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A critical fractional Choquard–Kirchhoff problem with magnetic field

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In this paper, we are interested in a fractional Choquard–Kirchhoff-type problem involving an external magnetic potential and a critical nonlinearity

$$M(\|u\|_{s,A}^2)[(-\Delta)_A^s u + u] = \lambda \int_{\mathbb{R}^N} \frac{F(|u|^2)}{|x-y|^\alpha} dy f(|u|^2)u + |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^N,$$

$$\|u\|_{s,A} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} |u|^2 dx \right)^{1/2},$$

where $N > 2s$ with $0 < s < 1$, M is the Kirchhoff function, A is the magnetic potential, $(-\Delta)_A^s$ is the fractional magnetic operator, f is a continuous function, $F(|u|) = \int_0^{|u|} f(t)dt$, $\lambda > 0$ is a parameter, $0 < \alpha < \min\{N, 4s\}$ and $2_s^* = \frac{2N}{N-2s}$ is the critical exponent of fractional Sobolev space. We first establish a fractional version of the concentration-compactness principle with magnetic field. Then, together with the

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mountain pass theorem, we obtain the existence of nontrivial radial solutions for the above problem in non-degenerate and degenerate cases.

Keywords: Choquard–Kirchhoff equation; fractional magnetic operator; variational methods; critical Sobolev exponent.

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1. Introduction and Main Results

In this paper, we consider the existence of solutions to the following Choquard–Kirchhoff-type problem:

$$M(\|u\|_{s,A}^2)[(-\Delta)_A^s u + u] = \lambda(\mathcal{K}_\alpha * F(|u|^2)f(|u|^2))u + |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $V : \mathbb{R}^N \to \mathbb{R}^+$ is the scalar potential, $\mathcal{K}_\alpha(x) = |x|^{-\alpha}$, $A : \mathbb{R}^N \to \mathbb{R}^N$ is the magnetic potential, $\lambda > 0$ is a real parameter and $(-\Delta)_A^s$ is the fractional magnetic operator which, up to normalization, may be defined as

$$(-\Delta)_A^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|^{N+2s}} dy, \quad \forall\, x \in \mathbb{R}^N,$$

whenever $u \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$, see [12] and the references therein for further details on these kinds of operators. Here $s \in (0, 1)$, $N > 2s$ and $B_\varepsilon(x)$ denotes a ball in $\mathbb{R}^N$ with radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^N$. Clearly, the operator $(-\Delta)_A^s$ is consistent with the definition of fractional Laplacian $(-\Delta)^s$ if $A \equiv 0$. For more details on the fractional Laplacian, we refer to [14]. The fractional Laplacian operator $(-\Delta)^s$ can be seen as the infinitesimal generators of Lévy stable diffusion processes (see [1]). Moreover, the fractional Laplacian allows to develop a generalization of quantum mechanics and also to describe the motion of a chain or array of particles that are connected by elastic springs and unusual diffusion processes in turbulent fluid motions and material transports in fractured media (for more details see for example [4, 9] and the references therein). In fact, a great attention has been focused on the study of fractional and nonlocal operators of elliptic type in recent years. This type of operators arises in a quite natural way in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastically stabilization of Lévy processes, see [4, 9]. The literature on nonlocal operators and on their applications is very interesting and quite large, we refer the interested readers to [10, 15, 28, 33] and the references therein. For the basic properties of fractional Sobolev spaces and their applications by variational methods, we refer the readers to [14, 27].

On the one hand, this paper is motivated by some works that appeared in recent years concerning the following fractional Schrödinger equations with magnetic field:

$$(-\Delta)_A^s u + V(x)u = f(u), \tag{1.2}$$

where the magnetic Schrödinger operator is defined as

$$-(\nabla - \mathrm{i}A)^2 u = -\Delta u + 2\mathrm{i}A(x)\cdot\nabla u + |A(x)|^2 u + \mathrm{i}u\,\mathrm{div}A(x).$$

As stated in [34], up to correcting the operator with factor $(1-s)$ it follows that $(-\Delta)^s_A u$ converges to $-(\nabla u - \mathrm{i}A)^2 u$ in the limit $s \uparrow 1$. Thus, up to normalization, we may think the nonlocal case as an approximation of the local case. If $A \equiv 0$, then (1.2) becomes the fractional Schrödinger equation, which was proposed by Laskin [21, 22] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. In the last 10 years, there has been a lot of interest in the study of Eq. (1.2), see for instance, [7, 10, 15, 33, 39]. If the interaction between the particles is considered, that is, $f(u) = (\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u$, this kind of problem is usually named Choquard equation and has been investigated by many authors, see [5, 11, 13, 20].

On the other hand, Lü in [25] studied the following Kirchhoff-type equation

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V_\lambda(x)u = (\mathcal{K}_\alpha * u^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

where $a \in \mathbb{R}^+$, $b \in \mathbb{R}^+_0$ are given numbers, $V_\lambda(x) = 1+\lambda g(x)$, $\lambda \in \mathbb{R}^+$ is a parameter and $g(x)$ is a continuous potential function on $\mathbb{R}^3$, $q \in (2, 6-\alpha)$. By using the Nehari manifold and the concentration-compactness principle, the author obtained the existence of ground state solutions for (1.3) if the parameter λ is large enough. Indeed, problem (1.1) is related to the Kirchhoff equation proposed by Kirchhoff in 1883 as a generalization of the well-known D'Alembert's wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{\lambda} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.4}$$

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and p_0 is the initial tension. In [1], it was pointed out that the problem (1.4) models several physical systems, where u describes a process which depends on the average of itself. Nonlocal effect also finds its applications in biological systems. A parabolic version of Eq. (1.4) can be used to describe the growth and movement of a particular species. The movement, modeled by the integral term, is assumed to be dependent on the energy of the entire system with u being its population density. In [17], Fiscella and Valdinoci first proposed a stationary Kirchhoff variational model, in bounded regular domains of $\mathbb{R}^N$, which takes into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. For more details about stationary Kirchhoff problems involving the fractional Laplacian, we refer the interested readers to [16, 30, 37]. Recently, the solvability or multiplicity of the Kirchhoff-type equations with critical growth has been paid much attention by many authors, see [6, 31].

In [26], Mingqi *et al.* first investigated the existence and multiplicity of solutions for fractional Schrödinger–Kirchhoff-type equation with an external magnetic potential. Subsequently, Pucci, Xiang and Zhang in [32] studied the following fractional p-Laplacian equation of *Schrödinger–Choquard–Kirchhoff* type:

$$\begin{aligned} &M(\|u\|_s^p)[(-\Delta)_p^s u + V(x)|u|^{p-2}u] \\ &\quad = \lambda f(x,u) + (\mathcal{K}_\alpha * |u|^{p^*_{\alpha,s}})|u|^{p^*_{\alpha,s}-2}u \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{1.5}$$

where $p^*_{\alpha,s} = (pN - p\alpha/2)/(N - ps)$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality. The authors obtained several existence results for Eq. (1.5) by using variational methods in non-degenerate and degenerate cases. As $M(t) = a + bt$ and $p = 2$, with the help of variational methods, Wang and Xiang in [36] investigated the existence of two solutions and infinitely many solutions for problem (1.5) with external magnetic operator in non-degenerate and degenerate cases.

Motivated by the above works, we are devoted to the study of radial solutions for problem (1.1) involving an external magnetic field and critical Sobolev exponent. For this purpose, we first establish a fractional version of concentration-compactness principle with magnetic field. Then, together with the mountain pass theorem, we obtain the existence of nontrivial radial solutions for problem (1.1). It is worth stressing that the appearance of the magnetic field also brings additional difficulties to the problem. For example, the effects of the magnetic fields on the linear spectral sets and on the solution structure, and the possible interactions between the magnetic fields and the linear potentials. Concerning the study of elliptic equations with critical Sobolev exponent, we refer to the pioneering contributions of Brézis and Nirenberg [8].

Now we begin with the assumptions on the Kirchhoff function M:

(M_1) $M : \mathbb{R}^+_0 \to \mathbb{R}^+$ is continuous and there exists $m_0 > 0$ such that $\inf_{t\geq 0} M(t) = m_0$.

(M_2) There exists $\theta \in [1, 2^*_s/2)$ such that

$$M(t)t \leq \theta \mathscr{M}(t), \quad \forall\, t \in \mathbb{R}^+_0,$$

where $\mathscr{M}(t) = \int_0^t M(\tau)d\tau$.

A typical example is given by $M(t) = m_0 + bt^{\theta-1}$, where $b \in \mathbb{R}^+_0$, $t \in \mathbb{R}^+_0$. If $M(t) = a + bt$ with $a > 0, b \geq 0$, for all $t \geq 0$, $f(u) = |u|^{p-2}u$ and $s \nearrow 1^-$, then (1.1) reduces to the following equation:

$$(a + b\|u\|^2)[-(\nabla - \mathrm{i}A)^2 u + u] = \lambda(\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u + |u|^{2^*-2}u, \tag{1.6}$$

where $2^* = 2N/(N-2)$. Hence problem (1.1) can be regarded as a fractional version of Eq. (1.6). In particular, when $b = 0$, Eq. (1.6) without the critical term has been studied by some authors recently, see for example [2, 25]. Here we call Eq. (1.1) is non-degenerate if $a > 0$, $b \geq 0$, while Eq. (1.1) is degenerate if $a = 0$, $b > 0$.

Moreover, we impose the following assumptions on the nonlinearity f:

(f_1) $f \in C(\mathbb{R}^+, \mathbb{R})$;
(f_2) There exist $C > 0$ and $p \in (2, \frac{2N-\alpha}{N-2s})$ such that

$$|f(t)| \le C(1 + t^{\frac{p-2}{2}}) \quad \text{for all}\, t \in \mathbb{R}_0^+;$$

(f_3) There exists $\sigma \in (2\theta, 2_s^*)$ such that $0 < \sigma F(t) \le f(t)t^2$ whenever $t \in \mathbb{R}^+$, where $F(t) = \int_0^t f(\tau)\tau d\tau$.

To state our main results, we first give the definition of (weak) solutions for problem (1.1).

Definition 1.1. We say that $u \in H_A^s(\mathbb{R}^N, \mathbb{C})$ is a (weak) solution of problem (1.1), if

$$M(\|u\|_{s,A}^2)\Re\left[\iint_{\mathbb{R}^{2N}} \frac{(u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y))\overline{(\varphi(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\varphi(y))}}{|x-y|^{N+2s}}dxdy\right.$$
$$\left. + \int_{\mathbb{R}^N} u\overline{\varphi}dx\right]$$
$$= \Re\int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u|^2))f(|u|^2)u\overline{\varphi}dx + \Re\int_{\mathbb{R}^N} |u|^{2_s^*-2}u\overline{\varphi}dx,$$

for any $\varphi \in H_A^s(\mathbb{R}^N, \mathbb{C})$.

Theorem 1.1 (Non-degenerate case). *Let $s \in (0,1)$, $N > 2s$ and $0 < \alpha < \min\{N, 4s\}$. Assume that $A \in C(\mathbb{R}^N, \mathbb{R}^N)$, M satisfies (M_1) and (M_2), f satisfies (f_1)–(f_3) with $2\theta < p < \frac{2N-\alpha}{N-2s}$. Then there exists $\lambda^* > 0$ such that for any $\lambda \ge \lambda^*$ problem (1.1) has a nontrivial radial solution in $H_A^s(\mathbb{R}^N, \mathbb{C})$.*

Next we consider the degenerate case for problem (1.1). To this aim, we also require:

(M_3) there exists $m_1 > 0$ such that $M(t) \ge m_1 t^{\theta-1}$ for all $t \in \mathbb{R}^+$ and $M(0) = 0$.

For the nonlinearity f, we also need the following hypothesis:

(f_4) there exist $C > 0$ and $\max\{2, \theta\} < p < 2_s^*$ such that

$$|f(t)| \le C|t|^{\frac{p-2}{2}} \quad \text{for all}\, t \in \mathbb{R}_0^+.$$

Our second result reads as follows.

Theorem 1.2 (Degenerate case). *Let $s \in (0,1)$, $N > 2s$ and $0 < \alpha < N$. Assume that $A \in C(\mathbb{R}^N, \mathbb{R}^N)$, M satisfies (M_2) and (M_3), f satisfies $(f_1), (f_3)$ and (f_4). Then there exists $\lambda^* > 0$ such that for any $\lambda \ge \lambda^*$ problem (1.1) has a nontrivial radial solution in $H_A^s(\mathbb{R}^N, \mathbb{C})$.*

Finally, we would like to point out that it remains an open problem to verify the multiplicity of solutions or the existence of sign-changing solutions for problem (1.1). In particular, the existence of infinitely many solutions for problem (1.1) would be interesting. All these problems will be investigated in a future work by the authors.

This paper is organized as follows. In Sec. 2, we recall some necessary definitions and properties of spaces $H^s(\mathbb{R}^N)$ and $H^s_A(\mathbb{R}^N,\mathbb{C})$. In Sec. 3, we establish the principle of concentration-compactness in fractional Sobolev space $H^s_A(\mathbb{R}^N,\mathbb{C})$. In Secs. 4 and 5, we give the proofs of Theorems 1.1 and 1.2, respectively.

