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Partial Differential Equations

Bifurcation and asymptotics for the Lane–Emden–Fowler equation

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Abstract

We are concerned with the Lane–Emden–Fowler equation $-\Delta u = \lambda f(u) + a(x)g(u)$ in Ω, subject to the Dirichlet boundary condition $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, λ is a positive parameter, $a : \overline{\Omega} \to [0, \infty)$ is a Hölder function, and f is a positive nondecreasing continuous function such that $f(s)/s$ is nonincreasing in $(0, \infty)$. The singular character of the problem is given by the nonlinearity g which is assumed to be unbounded around the origin. In this Note we discuss the existence and the uniqueness of a positive solution of this problem and we also describe the precise decay rate of this solution near the boundary. The proofs rely essentially on the maximum principle and on elliptic estimates. ***To cite this article: M. Ghergu, V.D. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Bifurcation et analyse asymptotique pour l'équation de Lane–Emden–Fowler. On considère l'équation de Lane–Emden–Fowler $-\Delta u = \lambda f(u) + a(x)g(u)$ dans Ω avec une condition de Dirichlet $u = 0$ sur $\partial\Omega$, où $\Omega \subset \mathbb{R}^N$ est un domaine borné régulier, λ est un paramètre positif, $a : \overline{\Omega} \to [0, \infty)$ est une fonction de Hölder et f est une fonction continue, positive et croissante telle que l'application $f(s)/s$ soit décroissante sur $(0, \infty)$. Le caractère singulier de ce problème est donné par la nonlinéarité g, qui est non bornée autour de l'origine. Dans cette Note nous étudions l'existence et l'unicité d'une solution positive et nous établissons également son taux de décroissance vers 0 autour du bord. La méthode de démonstration repose sur le principe du maximum et sur des estimations elliptiques. ***Pour citer cet article : M. Ghergu, V.D. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Soit Ω un domaine borné et régulier de $\mathbb{R}^N$. On considère le problème de Dirichlet suivant, associé à l'équation de Lane–Emden–Fowler

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$$\begin{cases} -\Delta u = \lambda f(u) + a(x)g(u) & \text{dans } \Omega, \\ u > 0 & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases} \tag{P$_\lambda$}$$

où $\lambda > 0$, $a \in C^{0,\alpha}(\overline{\Omega})$ et $a > 0$ dans Ω. On suppose que $f \in C^{0,\alpha}[0,\infty)$ est une fonction croissante telle que $f > 0$ sur $(0,\infty)$ et l'application $(0,\infty) \ni s \mapsto f(s)/s$ soit décroissante. De plus, on suppose que $g : (0,\infty) \to [0,\infty)$ est une fonction de Hölder décroissante qui satisfait les conditions suivantes :

(g1) $\lim_{s\searrow 0} g(s) = +\infty$;
(g2) il existe $C_0, \eta_0 > 0$ et $\beta \in (0,1)$ tels que $g(s) \leqslant C_0 s^{-\beta}$ pour tout $s \in (0,\eta_0)$;
(g3) il existe $\theta > 0$ et $t_0 \geqslant 1$ tels que $g(\xi t) \geqslant \xi^{-\theta} g(t)$ pour tous $\xi \in (0,1)$ et $t \leqslant t_0\xi$;
(g4) l'application $(0,\infty) \ni \xi \mapsto \Lambda(\xi) := \lim_{t\searrow 0} \frac{g(\xi t)}{\xi g(t)}$ est une fonction continue.

On suppose aussi qu'il existe $\delta_0 > 0$ et une fonction positive croissante $k \in C(0,\delta_0)$ tels que

(k1) $\lim_{d(x)\searrow 0} \frac{a(x)}{k(d(x))} = c_0 \in (0,\infty)$, où $d(x) := \mathrm{dist}(x,\partial\Omega)$;
(k2) $\lim_{t\searrow 0} k(t)g(t) = +\infty$.

