Weak solutions of quasilinear problems with nonlinear boundary condition

Florica Cîrstea\textsuperscript{a}, Dumitru Motreanu\textsuperscript{b}, Vicențiu Rădulescu\textsuperscript{a,}\textsuperscript{*}

\textsuperscript{a}Department of Mathematics, University of Craiova, 1100 Craiova, Romania
\textsuperscript{b}Department of Mathematics, University A. I. Cuza, 6600 Iași, Romania

Received 1 December 1998; accepted 2 March 1999

Keywords: Weak solution; Weighted Sobolev space; Unbounded domain; Quasilinear eigenvalue problem

1. Introduction

The growing attention for the study of the $p$-Laplacian operator $\Delta_p$ in the last few decades is motivated by the fact that it arises in various applications. For instance, in Fluid Mechanics, the shear stress $\tau$ and the velocity gradient $\nabla u$ of certain fluids obey a relation of the form

$$\tau(x) = a(x) \nabla u(x),$$

where $\nabla u = |\nabla u|^{p-2} \nabla u$. Here $p > 1$ is an arbitrary real number and the case $p = 2$ (respectively $p < 2$, $p > 2$) corresponds to a Newtonian (respectively pseudoplastic, dilatant) fluid. The resulting equations of motion then involve $\text{div}(a \nabla u)$, which reduces to $a \Delta_p u = a \text{div} \nabla u$, provided that $a$ is constant. The $p$-Laplacian also appears in the study of torsional creep (elastic for $p = 2$, plastic as $p \to \infty$, see [7]), flow through porous media ($p = \frac{3}{2}$, see [12]) or glacial sliding ($p \in (1, \frac{3}{2}]$, see [9]).

Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with (possible noncompact) smooth boundary $\Gamma$ and $n$ is the unit outward normal on $\Gamma$. We consider the nonlinear elliptic boundary value problem:

$$-\text{div}(a(x)|\nabla u|^{p-2} \nabla u) = \lambda (1 + |x|)^q |u|^{p-2} u + (1 + |x|)^q |u|^{q-2} u \quad \text{in} \ \Omega,$$

$$a(x)|\nabla u|^{p-2} \nabla u \cdot n + b(x) \cdot |u|^{p-2} u = g(x,u) \quad \text{on} \ \Gamma. \quad (A)$$

\textsuperscript{*}Corresponding author.
We assume throughout that $1 < p < N$, $p < q < p^*$, $-N < \alpha_1 < -p$, $-N < \alpha_2 < q \cdot (N - p)/p - N$, $0 < a_0 \leq a \in L^\infty(\Omega)$ and $b : \Gamma \to \mathbb{R}$ is a continuous function satisfying
\[
\frac{c}{(1 + |x|)^{p-1}} \leq b(x) \leq \frac{C}{(1 + |x|)^{p-1}},
\]
for constants $0 < c \leq C$.

Let $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that
\[
(A1) \quad |g(x,s)| \leq g_0(x) + g_1(x)|s|^{m-1}, \quad q < m < \frac{N-1}{N-p},
\]
where $g_i : \Gamma \to \mathbb{R}$ ($i = 0, 1$) are measurable functions satisfying $g_0 \in L^m(m-1)(\Gamma; w_3^{1/(1-m)})$, $0 \leq g_i \leq C_{g}w_3$ a.e. in $\Gamma$,

for a constant $C_{g} > 0$, with $w_3(x) = (1 + |x|)^{p_3}$, $x \in \Gamma$, and $-N < \alpha_3 < m \cdot (N - p)/p - N + 1$.

We also assume
\[
(A2) \quad \lim_{s \to 0} \frac{g(x,s)}{|s|^{m-1}} = 0 \text{ uniformly in } x.
\]

(A3) There exists $\mu \in (p, q]$ such that
\[
\mu G(x,s) \leq sg(x,s) \quad \text{for a.e. } x \in \Gamma \text{ and every } s \in \mathbb{R}.
\]

(A4) There is a non-empty open set $U \subset \Gamma$ with $G(x,s) > 0$ for $(x,s) \in U \times (0, \infty)$, where $G$ is the primitive function of $g$ with respect to the second variable, i.e., $G(x,s) = \int_0^s g(x,t) \, dt$.

Our first result asserts that, under the above hypotheses, problem (A) has at least a solution in an appropriate space.

Eigenvalue problems involving the $p$-Laplacian have been the subject of much recent interest (we refer only to \cite{[1,3,4,6]}). Our purpose is to prove the existence of an eigensolution for the following eigenvalue problem:

\[
\begin{align*}
-\text{div}(a(x) |\nabla u|^{p-2}\nabla u) &= \lambda[(1 + |x|)^{p_1}|u|^{p-2}u + (1 + |x|)^{p_2}|u|^{q-2}u] \quad \text{in } \Omega, \\
&= a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)|u|^{p-2}u = \lambda g(x,u) \quad \text{on } \Gamma.
\end{align*}
\]

In the study of this problem we drop assumptions (A2) and (A4).