2. Preliminaries

In this section, we first give some basic results of fractional Sobolev spaces that will be used in the next sections. Let $0 < s < 1$ be real number satisfying $2s < N$ and the fractional critical exponent 2^*_s be defined as $2^*_s = \frac{2N}{N-2s}$. Let $L^2(\mathbb{R}^N)$ denote the Lebesgue space of real-valued functions with $\int_{\mathbb{R}^N} |u|^2 dx < \infty$. The fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined as follows:

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_s < \infty\},$$

where $[u]_s$ denotes the Gagliardo semi-norm

$$[u]_s = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{\frac{1}{2}},$$

equipped with the inner product

$$(u,v)_s = \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} uvdx, \quad \forall\, u,v \in H^s(\mathbb{R}^N)$$

and the norm

$$\|u\|_s = (\|u\|^2_{L^2(\mathbb{R}^N)} + [u]^2_s)^{\frac{1}{2}}.$$

The embedding $H^s(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ is continuous for any $\nu \in [2, 2^*_s]$ by [14, Theorem 6.7], that is, there exists a positive constant C_ν such that

$$\|u\|_{L^\nu(\mathbb{R}^N)} \le C_\nu \|u\|_s \quad \text{for all}\, u \in H^s(\mathbb{R}^N). \tag{2.1}$$

To prove the existence of radial weak solutions of Eq. (1.1), we need the following embedding theorem due to Lions in [23, Théorème II.1]. For further comments we refer to [35].

Theorem 2.1. *Let $0 < s < 1$ be real numbers with $2s < N$. Then for any $2 < \nu < 2^*_s$, there is a compact embedding*

$$H^s_r(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^\nu(\mathbb{R}^N),$$

where

$$H^s_r(\mathbb{R}^N) = \{u \in H^s(\mathbb{R}^N) : u(x) = u(|x|),\ \forall\, x \in \mathbb{R}^N\}.$$

Let $L^2(\mathbb{R}^N,\mathbb{C})$ be the Lebesgue space of complex-valued functions with $\int_{\mathbb{R}^N}|u|^2dx<\infty$ endowed with the scalar product

$$\langle u,v\rangle_{L^2}:=\Re\int_{\mathbb{R}^N}u\bar{v}dx\quad\text{for all}\,u,v\in L^2(\mathbb{R}^N,\mathbb{C}),$$

where the bar denotes complex conjugation. Suppose that $A:\mathbb{R}^N\to\mathbb{R}^N$ is a continuous function. Consider the magnetic Gagliardo semi-norm defined by

$$[u]_{s,A}:=\left(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{1/2},$$

and set

$$\mathcal{H}=\{u\in L^2(\mathbb{R}^N,\mathbb{C}):[u]_{s,A}<\infty\},$$

endowed with the norm

$$\|u\|_{s,A}:=(\|u\|_{L^2}^2+[u]_{s,A}^2)^{1/2},$$

where $\|u\|_{L^2}=(\int_{\mathbb{R}^N}|u|^2dx)^{1/2}$. The scalar product on $\mathcal{H}$ defined by

$$(u,v)_{s,A}:=\langle u,v\rangle_{L^2}+\langle u,v\rangle_{s,A},$$

where

$$\langle u,v\rangle_{s,A}=\Re\iint_{\mathbb{R}^{2N}}\frac{(u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y))\overline{(v(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}v(y))}}{|x-y|^{N+2s}}dxdy.$$

By [12, Proposition 2.1], we know $(\mathcal{H},(\cdot,\cdot)_{s,A})$ is a real Hilbert space. Moreover, the space $C_c^\infty(\mathbb{R}^N,\mathbb{C})$ is a subspace of $\mathcal{H}$, see [12, Proposition 2.2].

Now we define $H_A^s(\mathbb{R}^N)$ as the closure of $C_c^\infty(\mathbb{R}^N,\mathbb{C})$ in $\mathcal{H}$.

Lemma 2.1. *For each $u\in H_A^s(\mathbb{R}^N)$ it holds $|u|\in H^s(\mathbb{R}^N)$. More precisely,*

$$\||u|\|_s\le\|u\|_{s,A},\quad \textit{for all } u\in H_A^s(\mathbb{R}^N).$$

Proof. The proof follows by using the pointwise diamagnetic inequality:

$$||u(x)|-|u(y)||\le|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|,$$

for a.e. $x,y\in\mathbb{R}^N$, see [12, Lemma 3.1, Remark 3.2]. □

Following Lemma 2.1 and using the same discussion as in [12, Lemma 3.5], we have the following embedding result.

Lemma 2.2. *The embedding $H_A^s(\mathbb{R}^N,\mathbb{C})\hookrightarrow L^p(\mathbb{R}^N,\mathbb{C})$ is continuous for all $p\in[2,2_s^*]$. Furthermore, $H_A^s(\mathbb{R}^N,\mathbb{C})\hookrightarrow L^p(K,\mathbb{C})$ is compact for all $1\le p<2_s^*$ and any compact set $K\subset\mathbb{R}^N$.*

By Lemma 2.1, Theorem 2.1 and the Brézis–Lieb Lemma, we have the following result.

Lemma 2.3. *Set*

$$H^s_{r,A}(\mathbb{R}^N,\mathbb{C}) := \{u \in H^s_A(\mathbb{R}^N,\mathbb{C}) : u(x) = u(|x|),\ \forall\, x \in \mathbb{R}^N\}.$$

Then for any $\nu \in (2, 2^*_s)$ the embedding $H^s_{r,A}(\mathbb{R}^N,\mathbb{C}) \hookrightarrow L^\nu(\mathbb{R}^N,\mathbb{C})$ is compact.

Lemma 2.4. *Assume $1 < r,\ t < \infty$ and $0 < \alpha < N$ with $\frac{1}{r} + \frac{1}{t} + \frac{\alpha}{N} = 2$. If $u \in L^r(\mathbb{R}^N)$ and $v \in L^t(\mathbb{R}^N)$, then there exists $C(N,\alpha,r,t) > 0$ such that*

$$\iint_{\mathbb{R}^N} \frac{|u(x)||v(y)|}{|x-y|^\alpha} dxdy \le C(N,\alpha,r,t)\|u\|_{L^r(\mathbb{R}^N)}\|v\|_{L^t(\mathbb{R}^N)}.$$

3. The Concentration-Compactness Principle with Magnetic Operator

In [24], Lions established the principle of concentration-compactness in classical Sobolev space, and then the concentration-compactness principle was well used to solve elliptic problems involving critical exponent, see also [18]. In [29], the authors established the principle of concentration-compactness in fractional Sobolev spaces by using profile decomposition. In [38], Xiang, Zhang and Zhang established the concentration-compactness principle in fractional Sobolev space, which can be regarded as the fractional version of the principle of concentration-compactness in classical Sobolev space. However, their version of concentration-compactness principle cannot be directly applied to solve our problem because of the presence of a magnetic field. To this end, we will establish the concentration-compactness principle in $H^s_A(\mathbb{R}^N)$ with magnetic operator.

Let $C_c(\mathbb{R}^N)$ be the functions in $C(\mathbb{R}^N)$ with compact support sets and denote by $C_0(\mathbb{R}^N)$ the closure of $C_c(\mathbb{R}^N)$ with respect to the norm $|\eta|_\infty = \sup_{x\in\mathbb{R}^N} |\eta(x)|$. As is known to all, a finite measure on $\mathbb{R}^N$ is a continuous linear functional on $C_0(\mathbb{R}^N)$. Now we give a norm for measure μ

$$\|\mu\| = \sup_{C_0(\mathbb{R}^N),|\eta|_\infty=1} |(\mu\cdot\eta)|,$$

where $(\mu,\eta) = \int_{\mathbb{R}^N} \eta d\mu$.

From now on, we shortly denote by $\|\cdot\|_q$ the norm of $L^q(\mathbb{R}^N)$.

Definition 3.1. Let $\mathcal{M}(\mathbb{R}^N)$ denote the finite non-negative Borel measure space on $\mathbb{R}^N$. For any $\mu \in \mathcal{M}(\mathbb{R}^N)$, $\mu(\mathbb{R}^N) = \|\mu\|$ holds. We say that $\mu_n \rightharpoonup \mu$ weakly * in $\mathcal{M}(\mathbb{R}^N)$, if $(\mu_n,\eta) \to (\mu,\eta)$ holds for all $\eta \in C_0(\mathbb{R}^N)$ as $n \to \infty$.

Theorem 3.1. *Let $\{u_n\}_n \subset H^s_A(\mathbb{R}^N,\mathbb{C})$ such that $\|u_n\|_{s,A} \le C$ for all $n \ge 1$, where C is a positive constant. Put $\mu_n(x) = \int_{\mathbb{R}^N} \frac{|u_n(x)-e^{\mathrm{i}(x-y)A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}}dy + |u_n(x)|^2$,*

$x \in \mathbb{R}^N,\ n \in \mathbb{N}$. *Assume*

$$u_n \rightharpoonup u \quad \textit{weakly in } H^s_A(\mathbb{R}^N, \mathbb{C}),$$
$$\mu_n \rightharpoonup \mu \quad \textit{weakly * in } \mathcal{M}(\mathbb{R}^N),$$
$$|u_n|^{2^*_s} \rightharpoonup \nu \quad \textit{weakly * in } \mathcal{M}(\mathbb{R}^N),$$

then

$$\mu = \int_{\mathbb{R}^N} \frac{|u(x) - e^{\mathrm{i}(x-y)A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}} dy + |u|^2 + \sum_{j\in J} \mu_j \delta_{x_j} + \widetilde{\mu}, \quad \mu(\mathbb{R}^N) \le C^2,$$

$$\nu = |u|^{2^*_s} + \sum_{j\in J} v_j \delta_{x_j}, \quad \nu(\mathbb{R}^N) \le S^{-\frac{2^*_s}{2}} C^2,$$

where J is at most countable, sequences $\{\mu_j\}_j, \{\nu_j\}_j \subset [0, \infty)$, $\{x_j\}_j \subset \mathbb{R}^N$, δ_{x_j} is the Dirac mass centered at $\{x_j\}_j$, $\widetilde{\mu}$ is a non-atomic measure,

$$\nu(\mathbb{R}^N)^{\frac{1}{2^*_s}} \le S^{-\frac{1}{2}} \mu(\mathbb{R}^N)^{\frac{1}{2}}, \quad \nu_j^{\frac{1}{2^*_s}} \le S^{-1/2} \mu_j^{\frac{1}{2}}, \quad \forall j \in J,$$

and

$$S = \inf\{\|u\|^2_{s,A} : \|u\|_{2^*_s} = 1\}.$$

Lemma 3.1. *Assume that $\{u_n\}_n \subset H^s_A(\mathbb{R}^N, \mathbb{C})$ is the sequence given by Theorem* 3.1, *let $x_0 \in \mathbb{R}^N$ fixed and let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \varphi \le 1$; $\varphi \equiv 1$ in $B(0,1)$, $\varphi \equiv 0$ in $\mathbb{R}^N \backslash B(0,2)$ and $|\nabla\varphi| \le 2$. For any $\varepsilon > 0$, set $\varphi_\varepsilon(x) = \varphi(\frac{x-x_0}{\varepsilon})$ for all $x \in \mathbb{R}^N$. Then*

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^2 |\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^{N+2s}} dxdy = 0.$$

Proof. The proof is completely similar to that of [38, Lemma 2.3], so we omit it. □

Proof of Theorem 3.1. We divide the proof into four parts.

Part 1. $\mu(\mathbb{R}^N) \le C^2$ and $\nu(\mathbb{R}^N) \le S^{-2^*_s/2} C^2$.

For $R > 0$, take $\eta \in C_0^\infty(B_{2R}(0))$ satisfying $0 \le \eta \le 1$ and $\eta \equiv 1$ on $B_R(0)$. Then

$$\int_{\mathbb{R}^N} \mu_n \eta dx \to \int_{\mathbb{R}^N} \eta d\mu.$$

Since $\|u_n\| \le C$, we obtain

$$\int_{\mathbb{R}^N} \mu_n \eta dx \le \int_{\mathbb{R}^N} \mu_n dx \le C^2.$$

Hence $\mu(B_R(0)) \le \int_{\mathbb{R}^N} \eta d\mu \le C^p$. Let $R \to \infty$, we get $\mu(\mathbb{R}^N) \le C^2$. Similarly, we have $\nu(\mathbb{R}^N) \le S^{-2^*_s/2} C^2$, since $\int_{\mathbb{R}^N} |u_n|^{2^*_s} dx \le S^{-2^*_s/2} C^2$ by the definition of S and $\|u_n\|_{s,A} \le C$.

Part 2. $\mu = \int_{\mathbb{R}^N} \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}dy + |u|^2 + \sum_{j\in J}\mu_j\delta_{x_j} + \widetilde{\mu}$, where $\{x_j\}_j \subset \mathbb{R}^N$, $\{\mu_j\} \subset [0,\infty)$, J is at most countable set, $\widetilde{\mu} \in M(\mathbb{R}^N)$ is a non-negative non-atomic measure and δ_{x_j} is the Dirac mass at x_j.

