int_{\mathbb{R}^N} u^2\,dx - \frac{N(N+\alpha)}{4N+2\alpha+4}\int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{2N+\alpha+2}{N}}\right)|u|^{\frac{2N+\alpha+2}{N}}\,dx = 0. \tag{1.10}$$

Multiplying (1.9) by $\frac{N}{2}$ and subtracting (1.10) yields

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx + (N+2)\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx = \frac{N(N+2)}{4N+2\alpha+4}\int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{2N+\alpha+2}{N}}\right)|u|^{\frac{2N+\alpha+2}{N}}\,dx.$$

It then follows that

$$c_3^* \le \left(\frac{4N+2\alpha+4}{N} \frac{\left(\int_{\mathbb{R}^N} u^2\,dx\right)^{\frac{\alpha+2}{N}} \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx}{\int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{2N+\alpha+2}{N}}\right)|u|^{\frac{2N+\alpha+2}{N}}\,dx} \right)^{\frac{N}{\alpha+2}} < \int_{\mathbb{R}^N} u^2\,dx = c,$$

thus contradicting the initial assumption $0 < c \le c_3^*$.

The global minimizer of Φ on $\mathcal{S}_c$, if it exists, is necessarily a normalized solution of (1.1) for some $\lambda \in \mathbb{R}$. While Theorem 1.3 establishes the nonexistence of global minimizers for $\frac{2N+\alpha+2}{N} \le p < \frac{2(N+\alpha)}{N-2}$, this does not preclude the existence of normalized solutions for (1.1) altogether. Indeed, Remark 1.4 demonstrates the absence of normalized solutions specifically when $p = \frac{2N+\alpha+2}{N}$ and $0 < c \le c_3^*$. For other parameter regimes, normalized solutions may still exist. To address this, we introduce the Pohožaev set $\mathcal{P}_c = \{u \in \mathcal{S}_c \,|\, P(u) = 0\}$ as in [53] and analyze the minimization problem

$$m_{c,\mathcal{P}} = \inf_{u \in \mathcal{P}_c} \Phi(u),$$

where

$$P(u) = \int_{\mathbb{R}^N} |\nabla u|^2\,dx + (N+2)\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx - \frac{Np-N-\alpha}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\,dx.$$

In the forthcoming result, we prove that the local minimizer of Φ on $\mathcal{P}_c$ does exist under certain conditions. Moreover, Lemma A.4 in the appendix shows that the local minimizer is indeed a normalized solution of (1.1) for some $\lambda \in \mathbb{R}$.

Theorem 1.5. *Suppose that $N = 3, 4$, $2N-6 < \alpha < N$ and $\frac{2N+\alpha+2}{N} \le p \le \frac{N+\alpha}{N-2}$. Set*

$$c_4^* = \begin{cases} c_3^*, & \text{when } p = \frac{2N+\alpha+2}{N}, \\ 0, & \text{when } \frac{2N+\alpha+2}{N} < p \le \frac{N+\alpha}{N-2}. \end{cases}$$

If $c > c_4^*$, then we have $\mathcal{P}_c \ne \emptyset$, $m_{c,\mathcal{P}} > 0$ and $m_{c,\mathcal{P}}$ has a minimizer.

Notations. Throughout this paper, C denotes a positive constant whose value may change from line to line, and $o_n(1)$ signifies a generic infinitesimal as $n \to \infty$. We shall work in the metric space

$$X = \left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx < \infty \right\}$$

equipped with the distance $d_X(u,v) = \|u - v\|_{H^1(\mathbb{R}^N)} + \|\nabla(u^2) - \nabla(v^2)\|_{L^2(\mathbb{R}^N)}$. While X is complete under this metric, it is not a vector space since it fails to be closed under the sum. We define $(s\star u)(x) = s^{\frac{N}{2}}u(sx)$ for $s > 0$. Then $\|s\star u\|_{L^2(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)}$ and

$$\int_{\mathbb{R}^N} |\nabla(s\star u)|^2\,dx = s^2\int_{\mathbb{R}^N}|\nabla u|^2\,dx, \tag{1.11}$$

$$\int_{\mathbb{R}^N}(s\star u)^2|\nabla(s\star u)|^2\,dx = s^{N+2}\int_{\mathbb{R}^N}u^2|\nabla u|^2\,dx, \tag{1.12}$$

$$\int_{\mathbb{R}^N}(I_\alpha * |s\star u|^p)|s\star u|^p\,dx = s^{Np-N-\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx. \tag{1.13}$$

The scaling properties (1.11)–(1.13) play a crucial role in our analysis.

2. Preliminaries

In this section, we state some useful lemmas. We first recall the Gagliardo–Nirenberg inequality (see [45, Theorem B]) and the Hardy–Littlewood–Sobolev inequality (see [26, Theorem 4.3]).

Lemma 2.1 (Gagliardo-Nirenberg inequality). *Assume that* $2 < p < \frac{2N}{N-2}$ *and denote by* $W_p \in H^1(\mathbb{R}^N)$ *the unique positive radial solution of the following semilinear elliptic equation*

$$-\frac{N(p-2)}{4}\Delta u + \left(1 - \frac{(p-2)(N-2)}{4}\right)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

Then any $u \in H^1(\mathbb{R}^N)$ *satisfies*

$$\int_{\mathbb{R}^N}|u|^p\,dx \le \frac{p}{2\|W_p\|^{p-2}_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}u^2\,dx\right)^{\frac{2p-N(p-2)}{4}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^{\frac{N(p-2)}{4}}.$$

Lemma 2.2 (Hardy–Littlewood–Sobolev inequality). *Let* $t > 1$, $r > 1$ *and* $0 < \mu < N$ *be such that* $\frac{1}{t} + \frac{\mu}{N} + \frac{1}{r} = 2$. *Then there is a sharp constant* $C(N,\mu,t,r)$ *such that, for* $f \in L^t(\mathbb{R}^N)$ *and* $h \in L^r(\mathbb{R}^N)$, *it holds*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)h(y)}{|x-y|^\mu}\,dxdy\right| \le C(N,\mu,t,r)\|f\|_{L^t(\mathbb{R}^N)}\|h\|_{L^r(\mathbb{R}^N)}.$$

The next result is a nonlocal version of Brézis-Lieb lemma, which was essentially proved in [1] (see also [5, 39]). We shall briefly sketch the proof in the appendix for the reader's convenience.

Lemma 2.3. *Assume that* $F \in C^1(\mathbb{R},\mathbb{R})$ *satisfies* $F(0) = 0$ *and there exists* $C > 0$ *such that* $|F'(t)| \le C\big(|t|^{\frac{\alpha}{N}} + |t|^{\frac{N+2\alpha+2}{N-2}}\big)$ *for* $t \in \mathbb{R}$. *If* $\{u_n\}$ *is a bounded sequence in* X *and* $u_n \to u$ *a.e. in* $\mathbb{R}^N$ *for some* $u \in X$, *then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}[(I_\alpha * F(u_n))F(u_n) - (I_\alpha * F(u_n-u))F(u_n-u) - (I_\alpha * F(u))F(u)]dx = 0.$$

3. Proof of Theorem 1.1

This section addresses the case $p = \frac{N+\alpha}{N}$. Recall that

$$Q_{\frac{N+\alpha}{N}}(x) = C_0 \left(\frac{\sigma}{\sigma^2 + |x-z|^2} \right)^{\frac{N}{2}}, \quad \text{for } x \in \mathbb{R}^N,$$

where $\sigma > 0$ and $z \in \mathbb{R}^N$ are parameters, and $C_0 > 0$ is a fixed constant such that

$$\int_{\mathbb{R}^N} \left(I_\alpha * \left| Q_{\frac{N+\alpha}{N}} \right|^{\frac{N+\alpha}{N}} \right) \left| Q_{\frac{N+\alpha}{N}} \right|^{\frac{N+\alpha}{N}} dx = 1.$$

Then any $u \in H^1(\mathbb{R}^N)$ satisfies

$$\int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{N+\alpha}{N}} \right) |u|^{\frac{N+\alpha}{N}} dx \le \frac{1}{\left\| Q_{\frac{N+\alpha}{N}} \right\|_{L^2(\mathbb{R}^N)}^{\frac{2(N+\alpha)}{N}}} \left(\int_{\mathbb{R}^N} u^2 \, dx \right)^{\frac{N+\alpha}{N}}, \tag{3.1}$$

with equality attained if and only if $u = Q_{\frac{N+\alpha}{N}}$.

Proof of Theorem 1.1. Let $c > 0$. Setting

$$\kappa = \frac{\sqrt{c}}{\left\| Q_{\frac{N+\alpha}{N}} \right\|_{L^2(\mathbb{R}^N)}},$$

we have $s \star \left(\kappa Q_{\frac{N+\alpha}{N}} \right) \in \mathcal{S}_c$ for all $s > 0$. Then, since $Q_{\frac{N+\alpha}{N}}$ optimizes (3.1), it holds

$$m_c \le \lim_{s \to 0^+} \Phi\left(s \star \left(\kappa Q_{\frac{N+\alpha}{N}} \right) \right) = - \frac{N c^{\frac{N+\alpha}{N}}}{2(N+\alpha) \left\| Q_{\frac{N+\alpha}{N}} \right\|_{L^2(\mathbb{R}^N)}^{\frac{2(N+\alpha)}{N}}}. \tag{3.2}$$

By (3.1) again, we have

$$\Phi(u) \ge - \frac{N}{2(N+\alpha) \left\| Q_{\frac{N+\alpha}{N}} \right\|_{L^2(\mathbb{R}^N)}^{\frac{2(N+\alpha)}{N}}} \left(\int_{\mathbb{R}^N} u^2 \, dx \right)^{\frac{N+\alpha}{N}} = - \frac{N c^{\frac{N+\alpha}{N}}}{2(N+\alpha) \left\| Q_{\frac{N+\alpha}{N}} \right\|_{L^2(\mathbb{R}^N)}^{\frac{2(N+\alpha)}{N}}}$$

for any $u \in \mathcal{S}_c$, which implies

$$m_c \ge - \frac{N c^{\frac{N+\alpha}{N}}}{2(N+\alpha) \left\| Q_{\frac{N+\alpha}{N}} \right\|_{L^2(\mathbb{R}^N)}^{\frac{2(N+\alpha)}{N}}}. \tag{3.3}$$

Then (1.5) follows directly by combining (3.2) and (3.3).