2. Preliminaries and the main results

Let $C_0^\infty(\Omega)$ be the space of $C_0^\infty(\mathbb{R}^N)$ — functions restricted on $\Omega$. We define the weighted Sobolev space $E$ as the completion of $C_0^\infty(\Omega)$ in the norm
\[
\|u\|_E = \left( \int_{\Omega} \left( |\nabla u(x)|^p + \frac{1}{(1 + |x|)^p} |u(x)|^p \right) \, dx \right)^{1/p}.
\]

Denote by $L^p(\Omega; w_1)$, $L^q(\Omega; w_2)$ and $L^m(\Gamma; w_3)$ the weighted Lebesgue spaces with weight functions
\[
w_i(x) = (1 + |x|)^{p_i}, \quad i = 1, 2, 3
\]
and the norms defined by

\[ ||u||_{p,w_1}^p = \int_{\Omega} w_1 |u(x)|^p \, dx, \]

\[ ||u||_{q,w_2}^q = \int_{\Omega} w_2 |u(x)|^q \, dx \quad \text{and} \quad ||u||_{m,w_3}^m = \int_{\Gamma} w_3 |u(x)|^m \, d\Gamma. \]

Then we have the following embedding and trace theorem.

**Theorem 1.** If

\[ p \leq r \leq \frac{pN}{N - p} \quad \text{and} \quad -N < a \leq r \cdot \frac{N - p}{p} - N, \]

then the embedding \( E \subseteq L^r(\Omega; w) \) is continuous, where \( w(x) = (1 + |x|^2). \) If the upper bounds for \( r \) in (1) are strict, then the embedding is compact. If

\[ p \leq m \leq p \cdot \frac{N - 1}{N - p} \quad \text{and} \quad -N < a_3 \leq m \cdot \frac{N - p}{p} - N + 1, \]

then the trace operator \( E \rightarrow L^m(\Gamma; w_3) \) is continuous. If the upper bounds for \( m \) in (2) are strict, then the trace is compact.

This theorem is a consequence of Theorem 2 and Corollary 6 of Pflüger [11].

**Lemma 1.** The quantity

\[ ||u||_b^p = \int_{\Omega} a(x)|\nabla u|^p \, dx + \int_{\Gamma} b(x)|u|^p \, d\Gamma \]

defines an equivalent norm on \( E. \)

For the proof of this result we refer to [10], Lemma 2.

We denote by \( N_g, N_G \) the corresponding Nemytskii operators.

**Lemma 2.** The operators

\[ N_g : L^m(\Gamma; w_3) \rightarrow L^{m/(m-1)}(\Gamma; w_3^{1/(1-m)}), \quad N_G : L^m(\Gamma; w_3) \rightarrow L^1(\Gamma) \]

are bounded and continuous.

**Proof.** Let \( m' = m/(m - 1) \) and \( u \in L^m(\Gamma; w_3). \) Then, by (A1) we have

\[
\int_{\Gamma} |N_g(u)|^{m'} \cdot w_3^{1/(1-m)} \, d\Gamma \leq 2^{m'-1} \left( \int_{\Gamma} g_0^{m'} \cdot w_3^{1/(1-m)} \, d\Gamma + \int_{\Gamma} g_1^{m'} |u|^m \cdot w_3^{1/(1-m)} \, d\Gamma \right)
\]

\[
\leq 2^{m'-1} \left( C + C_g \cdot \int_{\Gamma} |u|^m \cdot w_3 \, d\Gamma \right),
\]
which shows that \( N_G \) is bounded. In a similar way, we obtain

\[
\int_\Gamma |N_G(u)| \, d\Gamma \leq \int_\Gamma g_0 |u| \, d\Gamma + \int_\Gamma g_1 |u|^m \, d\Gamma \\
\leq \left( \int_\Gamma g_0' w_3^{1/(1-m)} \, d\Gamma \right)^{1/m'} \cdot \left( \int_\Gamma |u|^m \cdot w_3 \, d\Gamma \right)^{1/m} + C_1 \int_\Gamma |u|^m \cdot w_3 \, d\Gamma
\]

and we claim that \( N_G \) is bounded.

Now, from the usual properties of Nemytskii operators we deduce the continuity of these operators.

By weak solution of problem (A) we mean a function \( u \in \mathcal{E} \) such that

\[
\int_\Omega a(x)|\nabla u|^{p-2} \nabla u : \nabla v \, dx + \int_\Gamma b(x)|u|^{p-2}uv \, d\Gamma \\
= \lambda \int_\Omega w_1 |u|^{q-2} uv \, dx + \int_\Omega w_2 |u|^{q-2} uv \, dx + \int_\Gamma g(x,u)v \, d\Gamma, \quad \forall v \in \mathcal{E}.
\]

Define

\[
\lambda := \inf_{u \in \mathcal{E}, u \neq 0} \left( \frac{\int_\Omega a(x)|\nabla u|^p \, dx + \int_\Gamma b(x)|u|^p \, d\Gamma}{\int_\Omega |u|^p \cdot w_1 \, dx} \right).
\]

Our first result is

**Theorem 2.** Assume that conditions (A1)–(A4) hold. Then, for every \( \lambda < \lambda \), problem (A) has a nontrivial weak solution.