Take $0 \leq \eta \in C_0(\mathbb{R}^N)$ and set

$$\mathscr{F}(u) = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}dy + |u(x)|^2\right)\eta dx.$$

It is easy to verify that $\mathscr{F}$ is a continuously differentiable convex functional on $H^s_A(\mathbb{R}^N,\mathbb{C})$. So $\mathscr{F}$ is weakly lower semicontinuous on $H^s_A(\mathbb{R}^N,\mathbb{C})$. Thus,

$$\begin{aligned}&\liminf_{n\to\infty}\int_{\mathbb{R}^N}\mu_n\eta dx\\ &\quad\geq \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}dy + |u(x)|^2\right)\eta dx.\end{aligned}$$

It follows from $\mu_n \to \mu$ weakly * in $M(\mathbb{R}^N)$ that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\mu_n\eta dx = \int_{\mathbb{R}^N}\eta d\mu.$$

Hence

$$\int_{\mathbb{R}^N}\eta d\mu \geq \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}dy + |u(x)|^2\right)\eta dx.$$

The arbitrariness of $\eta \in C_0(\mathbb{R}^N)$ with $\eta \geq 0$ implies that

$$\mu \geq \int_{\mathbb{R}^N}\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}dy + |u|^2.$$

Therefore, we obtain

$$\mu - \int_{\mathbb{R}^N}\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}dy - |u|^2 = \sum_{j\in J}\mu_j\delta_{x_j} + \widetilde{\mu}.$$

Part 3. $\nu = |u|^{2^*_s} + \sum_{j\in J}\nu_j\delta_{x_j}$, where $\{x_j\}_j$ is as above and $\{\nu_j\}_j \subset [0,\infty)$.

Since $u_n \rightharpoonup u$ weakly in $H^s_A(\mathbb{R}^N,\mathbb{C})$, there exists a subsequence still denoted by $\{u_n\}_n$ such that $u_n \to u$ a.e. in $\mathbb{R}^N$. Take $\eta \in C_0(\mathbb{R}^N)$. It follows from the boundedness of $\{u_n\}_n$ in $L^{2^*_s}(\mathbb{R}^N,\mathbb{C})$ and the Brézis–Lieb Lemma that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(|u_n|^{2^*_s} - |u_n-u|^{2^*_s})\eta dx = \int_{\mathbb{R}^N}|u|^{2^*_s}\eta dx.$$

Set $\overline{\nu} = \nu - |u|^{2_s^*}$. By the fact that $\int_{\mathbb{R}^N} |u_n|^{2_s^*}\eta dx \to \int_{\mathbb{R}^N} \eta d\nu$ as $n \to \infty$, it yields

$$\int_{\mathbb{R}^N} \eta d\overline{\nu} = \int_{\mathbb{R}^N} \eta d\nu - \int_{\mathbb{R}^N} |u|^{2_s^*}\eta dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u|^{2_s^*}\eta dx, \tag{3.1}$$

so that $|u_n - u|^{2_s^*} \rightharpoonup \overline{\nu}$ weakly * in $\mathcal{M}(\mathbb{R}^N)$. Furthermore,

$$\overline{\nu} = \nu - |u|^{2_s^*} = \sum_{j\in J'} \nu_j \delta_{y_j} + \widetilde{\nu}.$$

Next, we prove that the atom of ν is that of μ and $\widetilde{\nu} = 0$. Let $x_0 \in \mathbb{R}^N$ fixed and let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \varphi \le 1$; $\varphi \equiv 1$ in $B(0,1)$, $\varphi \equiv 0$ in $\mathbb{R}^N \backslash B(0,2)$, and $|\nabla\varphi| \le 2$. Denote $\varphi_\varepsilon(x) = \varphi(\frac{x-x_0}{\varepsilon})$ for all $x \in \mathbb{R}^N$. Then,

$$\int_{\mathbb{R}^N} |u_n\varphi_\varepsilon|^{2_s^*} dx = \int_{\mathbb{R}^N} |u_n|^{2_s^*} \varphi_\varepsilon^{2_s^*} dx \to \int_{\mathbb{R}^N} \varphi_\varepsilon^{2_s^*} d\nu \quad \text{as } n \to \infty,$$

and

$$\int_{\mathbb{R}^N} \varphi_\varepsilon^{2_s^*} d\nu \to \nu(\{x_0\}) \quad \text{as } \varepsilon \to 0.$$

Similarly, we have

$$\int_{\mathbb{R}^N} \mu_n \varphi_\varepsilon^2 dx \to \int_{\mathbb{R}^N} \varphi_\varepsilon^2 d\mu \quad \text{as } n \to \infty$$

and

$$\int_{\mathbb{R}^N} \varphi_\varepsilon^2 d\mu \to \mu(\{x_0\}) \quad \text{as } \varepsilon \to 0.$$

Hence, we obtain

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n\varphi|^{2_s^*} dx = \nu(\{x_0\}) \tag{3.2}$$

and

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} \mu_n \varphi_\varepsilon^2 dx = \mu(\{x_0\}). \tag{3.3}$$

In view of the definition of S, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_n\varphi_\varepsilon|^{2_s^*} dx \\
&\quad \le S^{\frac{-2_s^*}{2}} \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)\varphi_\varepsilon(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u_n(y)\varphi_\varepsilon(y)|^2}{|x-y|^{N+2s}} dxdy \right. \\
&\qquad \left. + \int_{\mathbb{R}^N} |u_n\varphi_\varepsilon|^2 dx \right)^{2_s^*/2}
\end{aligned}$$

$$
\begin{aligned}
&= S^{\frac{-2_s^*}{2}} \Bigg(\iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^2 |\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^{N+2s}} dxdy \\
&\quad + 2\Re \iint_{\mathbb{R}^{2N}} \frac{\begin{array}{c}\varphi_\varepsilon(x)\overline{u_n(y)} e^{-\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}(u_n(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_n(y)) \\ \times (\varphi_\varepsilon(x) - \varphi_\varepsilon(y))\end{array}}{|x-y|^{N+2s}} dxdy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2 |\varphi_\varepsilon(x)|^2}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} |u_n \varphi_\varepsilon|^2 dx \Bigg)^{2_s^*/2},
\end{aligned} \tag{3.4}
$$

By the Hölder inequality, we have

$$
\begin{aligned}
&\left| \Re \iint_{\mathbb{R}^{2N}} \frac{\begin{array}{c}\varphi_\varepsilon(x)\overline{u_n(y)} e^{-\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}(u_n(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_n(y)) \\ \times (\varphi_\varepsilon(x) - \varphi_\varepsilon(y))\end{array}}{|x-y|^{N+2s}} dxdy \right| \\
&\quad \leq \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^2 (\varphi_\varepsilon(x) - \varphi_\varepsilon(y))^2}{|x-y|^{N+2s}} dxdy \right)^{1/2} \\
&\quad\quad \times \left(\iint_{\mathbb{R}^{2N}} \frac{|\varphi_\varepsilon(x)|^2 |u_n(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2} \\
&\quad \leq C \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^2 (\varphi_\varepsilon(x) - \varphi_\varepsilon(y))^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}.
\end{aligned}
$$

Therefore, in view of Lemma 3.1, we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \Re \iint_{\mathbb{R}^{2N}} \frac{\begin{array}{c}\varphi_\varepsilon(x)\overline{u_n(y)} e^{-\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}(u_n(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_n(y)) \\ \times (\varphi_\varepsilon(x) - \varphi_\varepsilon(y))\end{array}}{|x-y|^{N+2s}} dxdy \\
&\quad = 0.
\end{aligned} \tag{3.5}
$$

By (3.2)–(3.5), we deduce that

$$
\nu(\{x_0\}) \leq S^{-2_s^*/2} \mu(\{x_0\})^{2_s^*/2}. \tag{3.6}
$$

Then the arbitrary of x_0 implies that the atom of ν is that of μ, that is $\{y_j : j \in J'\} \subset \{x_j : j \in J\}$. Therefore, we get

$$
\nu - |u|^{2_s^*} = \sum_{j\in J} \nu_j \delta_{x_j} + \widetilde{\nu}.
$$

It remains to show that $\widetilde{\nu} = 0$. To this aim, let $\widetilde{u}_n = u_n - u$. Then $\widetilde{u}_n \rightharpoonup 0$ weakly in $H_A^s(\mathbb{R}^N, \mathbb{C})$. Hence there exists a subsequence of $\{\widetilde{u}_n\}_n$ still denoted by

$\{\widetilde{u}_n\}_n$ such that

$$\widetilde{\mu}_n := \int_{\mathbb{R}^N} \frac{|\widetilde{u}_n(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\widetilde{u}_n(y)|^2}{|x-y|^{N+2s}}dy + |\widetilde{u}_n(x)|^2 \rightharpoonup \overline{\mu} \quad \text{weakly in } \mathcal{M}(\mathbb{R}^N).$$

For any $0 < r < R$, take $\eta \in C_0^\infty(B_R(x_0))$ satisfying $0 \le \eta \le 1$ and $\eta \equiv 1$ on $B_r(x_0)$. It follows from the definition of S that

$$\begin{aligned}
&\int_{B_R(x_0)} \eta^{2_s^*}|\widetilde{u}_n|^{2_s^*}dx \\
&\quad = \int_{B_R(x_0)} |\eta\widetilde{u}_n|^{2_s^*}dx \\
&\quad \le S^{-2_s^*/2}\Bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\eta(x)\widetilde{u}_n(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\eta(y)\widetilde{u}_n(y)|^2}{|x-y|^{N+2s}}dy \\
&\qquad + |\eta\widetilde{u}_n|^2dx\Bigg)^{2_s^*/2} \\
&\quad = S^{-2_s^*/2}\Bigg[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\eta(x)-\eta(y))\widetilde{u}_n(x)|^2}{|x-y|^{N+2s}}dydx \\
&\qquad + 2\Re\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\begin{array}{c}\overline{\widetilde{u}_n(x)}\eta(y)(\widetilde{u}_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}\widetilde{u}_n(y)) \\ \times(\eta(x)-\eta(y))\end{array}}{|x-y|^{N+2s}}dydx \\
&\qquad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\eta(y)(\widetilde{u}_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}\widetilde{u}_n(y))|^2}{|x-y|^{N+2s}}dydx \\
&\qquad + \int_{\mathbb{R}^N}|\eta\widetilde{u}_n|^2dx\Bigg]^{2_s^*/2}.
\end{aligned} \tag{3.7}$$

Note that $|\eta(x)-\eta(y)|^2 \le (\|\eta\|_{C^1}+2)^2\min\{1,|x-y|^2\}$ for all $x, y \in B_R(x_0)$. Hence, by the compact embedding for fractional Sobolev spaces on bounded domains, we obtain that $\widetilde{u}_n \to 0$ strongly in $L^2(B_R(x_0), \mathbb{C})$. Furthermore,

$$\begin{aligned}
&\int_{B_R(x_0)}\int_{B_R(x_0)} \frac{|(\eta(x)-\eta(y))\widetilde{u}_n(x)|^2}{|x-y|^{N+2s}}dydx \\
&\quad \le (\|\eta\|_{C^1}+2)^2\int_{B_R(x_0)} \frac{\min\{1,|x-y|^2\}}{|x-y|^{N+2s}}dy\int_{B_R(x_0)}|\widetilde{u}_n(x)|^2dx \\
&\quad \le C(\|\eta\|_{C^1}+2)^2\int_{B_R(x_0)}|\widetilde{u}_n(x)|^2dx \to 0 \quad \text{as } n\to\infty,
\end{aligned}$$

and with a similar discussion as in [38] and [39], we obtain

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(x_0)}\int_{B_R(x_0)} \frac{|\eta(x)\widetilde{u}_n(x)|^2}{|x-y|^{N+2s}}dxdy = 0$$

so that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(\eta(x)-\eta(y))\widetilde{u}_n(y)|^2}{|x-y|^{N+2s}}dydx=0, \tag{3.8}$$

this together with the Hölder inequality implies that

$$\begin{aligned}&\limsup_{n\to\infty}\Re\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\overline{\widetilde{u}_n(x)}\eta(y)(\widetilde{u}_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}\widetilde{u}_n(y))(\eta(x)-\eta(y))}{|x-y|^{N+2s}}dydx\\ &\quad=0.\end{aligned} \tag{3.9}$$

Note that

$$\begin{aligned}&\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\eta^2(x)|\widetilde{u}_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}\widetilde{u}_n(y)|^2}{|x-y|^{N+2s}}dy+\eta^2(x)|\widetilde{u}_n(x)|^2dx\\ &\quad\le\int_{\mathbb{R}^N}\eta^2d\overline{\mu}\le\int_{\overline{B_R(x_0)}}d\overline{\mu}=\overline{\mu}(\overline{B_R(x_0)}).\end{aligned} \tag{3.10}$$