If there exists $u \in \mathcal{S}_c$ achieving m_c, then we see from (1.5) and (3.1) that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx &\le \Phi(u) + \frac{N}{2(N+\alpha)} \int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{N+\alpha}{N}} \right) |u|^{\frac{N+\alpha}{N}} dx \\ &\le m_c + \frac{N}{2(N+\alpha) \left\| Q_{\frac{N+\alpha}{N}} \right\|_{L^2(\mathbb{R}^N)}^{\frac{2(N+\alpha)}{N}}} \left(\int_{\mathbb{R}^N} u^2 \, dx \right)^{\frac{N+\alpha}{N}} \\ &= 0. \end{aligned}$$

This forces $u = 0$ in $\mathbb{R}^N$, contradicting $u \in \mathcal{S}_c$. The proof is finished. □

4. Proof of Theorem 1.2

Throughout this section, we always assume that $\frac{N+\alpha}{N} < p < \frac{2N+\alpha+2}{N}$.

4.1. *Some useful lemmas*

Lemma 4.1. *The functional Φ is coercive and bounded from below on $\mathcal{S}_c$.*

Proof. By (1.7), we have

$$\Phi(u) \geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx - \frac{C\cdot c^{\frac{2(N+\alpha)-p(N-2)}{N+2}}}{2p}\left(\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx\right)^{\frac{Np-N-\alpha}{N+2}}$$

for any $u \in \mathcal{S}_c$. The desired conclusion follows easily because $0 < \frac{Np-N-\alpha}{N+2} < 1$. □

As an immediate consequence of Lemma 4.1, one has $m_c > -\infty$ for any $c > 0$.

Lemma 4.2. *It holds that $m_c \leq 0$ for any $c > 0$.*

Proof. Fix an arbitrary $u \in \mathcal{S}_c$. Observe that $s \star u \in \mathcal{S}_c$ for all $s > 0$. Therefore,

$$m_c \leq \lim_{s\to 0^+} \Phi(s \star u) = 0.$$

This completes the proof. □

In the subsequent lemma, we establish the existence of a global minimizer under additional condition $m_c < 0$. Interestingly, as will be discussed later, the critical threshold case $m_c = 0$ exhibits a dichotomy: minimizers may or may not exist, depending on the specific values of c and p.

Lemma 4.3. *Let $c > 0$ be such that*

$$m_c < 0.$$

Then there exists a Schwartz symmetric function $u \in \mathcal{S}_c$ achieving m_c.

Proof. Let $\{u_n\} \subset \mathcal{S}_c$ be a minimizing sequence of m_c, i.e.

$$\lim_{n\to\infty} \Phi(u_n) = m_c < 0.$$

By [8, Lemma 4.3] and [26, Theorem 3.7], we may assume without loss of generality that $\{u_n\} \subset \mathcal{S}_c$ is a minimizing sequence of Schwartz symmetric functions. It follows from Lemma 4.1 that $\{u_n\}$ is bounded in X and, by the Sobolev inequality, is also bounded in $L^{\frac{4N}{N-2}}(\mathbb{R}^N)$. Assume by extracting a subsequence that $u_n \rightharpoonup u$ in X,

$u_n \to u$ in $L^q(\mathbb{R}^N)$ with $2 < q < \frac{4N}{N-2}$, and $u_n \to u$ a.e. in $\mathbb{R}^N$. Using (1.7) and Lemma 2.3 yields that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p \, dx = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx.$$

Then by [8, Lemma 4.3] again,

$$\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n) = m_c < 0. \tag{4.1}$$

To conclude that u is the desired minimizer of m_c, it suffices to prove $u \in \mathcal{S}_c$. Clearly, $u \neq 0$ by (4.1) and $\|u\|^2_{L^2(\mathbb{R}^N)} \le \liminf_{n\to\infty} \|u_n\|^2_{L^2(\mathbb{R}^N)} = c$. Suppose, for contradiction, that $\tau := \|u\|^2_{L^2(\mathbb{R}^N)} \in (0, c)$. We define the scaled function $\tilde{u}(x) = u\big((\frac{c}{\tau})^{-\frac{1}{N}} x\big)$ for $x \in \mathbb{R}^N$. Then $\tilde{u} \in \mathcal{S}_c$ and, since $\Phi(u) < 0$,

$$\begin{aligned} m_c &\le \Phi(\tilde{u}) \\ &= \left(\frac{c}{\tau}\right)^{\frac{N-2}{N}} \left(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx\right) \\ &\quad - \left(\frac{c}{\tau}\right)^{\frac{N+\alpha}{N}} \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx \\ &\le \left(\frac{c}{\tau}\right)^{\frac{N+\alpha}{N}} \Phi(u) \\ &< \Phi(u), \end{aligned}$$

which contradicts (4.1). The proof is finished. □

4.2. *The case $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$*

Lemma 4.4. *If $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$, then $m_c < 0$ for any $c > 0$.*

Proof. The assumption $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$ can be equivalently expressed as

$$0 < Np - N - \alpha < 2.$$

Fix an arbitrary function $u \in \mathcal{S}_c$. Then $s \star u \in \mathcal{S}_c$ for all $s > 0$ and thereby

$$\begin{aligned} m_c &\le \Phi(s \star u) \\ &= \frac{s^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + s^{N+2}\int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx - \frac{s^{Np-N-\alpha}}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx \\ &< 0, \end{aligned}$$

provided that $s > 0$ is sufficiently small. The proof is finished. □

Proof of Theorem 1.2(1). The result is a conjunction of Lemmas 4.3 and 4.4. □

4.3. *The case* $p = \frac{N+\alpha+2}{N}$

We first set $c_1^* = \|Q_{\frac{N+\alpha+2}{N}}\|^2_{L^2(\mathbb{R}^N)} > 0$, which is the threshold mass as shown below.

Lemma 4.5. *If* $p = \frac{N+\alpha+2}{N}$, *then* $m_c = 0$ *and* m_c *is not achieved for any* $0 < c \le c_1^*$.

Proof. Let $p = \frac{N+\alpha+2}{N}$. We see from (1.8) that, for any $u \in \mathcal{S}_c$,

$$\frac{N}{N+\alpha+2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{N+\alpha+2}{N}})|u|^{\frac{N+\alpha+2}{N}}\,dx \le \left(\frac{c}{c_1^*}\right)^{\frac{\alpha+2}{N}}\int_{\mathbb{R}^N}|\nabla u|^2\,dx$$

and thus

$$\Phi(u) \ge \frac{1}{2}\left(1-\left(\frac{c}{c_1^*}\right)^{\frac{\alpha+2}{N}}\right)\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \int_{\mathbb{R}^N}u^2|\nabla u|^2\,dx > 0 \tag{4.2}$$

when $0 < c \le c_1^*$. It follows $m_c \ge 0$. Combining this with Lemma 4.2 yields $m_c = 0$ for any $0 < c \le c_1^*$. The strict inequality in (4.2) precludes the existence of minimizers for m_c. □

Lemma 4.6. *If* $p = \frac{N+\alpha+2}{N}$, *then* $m_c < 0$ *and* m_c *admits a minimizer for any* $c > c_1^*$.

Proof. Let $p = \frac{N+\alpha+2}{N}$ and recall the constant

$$\begin{aligned} c_1^* &= \int_{\mathbb{R}^N}|Q_{\frac{N+\alpha+2}{N}}|^2\,dx = \int_{\mathbb{R}^N}|\nabla Q_{\frac{N+\alpha+2}{N}}|^2\,dx \\ &= \frac{N}{N+\alpha+2}\int_{\mathbb{R}^N}\left(I_\alpha * |Q_{\frac{N+\alpha+2}{N}}|^{\frac{N+\alpha+2}{N}}\right)|Q_{\frac{N+\alpha+2}{N}}|^{\frac{N+\alpha+2}{N}}\,dx. \end{aligned}$$

Write $\kappa = \sqrt{\frac{c}{c_1^*}}$ for simplicity. Then $s \star \left(\kappa Q_{\frac{N+\alpha+2}{N}}\right) \in \mathcal{S}_c$ for all $s > 0$ and so

$$\begin{aligned} m_c &\le \inf_{s>0}\Phi\left(s\star\left(\kappa Q_{\frac{N+\alpha+2}{N}}\right)\right) \\ &= \inf_{s>0}\left(-\frac{c}{2}\left(\kappa^{\frac{2\alpha+4}{N}}-1\right)s^2 + \kappa^4\int_{\mathbb{R}^N}|Q_{\frac{N+\alpha+2}{N}}|^2|\nabla Q_{\frac{N+\alpha+2}{N}}|^2\,dx\cdot s^{N+2}\right) \\ &= -\frac{Nc^{1-\frac{2}{N}}}{2(N+2)}\left(\kappa^{\frac{2\alpha+4}{N}}-1\right)^{1+\frac{2}{N}}\left(\frac{(c_1^*)^2}{(N+2)\int_{\mathbb{R}^N}|Q_{\frac{N+\alpha+2}{N}}|^2|\nabla Q_{\frac{N+\alpha+2}{N}}|^2\,dx}\right)^{\frac{2}{N}} \\ &< 0 \end{aligned}$$

when $c > c_1^*$. With this, Lemma 4.3 ensures the existence of a minimizer for m_c. □

Proof of Theorem 1.2(2). The result is a conjunction of Lemmas 4.5 and 4.6. □

4.4. *The case* $\frac{N+\alpha+2}{N} < p < \frac{2N+\alpha+2}{N}$

Lemma 4.7. *If* $\frac{N+\alpha+2}{N} < p < \frac{2N+\alpha+2}{N}$ *and* (1.6) *holds, then* m_c *is not achieved for small* $c > 0$.

Proof. Assume by contradiction that there exists a sequence of positive numbers $\{c_n\}$ such that $\lim_{n\to\infty} c_n = 0$ and m_{c_n} is achieved by some $u_n \in \mathcal{S}_{c_n}$. Since $\Phi(u_n) = m_{c_n} \le 0$ by Lemma 4.2, we have

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2\,dx \le \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p\,dx. \tag{4.3}$$

It follows from (1.7) that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p\,dx \le C \cdot c_n^{\frac{2(N+\alpha)-p(N-2)}{N+2}} \left(\int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2\,dx\right)^{\frac{Np-N-\alpha}{N+2}}. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2\,dx + \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p\,dx \le C \cdot c_n^{\frac{2(N+\alpha)-p(N-2)}{2N+\alpha+2-Np}}.$$

Since $\frac{2(N+\alpha)-p(N-2)}{2N+\alpha+2-Np} > 0$ and $\lim_{n\to\infty} c_n = 0$, there must be

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx &= \lim_{n\to\infty}\int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2\,dx \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p\,dx = 0.\end{aligned} \tag{4.5}$$

We shall divide into two cases and get a contradiction to (4.5) in each case.