We stress that for the following result of the paper we drop assumptions (A2) and (A4).

By weak solution of problem (B) we mean a function \( u \in \mathcal{E} \) such that

\[
\int_\Omega a(x)|\nabla u|^{p-2} \nabla u : \nabla v \, dx + \int_\Gamma b(x)|u|^{p-2}uv \, d\Gamma \\
= \lambda \int_\Omega w_1 |u|^{q-2} uv \, dx + \int_\Omega w_2 |u|^{q-2} uv \, dx + \int_\Gamma g(x,u)v \, d\Gamma, \quad \forall v \in \mathcal{E}.
\]

We now state the main result of solving problem (B).

**Theorem 3.** Assume that hypotheses (A1) and (A3) hold. Let \( d \) be an arbitrary real number such that \( 1/d \) is not an eigenvalue \( \lambda \) in problem (B), and satisfying

\[
d > \frac{1}{\lambda}.
\]

Then there exists \( \bar{\rho} > 0 \) such that for all \( r > \rho \geq \bar{\rho} \), eigenvalue problem (B) has an eigensolution \( (u, \lambda) = (u_d, \lambda_d) \in \mathcal{E} \times \mathbb{R} \) for which one has

\[
\lambda_d \in \left[ \frac{1}{d + \rho^2 \|u_d\|_{h}^{\alpha - p}} d + \rho^2 \|u_d\|_{h}^{\alpha - p} \right].
\]
3. Proof of Theorem 2

The key argument in the proof is the Mountain-Pass Theorem in the following variant (see [2]):

**Ambrosetti–Rabinowitz Theorem.** Let \( X \) be a real Banach space and \( F : X \to \mathbb{R} \) be a \( C^1 \)-functional. Suppose that \( F \) satisfies the Palais–Smale condition and the following geometric assumptions:

- there exist positive constants \( R \) and \( c_0 \) such that \( F(u) \geq c_0 \), for all \( u \in X \) with \( \|u\| = R \);
- \( F(0) < c_0 \) and there exists \( v \in X \) such that \( \|v\| > R \) and \( F(v) < c_0 \).

Then the functional \( F \) possesses at least a critical point.

Throughout this section we use the same notations as was previously done in the case of problem (A) excepting that \( h(x,s) = w_2(x)\lambda q^2 s, \forall x \in \Omega, s \in \mathbb{R} \).

The energy functional corresponding to (A) is defined as \( F : E \to \mathbb{R} \)

\[
F(u) = \frac{1}{p} \int_{\Omega} a(x) \cdot |\nabla u|^p \, dx + \frac{1}{p} \int_{\Gamma} b(x) \cdot |u|^p \, d\Gamma \\
- \frac{\lambda}{p} \int_{\Omega} w_1 \cdot |u|^p \, dx - \int_{\Gamma} G(x,u) \, d\Gamma - \int_{\Omega} H(x,u) \, dx,
\]

where \( H \) denotes the primitive function of \( h \) with respect to the second variable.

By Lemma 1 we have \( \| \cdot \|_b \simeq \| \cdot \|_p \). We may write

\[
F(u) = \frac{1}{p} \cdot \|u\|_b^p - \frac{\lambda}{p} \int_{\Omega} w_1 \cdot |u|^p \, dx - \int_{\Gamma} G(x,u) \, d\Gamma - \int_{\Omega} H(x,u) \, dx.
\]

We observe that

\[
|H(x,u)| = \frac{1}{q} w_2(x)|u|^q.
\]

Since \( p < q < p^* \), \(-N < \alpha_1 < -p \) and \(-N < \alpha_2 < q \cdot (N - p)/(p - N)\) we can apply Theorem 1 and we obtain that the embeddings \( E \subseteq L^p(\Omega; w_1) \) and \( E \subseteq L^q(\Omega; w_2) \) are compact. This and (6) imply that \( F \) is well defined.

Our hypothesis

\[
\lambda < \tilde{\lambda} := \inf_{u \in E, u \neq 0} \frac{\|u\|_b^p}{\|u\|_p^p, w_1}
\]

implies the existence of some \( C_0 > 0 \) such that, for every \( v \in E \)

\[
\|v\|_b^p - \lambda \|v\|_p^{\alpha_1} \geq C_0 \|v\|_b^p.
\]

We shall prove in what follows that \( F \) satisfies the hypotheses of the Mountain-Pass Theorem.