Inserting (3.8)–(3.10) in (3.7), we obtain

$$\overline{\nu}(\overline{B_r(x_0)})\le\int_{\mathbb{R}^N}\eta^2d\overline{\nu}=\lim_{n\to\infty}\int_{\mathbb{R}^N}|\widetilde{u}_n|^{2_s^*}\eta^2dx\le S^{-2_s^*/2}(\overline{\mu}(\overline{B_R(x_0)}))^{2_s^*/2}.$$

Let $r\to R^-$, we get

$$\overline{\nu}(\overline{B_R(x_0)})\le S^{-2_s^*/2}(\overline{\mu}(\overline{B_R(x_0)}))^{2_s^*/2}, \tag{3.11}$$

this means that $\overline{\nu}$ is absolutely continuous with respect to $\overline{\mu}$. Hence the Radon–Nikodym theorem implies that there exists a function $h\in L^1(\mathbb{R}^N,\overline{\mu})$ such that $d\overline{\nu}=hd\overline{\mu}$. Then we derive from Lebesgue's differential theorem and (3.11) that

$$\begin{aligned}h(x_0)&=\lim_{R\to0}\frac{\overline{\nu}(\overline{B_R(x_0)})}{\overline{\mu}(\overline{B_R(x_0)})}\\ &\le S^{-2_s^*/2}\lim_{R\to0}\overline{\mu}(\overline{B_R(x_0)})^{\frac{2_s^*}{2}-1}\\ &=S^{-2_s^*/2}\overline{\mu}\left(\{x_0\}\right)^{\frac{2_s^*}{2}-1}.\end{aligned} \tag{3.12}$$

Now we show that $\widetilde{\nu}=0$. For $\forall\, x\in\mathbb{R}^N\backslash\{x_j:j\in J\}$. If $h(x)\neq0$, then by (3.12) we know that $\overline{\mu}(\{x\})\neq0$, thus $\overline{\nu}(\{x\})\neq0$. Note that (3.1) implies that $\overline{\nu}$ and ν have the same atom, so that x is an atom of μ, which is a contradiction. Hence $h\equiv0$ on $\mathbb{R}^N\backslash\{x_j:j\in J\}$. Therefore, $\overline{\nu}=0$ on $\mathbb{R}^N\backslash\{x_j:j\in J\}$. In conclusion, $\widetilde{\nu}=0$, since $\widetilde{\nu}$ is a non-atomic measure.

Part 4. $\nu(\mathbb{R}^N) \le S^{-2_s^*/2}\mu(\mathbb{R}^N)^{\frac{2_s^*}{2}}$, and $\nu_j^{1/2_s^*} \le S^{-\frac{1}{2}}\mu_j^{1/2}$ for all $j \in J$.

Take $\eta \in C_0^\infty(B_{2R}(0))$ satisfying $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_R(0)$ and $|\nabla\eta| \le 2/R$. Observe that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \eta^{2_s^*}|u_n|^{2_s^*}dx \\
&\quad\le S^{-2_s^*/2}\left(\iint_{\mathbb{R}^{2N}} \frac{|(\eta(x)-\eta(y))u_n(x)|^2}{|x-y|^{N+2s}}dxdy\right.\\
&\qquad + 2\Re\iint_{\mathbb{R}^{2N}} \frac{\overline{u_n(x)}\eta(y)(u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y))(\eta(x)-\eta(y))}{|x-y|^{N+2s}}dxdy\\
&\qquad \left. + \iint_{\mathbb{R}^{2N}} \frac{|\eta(y)(u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y))|^2}{|x-y|^{N+2s}}dxdy + \int_{\mathbb{R}^N}|\eta u_n|^2dx\right)^{2_s^*/2},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^{2N}} \frac{\overline{u_n(x)}\eta(y)(u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y))(\eta(x)-\eta(y))}{|x-y|^{N+2s}}dxdy\right|\\
&\quad\le \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^2(\eta(x)-\eta(y))^2}{|x-y|^{N+2s}}dxdy\right)^{1/2}\\
&\qquad\times\left(\iint_{\mathbb{R}^{2N}} \frac{\eta(y)^2|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)^2|}{|x-y|^{N+2s}}dxdy\right)^{1/2}.
\end{aligned}$$

Since $\eta \equiv 1$ on $B_R(0)$, we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|(\eta(x)-\eta(y))u_n(x)|^2}{|x-y|^{N+2s}}dydx\\
&\quad= \int_{\mathbb{R}^N\setminus B_R(0)}\int_{\mathbb{R}^N\setminus B_R(0)} \frac{|(\eta(x)-\eta(y))u(x)|^2}{|x-y|^{N+2s}}dydx\\
&\qquad+\int_{B_R(0)}\int_{\mathbb{R}^N\setminus B_R(0)} \frac{|(\eta(x)-\eta(y))u_n(x)|^2}{|x-y|^{N+2s}}dydx\\
&\qquad+\int_{\mathbb{R}^N\setminus B_R(0)}\int_{B_R(0)} \frac{|(\eta(x)-\eta(y))u_n(x)|^2}{|x-y|^{N+2s}}dydx\\
&\quad\le CR^{-2s}\int_{\mathbb{R}^N}|u_n(x)|^2dx\\
&\quad\le CR^{-2s}\to 0 \quad \text{as } R\to\infty.
\end{aligned}\tag{3.13}$$

Letting $n \to \infty$, we have

$$\int_{\mathbb{R}^N}\eta^{2_s^*}d\nu \le S^{-\frac{2_s^*}{2}}\left(CR^{-2s}+\int_{\mathbb{R}^N}\eta^2d\mu\right)^{\frac{2_s^*}{2}}. \tag{3.14}$$

Using $\nu(\overline{B_R(0)}) \leq \int_{\mathbb{R}^N} \eta^{2_s^*} d\nu$ and letting $R \to \infty$ (3.14), we get

$$\nu(\mathbb{R}^N) \leq S^{-\frac{2_s^*}{2}} (\mu(\mathbb{R}^N))^{2_s^*/2}.$$

A similar discussion as (3.6) gives that $\nu_j^{1/2_s^*} \leq S^{-1/2} \mu_j^{1/2}$. Thus, the theorem is proved. □

Actually, Theorem 3.1 does not provide any information about the possible loss of mass at infinity of a weakly convergent sequence. The following theorem expresses this fact in quantitative terms.

Theorem 3.2. *Let* $\{u_n\}_n \subset H_A^s(\mathbb{R}^N, \mathbb{C})$ *such that*

$$\int_{\mathbb{R}^N} \frac{|u_n(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2}{|x-y|^{N+2s}} dy + \int_{\mathbb{R}^N} |u|^2 dx \rightharpoonup \mu \quad \textit{weakly * in } \mathcal{M}(\mathbb{R}^N),$$

$$|u_n|^{2_s^*} \rightharpoonup \nu \quad \textit{weakly * in } \mathcal{M}(\mathbb{R}^N),$$

and define

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} \int_{\mathbb{R}^N} \frac{|u_n(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2}{|x-y|^{N+2s}} dy dx,$$

and

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{2_s^*} dx.$$

Then the quantities μ_∞ *and* ν_∞ *are well defined and satisfy*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2}{|x-y|^{N+2s}} dy dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty,$$

and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty.$$

Proof. Let $\eta \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \eta \leq 1$; $\eta = 1$ in $\mathbb{R}^N \backslash B_2(0)$, $\eta \equiv 0$ in $B_1(0)$. For any $R > 0$, define $\eta_R(x) = \eta(x/R)$, then

$$\begin{aligned}
&\int_{|x|>2R} \int_{\mathbb{R}^N} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2}{|x-y|^{N+2s}} dy dx \\
&\leq \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2}{|x-y|^{N+2s}} dy \right) \eta_R dx \\
&\leq \int_{|x|>R} \left(\int_{\mathbb{R}^N} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2}{|x-y|^{N+2s}} dy \right) dx.
\end{aligned}$$

This means that

$$\mu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}}dy\right)\eta_R dx.$$

A similar discussion gives that

$$\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2_s^*}\eta_R dx = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n\eta_R|^{2_s^*}dx.$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}}dy\right)dx\\
&\quad=\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}}dy\right)\eta_R dx\\
&\qquad+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}}dy\right)(1-\eta_R)dx. \qquad (3.15)
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}}dy+|u_n(x)|^2\right)(1-\eta_R)dx\\
&\quad\to\int_{\mathbb{R}^N}(1-\eta_R)d\mu,
\end{aligned}$$

as $n\to\infty$. Hence, we get

$$\mu(\mathbb{R}^N)=\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}}dy\right)(1-\eta_R)dx.$$

Here we have used the fact that $\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n(x)|^2(1-\eta_R(x))dx=0$. It follows from (3.15) that

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}}dy\right)dx\\
&\quad=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}}dy\right)dx\\
&\quad=\lim_{R\to\infty}\limsup_{n\to\infty}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}}dy\right)\eta_R dx\right.\\
&\qquad\left.+\int_{\mathbb{R}^N}(1-\eta_R)d\mu\right]
\end{aligned}$$

$$= \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}}dy\right)\eta_R dx+\mu(\mathbb{R}^N)$$
$$= \mu_\infty+\mu(\mathbb{R}^N).$$

Similarly, we can obtain that $\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx=\nu(\mathbb{R}^N)+\nu_\infty$. □

4. Proof of Theorem 1.1

The functional associated with problem (1.1) is defined as

$$\mathcal{I}_\lambda(u)=\frac{1}{2}\mathscr{M}([u]_{s,A}^2)-\frac{\lambda}{4}\iint_{\mathbb{R}^{2N}}\frac{F(|u(x)|^2)F(|u(y)|^2)}{|x-y|^\alpha}dxdy-\frac{1}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}dx.$$

for all $u\in H_{r,A}^s(\mathbb{R}^N,\mathbb{C})$.

By (f_2), we have

$$F(|u|^2)\le C(|u|^2+|u|^p),\quad \forall\, u\in H_A^s(\mathbb{R}^N,\mathbb{C}).$$

Note that, by the Hardy–Littlewood–Sobolev inequality, the integral

$$\iint_{\mathbb{R}^N}\frac{F(|u(x)|^2)F(|u(y)|^2)}{|x-y|^\alpha}dxdy$$

is well defined if $F(|u|^2)\in L^r(\mathbb{R}^N)$ for some $r>1$ satisfying

$$\frac{2}{r}+\frac{\alpha}{N}=2,$$

that is $r=\frac{2N}{2N-\alpha}$. Actually, by $\alpha<\min\{N,4s\}$, it follows that $2<2r<2_s^*$. Moreover, from $2<pr<2_s^*$, we deduce

$$\begin{aligned}\int_{\mathbb{R}^N}|F(|u|^2)|^r dx &\le 2^{r-1}C^r\left(\int_{\mathbb{R}^N}|u|^{2r}dx+\int_{\mathbb{R}^N}|u|^{pr}dx\right)\\ &\le 2^{r-1}C^r(C_{2r}^{2r}\|u\|_{s,A}^{2r}+C_{pr}^{pr}\|u\|_{s,A}^{pr})<\infty,\quad \forall\, u\in H_A^s(\mathbb{R}^N,\mathbb{C}).\end{aligned}\tag{4.1}$$

By a standard discussion, one can show that $\mathcal{I}_\lambda$ is of class C^1 and

$$\begin{aligned}&\langle\mathcal{I}_\lambda'(u),v\rangle\\ &= M(\|u\|_{s,A}^2)\left[\Re\iint_{\mathbb{R}^{2N}}\frac{[u(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)]\times\overline{[v(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y)]}}{|x-y|^{N+2s}}dxdy\right.\\ &\quad\left.+\Re\int_{\mathbb{R}^N}u\overline{v}dx\right]-\lambda\Re\int_{\mathbb{R}^N}(\mathcal{K}_\alpha*F(|u|^2))f(|u|^2)u\overline{v}dx-\Re\int_{\mathbb{R}^N}|u|^{2_s^*-2}u\overline{v}dx,\end{aligned}$$

for all $u, v \in H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$. Hence a critical point of $\mathcal{I}_\lambda$ is a (weak) solution of (1.1).

Since the embedding $H^s_{r,A}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^{2^*_s}(\mathbb{R}^N, \mathbb{C})$ is not compact, we will use Theorem 3.1 to get the existence of solutions of (1.1). Some techniques for finding the solutions are borrowed from [19].

Lemma 4.1. *Let $\{u_n\}_n \subset H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$ be a Palais–Smale sequence of functional $\mathcal{I}_\lambda$, that is,*

$$\mathcal{I}_\lambda(u_n) \to c_\lambda \quad \text{and} \quad \mathcal{I}'_\lambda(u_n) \to 0 \quad \text{in } (H^s_{r,A}(\mathbb{R}^N, \mathbb{C}))',$$

as $n \to \infty$, where $(H^s_{r,A}(\mathbb{R}^N, \mathbb{C}))'$ is the dual of $H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$. If $2 < p < \frac{2N-\alpha}{N-2s}$ and

$$c_\lambda < \left(\frac{1}{2\theta} - \frac{1}{2^*_s}\right)(m_0 S)^{\frac{2^*_s}{2^*_s - 2}},$$

where S comes from Theorem 3.1, then there exists a subsequence of $\{u_n\}$ strongly convergent in $H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$.