Case 1. $\boldsymbol{p \le \frac{N+\alpha}{N-2}}$**.** If $p < \frac{N+\alpha}{N-2}$, then $\Phi(u_n) = m_{c_n} \le 0$ and (1.8) imply

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx &\le \frac{1}{p}\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p\,dx \\ &\le C\left(\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx\right)^{\frac{Np-N-\alpha}{2}} \left(\int_{\mathbb{R}^N} u_n^2\,dx\right)^{\frac{N+\alpha-(N-2)p}{2}}.\end{aligned}$$

We remark that the above inequality also holds when $p = \frac{N+\alpha}{N-2}$ by using Lemma 2.2 and the Sobolev inequality. Since $\frac{Np-N-\alpha}{2} > 1$ and $\frac{N+\alpha-(N-2)p}{2} \ge 0$, there exists a constant $\delta > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx \ge \delta,$$

directly contradicting (4.5).

Case 2. $p > \frac{N+\alpha}{N-2}$. This case occurs only when $N \geq 5$ or $N = 4$ and $0 < \alpha < 2$. In the following arguments, we will always keep this in mind and assume that (1.6) holds true. We claim that

$$u_n \in L^q(\mathbb{R}^N) \tag{4.6}$$

and $\{u_n\}$ is bounded in $L^q(\mathbb{R}^N)$ for any $q \geq \frac{4N}{N-2}$. The proof relies on the Moser iteration and is given in the appendix. Then we deduce from $\Phi(u_n) = m_{c_n} \leq 0$, Lemma 2.2 and the Hölder inequality that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx &\leq \frac{1}{p}\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p\,dx \\ &\leq C\left(\int_{\mathbb{R}^N} |u_n|^{\frac{2Np}{N+\alpha}}\,dx\right)^{\frac{N+\alpha}{N}} \\ &\leq C\left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N}{N-2}}\,dx\right)^{\frac{(N-2)(N+\alpha-2)p}{N[(N-2)p-2]}}\left(\int_{\mathbb{R}^N} |u_n|^{Np}\,dx\right)^{\frac{2[(N-2)p-N-\alpha]}{N[(N-2)p-2]}} \\ &\leq C\left(\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx\right)^{\frac{(N+\alpha-2)p}{(N-2)p-2}}\left(\int_{\mathbb{R}^N} |u_n|^{Np}\,dx\right)^{\frac{2[(N-2)p-N-\alpha]}{N[(N-2)p-2]}}.\end{aligned}$$

Since $\frac{(N+\alpha-2)p}{(N-2)p-2} > 1$, $Np > \frac{4N}{N-2}$ and $\frac{2[(N-2)p-N-\alpha]}{N[(N-2)p-2]} > 0$, there also exists $\delta > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx \geq \delta,$$

which contradicts (4.5) again. □

Remark 4.8. Combining Lemmas 4.2, 4.3 and 4.7, we conclude that

$$m_c = 0$$

when $c > 0$ is sufficiently small.

Lemma 4.9. *If $\frac{N+\alpha+2}{N} < p < \frac{2N+\alpha+2}{N}$, then $m_c < 0$ for large $c > 0$.*

Proof. Let $w \in X$ be such that

$$\int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p\,dx > 0.$$

Define the scaled function

$$u_c(x) = w\left(c^{-\frac{1}{N}}\|w\|_{L^2(\mathbb{R}^N)}^{\frac{2}{N}}\,x\right), \quad \text{for } x \in \mathbb{R}^N.$$

A simple calculation shows that $u_c \in \mathcal{S}_c$ and

$$\lim_{c\to+\infty} \Phi(u_c) = -\infty.$$

Then the desired conclusion follows easily. □

Next we study the monotonicity and continuity of m_c with respect to $c > 0$.

Lemma 4.10. *If $c > 0$ and $\theta > 1$, then $m_{\theta c} \leq \theta m_c$.*

Proof. Given $\varepsilon > 0$, there is a function $u \in \mathcal{S}_c$ such that

$$\Phi(u) \leq m_c + \varepsilon.$$

Define $\tilde{u}(x) = \theta^{\frac{1}{N+2}} u\big(\theta^{-\frac{1}{N+2}} x\big)$ for $x \in \mathbb{R}^N$. Then $\tilde{u} \in \mathcal{S}_{\theta c}$ and so

$$m_{\theta c} \leq \Phi(\tilde{u}) < \theta \Phi(u) \leq \theta(m_c + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we have $m_{\theta c} \leq \theta m_c$, concluding the proof. □

We remark that, if the infimum m_c is attained, then one can take $\varepsilon = 0$ and choose $u \in \mathcal{S}_c$ being a minimizer of m_c in the above arguments. Therefore, the strict inequality $m_{\theta c} < \theta m_c$ holds true in this special case.

As a direct consequence of Lemmas 4.2 and 4.10, we have

Lemma 4.11. *The function $c \mapsto m_c$ is non-increasing on $(0, +\infty)$.*

Lemma 4.12. *The function $c \mapsto m_c$ is continuous on $(0, +\infty)$.*

Proof. According to Heine Theorem, it suffices to prove that, for any given number $c > 0$ and any sequence $\{c_n\}$ of positive numbers satisfying $c_n \to c$ as $n \to \infty$, one has $\lim_{n\to\infty} m_{c_n} = m_c$.

Let $u_n \in \mathcal{S}_{c_n}$ be such that

$$\Phi(u_n) \leq m_{c_n} + \frac{1}{n} \leq \frac{1}{n}. \tag{4.7}$$

In view of Lemma 4.1, $\{u_n\}$ is bounded in X. Setting $\tilde{u}_n = \sqrt{\frac{c}{c_n}}\, u_n$, we have $\tilde{u}_n \in \mathcal{S}_c$ and so $m_c \leq \Phi(\tilde{u}_n)$. A direct computation shows that $\Phi(\tilde{u}_n) = \Phi(u_n) + o_n(1)$ as $n \to \infty$. Combining this with (4.7) leads to

$$m_c \leq \liminf_{n\to\infty} \Phi(\tilde{u}_n) = \liminf_{n\to\infty} \Phi(u_n) \leq \liminf_{n\to\infty} m_{c_n}. \tag{4.8}$$

Conversely, one can choose a minimizing sequence $\{v_n\} \subset \mathcal{S}_c$ of m_c and set $\tilde{v}_n = \sqrt{\frac{c_n}{c}}\, v_n$. Similar to above, we have $\tilde{v}_n \in \mathcal{S}_{c_n}$, $m_{c_n} \leq \Phi(\tilde{v}_n)$ and $\Phi(\tilde{v}_n) = \Phi(v_n) + o_n(1)$ as $n \to \infty$. Then

$$\limsup_{n\to\infty} m_{c_n} \leq \limsup_{n\to\infty} \Phi(\tilde{v}_n) = \limsup_{n\to\infty} \Phi(v_n) = m_c. \tag{4.9}$$

The desired result is a conjunction of (4.8) and (4.9). The proof is finished. □

We introduce the critical mass threshold, defined as

$$c_2^* = \inf\{c > 0 \,|\, m_c < 0\}.$$

It follows from Remark 4.8 and Lemma 4.9 that $0 < c_2^* < \infty$. Moreover, combining Lemmas 4.2, 4.11 and 4.12, we can conclude that $m_c = 0$ for all $0 < c \leq c_2^*$, whereas $m_c < 0$ for all $c > c_2^*$.

Lemma 4.13. *If* $\frac{N+\alpha+2}{N} < p < \frac{2N+\alpha+2}{N}$, *then* m_c *is not achieved for any* $0 < c < c_2^*$.

Proof. Assume that m_c has a minimizer for some $0 < c < c_2^*$. Then

$$0 = m_{c_2^*} < \frac{c_2^*}{c} m_c = 0$$

by the strict inequality in Lemma 4.10, which is absurd. The proof is finished. □

Lemma 4.14. *If* $\frac{N+\alpha+2}{N} < p < \frac{2N+\alpha+2}{N}$, *then* m_c *is achieved for any* $c > c_2^*$.

Proof. Since $m_c < 0$ for any $c > c_2^*$, the conclusion follows from Lemma 4.3. □

Lemma 4.15. *If* $\frac{N+\alpha+2}{N} < p < \frac{2N+\alpha+2}{N}$, *then* $m_{c_2^*}$ *admits a minimizer.*

Proof. The idea in the proof of Lemma 4.3 does not work any more, because we cannot show that the weak limit of the minimizing sequence is nontrivial due to $m_{c_2^*} = 0$. To surmount this obstacle, we adopt the dual approach introduced in [7, 33]. Let $u = \varphi(v)$ be the inverse function of $v = \frac{1}{2}u\sqrt{1+2u^2} + \frac{\sqrt{2}}{4}\ln\left(\sqrt{2}u + \sqrt{1+2u^2}\right)$. Consider the dual minimization problem

$$\overline{m}_c = \inf_{v \in \overline{\mathcal{S}}_c} \overline{\Phi}(v),$$

where

$$\overline{\mathcal{S}}_c = \left\{ v \in H^1(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} \varphi^2(v)dx = c \right\}$$

and

$$\overline{\Phi}(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\, dx - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |\varphi(v)|^p)|\varphi(v)|^p\, dx.$$

Since $\mathcal{S}_c = \{u = \varphi(v) \,|\, v \in \overline{\mathcal{S}}_c\}$ by [54, Lemma 2.2], one can conclude that $m_c = \overline{m}_c$.