On cherche des solutions du problème (P_λ) dans la classe

$$\mathcal{E} := \{u \in C^2(\Omega) \cap C^{1,1-\alpha}(\overline{\Omega});\ \Delta u \in L^1(\Omega)\}.$$

Soit $m := \lim_{s\to\infty} f(s)/s \in [0,\infty)$. On suppose partout dans ce travail que f a une croissance linéaire à l'infini, c'est-à-dire, $m > 0$. Soit λ_1 la première valeur propre de l'opérateur de Laplace dans $H_0^1(\Omega)$.

Le résultat principal de cette Note est le suivant.

Théorème 0.1. *Soit $\lambda^* = \lambda_1/m$. Sous les hypothèses précédentes on a*

(i) *pour tout $\lambda \geqslant \lambda^*$, le problème (P_λ) n'a pas de solutions dans la classe $\mathcal{E}$;*
(ii) *pour tout $0 < \lambda < \lambda^*$, le problème (P_λ) admet une solution unique $u_\lambda \in \mathcal{E}$. De plus, l'application $\lambda \mapsto u_\lambda$ est strictement croissante et $\lim_{\lambda\nearrow\lambda^*} u_\lambda = +\infty$, uniformément sur les sous-ensembles compacts de Ω ;*
(iii) *pour tout $0 < \lambda < \lambda^*$, le comportement asymptotique de u_λ autour de $\partial\Omega$ est donnée par*

$$\lim_{d(x)\searrow 0} \frac{u_\lambda(x)}{h(d(x))} = \xi_0,$$

où $\Lambda(\xi_0) = c_0^{-1}$ et $h \in C^2(0,\eta] \cap C[0,\eta]$ $(\eta < \delta_0)$ est définie par

$$\begin{cases} h''(t) = -k(t)g(h) & \text{dans } (0,\eta], \\ h > 0 & \text{dans } (0,\eta], \\ h(0) = 0. \end{cases}$$

1. Introduction and the main result

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Consider the following Dirichlet boundary value problem for the Lane–Emden–Fowler equation

$$\begin{cases} -\Delta u = \lambda f(u) + a(x)g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_\lambda$}$$

where λ is a positive parameter, and the variable potential $a(x)$ satisfies $a \in C^{0,\alpha}(\Omega)$, and $a > 0$ in Ω. We assume that $f \in C^{0,\alpha}[0,\infty)$ is a nondecreasing function which is positive on $(0,\infty)$ and such that

(f1) the mapping $(0,\infty) \ni s \mapsto \frac{f(s)}{s}$ is nonincreasing.

We suppose that $g \in C^{0,\alpha}(0,\infty)$ is a nonnegative and nonincreasing function which satisfies

(g1) $\lim_{s\searrow 0} g(s) = +\infty$;
(g2) there exists $C_0, \eta_0 > 0$ and $\beta \in (0,1)$ such that $g(s) \leqslant C_0 s^{-\beta}$ for all $s \in (0,\eta_0)$;
(g3) there exists $\theta > 0$ and $t_0 \geqslant 1$ such that $g(\xi t) \geqslant \xi^{-\theta} g(t)$ for all $\xi \in (0,1)$ and $t \leqslant t_0 \xi$;
(g4) the mapping $(0,\infty) \ni \xi \mapsto \Lambda(\xi) := \lim_{t\searrow 0} \frac{g(\xi t)}{\xi g(t)}$ is a continuous function.

The link between the potential a and the singular nonlinearity g is given by the following assumptions: we assume that there exists $\delta_0 > 0$ and a positive nondecreasing function $k \in C(0,\delta_0)$ such that

(k1) $\lim_{d(x)\searrow 0} \frac{a(x)}{k(d(x))} = c_0 \in (0,\infty)$, where $d(x) := \mathrm{dist}(x, \partial\Omega)$;
(k2) $\lim_{t\searrow 0} k(t) g(t) = +\infty$.