Proof. Denote $I(u) = (1/p)\|u\|_p^p$, $K_G(u) = \int_{\Gamma} G(x,u) \, d\Gamma$, $K_H(u) = \int_{\Omega} H(x,u) \, dx$ and $K_{\Phi}(u) = \int_{\Omega} (1/p)w_1 |u|^p \, dx$, where $\Phi(x,u) = (1/p)w_1(x) |u|^p$. Then the directional derivative of $F$ in the direction $v \in E$ is

$$
\langle F'(u), v \rangle = \langle I'(u), v \rangle - \lambda \langle K_{\Phi}'(u), v \rangle - \langle K_G'(u), v \rangle - \langle K_H'(u), v \rangle,
$$

where

$$
\langle I'(u), v \rangle = \int_{\Omega} a(x) |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \int_{\Gamma} b(x) |u|^{p-2} uv \, d\Gamma;
$$

$$
\langle K_G'(u), v \rangle = \int_{\Gamma} g(x,u) v \, d\Gamma;
$$

$$
\langle K_H'(u), v \rangle = \int_{\Omega} h(x,u) v \, dx; \quad (K_G'(u), v) = \int_{\Omega} w_1 |u|^{p-2} uv \, dx.
$$

Clearly, $I' : E \to E^*$ is continuous. The operator $K_G'$ is a composition of the operators

$$
K_G' : E \to L^n(\Gamma; w_3) \xrightarrow{N} L^{n/(m-1)}(\Gamma; w_3^{1/(1-m)}) \xrightarrow{l} E^*,
$$

where $\langle l(u), v \rangle = \int_{\Gamma} uv \, d\Gamma$. Since

$$
\int_{\Gamma} |uv| \, d\Gamma \leq \left( \int_{\Gamma} |u|^{m'} w_3^{1/(1-m)} \, d\Gamma \right)^{1/m} \cdot \left( \int_{\Gamma} |v|^{m} w_3 \, d\Gamma \right)^{1/m},
$$

then $l$ is continuous, by Theorem 1. As a composition of continuous operators, $K_G'$ is continuous, too. Moreover, by our assumptions on $w_3$, the trace operator $E \to L^m(\Gamma; w_3)$ is compact and therefore, $K_G'$ is also compact.

Set $\phi(u) = w_1 |u|^{p-2} u$. By the proof of Lemma 2 we deduce that the Nemytskii operator corresponding to any function which satisfies (A1) is bounded and continuous. Hence $N_{\phi}$ and $N_{\phi}'$ are bounded and continuous. We note that

$$
K_{\phi}' : E \subset L^p(\Omega; w_1) \xrightarrow{N} L^{p/(p-1)}(\Omega; w_1^{1/(1-p)}) \xrightarrow{\eta} E^*,
$$

where $\langle \eta(u), v \rangle = \int_{\Omega} uv \, dx$. Since

$$
\int_{\Omega} |uv| \, dx \leq \left( \int_{\Omega} |u|^{p/(p-1)} w_1^{1/(1-p)} \, dx \right)^{(p-1)/p} \cdot \left( \int_{\Omega} |v|^{p} w_1 \, dx \right)^{1/p},
$$

it follows that $\eta$ is continuous. But $K_{\phi}'$ is the composition of three continuous operators and by the assumptions on $w_1$, the embedding $E \subset L^p(\Omega; w_1)$ is compact. This implies that $K_{\phi}'$ is compact.

In a similar way, we obtain that $K_H'$ is compact and the continuous Fréchet-differentiability of $F$ follows.

Now, let $u_n \in E$ be a Palais–Smale sequence, i.e.,

$$
|F(u_n)| \leq C \quad \text{for all } n \quad (7)
$$
and
\[ \|F'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty. \]  \hfill (8)

We first prove that \((u_n)\) is bounded in \(E\). Note that (8) implies:
\[ |\langle F'(u_n), u_n \rangle| \leq \mu \cdot \|u_n\|_b \quad \text{for } n \text{ large enough.} \]

This and (7) imply that
\[ C + \|u_n\|_b \geq F(u_n) - \frac{1}{\mu} \cdot \langle F'(u_n), u_n \rangle. \]  \hfill (9)

But
\[
\langle F'(u_n), u_n \rangle = \int_\Omega a(x)|\nabla u_n|^p \, dx + \int_\Gamma b(x)|u_n|^p \, d\Gamma - \lambda \cdot \int_\Omega w_1|u_n|^p \, dx \\
- \int_\Omega h(x, u_n)u_n \, dx - \int_\Gamma g(x, u_n)u_n \, d\Gamma = \|u_n\|_b^p - \lambda \cdot \|u_n\|_{p, w_1}^p
\]
and
\[ F(u_n) = \frac{1}{p} \left( \|u_n\|_b^p - \lambda \cdot \|u_n\|_{p, w_1}^p \right) - \int_\Omega H(x, u_n) \, dx - \int_\Gamma G(x, u_n) \, d\Gamma. \]

We have
\[
F(u_n) - \frac{1}{\mu} \cdot \langle F'(u_n), u_n \rangle = \left( \frac{1}{p} - \frac{1}{\mu} \right) \left( \|u_n\|_b^p - \lambda \cdot \|u_n\|_{p, w_1}^p \right) \\
- \left( \int_\Omega H(x, u_n) \, dx - \frac{1}{\mu} \int_\Gamma h(x, u_n)u_n \, dx \right) \\
- \left( \int_\Gamma G(x, u_n) \, d\Gamma - \frac{1}{\mu} \int_\Gamma g(x, u_n)u_n \, d\Gamma \right).
\]