Let $c_n = c_2^* + \frac{1}{n} > c_2^*$. Then Lemma 4.14 ensures the existence of $u_n \in \mathcal{S}_{c_n}$ satisfying $\Phi(u_n) = m_{c_n} < 0$. The proof of Lemma 4.1 indicates that $\{u_n\}$ is bounded in X and, by the Sobolev inequality, is also bounded in $L^q(\mathbb{R}^N)$ for $2 \le q \le \frac{4N}{N-2}$. Write $v_n = \varphi^{-1}(u_n)$. Then $v_n \in \overline{\mathcal{S}}_{c_n}$ and $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. By Lemma 4.12, we have

$$\lim_{n\to\infty} \overline{\Phi}(v_n) = \lim_{n\to\infty} \Phi(u_n) = \lim_{n\to\infty} m_{c_n} = m_{c_2^*} = 0. \tag{4.10}$$

Assume up to a subsequence that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2\, dx = \frac{1}{p}\lim_{n\to\infty}\int_{\mathbb{R}^N} (I_\alpha * |\varphi(v_n)|^p)|\varphi(v_n)|^p\, dx \ge 0.$$

Claim. *There exists a constant* $\delta > 0$ *such that*

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} v_n^2\, dx \ge \delta. \tag{4.11}$$

If this holds true, then up to a subsequence there exist $\{y_n\} \subset \mathbb{R}^N$ and $v \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that $\tilde{v}_n := v_n(\cdot + y_n) \rightharpoonup v$ in $H^1(\mathbb{R}^N)$, $\tilde{v}_n \to v$ in $L^q_{\mathrm{Loc}}(\mathbb{R}^N)$

for $2 \leq q < \frac{2N}{N-2}$ and $\tilde{v}_n \to v$ a.e. in $\mathbb{R}^N$. Writing $u = \varphi(v) \in X$, we have

$$\begin{aligned}0 &< \|u\|^2_{L^2(\mathbb{R}^N)} = \|\varphi(v)\|^2_{L^2(\mathbb{R}^N)} \\ &\leq \lim_{n\to\infty} \|\varphi(\tilde{v}_n)\|^2_{L^2(\mathbb{R}^N)} = \lim_{n\to\infty} \|u_n\|^2_{L^2(\mathbb{R}^N)} = \lim_{n\to\infty} c_n = c_2^*.\end{aligned}$$

By (4.10), the definition of $\overline{m}_c$, Lemma 4.12 and Lemma A.3 in the appendix,

$$\begin{aligned}0 = \lim_{n\to\infty} \overline{\Phi}(\tilde{v}_n) &= \overline{\Phi}(v) + \lim_{n\to\infty} \overline{\Phi}(\tilde{v}_n - v) \\ &\geq \overline{m}_{\|\varphi(v)\|^2_{L^2(\mathbb{R}^N)}} + \overline{m}_{c_2^* - \|\varphi(v)\|^2_{L^2(\mathbb{R}^N)}} \\ &= m_{\|u\|^2_{L^2(\mathbb{R}^N)}} + m_{c_2^* - \|u\|^2_{L^2(\mathbb{R}^N)}} = 0,\end{aligned}$$

from which it follows

$$\Phi(u) = \overline{\Phi}(v) = m_{\|u\|^2_{L^2(\mathbb{R}^N)}}.$$

By Lemma 4.13, there must be $\|u\|^2_{L^2(\mathbb{R}^N)} = c_2^*$ and henceforth u is a minimizer of $m_{c_2^*}$.

Now we prove the claim by contradiction. If (4.11) were false, then $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < \frac{2N}{N-2}$ by P.-L. Lions' lemma (see [27, Lemma I.1] or [46, Lemma 1.21]). Since $|\varphi(t)| \leq \min\left\{|t|,\, 2^{\frac{1}{4}}\sqrt{|t|}\right\}$ for $t \in \mathbb{R}$, one can easily verify that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2\, dx = \frac{1}{p} \lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |\varphi(v_n)|^p)|\varphi(v_n)|^p\, dx = 0. \tag{4.12}$$

Since $\frac{N+\alpha+2}{N} < p < \frac{2N+\alpha+2}{N}$, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|\varphi(t)|^{\frac{2Np}{N+\alpha}} \leq \varepsilon |t|^{2+\frac{4}{N+\alpha}} + C_\varepsilon |t|^{\frac{2N}{N-2}}, \quad \text{for } t \in \mathbb{R}.$$

Then, by Lemmas 2.1 and 2.2,

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |\varphi(v_n)|^p)|\varphi(v_n)|^p\, dx &\leq C \left(\int_{\mathbb{R}^N} |\varphi(v_n)|^{\frac{2Np}{N+\alpha}}\, dx \right)^{\frac{N+\alpha}{N}} \\ &\leq C \left(\varepsilon \int_{\mathbb{R}^N} |v_n|^{2+\frac{4}{N+\alpha}}\, dx + C_\varepsilon \int_{\mathbb{R}^N} |v_n|^{\frac{2N}{N-2}}\, dx \right)^{\frac{N+\alpha}{N}} \\ &\leq C\varepsilon \int_{\mathbb{R}^N} |\nabla v_n|^2\, dx + C_\varepsilon \left(\int_{\mathbb{R}^N} |\nabla v_n|^2\, dx \right)^{\frac{N+\alpha}{N-2}}.\end{aligned}$$

In the last inequality we have used the boundedness of $\{v_n\}$ in $H^1(\mathbb{R}^N)$. Therefore,

$$\overline{\Phi}(v_n) \geq \left[\frac{1}{2} - C\varepsilon - C_\varepsilon \left(\int_{\mathbb{R}^N} |\nabla v_n|^2\, dx \right)^{\frac{\alpha+2}{N-2}} \right] \int_{\mathbb{R}^N} |\nabla v_n|^2\, dx.$$

Taking (4.12) into account, we conclude that

$$\overline{\Phi}(v_n) \geq 0, \quad \text{for sufficiently large } n,$$

contradicting $\overline{\Phi}(v_n) = \Phi(u_n) = m_{c_n} < 0$. The proof is finished. □

Proof of Theorem 1.2(3). It is a combination of Lemmas 4.13, 4.14 and 4.15. □

5. Proof of Theorems 1.3 and 1.5

This section concerns the case $\frac{2N+\alpha+2}{N} \le p < \frac{2(N+\alpha)}{N-2}$. Set

$$c_3^* = \left(\frac{4N+2\alpha+4}{N} \inf_{u\in X\setminus\{0\}} \frac{\left(\int_{\mathbb{R}^N} u^2\,dx\right)^{\frac{\alpha+2}{N}} \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx}{\int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{2N+\alpha+2}{N}}\right)|u|^{\frac{2N+\alpha+2}{N}}\,dx} \right)^{\frac{N}{\alpha+2}}.$$

As a particular case of (1.7), one has

$$\int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{2N+\alpha+2}{N}}\right)|u|^{\frac{2N+\alpha+2}{N}}\,dx \le C\left(\int_{\mathbb{R}^N} u^2\,dx\right)^{\frac{\alpha+2}{N}} \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx$$

for any $u \in X$, thereby establishing that c_3^* is indeed a positive constant.

Lemma 5.1. *If $p = \frac{2N+\alpha+2}{N}$, then $m_c = 0$ and m_c is not achieved for any $0 < c \le c_3^*$.*

Proof. Let $0 < c \le c_3^*$. By the definition of c_3^*, we have

$$\Phi(u) \ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \left(1 - \left(\frac{c}{c_3^*}\right)^{\frac{\alpha+2}{N}}\right)\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx > 0 \qquad (5.1)$$

for all $u \in \mathcal{S}_c$, which implies that $m_c \ge 0$. On the other hand, we also have $m_c \le 0$ as in Lemma 4.2. Then it must be $m_c = 0$. The strict inequality in (5.1) precludes the existence of minimizers for m_c. □

Lemma 5.2. *If $p = \frac{2N+\alpha+2}{N}$ and $c > c_3^*$, then*

$$\left\{ u \in \mathcal{S}_c \,\middle|\, \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx < \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\,dx \right\} \ne \emptyset.$$

Proof. By the definition of c_3^*, there exists $w \in X\setminus\{0\}$ such that

$$\left(\int_{\mathbb{R}^N} w^2\,dx\right)^{\frac{\alpha+2}{N}} \int_{\mathbb{R}^N} w^2|\nabla w|^2\,dx < \frac{Nc^{\frac{\alpha+2}{N}}}{4N+2\alpha+4}\int_{\mathbb{R}^N} \left(I_\alpha * |w|^{\frac{2N+\alpha+2}{N}}\right)|w|^{\frac{2N+\alpha+2}{N}}\,dx.$$

Define $u(x) = \frac{\sqrt{c}}{\|w\|_{L^2(\mathbb{R}^N)}}\,w(x)$ for $x \in \mathbb{R}^N$. Then $u \in \mathcal{S}_c$ and

$$\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx < \frac{N}{4N+2\alpha+4}\int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{2N+\alpha+2}{N}}\right)|u|^{\frac{2N+\alpha+2}{N}}\,dx.$$

The proof is finished. □

Lemma 5.3. *If $p = \frac{2N+\alpha+2}{N}$, then $m_c = -\infty$ for any $c > c_3^*$.*

Proof. Let $c > c_3^*$ and, by Lemma 5.2, fix some $u \in \mathcal{S}_c$ such that

$$\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx < \frac{1}{2p}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p\,dx.$$

By (1.11)–(1.13), we have

$$\Phi(s\star u) = \frac{s^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - s^{N+2} \times \left[\frac{1}{2p}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p\,dx - \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx\right] \to -\infty$$

as $s \to +\infty$. The desired conclusion follows easily. □

Lemma 5.4. *If $\frac{2N+\alpha+2}{N} < p < \frac{2(N+\alpha)}{N-2}$, then $m_c = -\infty$ for any $c > 0$.*

Proof. Let $u \in \mathcal{S}_c$ be fixed. By (1.11)–(1.13), we have

$$\Phi(s\star u) = \frac{s^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + s^{N+2}\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx - \frac{s^{Np-N-\alpha}}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\,dx.$$

Given that $Np - N - \alpha > N + 2 > 2$, it follows

$$\lim_{s\to+\infty} \Phi(s\star u) = -\infty,$$

which implies that $m_c = -\infty$. The proof is finished. □

Proof of Theorem 1.3. It follows immediately from Lemmas 5.1, 5.3 and 5.4. □

In the remainder of this section, we always assume that $N = 3, 4$, $2N - 6 < \alpha < N$ and $\frac{2N+\alpha+2}{N} \le p \le \frac{N+\alpha}{N-2}$. We also set $c_4^* = c_3^*$ if $p = \frac{2N+\alpha+2}{N}$ and $c_4^* = 0$ if $\frac{2N+\alpha+2}{N} < p \le \frac{N+\alpha}{N-2}$. Now we introduce the Pohožaev set

$$\mathcal{P}_c = \{u \in \mathcal{S}_c \,|\, P(u) = 0\}$$

for $c > c_4^*$, where P is defined by

$$P(u) = \int_{\mathbb{R}^N} |\nabla u|^2\,dx + (N+2)\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx - \frac{Np-N-\alpha}{2p} \times \int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p\,dx.$$

In order to show that $\mathcal{P}_c \neq \emptyset$, we need the following two elementary lemmas.