Proof. By $\mathcal{I}_\lambda(u_n) \to c_\lambda$ and $\mathcal{I}'_\lambda(u_n) \to 0$ in $(H^s_{r,A}(\mathbb{R}^N, \mathbb{C}))'$, there exists $C > 0$ such that

$$\begin{aligned}
C + C\|u_n\|_{s,A} \geq \mathcal{I}_\lambda(u_n) - \frac{1}{\sigma}\langle \mathcal{I}'_\lambda(u_n), u_n\rangle \\
&= \frac{1}{2}\mathscr{M}(\|u_n\|^2_{s,A}) - \frac{1}{\sigma}M(\|u_n\|^2_{s,A})\|u_n\|^2_{s,A} \\
&\quad - \frac{\lambda}{4}\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|u_n|^2))F(|u_n|^2)dx \\
&\quad + \frac{\lambda}{\sigma}\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|u_n|^2))f(|u_n|^2)|u_n|^2dx \\
&\quad + \left(\frac{1}{\sigma} - \frac{1}{2^*_s}\right)\int_{\mathbb{R}^N}|u_n|^{2^*_s}dx.
\end{aligned}$$

It follows from (M_2) and (f_3) that

$$C + C\|u_n\|_{s,A} \geq \left(\frac{1}{2\theta} - \frac{1}{\sigma}\right)M(\|u_n\|^2_{s,A})\|u_n\|^2_{s,A} \geq m_0\left(\frac{1}{2\theta} - \frac{1}{\sigma}\right)\|u_n\|^2_{s,A},$$

this together with $2 \leq 2\theta < \sigma$ implies that $\{u_n\}$ is bounded in $H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$. Then there exists $u \in H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$ such that, up to a subsequence, it follows that

$$\begin{aligned}
u_n \rightharpoonup u \quad & \text{in } H^s_{r,A}(\mathbb{R}^N, \mathbb{C}) \text{ and in } L^{2^*_s}(\mathbb{R}^N, \mathbb{C}), \\
u_n \to u \quad & \text{a.e. in } \mathbb{R}^N, \\
|u_n|^{2^*_s-2}u_n \rightharpoonup |u|^{2^*_s-2}u \quad & \text{weakly in } L^{\frac{2^*_s}{2^*_s-1}}(\mathbb{R}^N, \mathbb{C}), \\
\|u_n\|_{s,A} \to \beta. &
\end{aligned} \tag{4.2}$$

Since $2 < p < \frac{2N-\alpha}{N-2s} < 2_s^*$ and $2 < 4N/(2N-\alpha) < 2_s^*$, by Theorem 2.1 we get that $|u_n| \to |u|$ strongly in $L^{\frac{2Np}{2N-\alpha}}(\mathbb{R}^N) \cap L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N)$. Hence the Brézis–Lieb Lemma implies that $u_n \to u$ strongly in $L^{\frac{2Np}{2N-\alpha}}(\mathbb{R}^N,\mathbb{C}) \cap L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N,\mathbb{C})$. By (f_2), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |F(|u_n|^2) - F(|u|^2)|^{\frac{2N}{2N-\alpha}} dx \\
&\quad \le \int_{\mathbb{R}^N} |f(|u|^2 + \vartheta(|u_n|^2 - |u|^2))|^{\frac{2N}{2N-\alpha}} ||u_n|^2 - |u|^2|^{\frac{2N}{2N-\alpha}} dx \\
&\quad \le \int_{\mathbb{R}^N} [C(1 + (|u_n| + |u|)^{p-2})]^{\frac{2N}{2N-\alpha}} (|u_n| + |u|)^{\frac{2N}{2N-\alpha}} |u_n - u|^{\frac{2N}{2N-\alpha}} dx \\
&\quad \le C^{\frac{2N}{2N-\alpha}} 2^{\frac{\alpha}{2N-\alpha}} \int_{\mathbb{R}^N} (|u_n| + |u|)^{\frac{2N}{2N-\alpha}} |u_n - u|^{\frac{2N}{2N-\alpha}} dx \\
&\qquad + 2^{\frac{\alpha}{2N-\alpha}} C^{\frac{2N}{2N-\alpha}} \int_{\mathbb{R}^N} (|u_n| + |u|)^{(p-1)\frac{2N}{2N-\alpha}} |u_n - u|^{\frac{2N}{2N-\alpha}} dx.
\end{aligned}$$

By the Hölder inequality, we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}^N} |F(|u_n|^2) - F(|u|^2)|^{\frac{2N}{2N-\alpha}} dx \\
&\quad \le C^{\frac{2N}{2N-\alpha}} 2^{\frac{\alpha}{2N-\alpha}} \|(|u_n| + |u|)^{\frac{2N}{2N-\alpha}}\|_{L^2(\mathbb{R}^N)} \||u_n - u|^{\frac{2N}{2N-\alpha}}\|_{L^2(\mathbb{R}^N)} \\
&\qquad + 2^{\frac{\alpha}{2N-\alpha}} C^{\frac{2N}{2N-\alpha}} \|(|u_n| + |u|)^{(p-1)\frac{2N}{2N-\alpha}}\|_{L^{\frac{p}{p-1}}(\mathbb{R}^N)} \||u_n - u|^{\frac{2N}{2N-\alpha}}\|_{L^p(\mathbb{R}^N)} \\
&\quad \le C\||u_n - u|^{\frac{2N}{2N-\alpha}}\|_{L^2(\mathbb{R}^N)} + C\||u_n - u|^{\frac{2N}{2N-\alpha}}\|_{L^p(\mathbb{R}^N)} \\
&\quad \to 0,
\end{aligned}$$

as $n \to \infty$, where $C > 0$ independent of n. Thus, we obtain that $F(|u_n|^2) \to F(|u|^2)$ in $L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$. Note that by the Hardy–Littlewood–Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$. Then

$$(\mathcal{K}_\alpha * F(|u_n|^2)) \to (\mathcal{K}_\alpha * F(|u|^2)) \quad \text{in } L^{\frac{2N}{\alpha}}(\mathbb{R}^N) \tag{4.3}$$

as $n \to \infty$.

For $\varphi \in H^s_{r,A}(\mathbb{R}^N,\mathbb{C})$ fixed, by (f_2) with $\varepsilon = 1$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f(|u_n|^2) u_n \overline{\varphi}|^{\frac{2N}{2N-\alpha}} dx \\
&\quad \le 2^{\frac{\alpha}{2N-\alpha}} C^{\frac{2N}{2N-\alpha}} \left(\int_{\mathbb{R}^N} (|u_n||\varphi|)^{\frac{2N}{2N-\alpha}} dx + \int_{\mathbb{R}^N} |u_n|^{(p-1)\frac{2N}{2N-\alpha}} |\varphi|^{\frac{2N}{2N-\alpha}} dx \right) \\
&\quad \le 2^{\frac{\alpha}{2N-\alpha}} C^{\frac{2N}{2N-\alpha}} \Big(\||u_n|^{\frac{2N}{2N-\alpha}}\|_{L^2(\mathbb{R}^N)} \||\varphi|^{\frac{2N}{2N-\alpha}}\|_{L^2(\mathbb{R}^N)} \\
&\qquad + \||u_n|^{(p-1)\frac{2N}{2N-\alpha}}\|_{L^{\frac{p}{p-1}}(\mathbb{R}^N)} \||\varphi|^{\frac{2N}{2N-\alpha}}\|_{L^p(\mathbb{R}^N)} \Big) \\
&\quad \le C,
\end{aligned}$$

thanks to $2 < \frac{4N}{2N-\alpha} < 2_s^*$ and $2 < p\frac{2N}{2N-\alpha} < 2_s^*$, where $C > 0$ denotes various constants. Clearly, $f(|u_n|^2)u_n\overline{\varphi} \to f(|u|^2)u\overline{\varphi}$ a.e. in $\mathbb{R}^N$. Hence, up to a subsequence, $\Re f(|u_n|^2)u_n\overline{\varphi}$ weakly converges to $\Re f(|u|^2)u\overline{\varphi}$ in $L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$. This together with (4.3) yields that

$$\lim_{n\to\infty} \Re \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u_n|^2))f(|u_n|^2)u_n\overline{\varphi}dx = \Re \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u|^2))f(|u|^2)u\overline{\varphi}dx \tag{4.4}$$

for each $\varphi \in H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$.

Now we claim that

$$u_n \to u \quad \text{in } H^s_{r,A}(\mathbb{R}^N, \mathbb{C}) \tag{4.5}$$

as $n \to \infty$. In fact, it follows from Theorem 3.1 that there exist an at most countable set of distinct points $\{x_i\}_{i\in J}$, non-negative numbers $\{\mu_i\}_{i\in J}, \{\nu_i\}_{i\in J} \subset [0,\infty)$ and a non-atomic measure $\widetilde{\mu}$ such that

$$\begin{aligned} \mu &= \int_{\mathbb{R}^N} \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}dy + |u|^2 + \sum_{i\in J} \mu_i\delta_{x_i} + \widetilde{\mu}, \\ \nu &= |u(x)|^{2_s^*} + \sum_{i\in J} \nu_i\delta_{x_i}. \end{aligned} \tag{4.6}$$

Now, in order to prove (4.5) we proceed by steps.

Step 1. Fix $i_0 \in J$. Then we prove that either $\nu_{i_0} = 0$ or

$$\nu_{i_0} \geq (m_0 S)^{2_s^*/(2_s^*-2)}. \tag{4.7}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N; [0,1])$ be a radial symmetric function satisfying $\varphi = 1$ in $B(0,1)$; $\varphi = 0$ in $\mathbb{R}^N \setminus B(0,2)$ and $|\nabla\varphi| \leq 2$. For any $\varepsilon > 0$ we set $\varphi_\varepsilon = \varphi(\frac{x-x_{i_0}}{\varepsilon})$. Clearly $\{\varphi_\varepsilon u_n\}$ is bounded in $H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$ and $\langle \mathcal{I}'_\lambda(u_n), \varphi_\varepsilon u_n\rangle \to 0$ as $n \to \infty$. Hence

$$\begin{aligned} &M(\|u_n\|^2_{s,A})(u_n, \varphi_\varepsilon u_n)_{s,A} \\ &\quad = \lambda\Re \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u_n|^2))f(|u_n|^2)u_n\varphi_\varepsilon\overline{u_n}dx + \Re \int_{\mathbb{R}^N} |u_n|^{2_s^*-2}u_n\varphi_\varepsilon\overline{u_n}dx. \end{aligned} \tag{4.8}$$

It is easy to verify that

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u_n|^2))f(|u_n|^2)|u_n|^2\varphi_\varepsilon dx = 0,$$

since

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u_n|^2))f(|u_n|^2)|u_n|^2\varphi_\varepsilon dx = \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u|^2))f(|u|^2)|u|^2\varphi_\varepsilon dx$$

and $\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u|^2))f(|u|^2)|u|^2\varphi_\varepsilon dx = 0$.

Observe that

$$
\begin{aligned}
(u_n, \varphi_\varepsilon u_n)_{s,A} &= \langle u_n, \varphi_\varepsilon u_n\rangle_{s,A} + \langle u_n, \varphi_\varepsilon u_n\rangle_{L^2,V} \\
&= \iint_{\mathbb{R}^N} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2\varphi_\varepsilon(x)}{|x-y|^{N+2s}}dxdy \\
&\quad + \Re \iint_{\mathbb{R}^N} \frac{(u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y))(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))\overline{u_n(y)}}{|x-y|^{N+2s}}dxdy \\
&\quad + \int_{\mathbb{R}^N} |u_n|^2\varphi_\varepsilon dx. \qquad (4.9)
\end{aligned}
$$

First, it is easy to see that

$$
\iint_{\mathbb{R}^N} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2\varphi_\varepsilon(x)}{|x-y|^{N+2s}}dxdy + \int_{\mathbb{R}^N} |u_n|^2\varphi_\varepsilon dx \to \int_{\mathbb{R}^N} \varphi_\varepsilon(x)d\mu \qquad (4.10)
$$

as $n \to \infty$ and

$$
\int_{\mathbb{R}^N} \varphi_\varepsilon(x)d\mu \to \mu(x_{i_0}) = \mu_{i_0} \qquad (4.11)
$$

as $\varepsilon \to 0$. Similarly,

$$
\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2_s^*}\varphi_\varepsilon(x)dx \to \nu(\{x_{i_0}\}) = \nu_{i_0}. \qquad (4.12)
$$

Note that the Hölder inequality implies

$$
\begin{aligned}
&\left|\iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y))(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))\overline{u_n(y)}}{|x-y|^{N+2s}}dxdy\right| \\
&\quad \le \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{1/2} \\
&\qquad \times \left(\iint_{\mathbb{R}^{2N}} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2|u_n(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{1/2} \\
&\quad \le C\left(\iint_{\mathbb{R}^{2N}} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2|u_n(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{1/2}.
\end{aligned}
$$

By Lemma 3.1, we have

$$
\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2|u_n(y)|^2}{|x-y|^{N+2s}}dxdy = 0.
$$

Hence,

$$
\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y))(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))\overline{u_n(y)}}{|x-y|^{N+2s}}dxdy = 0. \qquad (4.13)
$$

Combining (4.9), (4.11) with (4.13), we obtain by $\|u_n\|_{s,A} \to \beta$ that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} M(\|u_n\|^2_{s,A})(u_n, \varphi_\varepsilon u_n)_{s,A} = M(\beta^2)\mu_{i_0}.$$

Inserting this into (4.8) and using (4.12), we deduce

$$M(\beta^2)\mu_{i_0} = \nu_{i_0}.$$

By (M_1), we get $m_0\mu_{i_0} \le \nu_{i_0}$. It follows from $\nu_i \le S^{-2_s^*/2}(\mu_i)^{2_s^*/2}$ for all $i \in J$ that

$$\nu_{i_0} \le S^{-2_s^*/2}\left(\frac{\nu_{i_0}}{m_0}\right)^{2_s^*/2}.$$

Hence $\nu_{i_0} = 0$ or $\nu_{i_0} \ge (m_0 S)^{2_s^*/(2_s^*-2)}$.