By (A3) we deduce that
\[ \int_\Gamma G(x, u_n) \, d\Gamma \leq \frac{1}{\mu} \int_\Gamma g(x, u_n)u_n \, d\Gamma. \]  \hfill (10)

A simple computation yields
\[ \int_\Omega H(x, u_n) \, dx = \frac{1}{q} \int_\Omega h(x, u_n)u_n \, dx \leq \frac{1}{\mu} \int_\Omega h(x, u_n)u_n \, dx. \]  \hfill (11)

By (10) and (11) we obtain that
\[ F(u_n) - \frac{1}{\mu} \cdot \langle F'(u_n), u_n \rangle \geq \left( \frac{1}{p} - \frac{1}{\mu} \right) C_0 \|u_n\|_b^p. \]  \hfill (12)

Relations (9) and (12) imply that
\[ C + \|u_n\|_b \geq \left( \frac{1}{p} - \frac{1}{\mu} \right) C_0 \|u_n\|_b^p. \]

This shows that \((u_n)\) is bounded in \(E\).
To prove that \((u_n)\) contains a Cauchy sequence we use the following inequalities for \(\xi, \zeta \in \mathbb{R}^N\) (see [5, Lemma 4.10]):

\[
|\xi - \zeta|^p \leq C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta) (\xi - \zeta) \quad \text{for } p \geq 2,
\]

\[
|\xi - \zeta|^2 \leq C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta) (|\xi| + |\zeta|)^{2-p} \quad \text{for } 1 < p < 2.
\]

Then we obtain in the case \(p \geq 2\):

\[
\|u_n - u_k\|_b^p = \int_\Omega a(x)|\nabla u_n - \nabla u_k|^p \, dx + \int_\Gamma b(x)|u_n - u_k|^p \, d\Gamma
\]

\[
\leq C((I'(u_n), u_n - u_k) - (I'(u_k), u_n - u_k))
\]

\[
= C(F'(u_n), u_n - u_k) - (F'(u_k), u_n - u_k)
\]

\[
+ \lambda (K'_b(u_n), u_n - u_k) - \lambda (K'_b(u_k), u_n - u_k)
\]

\[
+ (K'_G(u_n), u_n - u_k) - (K'_G(u_k), u_n - u_k)
\]

\[
+ (K'_{H^2}(u_n), u_n - u_k) - (K'_{H^2}(u_k), u_n - u_k)
\]

\[
\leq C(\|F'(u_n)\|_{E*} + \|F'(u_k)\|_{E*}
\]

\[
+ |\lambda| \cdot \|K'_b(u_n) - K'_b(u_k)\|_{E*} + \|K'_G(u_n) - K'_G(u_k)\|_{E*}
\]

\[
+ \|K'_{H^2}(u_n) - K'_{H^2}(u_k)\|_{E*}) \|u_n - u_k\|_b.
\]

Since \(F'(u_n) \to 0\) and \(K'_b, K'_G, K'_{H^2}\) are compact, we can assume, passing eventually to a subsequence, that \((u_n)\) converges in \(E\).

If \(1 < p < 2\), then we use the estimate

\[
\|u_n - u_k\|_b^2 \leq C(|I'(u_n), u_n - u_k) - (I'(u_k), u_n - u_k)| (\|u_n\|^{2-p} + \|u_k\|^{2-p}).
\]

Since \(\|u_n\|_b\) is bounded, the same arguments lead to a convergent subsequence. In order to prove estimate (15) we recall the following result: for all \(s \in (0, \infty)\) there is a constant \(C_s > 0\) such that

\[
(x + y)^s \leq C_s (x^s + y^s) \quad \text{for any } x, y \in (0, \infty).
\]

Then, we obtain

\[
\|u_n - u_k\|_b^2 = \left(\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p \, dx + \int_\Gamma b(x)|u_n - u_k|^p \, d\Gamma\right)^{2/p}
\]

\[
\leq C_p \left[\left(\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p \, dx\right)^{2/p} + \left(\int_\Gamma b(x)|u_n - u_k|^p \, d\Gamma\right)^{2/p}\right].
\]
Using (14) and (16) and the Hölder inequality we find