Lemma 5.5. *Let $p = \frac{2N+\alpha+2}{N}$. Then, for each fixed $u \in \mathcal{S}_c$ satisfying*

$$\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx < \frac{1}{2p}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p\,dx, \tag{5.2}$$

there is a unique constant $s_u > 0$ such that $s_u \star u \in \mathcal{P}_c$ and $\Phi(s_u \star u) = \max_{s>0} \Phi(s\star u)$.

Proof. Let $u \in \mathcal{S}_c$ satisfy (5.2) and define $\varphi : (0, +\infty) \to \mathbb{R}$ by

$$\begin{aligned}\varphi(s) &= \Phi(s \star u)\\ &= \frac{s^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - s^{N+2}\left[\frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^p\,dx - \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx\right].\end{aligned}$$

Clearly, φ has a unique maximum point $s_u > 0$, i.e.

$$\Phi(s_u \star u) = \varphi(s_u) = \max_{s>0} \varphi(s) = \max_{s>0} \Phi(s \star u).$$

Then $P(s_u \star u) = s_u\varphi'(s_u) = 0$, which means that $s_u \star u \in \mathcal{P}_c$. The proof is finished. □

Lemma 5.6. *Assume that* $\frac{2N+\alpha+2}{N} < p \le \frac{N+\alpha}{N-2}$. *Then, for each fixed* $u \in \mathcal{S}_c$, *there exists a unique constant* $s_u > 0$ *such that* $s_u \star u \in \mathcal{P}_c$ *and* $\Phi(s_u \star u) = \max_{s>0} \Phi(s \star u)$.

Proof. The proof is similar to that of Lemma 5.5 and is thus dropped. □

As a direct consequence of Lemmas 5.2, 5.5 and 5.6, we have

Lemma 5.7. *The Pohožaev set* $\mathcal{P}_c \ne \emptyset$ *for any* $c > c_4^*$.

Next we investigate the properties of Φ on the Pohožaev set $\mathcal{P}_c$.

Lemma 5.8. *Let* $p = \frac{2N+\alpha+2}{N}$. *We have the following conclusions*:

(1) Φ *is coercive and bounded from below on* $\mathcal{P}_c$.
(2) *There exists a constant* $\delta > 0$ *such that*

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx \ge \delta, \quad \int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^p\,dx \ge \delta, \quad \Phi(u) \ge \delta, \quad \text{for } u \in \mathcal{P}_c.$$

Proof. (1) It holds that

$$\Phi(u) = \Phi(u) - \frac{1}{N+2}P(u) = \frac{N}{2(N+2)}\int_{\mathbb{R}^N} |\nabla u|^2\,dx, \quad \text{for } u \in \mathcal{P}_c. \tag{5.3}$$

Note that, when $N = 3, 4$ and $2N - 6 < \alpha < N$, one has

$$p = \frac{2N+\alpha+2}{N} < \frac{N+\alpha}{N-2}.$$

Then, for $u \in \mathcal{P}_c$, we deduce from $P(u) = 0$ and (1.8) that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx &\le C\int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^p\,dx\\ &\le C\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\frac{N+2}{2}}.\end{aligned} \tag{5.4}$$

Combining this with (5.3), we see that Φ is coercive and bounded from below on $\mathcal{P}_c$.

(2) As a consequence of (5.4), one has

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx \le C\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\frac{N+2}{2}}, \quad \text{for } u \in \mathcal{P}_c.$$

Since $\frac{N+2}{2} > 1$, there exists a constant $\delta > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx \ge \delta, \quad \text{for } u \in \mathcal{P}_c. \tag{5.5}$$

The other two inequalities follow directly from (5.3)–(5.5). □

Lemma 5.9. *Let* $\frac{2N+\alpha+2}{N} < p \le \frac{N+\alpha}{N-2}$. *We have the following conclusions*:

(1) Φ *is coercive and bounded from below on* $\mathcal{P}_c$.
(2) *There exists a constant* $\delta > 0$ *such that*

$$\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx \ge \delta, \quad \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\,dx \ge \delta, \quad \Phi(u) \ge \delta, \quad \textit{for } u \in \mathcal{P}_c.$$

Proof. Observe that

$$\begin{aligned}\Phi(u) &= \Phi(u) - \frac{1}{Np-N-\alpha}P(u)\\ &= \frac{Np-N-\alpha-2}{2(Np-N-\alpha)}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{Np-2N-\alpha-2}{Np-N-\alpha}\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx\end{aligned} \tag{5.6}$$

for $u \in \mathcal{P}_c$ and, by (1.7),

$$\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx \le C\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\,dx \le C\left(\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx\right)^{\frac{Np-N-\alpha}{N+2}} \tag{5.7}$$

for $u \in \mathcal{P}_c$. The conclusion follows easily from (5.6) and (5.7). □

Define the infimum

$$m_{c,\mathcal{P}} = \inf_{u\in\mathcal{P}_c} \Phi(u).$$

Clearly, Lemmas 5.8 and 5.9 imply that $m_{c,\mathcal{P}} > 0$.

Lemma 5.10. $m_{c,\mathcal{P}}$ *is strictly decreasing with respect to* $c \in (c_4^*, +\infty)$.

Proof of Lemma 5.10 for the Case $p = \frac{2N+\alpha+2}{N}$. We assume that $d > c > c_3^*$. By the definition of $m_{c,\mathcal{P}}$ and Lemma 5.5, for each $n \in \mathbb{N}$ there is a function $u_n \in \mathcal{P}_c$ such that

$$\Phi(u_n) \le m_{c,\mathcal{P}} + \frac{1}{n} \tag{5.8}$$

and $\Phi(u_n) = \max_{s>0} \Phi(s \star u_n)$. By Lemma 5.8 and (5.8), $\{u_n\}$ is bounded in X. We set $\kappa = \sqrt{c/d} < 1$ and define the scaled function $w_n(x) = \kappa^{\frac{N-2}{2}} u_n(\kappa x)$ for $x \in \mathbb{R}^N$.

Then $\|w_n\|^2_{L^2(\mathbb{R}^N)} = d$ and

$$\int_{\mathbb{R}^N} |\nabla w_n|^2\,dx = \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx, \tag{5.9}$$

$$\int_{\mathbb{R}^N} w_n^2|\nabla w_n|^2\,dx = \kappa^{N-2}\int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2\,dx, \tag{5.10}$$

$$\int_{\mathbb{R}^N} \left(I_\alpha * |w_n|^p\right)|w_n|^p\,dx = \kappa^{(N-2)p-N-\alpha}\int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^p\right)|u_n|^p\,dx. \tag{5.11}$$

Therefore, by (5.10) and (5.11), we see that $w_n \in \mathcal{S}_d$ satisfies

$$\begin{aligned}\int_{\mathbb{R}^N} w_n^2|\nabla w_n|^2\,dx &< \int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2\,dx \\ &< \frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^p\right)|u_n|^p\,dx < \frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha * |w_n|^p\right)|w_n|^p\,dx.\end{aligned}$$

Here we have used the fact $p = \frac{2N+\alpha+2}{N} < \frac{N+\alpha}{N-2}$ when $N = 3, 4$ and $2N-6 < \alpha < N$. By Lemma 5.5, there is a unique constant $s_n := s_{w_n} > 0$ such that $s_n \star w_n \in \mathcal{P}_d$ and $\Phi(s_n \star w_n) = \max_{s>0}\Phi(s \star w_n)$. Since $\{w_n\}$ is bounded in X and $\Phi(s_n \star w_n) \geq m_{d,\mathcal{P}} > 0$, there exists $\delta > 0$ such that $s_n \geq \delta$ for all $n \in \mathbb{N}$. Then, since $u_n \in \mathcal{P}_c$, we see from Lemma 5.8 that

$$\frac{s_n^{N+2}}{2p}\left(\kappa^{(N-2)p-N-\alpha} - 1\right)\int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^p\right)|u_n|^p\,dx \geq \delta'$$

for some $\delta' > 0$. By (5.8)–(5.11) and $u_n \in \mathcal{P}_c$, we have

$$\begin{aligned}m_{d,\mathcal{P}} &\leq \Phi(s_n \star w_n) \\ &\leq \Phi(s_n \star u_n) - \frac{s_n^{N+2}}{2p}[\kappa^{(N-2)p-N-\alpha} - 1]\int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^p\right)|u_n|^p\,dx \\ &\leq \Phi(u_n) - \delta' \leq m_{c,\mathcal{P}} + \frac{1}{n} - \delta' < m_{c,\mathcal{P}},\end{aligned}$$

for n sufficiently large. The proof is finished. □

Proof of Lemma 5.10 for the Case $\frac{2N+\alpha+2}{N} < p \leq \frac{N+\alpha}{N-2}$. Let $d > c > 0$ and denote $\kappa = \left(\frac{d}{c}\right)^{\frac{1}{N}} > 1$. We first claim that, for each fixed function $u \in \mathcal{P}_c$, there exists $s_u \in \mathbb{R}$ such that

$$\frac{\alpha+2}{2(2N+\alpha+2-Np)} < s_u < \frac{\alpha+2}{2(N+\alpha+2-Np)} \quad \text{and}$$
$$w := \kappa^{Ns_u}u(\kappa^{2s_u-1}\cdot) \in \mathcal{P}_d.$$

Indeed, $\|w\|_{L^2(\mathbb{R}^N)}^2 = \kappa^N \|u\|_{L^2(\mathbb{R}^N)}^2 = d$. Define the function $\varphi_u : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}\varphi_u(s) &= P\big(\kappa^{Ns} u(\kappa^{2s-1}x)\big)\\ &= \kappa^{4s+N-2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + (N+2)\kappa^{2s(N+2)+N-2}\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx\\ &\quad - \frac{Np-N-\alpha}{2p}\kappa^{2s(Np-N-\alpha)+N+\alpha}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^p\big)|u|^p\,dx,\end{aligned}$$

which is clearly continuous with respect to s. Let $s_* = \frac{\alpha+2}{2(2N+\alpha+2-Np)} < 0$. Since

$$2s_*(N+2)+N-2 = 2s_*(Np-N-\alpha)+N+\alpha,$$

we see from $P(u) = 0$ that

$$\varphi_u(s_*) = \kappa^{4s_*+N-2}(1-\kappa^{2Ns_*})\int_{\mathbb{R}^N}|\nabla u|^2\,dx > 0.$$

Let $s^* = \frac{\alpha+2}{2(N+\alpha+2-Np)} < 0$. Since

$$4s^* + N - 2 = 2s^*(Np-N-\alpha)+N+\alpha,$$

we use $P(u) = 0$ again to obtain

$$\varphi_u(s^*) = (N+2)\kappa^{4s^*+N-2}(\kappa^{2Ns^*}-1)\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx < 0.$$

Then there exists $s_u \in (s_*, s^*)$ such that $\varphi_u(s_u) = 0$, which implies $w := \kappa^{Ns_u}u(\kappa^{2s_u-1}\cdot) \in \mathcal{P}_d$.