Step 2. We claim that (4.7) cannot occur, hence $\nu_i = 0$ for all $i \in J$.

By contradiction we assume that there exists a i_0 such that (4.7) holds true. By $\mathcal{I}_\lambda(u_n) \to c_\lambda$ and $\mathcal{I}'_\lambda(u_n) \to 0$ as $n \to \infty$, it follows that

$$c_\lambda = \lim_{n\to\infty}\left(\mathcal{I}_\lambda(u_n) - \frac{1}{2\theta}\langle \mathcal{I}'_\lambda(u_n), u_n\rangle\right). \tag{4.14}$$

Moreover, by (M_2) and (f_3), we have

$$\begin{aligned}
&\mathcal{I}_\lambda(u_n) - \frac{1}{2\theta}\langle \mathcal{I}'_\lambda(u_n), u_n\rangle \\
&\quad\ge \frac{1}{2}\mathscr{M}(\|u_n\|^2_{s,A}) - \frac{1}{2\theta}M(\|u_n\|^2_{s,A})\|u_n\|^2_{s,A} \\
&\qquad + \frac{\lambda}{2\theta}\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|u_n|^2))f(|u_n|^2)|u_n|^2dx - \frac{\lambda}{4}\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|u_n|^2))F(|u_n|^2)dx \\
&\qquad + \left(\frac{1}{2\theta} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx \\
&\quad\ge \left(\frac{1}{2\theta} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N}|u_n|^{2_s^*}\varphi_\varepsilon dx,
\end{aligned} \tag{4.15}$$

thanks to $\theta \ge 1$, $2\theta < \sigma < 2_s^*$ and $0 \le \varphi_\varepsilon \le 1$, where $\varphi_\varepsilon = \varphi(\frac{x-x_{i_0}}{\varepsilon})$ is defined as above. Combining (4.14) with (4.15), we have

$$c_\lambda = \lim_{n\to\infty}\mathcal{I}_\lambda(u_n) = \lim_{n\to\infty}\left(\mathcal{I}_\lambda(u_n) - \frac{1}{2\theta}\langle \mathcal{I}'_\lambda(u_n), u_n\rangle\right) \ge \left(\frac{1}{2\theta} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N}\varphi_\varepsilon d\nu,$$

from which, by letting $\varepsilon \to 0$ and using (4.7), it yields that

$$c_\lambda \ge \left(\frac{1}{2\theta} - \frac{1}{2_s^*}\right)\nu_{i_0} \ge \left(\frac{1}{2\theta} - \frac{1}{2_s^*}\right)(m_0 S)^{2_s^*/(2_s^*-2)},$$

which contradicts the assumption. Hence $\nu_i = 0$ for any $i \in J$.

Step 3. The assertion (4.5) holds

We first show that $u_n \to u$ in $L^{2_s^*}(\mathbb{R}^N, \mathbb{C})$. Assume that $\chi_R \in C^\infty(\mathbb{R}^{\mathbb{N}})$ satisfies $\chi_R \in [0,1]$ and $\chi_R(x) = 0$ for $|x| < R$, $\chi_R(x) = 1$ for $|x| > 2R$, and $|\nabla \chi_R| \leq 2/R$. With a similar discussion as in the proof of Theorem 3.2, we have

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \iint_{\mathbb{R}^N} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2 \chi_R(x)}{|x-y|^{N+2s}} dydx \tag{4.16}$$

and

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2_s^*} \chi_R(x) dx. \tag{4.17}$$

Thus a similar discussion as in the proof of Theorem 3.1 (see Part 2 of the proof of Theorem 3.1) gives that

$$\nu_\infty \leq S^{-2_s^*/2} \mu_\infty^{2_s^*/2}. \tag{4.18}$$

It follows from $\langle \mathcal{I}_\lambda'(u_n), \chi_R u_n \rangle \to 0$ as $n \to \infty$ that

$$\begin{aligned} M(\|u_n\|_{s,A}^2) \Bigg[&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2 \chi_R(x)}{|x-y|^{N+2s}} dxdy \\ &+ \Re \iint_{\mathbb{R}^N} \frac{(u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y))(\chi_R(x) - \chi_R(y))\overline{u_n(y)}}{|x-y|^{N+2s}} dxdy \\ &+ \int_{\mathbb{R}^N} |u_n|^2 \chi_R dx \Bigg] \\ = \lambda &\int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u_n|^2)) f(|u_n|^2)|u_n|^2 \chi_R dx + \int_{\mathbb{R}^N} |u_n|^{2_s^*} \chi_R dx + o(1). \end{aligned} \tag{4.19}$$

With a similar discussion as in Lemma 3.1, we have

$$\begin{aligned} &\lim_{R\to\infty} \limsup_{n\to\infty} M(\|u_n\|_{s,A}^2) \\ &\quad \times \left| \iint_{\mathbb{R}^N} \frac{(u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y))(\chi_R(x) - \chi_R(y))\overline{u_n(y)}}{|x-y|^{N+2s}} dxdy \right| = 0. \end{aligned}$$

Hence we deduce from (4.16) and (4.17) that

$$\begin{aligned} &\lim_{R\to\infty} \limsup_{n\to\infty} M(\|u_n\|_{s,A}^2) \left[\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2 \chi_R(x)}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} |u_n|^2 \chi_R dx \right] \\ &\quad \geq \lim_{R\to\infty} \limsup_{n\to\infty} m_0 \Bigg[\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2 \chi_R(x)}{|x-y|^{N+2s}} dxdy \\ &\qquad + \int_{\mathbb{R}^N} |u_n|^2 \chi_R(x) dx \Bigg] \\ &\quad = m_0 \mu_\infty. \end{aligned} \tag{4.20}$$

It is easy to see that

$$\begin{aligned}\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}&(\mathcal{K}_\alpha * F(|u_n|^2))F(|u_n|^2)\chi_R dx\\ &=\lim_{R\to\infty}\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|u|^2))F(|u|^2)\chi_R dx = 0.\end{aligned} \tag{4.21}$$

Therefore, we conclude from (4.19)–(4.21) and (4.17) that

$$m_0\mu_\infty \le \nu_\infty,$$

this together with (4.18) yields

$$m_0 S\nu_\infty^{\frac{2}{2_s^*}} \le \nu_\infty,$$

which implies that $\nu_\infty = 0$ or

$$\nu_\infty \ge (m_0 S)^{\frac{2_s^*}{2_s^*-2}}. \tag{4.22}$$

Assume (4.22) holds. Since $\|u_n\|_{s,A}^2$ and $\|u_n\|_{2_s^*}^{2_s^*}$ are bounded, up to a subsequence, we can assume that $\|u_n\|_{s,A}^2$ and $\|u_n\|_{2_s^*}^{2_s^*}$ are both convergent. Then by (4.16) and (4.17), we obtain

$$\lim_{n\to\infty}\|u_n\|_{s,A}^2 = \int_{\mathbb{R}^N} d\mu + \mu_\infty$$

and

$$\lim_{n\to\infty}\|u_n\|_{2_s^*}^{2_s^*} = \int_{\mathbb{R}^N} d\nu + \nu_\infty.$$

Thus, we have

$$\begin{aligned}c_\lambda &= \lim_{n\to\infty}\left(\mathcal{I}_\lambda(u_n) - \frac{1}{\sigma}\langle \mathcal{I}_\lambda'(u_n), u_n\rangle\right)\\
&\ge \lim_{n\to\infty}\left[\left(\frac{1}{2\theta}-\frac{1}{\sigma}\right)m_0\|u_n\|_{s,A}^2 + \left(\frac{1}{\sigma}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx\right.\\
&\qquad\left.+\frac{3\lambda}{4}\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|u_n|^2))F(|u_n|^2)dx\right]\\
&\ge m_0\left(\frac{1}{2\theta}-\frac{1}{\sigma}\right)\int_{\mathbb{R}^N}d\mu + m_0\left(\frac{1}{2\theta}-\frac{1}{\sigma}\right)\mu_\infty + \left(\frac{1}{\sigma}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^N}d\nu + \left(\frac{1}{\sigma}-\frac{1}{2_s^*}\right)\nu_\infty\\
&\ge m_0\left(\frac{1}{2\theta}-\frac{1}{\sigma}\right)\mu_\infty + \left(\frac{1}{\sigma}-\frac{1}{2_s^*}\right)\nu_\infty\\
&\ge m_0\left(\frac{1}{2\theta}-\frac{1}{\sigma}\right)S\nu_\infty^{2/(2_s^*)} + \left(\frac{1}{\sigma}-\frac{1}{2_s^*}\right)\nu_\infty \ge \left(\frac{1}{2\theta}-\frac{1}{2_s^*}\right)(m_0 S)^{2_s^*/(2_s^*-2)},\end{aligned}$$

due to $2\theta < \sigma$ and (4.22), which is a contradiction. Hence $\nu_\infty = 0$. In view of $J = \emptyset$, we have $\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx \to \int_{\mathbb{R}^N}|u|^{2_s^*}dx$ as $n\to\infty$. Furthermore, the Brèzis–Lieb

Lemma implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u|^{2_s^*}dx=0. \tag{4.23}$$

Now we define an operator as follows

$$\langle\mathcal{L}(v),w\rangle=\Re\iint_{\mathbb{R}^{2N}}\frac{(v(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y))\overline{(\omega(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}\omega(y))}}{|x-y|^{N+2s}}dxdy$$
$$+\Re\int_{\mathbb{R}^N}v\overline{\omega}dx,$$

for all $v,w\in H^s_{r,A}(\mathbb{R}^N,\mathbb{C})$. Obviously, $\mathcal{L}$ is a bounded bi-linear operator, being

$$|\langle\mathcal{L}(v),w\rangle|\le\|v\|_{s,A}\|w\|_{s,A},$$

by the Hölder inequality. Hence the weak convergence of $u_n\rightharpoonup u$ in $H^s_{r,A}(\mathbb{R}^N,\mathbb{C})$ implies

$$\lim_{n\to\infty}\langle\mathcal{L}u_n,u\rangle=\langle\mathcal{L}u,u\rangle\quad\text{and}\quad\lim_{n\to\infty}\langle\mathcal{L}u,u_n-u\rangle=0. \tag{4.24}$$

Clearly, $\langle\mathcal{I}_\lambda(u_n),u_n-u\rangle\to0$ as $n\to\infty$. Hence, by (4.24), one has

$$M(\|u_n\|^2_{s,A})\langle\mathcal{L}(u_n)-\mathcal{L}(u),u_n-u\rangle=\lambda\Re\int_{\mathbb{R}^N}(\mathcal{K}_\alpha*F(|u_n|^2))f(|u_n|^2)u_n(\overline{u_n-u})dx$$
$$+\Re\int_{\mathbb{R}^N}|u_n|^{2_s^*-2}u_n(\overline{u_n-u})dx+o(1).$$

Thus, we deduce from (4.2) and (4.23) that

$$M(\beta^2)\lim_{n\to\infty}\|u_n-u\|^2_{s,A}=0.$$

It follows from (M_1) that $u_n\to u$ in $H^s_{r,A}(\mathbb{R}^N)$ as $n\to\infty$. Therefore, (4.5) holds true. □

Now we state the general version of the mountain pass theorem in [3], which will be used later.

Theorem 4.1. *Let K be a functional on a Banach space E and $K\in C^1(E,\mathbb{R})$. Let us assume that there exist $\alpha,\rho>0$ such that*

(i) $K(u)\ge\alpha$, $\forall\,u\in E$ *with* $\|u\|=\rho$,
(ii) $K(0)=0$ *and* $K(e)<\alpha$ *for some* $e\in E$ *with* $\|e\|>\rho$.

Let us define $\Gamma=\{\gamma\in C([0,1];E):\gamma(0)=0,\gamma(1)=e\}$, *and*

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}K(\gamma(t)).$$

Then there exists a sequence $\{u_n\}_n\subset E$ *such that* $K(u_n)\to c$ *and* $K'(u_n)\to0$ *in* E' *(dual of* E*).*

In the following, we show that $\mathcal{I}_\lambda$ satisfies geometric properties (i) and (ii) of mountain pass.