\[
\int_{\Omega} a(x)|\nabla u_n - \nabla u_k|^p \, dx \\
\leq \int_{\Omega} a(x)(|\nabla u_n - \nabla u_k|^2)^{p/2} \, dx \\
\leq C \int_{\Omega} a(x)((|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k) \\
(\nabla u_n - \nabla u_k))^{p/2}(|\nabla u_n| + |\nabla u_k|)^{(p-2)/2} \, dx \\
= C \int_{\Omega} a(x)((|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k))^{p/2} \\
\times (a(x)(|\nabla u_n| + |\nabla u_k|)^p)^{(2-p)/2} \, dx \\
\leq C \left( \int_{\Omega} a(x)((|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\, dx \right)^{p/2} \\
\left( \int_{\Omega} a(x)(|\nabla u_n| + |\nabla u_k|)^p \, dx \right)^{(2-p)/2} \\
\leq \tilde{C}_p \left( \int_{\Omega} a(x)|\nabla u_n|^p \, dx + \int_{\Omega} a(x)|\nabla u_k|^p \, dx \right)^{(2-p)/2} \\
\left( \int_{\Omega} a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\, dx \right)^{p/2} \\
\leq \tilde{C}_p \left[ \left( \int_{\Omega} a(x)|\nabla u_n|^p \, dx \right)^{(2-p)/2} + \left( \int_{\Omega} a(x)|\nabla u_k|^p \, dx \right)^{(2-p)/2} \right] \\
\left( \int_{\Omega} a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\, dx \right)^{p/2} \\
\leq \tilde{C}_p \left( \int_{\Omega} a(x)((|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\, dx \right)^{p/2} \\
\times ((\|u_n\|_h^{(2-p)p/2} + \|u_k\|_h^{(2-p)p/2})^2) \\
\end{align*}

Using the last inequality and (16) we have the estimate

\[
\left( \int_{\Omega} a(x)|\nabla u_n - \nabla u_k|^p \, dx \right)^{2/p} \\
\leq C_p' \left( \int_{\Omega} a(x)((|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k) \cdot (\nabla u_n - \nabla u_k)\, dx \right) \\
\times ((\|u_n\|_h^{2-p} + \|u_k\|_h^{2-p})^2. \tag{18}
\]
In a similar way, we can obtain the estimate
\[
\left( \int b(x)|u_n - u_k|^p \, d\Gamma \right)^{2/p} \leq C_p' \left( \int b(x)(|u_n|^{p-2}u_n - |u_k|^{p-2}u_k)(u_n - u_k) \, dx \right)
\times (\|u_n\|_{b}^{2-p} + \|u_k\|_{b}^{2-p}).
\] (19)

It is now easy to observe that inequalities (17)–(19) imply estimate (15). The proof of Lemma 3 is complete.

**Verification of (4):** Using (6) we have
\[
\int_{\Omega} H(x,u) \, dx \leq \int_{\Omega} |H(x,u)| \, dx \leq \frac{1}{q} \|u\|_{q,w}^q
\]
and by Theorem 1 we have that there exists \( A > 0 \) such that
\[
\|u\|_{q,w}^q \leq A \|u\|_{b}^q \quad \text{for all } u \in E.
\]
This fact implies that
\[
F(u) = \frac{1}{p} (\|u\|_{b}^p - \lambda \|u\|_{p,w}^p) - \int_{\Omega} H(x,u) \, dx - \int_{\Gamma} G(x,u) \, d\Gamma
\geq \frac{C_0}{p} \|u\|_{b}^p - A \frac{1}{q} \|u\|_{b}^q - \int_{\Gamma} G(x,u) \, d\Gamma.
\]
By (A1) and (A2) we deduce that for every \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) such that
\[
|G(x,u)| \leq \epsilon b(x)|u|^p + C_\epsilon w_3(x)|u|^m.
\]
Consequently,
\[
F(u) \geq \frac{C_0}{p} \|u\|_{b}^p - A \frac{1}{q} \|u\|_{b}^q - \epsilon c_1 \|u\|_{b}^p - C_\epsilon C_2 \|u\|_{b}^m.
\]
For \( \epsilon > 0 \) and \( R > 0 \) small enough, we deduce that, for every \( u \in E \) with \( \|u\|_{b} = R, F(u) \geq c_0 > 0 \).

**Verification of (5):** We choose a nonnegative function \( \psi \in C_\infty(\Omega) \) such that \( \emptyset \neq \text{supp} \psi \cap \Gamma \subset U \). From \( G(x,s) \geq c_3 s^m - c_4 \) on \( U \times (0, \infty) \) and (A1) we claim that
\[
F(t\psi) = \frac{t^p}{p} (\|\psi\|_{b}^p - \lambda \|\psi\|_{p,w}^p) - \int_{\Omega} H(x,t\psi) \, dx - \int_{\Gamma} G(x,t\psi) \, d\Gamma
\leq \frac{t^p}{p} (\|\psi\|_{b}^p - \lambda \|\psi\|_{p,w}^p) - c_3 t^p \int_{U} \psi^m \, d\Gamma + c_4 |U| - \frac{t^q}{q} \int_{\Omega} w_2 \psi^q \, dx.
\]
Since \( q \geq p > \mu \), we obtain \( F(t\psi) \to -\infty \) as \( t \to \infty \). It follows that if \( t > 0 \) is large enough, \( F(t\psi) < 0 \). By Ambrosetti–Rabinowitz Theorem, problem (A) has a nontrivial weak solution.

4. Proof of Theorem 3

We start with the following auxiliary result.

Lemma 4. Under assumption (A1), if \( q \leq m \), there exists a number \( \bar{\rho} > 0 \) such that for each \( \rho \geq \bar{\rho} \) the function

\[
v \mapsto \frac{\rho^2}{m}\|v\|_b^m - \frac{1}{p}\|v\|_{p,w_1}^p - \int_{\Omega} H(x,v) \, dx - \int_{\Gamma} G(x,v) \, d\Gamma, \quad \forall v \in E,
\]

is bounded from below on \( E \).