By the definition of $m_{c,\mathcal{P}}$, for each $n \in \mathbb{N}$ there exists $u_n \in \mathcal{P}_c$ such that

$$\Phi(u_n) \le m_{c,\mathcal{P}} + \frac{1}{n}.$$

Let $s_n := s_{u_n}$ be the constant such that

$$\frac{\alpha+2}{2(2N+\alpha+2-Np)} < s_n < \frac{\alpha+2}{2(N+\alpha+2-Np)}, \quad w_n := \kappa^{Ns_n}u_n(\kappa^{2s_n-1}\cdot) \in \mathcal{P}_d.$$

Then

$$m_{d,\mathcal{P}} \le \Phi(w_n) = \Phi(u_n) - (\Phi(u_n)-\Phi(w_n)) \le m_{c,\mathcal{P}} + \frac{1}{n} - (\Phi(u_n)-\Phi(w_n)).$$

Since $P(u_n) = 0$ and $P(w_n) = 0$, one has

$$\begin{aligned}&\big(1-\kappa^{4s_n+N-2}\big)\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx + (N+2)\left(1-\kappa^{2s_n(N+2)+N-2}\right)\int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2\,dx\\ &\quad = \frac{Np-N-\alpha}{2p}\left(1-\kappa^{2s_n(Np-N-\alpha)+N+\alpha}\right)\int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^p\big)|u_n|^p\,dx.\end{aligned}$$

Then $\Phi(u_n) - \Phi(w_n)$ can be expressed as

$$\begin{aligned}\Phi(u_n) - \Phi(w_n) &= \frac{1}{2}\left(1 - \kappa^{4s_n+N-2}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx \\ &\quad + \left(1 - \kappa^{2s_n(N+2)+N-2}\right)\int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2\,dx \\ &\quad - \frac{1}{2p}\left(1 - \kappa^{2s_n(Np-N-\alpha)+N+\alpha}\right)\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p\,dx \\ &= \frac{Np-N-\alpha-2}{2(Np-N-\alpha)}\left(1 - \kappa^{4s_n+N-2}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx \\ &\quad + \frac{Np-2N-\alpha-2}{Np-N-\alpha}\left(1 - \kappa^{2s_n(N+2)+N-2}\right)\int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2\,dx.\end{aligned}$$

Since $s_n < \frac{\alpha+2}{2(N+\alpha+2-Np)} < 0$ and $\frac{2N+\alpha+2}{N} < p \le \frac{N+\alpha}{N-2}$, there hold

$$4s_n + N - 2 < \frac{N[N+\alpha-(N-2)p]}{N+\alpha+2-Np} \le 0$$

and

$$2s_n(N+2) + N - 2 < \frac{N[N+2\alpha+2-(N-2)p]}{N+\alpha+2-Np} < 0.$$

Then we see from Lemma 5.9 that $\Phi(u_n)-\Phi(w_n)$ has a positive lower bound and so

$$m_{d,\mathcal{P}} \le m_{c,\mathcal{P}} + \frac{1}{n} - (\Phi(u_n) - \Phi(w_n)) < m_{c,\mathcal{P}}$$

for n sufficiently large. The proof is finished. □

Proof of Theorem 1.5. Let $c > c_4^*$. We have already proved that $\mathcal{P}_c \neq \emptyset$ in Lemma 5.7 and, as a corollary of Lemmas 5.8 and 5.9, $m_{c,\mathcal{P}} > 0$. Let $\{u_n\} \subset \mathcal{P}_c$ be a minimizing sequence for $m_{c,\mathcal{P}}$, i.e.

$$\lim_{n\to\infty} \Phi(u_n) = m_{c,\mathcal{P}}.$$

According to Lemmas 5.8 and 5.9, $\{u_n\}$ is bounded in X. Denote by u_n^* the symmetric decreasing rearrangement of u_n. Clearly, u_n^* satisfies (see, for example, [26, Secs. 3.3, 3.7 and 7.17])

$$\int_{\mathbb{R}^N} |\nabla u_n^*|^2\,dx \le \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx, \quad \int_{\mathbb{R}^N} |u_n^*|^2|\nabla u_n^*|^2\,dx \le \int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2\,dx$$

and

$$\int_{\mathbb{R}^N} |u_n^*|^2\,dx = \int_{\mathbb{R}^N} u_n^2\,dx, \quad \int_{\mathbb{R}^N} (I_\alpha * |u_n^*|^p)|u_n^*|^p\,dx \ge \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p\,dx.$$

Then $\{u_n^*\}$ is also bounded in X, $P(u_n^*) \le P(u_n) = 0$ and $\Phi(u_n^*) \le \Phi(u_n)$. Assume up to a subsequence that $u_n^* \rightharpoonup u^*$ in X, $u_n^* \to u^*$ in $L^q(\mathbb{R}^N)$ with $2 < q < \frac{4N}{N-2}$, and $u_n^* \to u^*$ a.e. in $\mathbb{R}^N$. Using (1.7) and Lemma 2.3 yields that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n^*|^p)|u_n^*|^p \, dx = \int_{\mathbb{R}^N} (I_\alpha * |u^*|^p)|u^*|^p \, dx. \tag{5.12}$$

Combining this with the weak lower semi-continuity, we have

$$P(u^*) \le \liminf_{n\to\infty} P(u_n^*) \le \liminf_{n\to\infty} P(u_n) = 0$$

and

$$\Phi(u^*) \le \liminf_{n\to\infty} \Phi(u_n^*) \le \liminf_{n\to\infty} \Phi(u_n) = m_{c,\mathcal{P}}. \tag{5.13}$$

In order to conclude that u^* is indeed a minimizer of $m_{c,\mathcal{P}}$, it suffices to show $u^* \in \mathcal{P}_c$.

By Lemmas 5.8 and 5.9, there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^N} \left(I_\alpha * |u_n^*|^p\right)|u_n^*|^p \, dx \ge \int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^p\right)|u_n|^p \, dx \ge \delta, \quad \text{for } n \in \mathbb{N}.$$

Then (5.12) implies $u^* \neq 0$ and thus

$$0 < \|u^*\|^2_{L^2(\mathbb{R}^N)} \le \liminf_{n\to\infty} \|u_n^*\|^2_{L^2(\mathbb{R}^N)} = \liminf_{n\to\infty} \|u_n\|^2_{L^2(\mathbb{R}^N)} = c.$$

Suppose that $\beta := \|u^*\|^2_{L^2(\mathbb{R}^N)} < c$. If $P(u^*) = 0$, then by (5.13) and Lemma 5.10, $m_{\beta,\mathcal{P}} \le \Phi(u^*) \le m_{c,\mathcal{P}} < m_{\beta,\mathcal{P}}$, which is absurd. If $P(u^*) < 0$, then by Lemmas 5.5 and 5.6, there exists a unique constant $s^* := s_{u^*} \in (0,1)$ such that $s^* \star u^* \in \mathcal{P}_\beta$. Using Lemma 5.10 and the weak lower semi-continuity yields that

$$\begin{aligned}
m_{\beta,\mathcal{P}} &\le \Phi(s^* \star u^*) - \frac{1}{Np-N-\alpha} P(s^* \star u^*) \\
&< \frac{Np-N-\alpha-2}{2(Np-N-\alpha)} \int_{\mathbb{R}^N} |\nabla u^*|^2 \, dx + \frac{Np-2N-\alpha-2}{Np-N-\alpha} \int_{\mathbb{R}^N} |u^*|^2 |\nabla u^*|^2 \, dx \\
&\le \liminf_{n\to\infty} \left(\Phi(u_n) - \frac{1}{Np-N-\alpha} P(u_n) \right) = m_{c,\mathcal{P}} < m_{\beta,\mathcal{P}},
\end{aligned}$$

which is also absurd. Here we remark that $P(u^*) \le 0$ implies $\mathcal{P}_\beta \neq \emptyset$, regardless of whether $\beta > c_4^*$. Then, as seen from the proof, the conclusion of Lemma 5.10 remains valid and thus $m_{c,\mathcal{P}} < m_{\beta,\mathcal{P}}$ provided that $0 < \beta < c$. Therefore, it must be $\|u^*\|^2_{L^2(\mathbb{R}^N)} = c$. Repeating the above arguments, one can also conclude that $P(u^*) = 0$ and henceforth $u^* \in \mathcal{P}_c$. Therefore, u^* is indeed a minimizer of $m_{c,\mathcal{P}}$. □

Appendix A.

A.1. *Proof of Lemma* 2.3

The following result is taken from [47, Proposition 5.4.7].

Lemma A.1. *Let $q > 1$ and Ω be a domain in $\mathbb{R}^N$. If $\{u_n\}$ is bounded in $L^q(\Omega)$ and*

$$u_n \to u \quad a.e.\ in\ \Omega$$

for some $u \in L^q(\Omega)$, then we have $u_n \rightharpoonup u$ in $L^q(\Omega)$.