Lemma 4.2. *The functional $\mathcal{I}_\lambda$ satisfies the assumptions* (i)–(ii) *in Theorem* 4.1.

Proof. For each $\lambda > 0$, by the fractional Sobolev embedding $H^s_{r,A}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^p(\mathbb{R}^N, \mathbb{C})$, we have for all $u \in H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$ that

$$\begin{aligned}\mathcal{I}_\lambda(u) &\geq \frac{m_0}{2}\|u\|^2_{s,A} - \frac{\lambda}{4}C(N,\alpha)\|F(|u|^2)\|^2_{L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)} - \frac{1}{2^*_s}S^{-2^*_s/2}\|u\|^{2^*_s}_{s,A}\\ &\geq \frac{m_0}{2}\|u\|^2_{s,A} - \frac{\lambda}{4}C(N,\alpha)2^{\frac{\alpha}{2N-\alpha}}C^{\frac{2N}{2N-\alpha}}(C^{\frac{4N}{2N-\alpha}}_{\frac{4N}{2N-\alpha}}\|u\|^{\frac{4N}{2N-\alpha}}_{s,A}\\ &\quad + C^{\frac{2Np}{2N-\alpha}}_{\frac{2Np}{2N-\alpha}}\|u\|^{\frac{2Np}{2N-\alpha}}_{s,A})^{\frac{2N-\alpha}{N}} - \frac{1}{2^*_s}S^{-2^*_s/2}\|u\|^{2^*_s}_{s,A}\\ &\geq \frac{m_0}{2}\|u\|^2_{s,A} - \lambda C(\|u\|^4_{s,A} + \|u\|^{2p}_{s,A}) - \frac{1}{2^*_s}S^{-2^*_s/2}\|u\|^{2^*_s}_{s,A}.\end{aligned}$$

It follows from $2 < p$ that there exist $\rho > 0$ small enough and $\alpha_0 > 0$ such that $\mathcal{I}_\lambda(u) \geq \alpha_0 > 0$ for all $u \in H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$ with $\|u\|_{s,A} = \rho$, and all $\lambda > 0$. Hence (i) in Theorem 4.1 holds true.

Now we verify condition (ii) in Theorem 4.1. Let $\varphi_0 \in C^\infty_0(\mathbb{R}^N.\mathbb{C})$ be a radial symmetric function, with $\|\varphi_0\|_{s,A} = 1$. By (M_2), we have

$$\mathscr{M}(t) \leq \mathscr{M}(1)t^\theta \quad \text{for all}\, t \geq 1.$$

Then by (f_3), there holds

$$\begin{aligned}\mathcal{I}_\lambda(t\varphi_0) &\leq \mathscr{M}(1)t^{2\theta} - \frac{\lambda}{4}\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|t\varphi_0|^2))F(|t\varphi_0|^2)dx - \frac{t^{2^*_s}}{2^*_s}\int_{\mathbb{R}^N}|\varphi_0|^{2^*_s}dx\\ &\leq \mathscr{M}(1)t^{2\theta} - \frac{t^{2^*_s}}{2^*_s}\int_{\mathbb{R}^N}|\varphi_0|^{2^*_s}dx,\end{aligned}$$

and hence $\mathcal{I}_\lambda(t\varphi_0) \to -\infty$ as $t \to \infty$, since $2\theta < 2^*_s$. Therefore, there exists t_0 large enough such that $\mathcal{I}_\lambda(t_0\varphi_0) < 0$. Then we take $e = t_0\varphi_0$ and $\mathcal{I}_\lambda(e) < 0$. Hence (ii) of Theorem 4.1 holds true. This completes the proof. □

Proof of Theorem 1.1. We claim that

$$\begin{aligned}c_\lambda &= \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{I}_\lambda(\gamma(t))\\ &< \left(\frac{1}{2\theta} - \frac{1}{2^*_s}\right)(m_0S)^{\frac{2^*_s}{2^*_s-2}}.\end{aligned} \tag{4.25}$$

Now we assume (4.25) holds true, then Lemmas 4.1, 4.2 and Theorem 4.1 give the existence of nontrivial critical points of $\mathcal{I}_\lambda$.

To prove (4.25), we choose $v_0 \in H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$ such that

$$\|v_0\|_{s,A} = 1, \quad \|v_0\|_{2_s^*} > 0, \quad \lim_{t\to\infty} \mathcal{I}_\lambda(tv_0) = -\infty,$$

then $\sup_{t\geq 0} \mathcal{I}_\lambda(tv_0) = \mathcal{I}_\lambda(t_\lambda v_0)$ for some $t_\lambda > 0$. Hence t_λ satisfies

$$M(t_\lambda^2)t_\lambda^2 = \lambda \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|t_\lambda v_0|^2))f(|t_\lambda v_0|^2)|t_\lambda v_0|^2 dx + \int_{\mathbb{R}^N} |t_\lambda v_0|^{2_s^*} dx. \quad (4.26)$$

Furthermore, by (M_2) and (f_3), we get

$$\begin{aligned} \theta\mathscr{M}(\|t_\lambda v_0\|_{s,A}^2) &\geq M(\|t_\lambda v_0\|^2)\|v_0\|_{s,A}^2 \\ &= \lambda \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|t_\lambda v_0|^2))f(|t_\lambda v_0|^2)|t_\lambda v_0|^2 dx + t_\lambda^{2_s^*} \int_{\mathbb{R}^N} |v_0|^{2_s^*} dx. \\ &\geq t_\lambda^{2_s^*} \int_{\mathbb{R}^N} |v_0|^{2_s^*} dx. \end{aligned} \quad (4.27)$$

Now we show that $\{t_\lambda\}_\lambda$ is bounded. Without loss of generality, we assume that $t_\lambda \geq 1$ for all $\lambda > 0$. Using (M_2) again, we deduce from (4.27) that

$$\theta\mathscr{M}(1)t_\lambda^{2\theta} \geq t_\lambda^{2_s^*} \int_{\mathbb{R}^N} |v_0|^{2_s^*} dx.$$

It follows from $\theta < 2_s^*/2$ that $\{t_\lambda\}_\lambda$ is bounded.

We claim that $t_\lambda \to 0$ as $\lambda \to \infty$. Arguing by contradiction, we can assume that there exist $t_0 > 0$ and a sequence λ_n with $\lambda_n \to \infty$ as $n \to \infty$ such that $t_{\lambda_n} \to t_0$ as $n \to \infty$. By (f_2) and Lebesgue's dominated convergence theorem, we deduce

$$\begin{aligned} &\int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|t_{\lambda_n} v_0|^2))f(|t_{\lambda_n} v_0|^2)|t_{\lambda_n} v_0|^2 dx \\ &\quad \to \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|t_0 v_0|^2))f(|t_0 v_0|^2)|t_0 v_0|^2 dx \end{aligned}$$

as $n \to \infty$. From which it follows that

$$\lambda_n \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|t_0 v_0|^2))f(|t_0 v_0|^2)|t_0 v_0|^2 dx \to \infty \quad \text{as } n \to \infty.$$

Hence, (4.26) implies that

$$M(t_0^2)t_0^2 = \infty,$$

which is absurd. Therefore, $t_\lambda \to 0$ as $\lambda \to \infty$. Further, we deduce from (4.26) that

$$\lim_{\lambda\to\infty} \lambda \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|t_\lambda v_0|^2))f(|t_\lambda v_0|^2)|t_\lambda v_0|^2 dx = 0.$$

Moreover,

$$\lim_{\lambda\to\infty} \lambda \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|t_\lambda v_0|^2))F(|t_\lambda v_0|^2) dx = 0,$$

from this, $t_\lambda \to 0$ as $\lambda \to \infty$ and the definition of $\mathcal{I}_\lambda$, we get

$$\lim_{\lambda\to\infty}\left(\sup_{t\ge 0}\mathcal{I}_\lambda(tv_0)\right) = \lim_{\lambda\to\infty}\mathcal{I}_\lambda(t_\lambda v_0) = 0.$$

Then there exists $\lambda_* > 0$ such that for any $\lambda \ge \lambda_*$,

$$\sup_{t\ge 0}\mathcal{I}_\lambda(tv_0) < \left(\frac{1}{2\theta} - \frac{1}{2_s^*}\right)(m_0 S)^{\frac{2_s^*}{2_s^*-2}}.$$

If we take $e = Tv_0$, with T large enough to verify $\mathcal{I}_\lambda(e) < 0$, then we obtain

$$c_\lambda \le \max_{t\in[0,1]}\mathcal{I}_\lambda(\gamma(t)) \quad \text{by taking } \gamma(t) = tTv_0.$$

Therefore, $c_\lambda \le \sup_{t\ge 0}\mathcal{I}_\lambda(tv_0) < (\frac{1}{2\theta} - \frac{1}{2_s^*})(m_0 S)^{\frac{2_s^*}{2_s^*-2}}$ for λ large enough. □

5. Proof of Theorem 1.2

In this section, we start with the study of the degenerate case of (1.1). To this end, we always assume that $s \in (0,1)$, $N > 2s$, $0 < \alpha < N$, $\theta \in [1, 2_s^*)$, $A \in C(\mathbb{R}^N, \mathbb{R}^N)$, M satisfies (M_2) and (M_3), and f satisfies (f_1), (f_3) and (f_4). We first give a crucial lemma in the proof of existence of solutions for problem (1.1).

Lemma 5.1. *Let $\{u_n\}_n \subset H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$ be a Palais–Smale sequence of functional $\mathcal{I}_\lambda$, that is,*

$$\mathcal{I}_\lambda(u_n) \to c_\lambda \quad \text{and} \quad \mathcal{I}'_\lambda(u_n) \to 0 \quad \text{in } (H^s_{r,A}(\mathbb{R}^N, \mathbb{C}))',$$

as $n \to \infty$, where $(H^s_{r,A}(\mathbb{R}^N, \mathbb{C}))'$ is the dual of $H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$. If

$$c_\lambda < \left(\frac{1}{2\theta} - \frac{1}{2_s^*}\right)(m_1 S^\theta)^{2_s^*/(2_s^*-2\theta)},$$

where S is the number given in Theorem 3.1, then there exists a subsequence of $\{u_n\}_n$ strongly convergent in $H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$.

Proof. If $\inf_{n\ge 1}\|u_n\|_{s,A} = 0$, then there exists a subsequence of $\{u_n\}_n$ still denoted by $\{u_n\}_n$ such that $u_n \to 0$ in $H^s_{r,A}(\mathbb{R}^N)$ as $n \to \infty$. Thus, we assume that $d := \inf_{n\ge 1}\|u_n\|_{s,A} > 0$ in the following proof.

By $\mathcal{I}_\lambda(u_n) \to c_\lambda$ and $\mathcal{I}'_\lambda(u_n) \to 0$ in $(H^s_{r,A}(\mathbb{R}^N, \mathbb{C}))'$, there exists $C > 0$ such that

$$\begin{aligned} &C + C\|u_n\|_{s,A} \\ &\quad \ge \mathcal{I}_\lambda(u_n) - \frac{1}{\sigma}\langle \mathcal{I}'_\lambda(u_n), u_n\rangle \end{aligned}$$

$$= \frac{1}{2}\mathscr{M}(\|u_n\|_{s,A}^2) - \frac{1}{\sigma}M(\|u_n\|_{s,A}^2)\|u_n\|_{s,A}^2 - \frac{\lambda}{4}\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|u_n|^2))F(|u_n|^2)dx$$
$$+ \frac{\lambda}{\sigma}\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|u_n|^2))f(|u_n|^2)|u_n|^2dx + \left(\frac{1}{\sigma} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx.$$

It follows from $(M_2), (M_3)$ and (f_3) that

$$\begin{aligned} C + C\|u_n\|_{s,A} &\geq \left(\frac{1}{2\theta} - \frac{1}{\sigma}\right)M(\|u_n\|_{s,A}^2)\|u_n\|_{s,A}^2 \\ &\geq m_1\left(\frac{1}{2\theta} - \frac{1}{\sigma}\right)\|u_n\|_{s,A}^{2\theta}, \end{aligned}$$

this together with $2 \leq 2\theta < \sigma$ implies that $\{u_n\}_n$ is bounded in $H_{r,A}^s(\mathbb{R}^N, \mathbb{C})$. Then there exists $u \in H_{r,A}^s(\mathbb{R}^N, \mathbb{C})$ such that, up to a subsequence, it follows that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H_{r,A}^s(\mathbb{R}^N, \mathbb{C}) \text{ and in } L^{2_s^*}(\mathbb{R}^N, \mathbb{C}), \\ u_n &\to u \quad \text{a.e. in } \mathbb{R}^N, \\ |u_n|^{2_s^*-2}u_n &\rightharpoonup |u|^{2_s^*-2}u \quad \text{weakly in } L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^N, \mathbb{C}), \\ \|u_n\|_{s,A} &\to \beta. \end{aligned} \tag{5.1}$$

Similar to Lemma 4.1, we have as $n \to \infty$

$$(\mathcal{K}_\alpha * F(|u_n|^2)) \to (\mathcal{K}_\alpha * F(|u|^2)) \quad \text{in } L^{\frac{2N}{\alpha}}(\mathbb{R}^N) \tag{5.2}$$

and

$$\lim_{n\to\infty}\Re\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|u_n|^2))f(|u_n|^2)u_n\overline{\varphi}dx = \Re\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|u|^2))f(|u|^2)u\overline{\varphi}dx \tag{5.3}$$

for each $\varphi \in H_{r,A}^s(\mathbb{R}^N, \mathbb{C})$.