Due to the singular character of (P_λ), we cannot expect to find solutions in $C^2(\overline{\Omega})$ (see [6]). For this purpose we introduce the class

$$\mathcal{E} := \left\{ u \in C^2(\Omega) \cap C^{1,1-\alpha}\big(\overline{\Omega}\big);\ \Delta u \in L^1(\Omega) \right\}.$$

We first observe that, in view of (f1), there exists $m := \lim_{s\to\infty} f(s)/s \in [0,\infty)$. This number plays a crucial role in our analysis. Throughout this paper we shall assume that f has asymptotically linear growth at infinity, that is, $m > 0$. Let λ_1 be the first eigenvalue of the Laplace operator with the Dirichlet boundary condition and $\lambda^* := \lambda_1/m$. The main result of this Note is

Theorem 1.1. *Assume* (f1), (g1)–(g4), (k1)–(k2). *The following assertions hold true*:

(i) *if* $\lambda \geqslant \lambda^*$, *then* (P_λ) *has no solutions in* $\mathcal{E}$;
(ii) *if* $0 < \lambda < \lambda^*$, *then* (P_λ) *has a unique solution* $u_\lambda \in \mathcal{E}$ *which is strictly increasing with respect to* λ. *Moreover,* $\lim_{\lambda\nearrow\lambda^*} u_\lambda = +\infty$, *uniformly on compact subsets of* Ω;
(iii) *for any* $0 < \lambda < \lambda^*$, *the asymptotic behaviour of* u_λ *near* $\partial\Omega$ *is given by*

$$\lim_{d(x)\searrow 0} \frac{u_\lambda(x)}{h(d(x))} = \xi_0,$$

where $\Lambda(\xi_0) = c_0^{-1}$, *and* $h \in C^2(0,\eta] \cap C[0,\eta]$ $(\eta < \delta_0)$ *is defined by*

$$\begin{cases} h''(t) = -k(t)g(h) & \text{in } (0,\eta], \\ h > 0 & \text{in } (0,\eta], \\ h(0) = 0. \end{cases} \tag{1}$$

We conclude this section with some examples of functions g and k that fulfill the above assumptions: (i) $g(t) = t^{-p},\ k(t) = t^q,\ 0 < q < p < 1$; (ii) $g(t) = \frac{\ln(1+t)}{t^p},\ k(t) = t^q,\ 0 < q < p - 1 < 1$; (iii) $g(t) = \frac{1}{\ln(1+t^p)},\ k(t) = t^q$ or $k(t) = \ln(1+t^q),\ 0 < q < p < 1$; (iv) $g(t) = \frac{1}{\arctan t^p},\ k(t) = \arctan^q t$ or $k(t) = \ln(1+t^q),\ 0 < q < p < 1$.

2. Proof of Theorem 1.1

(i) Let $\phi_1 > 0$ be the first eigenfunction of the Laplace operator in Ω with Dirichlet boundary condition. Arguing by contradiction, let us suppose that there exists $\lambda \geqslant \lambda^*$ such that (P_λ) has a solution $u_\lambda \in \mathcal{E}$. Multiplying by ϕ_1 in (P_λ) and integrating we find

$$-\int_\Omega u_\lambda \Delta\phi_1 = -\int_\Omega \Delta u_\lambda \phi_1 = \lambda \int_\Omega f(u_\lambda) + \int_\Omega a(x)g(u_\lambda) > \lambda \int_\Omega f(u_\lambda) \geqslant \lambda_1 \int_\Omega u_\lambda.$$

This is clearly a contradiction since $-\Delta\phi_1 = \lambda_1\phi_1$ in Ω. Hence (P_λ) has no solutions in $\mathcal{E}$ for $\lambda \geqslant \lambda^*$.

(ii) For any $0 < \lambda < \lambda^*$ define $\Phi_\lambda(x,s) = \lambda f(s) + a(x)g(s)$, for all $(x,s) \in \overline{\Omega} \times (0,\infty)$. Then Φ_λ fulfills the assumptions in Lemmas 2.2 and 2.3 in [3] (see also [8]). So, with arguments similar to those given in the proof of Theorem 1.3 in [3], we obtain the existence and the uniqueness of the solution u_λ to problem (P_λ). The regularity of u_λ in the class $\mathcal{E}$ follows by Theorem 1 in [4] and using our assumption (g2).