Lemma A.2. *Under the assumptions of Lemma* 2.3, *there holds*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left|F(u_n)-F(u_n-u)-F(u)\right|^{\frac{2N}{N+\alpha}}dx = 0. \tag{A.1}$$

Proof. Since $F(0) = 0$ and $|F'(t)| \le C\big(|t|^{\frac{\alpha}{N}} + |t|^{\frac{N+2\alpha+2}{N-2}}\big)$ for $t \in \mathbb{R}$, we have

$$|F(u)| \le C\left(|u|^{\frac{N+\alpha}{N}} + |u|^{\frac{2(N+\alpha)}{N-2}}\right) \tag{A.2}$$

and

$$\begin{aligned}|F(u_n)-F(u_n-u)| &= \left|\int_0^1 F'(u_n-\tau u)u\,d\tau\right|\\ &\le C\int_0^1\left(|u_n-\tau u|^{\frac{\alpha}{N}} + |u_n-\tau u|^{\frac{N+2\alpha+2}{N-2}}\right)|u|\,d\tau\\ &\le C\left(|u_n|^{\frac{\alpha}{N}}|u| + |u|^{\frac{N+\alpha}{N}} + |u_n|^{\frac{N+2\alpha+2}{N-2}}|u| + |u|^{\frac{2(N+\alpha)}{N-2}}\right).\end{aligned}$$

By the Young inequality, we see that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}&|F(u_n)-F(u_n-u)-F(u)|^{\frac{2N}{N+\alpha}}\\ &\quad\le C\left(|u_n|^{\frac{\alpha}{N}}|u| + |u|^{\frac{N+\alpha}{N}} + |u_n|^{\frac{N+2\alpha+2}{N-2}}|u| + |u|^{\frac{2(N+\alpha)}{N-2}}\right)^{\frac{2N}{N+\alpha}}\\ &\quad\le C\left(|u_n|^{\frac{2\alpha}{N+\alpha}}|u|^{\frac{2N}{N+\alpha}} + u^2 + |u_n|^{\frac{2N(N+2\alpha+2)}{(N-2)(N+\alpha)}}|u|^{\frac{2N}{N+\alpha}} + |u|^{\frac{4N}{N-2}}\right)\\ &\quad\le \varepsilon\left(u_n^2 + |u_n|^{\frac{4N}{N-2}} - u^2 - |u|^{\frac{4N}{N-2}}\right) + C_\varepsilon\left(u^2 + |u|^{\frac{4N}{N-2}}\right).\end{aligned}$$

Define

$$\begin{aligned}g_{\varepsilon,n} = \max\Big\{&|F(u_n)-F(u_n-u)-F(u)|^{\frac{2N}{N+\alpha}}\\ &-\varepsilon\left(u_n^2 + |u_n|^{\frac{4N}{N-2}} - u^2 - |u|^{\frac{4N}{N-2}}\right), 0\Big\}.\end{aligned}$$

Then $g_{\varepsilon,n} \to 0$ a.e. in $\mathbb{R}^N$ and

$$0 \le g_{\varepsilon,n} \le C_\varepsilon\left(u^2 + |u|^{\frac{4N}{N-2}}\right) \in L^1(\mathbb{R}^N).$$

By the Lebesgue dominated convergence theorem, we have $g_{\varepsilon,n} \to 0$ in $L^1(\mathbb{R}^N)$. Since

$$|F(u_n)-F(u_n-u)-F(u)|^{\frac{2N}{N+\alpha}} \le g_{\varepsilon,n} + \varepsilon\left(u_n^2 + |u_n|^{\frac{4N}{N-2}}\right),$$

and $\{u_n\}$ is bounded in X, we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left|F(u_n)-F(u_n-u)-F(u)\right|^{\frac{2N}{N+\alpha}}dx\le C\cdot\varepsilon.$$

Since ε is arbitrary, we arrive at the conclusion of (A.1). The proof is finished. □

Proof of Lemma 2.3. We observe that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[(I_\alpha*F(u_n))F(u_n)-(I_\alpha*F(u_n-u))F(u_n-u)-(I_\alpha*F(u))F(u)\right]dx\\ &\quad=\int_{\mathbb{R}^N}(I_\alpha*F(u_n))\left[F(u_n)-F(u_n-u)-F(u)\right]dx\\ &\qquad+\int_{\mathbb{R}^N}(I_\alpha*F(u_n-u))\left[F(u_n)-F(u_n-u)-F(u)\right]dx\\ &\qquad+\int_{\mathbb{R}^N}(I_\alpha*F(u))\left[F(u_n)-F(u_n-u)-F(u)\right]dx\\ &\qquad+2\int_{\mathbb{R}^N}(I_\alpha*F(u))F(u_n-u)dx\\ &\quad\triangleq(\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV}).\end{aligned}$$

It suffices to show that each term on the right hand side is an infinitesimal as $n\to\infty$.

Step 1. (I) $=o_n(1)$, (II) $=o_n(1)$ and (III) $=o_n(1)$ as $n\to\infty$. Since $\{u_n\}$ is bounded in X, we see from (A.2) that $\{F(u_n)\}$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Then, using Lemmas 2.2 and A.2, we have

$$\begin{aligned}|(\mathrm{I})|&=\left|\int_{\mathbb{R}^N}(I_\alpha*F(u_n))\left[F(u_n)-F(u_n-u)-F(u)\right]dx\right|\\ &\le C\|F(u_n)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}\|F(u_n)-F(u_n-u)-F(u)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}\\ &=o_n(1)\end{aligned}$$

and, similarly, (II) $=o_n(1)$ and (III) $=o_n(1)$ as $n\to\infty$.

Step 2. (IV) $=o_n(1)$ as $n\to\infty$. Clearly, $\{F(u_n-u)\}$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ and $F(u_n-u)\to 0$ a.e. in $\mathbb{R}^N$. Then by Lemma A.1 it holds $F(u_n-u)\rightharpoonup 0$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Since $I_\alpha*F(u)\in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, we have

$$(\mathrm{IV})=2\int_{\mathbb{R}^N}(I_\alpha*F(u))F(u_n-u)dx=o_n(1)$$

as $n\to\infty$, completing the proof. □

Similar to Lemma 2.3, we also have

Lemma A.3. *Assume that $F \in C^1(\mathbb{R},\mathbb{R})$ satisfies $F(0)=0$ and there exists $C>0$ such that $|F'(t)| \le C\big(|t|^{\frac{\alpha}{N}} + |t|^{\frac{\alpha+2}{N-2}}\big)$ for $t \in \mathbb{R}$. If $\{u_n\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$ for some $u \in H^1(\mathbb{R}^N)$, then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}[(I_\alpha * F(u_n))F(u_n) - (I_\alpha * F(u_n-u))F(u_n-u) - (I_\alpha * F(u))F(u)]dx = 0.$$

A.2. *Proof of* (4.6)

We prove the result by using the Moser iteration.

Proof of (4.6)**.** We divide the proof into two steps.

Step 1. There exists a constant $C>0$ such that

$$\|I_\alpha * |u_n|^p\|_{L^\infty(\mathbb{R}^N)} \le C, \quad \text{for } n \in \mathbb{N}. \tag{A.3}$$

Indeed, the assumption (1.6) indicates that

$$2 \le \frac{N+\alpha}{N-2} < p < \frac{2N+\alpha+2}{N} \le \frac{4\alpha}{N-2}.$$

Let $r \in \mathbb{R}$ such that $1 < \frac{N}{\alpha} < r < \frac{4N}{(N-2)p}$. Then

$$2 < p < pr < \frac{4N}{N-2} \quad \text{and} \quad 0 < \frac{(N-\alpha)r}{r-1} < N.$$

Set $A_\alpha = \Gamma(\frac{N-\alpha}{2})/\big(2^\alpha \pi^{\frac{N}{2}}\Gamma(\frac{\alpha}{2})\big)$. Using the Hölder inequality yields that

$$\begin{aligned}
&(I_\alpha * |u_n|^p)(x)\\
&\quad = A_\alpha \int_{B_1(x)} \frac{|u_n(y)|^p}{|x-y|^{N-\alpha}}\,dy + A_\alpha \int_{\mathbb{R}^N\setminus B_1(x)} \frac{|u_n(y)|^p}{|x-y|^{N-\alpha}}\,dy\\
&\quad \le A_\alpha \left(\int_{B_1(x)} |u_n|^{pr}\,dy\right)^{\frac{1}{r}} \left(\int_{B_1(0)} |y|^{-\frac{(N-\alpha)r}{r-1}}\,dy\right)^{\frac{r-1}{r}} + A_\alpha \int_{\mathbb{R}^N\setminus B_1(x)} |u_n|^p\,dy\\
&\quad \le C\left(\|u_n\|^p_{L^{pr}(\mathbb{R}^N)} + \|u_n\|^p_{L^p(\mathbb{R}^N)}\right).
\end{aligned}$$

Then the conclusion of (A.3) follows easily, since $\{u_n\}$ is bounded in X.

Step 2. $u_n \in L^q(\mathbb{R}^N)$ and $\{u_n\}$ is bounded in $L^q(\mathbb{R}^N)$ for any $q \ge \frac{4N}{N-2}$. Since $u_n \in \mathcal{S}_{c_n}$ is a minimizer of m_{c_n}, there exists $\lambda_n \in \mathbb{R}$ such that u_n weakly solves the equation $\Phi'(u) + \lambda_n u = 0$, i.e.

$$\int_{\mathbb{R}^N}\left[(1+2u_n^2)\nabla u_n \cdot \nabla\phi + 2|\nabla u_n|^2 u_n\phi + \lambda_n u_n\phi - (I_\alpha * |u_n|^p)|u_n|^{p-2}u_n\phi\right]dx = 0$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$. Using an approximation argument, one can take any $\phi \in X$ satisfying $\int_{\mathbb{R}^N} u_n^2|\nabla\phi|^2\,dx < \infty$ and $\int_{\mathbb{R}^N} |\nabla u_n|^2\phi^2\,dx < \infty$ as test functions. A standard argument shows that the following Pohožaev-type identity holds:

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx+(N-2)\int_{\mathbb{R}^N}u_n^2|\nabla u_n|^2\,dx$$
$$+\frac{\lambda_n N}{2}\int_{\mathbb{R}^N}u_n^2\,dx-\frac{N+\alpha}{2p}\int_{\mathbb{R}^N}\big(I_\alpha*|u_n|^p)|u_n|^p\,dx=0.$$

Then

$$\begin{aligned}0\geq m_{c_n}&=\Phi(u_n)\\&=\frac{2+\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx+\frac{2+\alpha}{N+\alpha}\int_{\mathbb{R}^N}u_n^2|\nabla u_n|^2\,dx-\frac{\lambda_n N c_n}{2(N+\alpha)},\end{aligned}$$

from which it follows that $\lambda_n>0$.