Now we claim that

$$u_n \to u \quad \text{in } H_{r,A}^s(\mathbb{R}^N, \mathbb{C}) \tag{5.4}$$

as $n \to \infty$. By Theorem 3.1, there exist an at most countable set of distinct points $\{x_i\}_{i\in J}$, non-negative numbers $\{\mu_i\}_{i\in J}, \{\nu_i\}_{i\in J} \subset [0, \infty)$ and a non-atomic measure $\widetilde{\mu}$ such that

$$\begin{aligned} \mu &= \int_{\mathbb{R}^N}\frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}dy + |u|^2 + \sum_{i\in J}\mu_i\delta_{x_i} + \widetilde{\mu}, \\ \nu &= |u(x)|^{2_s^*} + \sum_{i\in J}\nu_i\delta_{x_i}, \end{aligned} \tag{5.5}$$

Next, in order to prove (5.4) we proceed by steps.

Step 1. Fix $i_0 \in J$. Then we prove that either $\nu_{i_0} = 0$ or

$$\nu_{i_0} \geq (m_1 S)^{2_s^*/(2_s^*-2)}. \tag{5.6}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N; [0,1])$ be a radial symmetric function satisfying $\varphi = 1$ in $B(0,1)$; $\varphi = 0$ in $\mathbb{R}^N \backslash B(0,2)$ and $|\nabla \varphi| \leq 2$. For any $\varepsilon > 0$ we set $\varphi_\varepsilon = \varphi(\frac{x - x_{i_0}}{\varepsilon})$. Clearly $\{\varphi_\varepsilon u_n\}_n$ is bounded in $H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$ and $\langle \mathcal{I}'_\lambda(u_n), \varphi_\varepsilon u_n \rangle \to 0$ as $n \to \infty$. Hence

$$\begin{aligned} &M(\|u_n\|^2_{s,A})(u_n, \varphi_\varepsilon u_n)_{s,A} \\ &\quad = \lambda \Re \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u_n|^2)) f(|u_n|^2) u_n \varphi_\varepsilon \overline{u_n} dx + \Re \int_{\mathbb{R}^N} |u_n|^{2_s^*-2} u_n \varphi_\varepsilon \overline{u_n} dx. \end{aligned} \tag{5.7}$$

It is easy to verify that

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u_n|^2)) f(|u_n|^2) |u_n|^2 \varphi_\varepsilon dx = 0. \tag{5.8}$$

Note that by (M_3), there holds

$$\begin{aligned} &M(\|u_n\|^2_{s,A}) \iint_{\mathbb{R}^N} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2 \varphi_\varepsilon(x)}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} |u_n|^2 \varphi_\varepsilon dx \\ &\quad \geq m_1 \left(\iint_{\mathbb{R}^N} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2 \varphi_\varepsilon(x)}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} |u_n|^2 \varphi_\varepsilon dx \right)^\theta. \end{aligned}$$

Using a similar discussion as in Lemma 4.1, we deduce that

$$\lim_{\varepsilon \to 0} \sup_{n \to \infty} M(\|u_n\|^2_{s,A})(u_n, \varphi_\varepsilon u_n)_{s,A} \geq m_1 \mu_{i_0}^\theta. \tag{5.9}$$

Similar to Lemma 4.1, we have

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2_s^*-2} u_n \varphi_\varepsilon \overline{u_n} dx = 0. \tag{5.10}$$

Inserting (5.8)–(5.10) into (5.7), we obtain

$$m_1 \mu_{i_0}^\theta \leq \nu_{i_0}.$$

It follows from $\nu_i \leq S^{-2_s^*/2} (\mu_i)^{2_s^*/2}$ for all $i \in J$ that $\nu_{i_0} = 0$ or

$$\nu_{i_0} \geq (m_1 S^\theta)^{2_s^*/(2_s^*-2\theta)}. \tag{5.11}$$

Step 2. We claim that (5.11) cannot occur, hence $\nu_i = 0$ for all $i \in J$.

By contradiction we assume that there exists a i_0 such that (5.11) holds true. Similar to Lemma 4.1, by (M_2) and (f_3), we deduce

$$c_\lambda \geq \left(\frac{1}{2\theta} - \frac{1}{2_s^*} \right) \int_{\mathbb{R}^N} |u_n|^{2_s^*} \varphi_\varepsilon dx. \tag{5.12}$$

It follows from (5.11) that

$$c_\lambda \geq \left(\frac{1}{2\theta} - \frac{1}{2_s^*} \right) (m_1 S^\theta)^{2_s^*/(2_s^*-2\theta)},$$

which contradicts the assumption. Hence $\nu_i = 0$ for any $i \in J$.

Step 3. The assertion (5.4) holds.

We first show that $u_n \to u$ in $L^{2_s^*}(\mathbb{R}^N, \mathbb{C})$ as $n \to \infty$. Assume that $\chi_R \in C^\infty(\mathbb{R}^{\mathbb{N}})$ satisfies $\chi_R \in [0,1]$ and $\chi_R(x) = 0$ for $|x| < R$, $\chi_R(x) = 1$ for $|x| > 2R$, and $|\nabla \chi_R| \leq 2/R$. With a similar discussion as in the proof of Theorem 3.2, we have

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \iint_{\mathbb{R}^N} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2 \chi_R(x)}{|x-y|^{N+2s}} dydx, \tag{5.13}$$

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2_s^*}(x) \chi_R(x) dx \tag{5.14}$$

and

$$\nu_\infty \leq S^{-2_s^*/2} \mu_\infty^{2_s^*/2}. \tag{5.15}$$

It follows from the fact that $\langle \mathcal{I}'_\lambda(u_n), \chi_R u_n\rangle \to 0$ as $n \to \infty$ that

$$\begin{aligned}
&M(\|u_n\|_{s,A}^2) \left[\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2 \chi_R(x)}{|x-y|^{N+2s}} dxdy \right. \\
&\quad + \Re \iint_{\mathbb{R}^N} \frac{(u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y))(\chi_R(x) - \chi_R(y))\overline{u_n(y)}}{|x-y|^{N+2s}} dxdy \\
&\quad \left. + \int_{\mathbb{R}^N} |u_n|^2 \chi_R dx \right] \\
&= \lambda \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u_n|^2)) f(|u_n|^2)|u_n|^2 \chi_R dx + \int_{\mathbb{R}^N} |u_n|^{2_s^*} \chi_R dx + o(1).
\end{aligned} \tag{5.16}$$

With a similar discussion as in Lemma 3.1, we have

$$\begin{aligned}
&\lim_{R\to\infty} \limsup_{n\to\infty} M(\|u_n\|_{s,A}^2) \\
&\quad \times \left| \iint_{\mathbb{R}^N} \frac{(u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y))(\chi_R(x) - \chi_R(y))\overline{u_n(y)}}{|x-y|^{N+2s}} dxdy \right| = 0.
\end{aligned}$$

Hence we deduce from (5.13) that

$$\begin{aligned}
&\lim_{R\to\infty} \limsup_{n\to\infty} M(\|u_n\|_{s,A}^2) \left[\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2 \chi_R(x)}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} |u_n|^2 \chi_R dx \right] \\
&\quad \geq \lim_{R\to\infty} \limsup_{n\to\infty} m_1 \left[\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_n(y)|^2 \chi_R(x)}{|x-y|^{N+2s}} dxdy \right. \\
&\quad \left. + \int_{\mathbb{R}^N} |u_n|^2 \chi_R(x) dx \right]^\theta \\
&\quad = m_1 \mu_\infty^\theta.
\end{aligned} \tag{5.17}$$

Similar to Lemma 4.1 we can obtain that

$$m_1\mu_\infty^\theta \leq \nu_\infty,$$

this together with (5.15) yields

$$m_1 S\nu_\infty^{\frac{2}{2_s^*}} \leq \nu_\infty,$$

which implies that $\nu_\infty = 0$ or

$$\nu_\infty \geq (m_1 S^\theta)^{\frac{2_s^*}{2_s^*-2\theta}}. \tag{5.18}$$

Assume (5.18) holds. Since $\|u_n\|_{s,A}^2$ and $\|u_n\|_{2_s^*}^{2_s^*}$ are bounded, up to a subsequence, we can assume that $\|u_n\|_{s,A}^2$ and $\|u_n\|_{2_s^*}^{2_s^*}$ are both convergent. Then by (5.13) and (5.14), we obtain

$$\lim_{n\to\infty} \|u_n\|_{s,A}^2 = \int_{\mathbb{R}^N} d\mu + \mu_\infty$$

and

$$\lim_{n\to\infty} \|u_n\|_{2_s^*}^{2_s^*} = \int_{\mathbb{R}^N} d\nu + \nu_\infty.$$

Thus, we have

$$\begin{aligned}
c_\lambda &= \lim_{n\to\infty}\left(\mathcal{I}_\lambda(u_n) - \frac{1}{\sigma}\langle \mathcal{I}_\lambda'(u_n), u_n\rangle\right)\\
&\geq \lim_{n\to\infty}\left[\left(\frac{1}{2\theta}-\frac{1}{\sigma}\right)m_1\|u_n\|_{s,A}^{2\theta} + \left(\frac{1}{\sigma}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx\right.\\
&\quad \left. + \frac{3\lambda}{4}\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|u_n|^2))F(|u_n|^2)dx\right]\\
&\geq m_1\left(\frac{1}{2\theta}-\frac{1}{\sigma}\right)\left(\int_{\mathbb{R}^N} d\mu + \mu_\infty\right)^\theta + \left(\frac{1}{\sigma}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^N} d\nu + \left(\frac{1}{\sigma}-\frac{1}{2_s^*}\right)\nu_\infty\\
&\geq m_1\left(\frac{1}{2\theta}-\frac{1}{\sigma}\right)\mu_\infty^\theta + \left(\frac{1}{\sigma}-\frac{1}{2_s^*}\right)\nu_\infty\\
&\geq m_1\left(\frac{1}{2\theta}-\frac{1}{\sigma}\right)S\nu_\infty^{2/(2_s^*)} + \left(\frac{1}{\sigma}-\frac{1}{2_s^*}\right)\nu_\infty \geq \left(\frac{1}{2\theta}-\frac{1}{2_s^*}\right)(m_1S^\theta)^{2_s^*/(2_s^*-2\theta)},
\end{aligned}$$

thanks to $2\theta < \sigma$ and (5.18), which is a contradiction. Hence $\nu_\infty = 0$. In view of $J = \emptyset$, we have $\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx \to \int_{\mathbb{R}^N}|u|^{2_s^*}dx$ as $n\to\infty$. Furthermore, the Brèzis–Lieb Lemma implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u|^{2_s^*}dx = 0. \tag{5.19}$$

A similar discussion as in Lemma 4.1 yields that $u_n \to u$ in $H_{r,A}^s(\mathbb{R}^N)$ as $n\to\infty$. Therefore, (5.4) holds true. $\square$

In what follows, we prove that $\mathcal{I}_\lambda$ satisfies geometric properties (i) and (ii) of mountain pass.

Lemma 5.2. *The functional $\mathcal{I}_\lambda$ satisfies the conditions* (i) *and* (ii) *in Theorem* 4.1.

Proof. For each $\lambda > 0$, by the fractional Sobolev embedding $H^s_{r,A}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^p(\mathbb{R}^N, \mathbb{C})$, we have by (M_3) and (f_4) that

$$\mathcal{I}_\lambda(u) \geq \frac{m_1}{\theta}\|u\|^{2\theta}_{s,A} - \lambda C\|u\|^{2p}_{s,A} - \frac{1}{2^*_s} S^{-2^*_s/2}\|u\|^{2^*_s}_{s,A},$$

for all $u \in H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$. It follows from $\max\{2, \theta\} < p$ that there exist $\rho_1 > 0$ small enough and $\alpha_1 > 0$ such that $\mathcal{I}_\lambda(u) \geq \alpha_1 > 0$ for all $u \in H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$ with $\|u\|_{s,A} = \rho_1$, and all $\lambda > 0$. Hence (i) in Theorem 4.1 holds true. Similar to Lemma 4.2, we can show that (ii) in Theorem 4.1 holds true. □

Proof of Theorem 1.2. By using the same discussion as the proof of Theorem 1.1, we deduce that

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) < \left(\frac{1}{2\theta} - \frac{1}{2^*_s}\right)(m_1 S^\theta)^{\frac{2^*_s}{2^*_s - 2\theta}}.$$

The rest of the proof is the same as in the proof to Theorem 1.1. □

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