In what follows we shall apply some ideas developed in [7]. Due to the special character of our problem, we will be able to prove that, in certain cases, L^2-boundedness implies H_0^1-boundedness!

Let $u_\lambda \in \mathcal{E}$ be the unique solution of (P_λ) for $\lambda < \lambda^*$. We prove that $\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$, uniformly on compact subsets of Ω. Suppose the contrary. Since $(u_\lambda)_{\lambda<\lambda^*}$ is a sequence of nonnegative superharmonic functions then, by Theorem 4.1.9 in [5], there exists a subsequence of $(u_\lambda)_{\lambda<\lambda^*}$ (still denoted by $(u_\lambda)_{\lambda<\lambda^*}$) which converges in $L^1_{\mathrm{loc}}(\Omega)$ to some u^*. The monotonicity of u_λ yields (up to a subsequence) $u_\lambda \nearrow u^*$ a.e. in Ω. We first prove that $(u_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. Arguing by contradiction and passing eventually at a subsequence, we have $u_\lambda = M(\lambda)w_\lambda$, where $M(\lambda) = ||u_\lambda||_{L^2(\Omega)} \to \infty$ as $\lambda \nearrow \lambda^*$ and $w_\lambda \in L^2(\Omega)$, $\|w_\lambda\|_{L^2(\Omega)} = 1$. Using (f1), (g2) and the monotonicity assumption on g, we deduce the existence of some constants $A > m$, and $B,\ C,\ D > 0$ such that $f(t) \leqslant At + B,\ g(t) \leqslant Ct^{-\alpha} + D$, for all $t > 0$. This implies $\frac{1}{M(\lambda)}(\lambda f(u_\lambda) + a(x)g(u_\lambda)) \to 0$ in $L^1_{\mathrm{loc}}(\Omega)$, so $-\Delta w_\lambda \to 0$ in $L^1_{\mathrm{loc}}(\Omega)$.

By definition, the sequence $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. We claim that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$. Indeed, by the growth assumptions on f and g,

$$\begin{aligned}
\int_\Omega |\nabla w_\lambda|^2 &= -\int_\Omega w_\lambda \Delta w_\lambda = \frac{1}{M(\lambda)}\int_\Omega w_\lambda \Delta u_\lambda = \frac{1}{M(\lambda)}\int_\Omega \big[\lambda w_\lambda f(u_\lambda) + a(x)g(u_\lambda)w_\lambda\big] \\
&\leqslant \frac{\lambda}{M(\lambda)}\int_\Omega w_\lambda(Au_\lambda + B) + \frac{\|a\|_\infty}{M(\lambda)}\int_\Omega w_\lambda\big(Cu_\lambda^{-\alpha} + D\big) = \lambda A\int_\Omega w_\lambda^2 + \frac{\|a\|_\infty C}{M(\lambda)^{1+\alpha}}\int_\Omega w_\lambda^{1-\alpha} \\
&\quad + \frac{\lambda B + \|a\|_\infty D}{M(\lambda)}\int_\Omega w_\lambda \leqslant \lambda^* A + \frac{||a||_\infty C}{M(\lambda)^{1+\alpha}}|\Omega|^{(1+\alpha)/2} + \frac{\lambda B + \|a\|_\infty D}{M(\lambda)}|\Omega|^{1/2}.
\end{aligned}$$

It follows that the sequence $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$, so the claim is proved. So, there exists $w \in H_0^1(\Omega)$ such that $w_\lambda \rightharpoonup w$ weakly in $H_0^1(\Omega)$ and $w_\lambda \to w$ strongly in $L^2(\Omega)$. Thus we obtain that $w \in H_0^1(\Omega)$ satisfies $\Delta w = 0$ in $\mathcal{D}'(\Omega)$, which contradicts $||w_\lambda||_{L^2(\Omega)} = 1$. Hence $(u_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$, so in $L^1(\Omega)$.