For $M>0$, let u_n^M be the truncation of u_n defined by $u_n^M=\max\{-M,\min\{u_n,M\}\}$. Choosing $\phi=|u_n^M|^\gamma u_n^M$ with $\gamma>1$ as a test function and using the facts $|u_n^M|\leq|u_n|$, $\nabla u_n\nabla u_n^M=|\nabla u_n^M|^2$ and $\lambda_n>0$, we deduce that

$$(\gamma+1)\int_{\mathbb{R}^N}(1+2|u_n^M|^2)|u_n^M|^\gamma|\nabla u_n^M|^2\,dx\leq\int_{\mathbb{R}^N}(I_\alpha*|u_n|^p)|u_n|^{p+\gamma}\,dx.$$

Since $\|I_\alpha*|u_n|^p\|_{L^\infty(\mathbb{R}^N)}\leq C$ by (A.3), we have

$$2(\gamma+1)\int_{\mathbb{R}^N}|u_n^M|^{\gamma+2}|\nabla u_n^M|^2\,dx\leq C\int_{\mathbb{R}^N}|u_n|^{p+\gamma}\,dx.$$

Then by the Sobolev inequality,

$$\left(\int_{\mathbb{R}^N}|u_n^M|^{\frac{N(\gamma+4)}{N-2}}\,dx\right)^{\frac{N-2}{N}}\leq C\gamma\int_{\mathbb{R}^N}|u_n|^{p+\gamma}\,dx.$$

Letting $M\to+\infty$, we obtain

$$\left(\int_{\mathbb{R}^N}|u_n|^{\frac{N(\gamma+4)}{N-2}}\,dx\right)^{\frac{N-2}{N}}\leq C\gamma\int_{\mathbb{R}^N}|u_n|^{p+\gamma}\,dx.\tag{A.4}$$

Now we carry out an iteration process by setting $\gamma_0=\frac{4N}{N-2}-p>1$ and

$$\gamma_j=\frac{N-2}{2}\left(\left(\frac{N}{N-2}\right)^{j+1}-1\right)\gamma_0>1,\quad\text{for } j=1,2,\ldots,$$

Clearly, $p+\gamma_j=\frac{N(\gamma_{j-1}+4)}{N-2}$ for $j=1,2,\ldots$. Then it follows from (A.4) that

$$\|u_n\|_{L^{p+\gamma_j}(\mathbb{R}^N)}\leq(C\gamma_{j-1})^{\frac{1}{\gamma_{j-1}+4}}\|u_n\|_{L^{p+\gamma_{j-1}}(\mathbb{R}^N)}^{\frac{\gamma_{j-1}+p}{\gamma_{j-1}+4}}.$$

Doing iteration yields that

$$\|u_n\|_{L^{p+\gamma_j}(\mathbb{R}^N)}\leq\prod_{k=0}^{j-1}(C\gamma_k)^{\frac{1}{\gamma_{j-1}+4}\cdot\left(\frac{N}{N-2}\right)^{j-1-k}}\|u_n\|_{L^{p+\gamma_0}(\mathbb{R}^N)}^{\frac{\gamma_0+p}{\gamma_{j-1}+4}\cdot\left(\frac{N}{N-2}\right)^{j-1}}$$

Note that $\gamma_j\to+\infty$ as $j\to\infty$, $p+\gamma_0=\frac{4N}{N-2}$ and $\{u_n\}$ is bounded in X. Using the Hölder inequality, we conclude that $u_n\in L^q(\mathbb{R}^N)$ and $\{u_n\}\subset L^q(\mathbb{R}^N)$ is bounded for any $q\geq\frac{4N}{N-2}$. The proof is finished. □

A.3. *From local minimizer to solution of* (1.1)

Lemma A.4. *If $u \in \mathcal{P}_c$ is minimizer of $m_{c,\mathcal{P}}$, then u is a normalized solution of* (1.1), *i.e.* $\langle \Phi'(u), \phi\rangle + \lambda \int_{\mathbb{R}^N} u\phi\, dx = 0$ *for all* $\phi \in C_0^\infty(\mathbb{R}^N)$, *where the Lagrange multiplier λ is given by* $\lambda = -\frac{\langle \Phi'(u), u\rangle}{c}$.

Proof. Suppose by contradiction that there exists $\phi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\langle \Phi'(u), \phi\rangle + \lambda \int_{\mathbb{R}^N} u\phi\, dx = -3. \tag{A.5}$$

Let $\zeta \in C_0^\infty(\mathbb{R}, [0,1])$ be a cut-off function satisfying $\zeta(s) = 1$ if $|s-1| \le \frac{1}{4}$ and $\zeta(s) = 0$ if $|s-1| \ge \frac{1}{2}$. Let $\varepsilon_1 > 0$ be sufficiently small and define a continuous map $g(s,t) : \left[\frac{1}{4}, \frac{7}{4}\right] \times [0, \varepsilon_1] \to \mathcal{S}_c$ by

$$g(s,t) = \frac{\sqrt{c}\,(s \star (u + t\zeta(s)\phi))}{\|s \star (u + t\zeta(s)\phi)\|_{L^2(\mathbb{R}^N)}}.$$

Then $g(s,0) = s \star u$ for $s \in \left[\frac{1}{4}, \frac{7}{4}\right]$.

By relation (A.5), there exist $\varepsilon_0 \in \left(0, \frac{1}{4}\right)$ and $\varepsilon_2 \in (0, \varepsilon_1)$ such that

$$\frac{\partial}{\partial t}\Phi(g(s,t)) = \left\langle \Phi'(g(s,t)), \frac{\partial}{\partial t} g(s,t)\right\rangle \le -2$$

for $(s,t) \in [1-\varepsilon_0, 1+\varepsilon_0] \times [0, \varepsilon_2]$. Choose a new cut-off function $\eta \in C_0^\infty(\mathbb{R}, [0,1])$ such that $\eta(s) = 1$ if $|s-1| \le \varepsilon_0$ and $\eta(s) = 0$ if $|s-1| \ge 2\varepsilon_0$. Define a new continuous map $h(s,t) : [1-2\varepsilon_0, 1+2\varepsilon_0] \times [0, \varepsilon_2] \to \mathcal{S}_c$ by

$$h(s,t) = \frac{\sqrt{c}\,(s \star (u + t\eta(s)\phi))}{\|s \star (u + t\eta(s)\phi)\|_{L^2(\mathbb{R}^N)}}.$$

Clearly, $h(s,0) = s \star u$ for $s \in [1-2\varepsilon_0, 1+2\varepsilon_0]$ and

$$\frac{\partial}{\partial t}\Phi(h(s,t)) = \left\langle \Phi'(h(s,t)), \frac{\partial}{\partial t} h(s,t)\right\rangle = \left\langle \Phi'(g(s,t)), \frac{\partial}{\partial t} g(s,t)\right\rangle \le -2 \tag{A.6}$$

for $(s,t) \in [1-\varepsilon_0, 1+\varepsilon_0] \times [0, \varepsilon_2]$.

Claim. *There exists $\varepsilon_3 \in (0, \varepsilon_2)$ such that*

$$\Phi(h(s, \varepsilon_3)) < m_{c,\mathcal{P}}, \quad \textit{for } s \in [1-2\varepsilon_0, 1+2\varepsilon_0]. \tag{A.7}$$

Indeed, combining Lemmas 5.5 and 5.6 with (A.6) yields

$$\Phi(h(s,t)) = \Phi(h(s,0)) + \int_0^t \left\langle \Phi'(h(s,\tau)), \frac{\partial}{\partial \tau} h(s,\tau)\right\rangle d\tau \le \Phi(u) - 2t < m_{c,\mathcal{P}}$$

for $(s,t) \in [1-\varepsilon_0, 1+\varepsilon_0] \times (0, \varepsilon_2]$. By Lemmas 5.5 and 5.6 again, there exists a constant $\delta > 0$ independent of s such that $\Phi(h(s,0)) = \Phi(s \star u) \le m_{c,\mathcal{P}} - 2\delta$ if $\varepsilon_0 \le |s-1| \le 2\varepsilon_0$. Using the continuity of $\Phi \circ h$, we conclude that there exists $\varepsilon_3 \in (0, \varepsilon_2)$ such that

$$\Phi(h(s,t)) \le m_{c,\mathcal{P}} - \delta < m_{c,\mathcal{P}}$$

for $(s,t) \in ([1-2\varepsilon_0, 1-\varepsilon_0] \cup [1+\varepsilon_0, 1+2\varepsilon_0]) \times [0, \varepsilon_3]$. Therefore, the claim is proved.

We define the homotopy $H : [1 - 2\varepsilon_0, 1 + 2\varepsilon_0] \times [0, 1] \to \mathbb{R}$ by

$$H(s, \tau) = P(h(s, \tau\varepsilon_3)).$$

It follows from the definition of h and Lemmas 5.5 and 5.6 that

$$H(1 - 2\varepsilon_0, \tau) = P((1 - 2\varepsilon_0) \star u) > 0, \quad H(1 + 2\varepsilon_0, \tau) = P((1 + 2\varepsilon_0) \star u) < 0$$

for all $\tau \in [0, 1]$. The homotopy invariance gives

$$\deg(H(\cdot, 1), (1 - 2\varepsilon_0, 1 + 2\varepsilon_0), 0) = \deg(H(\cdot, 0), (1 - 2\varepsilon_0, 1 + 2\varepsilon_0), 0) = -1.$$

Then, by the existence property, there exists $s_0 \in (1 - 2\varepsilon_0, 1 + 2\varepsilon_0)$ such that

$$P(h(s_0, \varepsilon_3)) = H(s_0, 1) = 0,$$

i.e. $h(s_0, \varepsilon_3) \in \mathcal{P}_c$.

Consequently, $\Phi(h(s_0, \varepsilon_3)) \geq m_{c,\mathcal{P}}$, which contradicts (A.7). □

ORCID

Haidong Liu https://orcid.org/0000-0001-6453-2872
Vicenţiu D. Rădulescu https://orcid.org/0000-0003-4615-5537

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