Proof. The growth condition for \( g \) implies that

\[
\int_{\Gamma} G(x,v) \, d\Gamma \leq \int_{\Gamma} \left( \theta_0(x)v \right)^m + \frac{1}{m} G_1(x) |v|^m \right) \, d\Gamma
\]

\[
\leq \left( \int_{\Gamma} \theta_0(x)v \, d\Gamma \right)^{(m-1)/m} \|v\|_{L^q(\Gamma;\omega_3)}^q + C_\mu \|v\|_b^m \quad \forall v \in E,
\]

with constants \( C_\mu > 0, \mu > 0 \). One also obtains that

\[
\int_{\Omega} H(x,v) \, dx = \frac{1}{q} \|v\|_{L^q(\Omega;\omega_3)}^q \leq C_2 \|v\|_b^q \quad \forall v \in E,
\]

with constants \( \bar{C}_2 > 0, \bar{C} > 0 \). Clearly, we can choose now the positive number \( \bar{\rho} \) as desired.

In view of Lemma 4, one can find numbers \( b_0 > 0 \) and \( \alpha > 0 \) such that

\[
\frac{\rho^2}{m}\|v\|_b^m + \frac{2}{m} b_0 - \frac{1}{p}\|v\|_{p,w_1}^p - \int_{\Omega} H(x,v) \, dx - \int_{\Gamma} G(x,v) \, d\Gamma \geq \alpha > 0, \quad \forall v \in E.
\]

With \( b_0 > 0 \) and \( \bar{\rho} > 0 \) as above we consider numbers \( r > \rho \geq \bar{\rho} \) and a function \( \beta \in C^1(\mathbb{R}) \) such that

\[
\beta(0) = \beta(r) = 0, \quad \beta(\rho) = b_0, \quad \beta'(t) < 0 \iff t < 0 \quad \text{or} \quad \rho < t < r,
\]

\[
\lim_{|t| \to +\infty} \beta(t) = +\infty.
\]
Lemma 5. Assume that conditions (A1) and (A3) are fulfilled. Then, for any $d > 0$ satisfying (3), the functional $J : E \times \mathbb{R} \to \mathbb{R}$ is defined by

$$J(v, t) = \frac{t^2}{m} \|v\|_b^m + \frac{2}{m} \beta(t) - \frac{1}{p} \|v\|_{p, \psi}^p - \int_\Omega H(x, v) \, dx$$

$$- \int_\Gamma G(x, v) \, dx + \frac{d}{p} \|v\|_b^p, \quad \forall (v, t) \in E \times \mathbb{R}$$

is of class $C^1$ and satisfies the Palais–Smale condition.

Proof. The property of $J$ which is continuously differentiable has been already justified in the proof of Theorem 2.

In order to check the Palais–Smale condition let the sequences $\{v_n\} \subset E$ and $\{t_n\} \subset \mathbb{R}$ satisfy

$$|J(v_n, t_n)| \leq M, \quad \forall n \geq 1 \quad (25)$$

$$J'_e(t_n, t_n) = \frac{t_n^2}{m} \|v_n\|_b^m - \frac{2}{m} \beta(t_n) - \frac{1}{p} \|v_n\|_{p, \psi}^p$$

$$- K'_\phi(v_n) - K'_H(v_n) - K'_G(v_n) + dI'(v_n) \to 0 \quad \text{as } n \to \infty, \quad (26)$$

$$J'_e(v_n, t_n) = \frac{2}{m} (t_n \|v_n\|_b^m + \beta'(t_n) \to 0, \quad (27)$$

where $I, K_\phi, K_H, K_G$ have been introduced in the proof of Lemma 3.

From (20), (21), (24), and (25) we infer that

$$M \geq \frac{t_n^2}{m} \|v_n\|_b^m + \frac{2}{m} \beta(t_n) - \frac{1}{p} \|v_n\|_{p, \psi}^p$$

$$- \int_\Omega H(x, v_n) \, dx - \int_\Gamma G(x, v_n) \, dx + \frac{d}{p} \|v_n\|_b^p$$

$$\geq \frac{t_n^2 - \rho^2}{m} \|v_n\|_b^m + \frac{2}{m} (\beta(t_n) - \beta(\rho)) + \frac{d}{p} \|v_n\|_b^p.$$

Condition (23) in conjunction with the inequality above yields the boundedness of $\{t_n\}$.