Next, multiplying by ϕ_1 in (P_λ) we obtain $-\int_\Omega \Delta u_\lambda \varphi_1 = \lambda\int_\Omega f(u_\lambda)\varphi_1 + \int_\Omega a(x)g(u_\lambda)\varphi_1$, for all $0 < \lambda < \lambda^*$. On the other hand, by (f1) it follows that $f(u_\lambda) \geqslant mu_\lambda$ in Ω, for all $\lambda < \lambda^*$. Thus we obtain $\lambda_1\int_\Omega u_\lambda\varphi_1 \geqslant \lambda m\int_\Omega u_\lambda\varphi_1 + \int_\Omega a(x)g(u_\lambda)\varphi_1$, for all $0 < \lambda < \lambda^*$. Passing to the limit as $\lambda \nearrow \lambda^*$, we can use the monotone convergence theorem to get $\lambda_1\int_\Omega u^*\varphi_1 \geqslant \lambda_1\int_\Omega u^*\varphi_1 + \int_\Omega a(x)g(u^*)\varphi_1$. Hence $\int_\Omega a(x)g(u^*)\varphi_1 = 0$, which is a contradiction. This shows that $\lim_{\lambda\nearrow\lambda^*} u_\lambda = +\infty$, uniformly on compact subsets of Ω.

(iii) We apply in the proof some ideas developed in [1,2]. However, we point out that the framework in the present case is different: in [1,2] it is applied Karamata's regular variation theory in order to establish the asymptotic behaviour of blow-up boundary solutions, while in what follows we use a direct approach for a Dirichlet boundary value problem with singular nonlinearity. We also mention that the function h defined in (1) plays a similar role to the comparison function that describes the asymptotic behaviour of the blow-up boundary solution in [2]. We start with two auxiliary results.

Lemma 2.1. *The function* $\Lambda : (0,\infty) \to (0,\infty)$ *is bijective.*

Proof. Notice first that $\Lambda(1/\xi) = 1/\Lambda(\xi)$, for any $\xi \in (0,\infty)$. Taking into account this fact and the assumption (g4), it is enough to show that $\lim_{\xi\searrow 0} \Lambda(\xi) = \infty$, in order to achieve the surjectivity of Λ. Fix $\xi \in (0,1)$. According to (g3), we have $\frac{g(\xi t)}{\xi g(t)} \geqslant \xi^{-1-\theta}$, for all $0 < t \leqslant t_0\xi$. Passing to the limit as $t \searrow 0$ we obtain $\Lambda(\xi) \geqslant \xi^{-1-\theta}$. Since $\xi \in (0,1)$ is chosen arbitrarily, we conclude that $\lim_{\xi\searrow 0} \Lambda(\xi) = \infty$.

To complete the proof we only need to show that Λ is decreasing on $(0,\infty)$. Fix $0 < \xi_1 < \xi_2$. So, by (g3),

$$g(\xi_1 t) = g\left(\frac{\xi_1}{\xi_2}\xi_2 t\right) \geqslant \left(\frac{\xi_1}{\xi_2}\right)^{-\theta} g(\xi_2 t), \quad \text{for all } 0 < t \leqslant t_0 \frac{\xi_1}{\xi_2^2}.$$

Hence

$$\frac{g(\xi_1 t)}{\xi_1 g(t)} \geqslant \left(\frac{\xi_1}{\xi_2}\right)^{-1-\theta} \frac{g(\xi_2 t)}{\xi_2 g(t)}, \quad \text{for all } 0 < t \leqslant t_0 \frac{\xi_1}{\xi_2^2}.$$

Letting now $t \searrow 0$ we deduce that

$$\Lambda(\xi_1) \geqslant \left(\frac{\xi_1}{\xi_2}\right)^{-1-\theta} \Lambda(\xi_2) > \Lambda(\xi_2),$$

which concludes our proof. □

Lemma 2.2. *The function* h *defined in* (1) *satisfies*

(i) $h \in C^1[0,\eta]$;
(ii) $\lim_{t\searrow 0} h''(t) = -\infty$.

Proof. (i) Since h is concave, we deduce that $h'(0+)$ exists in $\mathbb{R} \cup \{+\infty\}$. Using now the fact that $h > 0$ in $(0,\eta]$, it follows that $h'(0+) > 0$. In order to prove that $h'(0+) < \infty$, we have $\frac{1}{2}[(h')^2]'(t) = -k(t)g(h(t))h'(t)$ in $(0,\eta]$. Integrating on $[t,\eta]$, $0 < t < \eta$, we get $h'(t) - h'(\eta) = 2\int_t^\eta k(s)g(h(s))h'(s)\,\mathrm{d}s \leqslant 2k(\eta)\int_{h(t)}^{h(\eta)} g(s)\,\mathrm{d}s$. Hence $h'(t) \leqslant C_1 G(h(t)) + C_2$, for $0 < t < \eta$, where $G(t) := \int_t^{h(\eta)} g(s)\,\mathrm{d}s$. By virtue of (g2), we conclude that $h'(0+) < \infty$, so $h \in C^1[0,\eta]$.

(ii) Let $b = h'(0) > 0$. Since h' is decreasing on $(0,\eta)$, the Lagrange mean value theorem yields $h(t)/t = [h(t) - h(0)]/t = h'(c_t) < b$, for all $0 < t < \eta$. Thus $h(t) < bt$ for all $t \in (0,\eta)$ and so $g(h(t)) \geqslant g(bt)$ for all $t \in (0,\eta)$. It follows that

$$h''(t) = -k(t)g\big(h(t)\big) \leqslant -k(t)g(bt) = -k(t)g(t)\frac{g(bt)}{bg(t)}b.$$

Passing to the limit as $t \searrow 0$ in the above inequality, in view of (k2) and Lemma 2.1 we get $\lim_{t\searrow 0} h''(t) = -\infty$, which concludes the proof of Lemma 2.2. □

We are now in position to complete the proof of Theorem 1.1. Define $\Psi : (0,\infty) \to (0,\infty)$ by

$$\Psi(\xi) = \lim_{d(x)\searrow 0} \frac{a(x)g(h(d(x))\xi)}{k(d(x))g(h(d(x)))\xi}, \quad \text{for all } \xi > 0.$$

The definition of Ψ implies that for any $\xi > 0$ we have $\Psi(\xi) = \lim_{d(x)\searrow 0} \frac{a(x)}{k(d(x))} \frac{g(h(d(x))\xi)}{g(h(d(x)))\xi} = c_0\Lambda(\xi)$. So, by Lemma 2.1, Ψ is bijective. Set $\xi_0 := \Psi^{-1}(1)$, that is $\Lambda(\xi_0) = \frac{1}{c_0}$.

Fix $\varepsilon \in (0, 1/2)$ and let $\xi_1, \xi_2 > 0$ be such that $\Psi(\xi_1) = 1 - 2\varepsilon$ and $\Psi(\xi_2) = 1 + 2\varepsilon$. The monotonicity of Ψ implies $\xi_1 > \xi_2$. For any $\delta > 0$, we define $\Omega_\delta = \{x \in \Omega;\ d(x) \leqslant \delta\}$. By the regularity of $\partial\Omega$ and Lemmas 2.1, 2.2 we can choose $\delta \in (0, \eta)$ sufficiently small such that

(i) $d(x) \in C^2(\Omega_\delta)$;

(ii) $\left|\frac{h'(s)}{h''(s)}\Delta d(x) + \lambda \frac{f(\xi_k h(s))}{\xi_k h''(s)}\right| < \varepsilon$ for all $(x, s) \in \Omega_\delta \times (0, \delta)$, $k = 1, 2$;

(iii) $\frac{k(d(x))g(h(d(x)))\xi_2}{g(h(d(x))\xi_2)}(\Psi(\xi_2) - \varepsilon) < a(x) < \frac{k(d(x))g(h(d(x)))\xi_1}{g(h(d(x))\xi_1)}(\Psi(\xi_1) + \varepsilon)$ in Ω_δ.