Let us check the boundedness of $\{v_n\}$ along a subsequence. Without loss of generality, we may admit that $\{v_n\}$ is bounded away from 0. From (22) we deduce that the sequence $\{(t_n \|v_n\|_b^m)\}$ is bounded. Therefore, it is sufficient to argue in the case where $t_n \to 0$. From (24) it turns out that

$$\frac{1}{p} \|v_n\|_{p, \psi}^p + \int_\Omega H(x, v_n) \, dx + \int_\Gamma G(x, v_n) \, dx - \frac{d}{p} \|v_n\|_b^p$$

is bounded. By (26) it is seen that

$$\frac{1}{\|v_n\|_b^m} (\langle K'_\phi(v_n), v_n \rangle - \langle K'_H(v_n), v_n \rangle - \langle K'_G(v_n), v_n \rangle + d\|v_n\|_b^p) \to 0 \quad \text{as } n \to \infty.$$
Then, for $n$ sufficiently large, assumption (A3) allows to write

$$M + 1 + \|v_n\|_b \geq d \left( \frac{1}{p} - \frac{1}{\mu} \right) \|v_n\|_b^p + \left( \frac{1}{\mu} - \frac{1}{q} \right) \|v_n\|_{L^q(\Omega, w_2)}^q$$

$$+ \int_{\Omega} \left( \frac{1}{\mu} g(x, v_n) v_n - G(x, v_n) \right) \ d\Gamma + \left( \frac{1}{\mu} - \frac{1}{p} \right) \|v_n\|_{p, w_1}^p$$

$$\geq \left( \frac{1}{p} - \frac{1}{\mu} \right) (d \|v_n\|_b^p - \|v_n\|_{p, w_1}^p) \geq \left( \frac{1}{p} - \frac{1}{\mu} \right) \left( d - \frac{1}{p} \right) \|v_n\|_b^p.$$

By (3), this establishes the boundedness of $\{v_n\}$ in $E$.

In view of the compactness of the mappings $K'_0$, $K'_H$, $K'_G$ (see the proof of Lemma 3), by (26) we get that

$$(d + t_n^2 \|v_n\|_b^{m-p}) I'(v_n)$$

converges in $E$ as $n \to \infty$. The boundedness of $\{u_n\}$ and $\{v_n\}$ ensures that $\{I'(v_n)\}$ is convergent in $E^*$ along a subsequence. Assume that $p \geq 2$. Inequality (13) shows that

$$\|u_n - u_k\|_b^p \leq C \left[ \int_{\Omega} a(x) (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_k|^{p-2} \nabla u_k) \cdot (\nabla u_n - \nabla u_k) \ dx 
+ \int_{\Gamma} b(x) (|u_n|^{p-2} u_n - |u_k|^{p-2} u_k) (u_n - u_k) \ d\Gamma \right]$$

$$= C \|I'(u_n) - I'(u_k), u_n - u_k\| \leq C \|I'(u_n) - I'(u_k)\| \|u_n - u_k\|_b$$

if $p \geq 2$.

Consequently, if $p \geq 2$, $\{v_n\}$ possesses a convergent subsequence. Proceeding in the same way with inequality (14) in place of (13) we obtain the result for $1 < p < 2$.

In the proof of Theorem 3 we shall make use of the following variant of the Mountain-Pass Theorem (see [8]):

**Lemma 6.** Let $E$ be a Banach space and let $J : E \times \mathbb{R} \to \mathbb{R}$ be a $C^1$ functional verifying the hypotheses

(a) there exist constants $\rho > 0$ and $\varepsilon > 0$ such that $J(v, \rho) \geq \varepsilon$, for every $v \in E$;

(b) there is some $r > \rho$ with $J(0, 0) = J(0, r) = 0$.

Then the number

$$c := \inf_{g \in \mathcal{P}} \max_{0 \leq \tau \leq 1} J(g(\tau))$$

is a critical value of $J$, where

$$\mathcal{P} := \{g \in C([0, 1], E \times \mathbb{R}); \ g(0) = (0, 0), \ g(1) = (0, r)\}.$$
Proof of Theorem 3. We apply Lemma 6 to the function $J$ defined in (24). It is clear that assertion (a) is verified with $\rho > 0$ and $\alpha > 0$ described in Lemma 4 and (20). Due to relation (21), condition (b) in Lemma 6 holds. Lemma 5 ensures that the functional $(u,t) \in E \times \mathbb{R}$ such that

$$J'(u,t) = (d + \frac{t^2}{\|u\|^m} I'(u) - K'(u) - K_{d'}(u) = 0,$$

(28)

$$J'(u,t) = \frac{2}{m} (t \|u\|^m + \beta'(t)) = 0.$$

(29)

From (29) it follows that

$$tf'(t) \leq 0.$$

(30)

Combining (30) and (22) we derive that if $t \neq 0$, then $u \neq 0$ and

$$\rho \leq t \leq r.$$

(31)

Therefore, for each $d$ in (3) such that $1/d$ is not an eigenvalue in (B) and each $r > \rho \geq \beta$ we deduce that there exists a critical point $(u,t) = (u_d,t_d) \in E \times \mathbb{R}_+$ of $J$, where $t = t_d$ verifies (31). Consequently, relation (28) establishes that $u_d \in E$ is an eigenfunction in problem (B) where the corresponding eigenvalue is

$$\lambda_d = \frac{1}{d + \frac{t_d^2}{\|u_d\|^m}}.$$

with $t = t_d$ satisfying (31). This completes the proof. \(\square\)

References