For any $x \in \overline{\Omega}_\delta$ define $\bar{v}(x) = h(d(x))\xi_1$ and $\bar{v}(x) = h(d(x))\xi_2$. Then

$$\Delta\underline{v}(x) = \xi_2\big[h'\big(d(x)\big)\Delta d(x) + h''\big(d(x)\big)\big|\nabla d(x)\big|^2\big], \quad \text{for all } x \in \Omega_\delta.$$

We now use Lemma 2.2 and the assumption (g2) in order to obtain $\Delta\underline{v} \in L^1(\Omega_\delta)$. Taking into account the fact that $h'' < 0$, by (i), (ii) and the first inequality in (iii) we have (since $|\nabla d(x)| = 1$)

$$\begin{aligned}
\Delta\bar{v}(x) + \Phi_\lambda\big(x, \bar{v}(x)\big) &= \xi_1 h'\big(d(x)\big)\Delta d(x) + h''\big(d(x)\big)\xi_1 + \lambda f\big(h\big(d(x)\big)\xi_1\big) + a(x)g\big(h\big(d(x)\big)\xi_1\big)\\
&= h''\big(d(x)\big)\xi_1\left[\frac{h'(d(x))}{h''(d(x))}\Delta d(x) + \lambda\frac{f(\xi_1 h(d(x)))}{\xi_1 h''(d(x))} + 1 - \frac{a(x)g(h(d(x))\xi_1)}{k(d(x))g(h(d(x)))\xi_1}\right]\\
&\leqslant h''\big(d(x)\big)\xi_1\left[\frac{h'(d(x))}{h''(d(x))}\Delta d(x) + \lambda\frac{f(\xi_1 h(d(x)))}{\xi_1 h''(d(x))} + 1 - \big(\Psi(\xi_1) + \varepsilon\big)\right] \leqslant 0.
\end{aligned}$$

In a similar manner, by (i), (ii) and the second inequality in (iii) we have

$$\begin{aligned}
\Delta\underline{v}(x) + \Phi_\lambda\big(x, \underline{v}(x)\big) &= \xi_2 h'\big(d(x)\big)\Delta d(x) + h''\big(d(x)\big)\xi_2 + \lambda f\big(h\big(d(x)\big)\xi_2\big) + a(x)g\big(h\big(d(x)\big)\xi_2\big)\\
&= h''\big(d(x)\big)\xi_2\left[\frac{h'(d(x))}{h''(d(x))}\Delta d(x) + \lambda\frac{f(\xi_2 h(d(x)))}{\xi_2 h''(d(x))} + 1 - \frac{a(x)g(h(d(x))\xi_2)}{k(d(x))g(h(d(x)))\xi_2}\right]\\
&\geqslant h''\big(d(x)\big)\xi_2\left[\frac{h'(d(x))}{h''(d(x))}\Delta d(x) + \lambda\frac{f(\xi_2 h(d(x)))}{\xi_2 h''(d(x))} + 1 - \big(\Psi(\xi_2) - \varepsilon\big)\right] \geqslant 0.
\end{aligned}$$

Hence $\Delta\bar{v} + \Phi_\lambda(x, \bar{v}) \leqslant 0 \leqslant \Delta\underline{v} + \Phi_\lambda(x, \underline{v})$ in Ω_δ, $\underline{v} \leqslant \bar{v}$ on $\partial\Omega_\delta$, and $\Delta\underline{v} \in L^1(\Omega_\delta)$. Now, by Lemma 2.3 in [3] (see also [8]), $\underline{v} \leqslant \bar{v}$ in $\overline{\Omega}_\delta$. This yields $\xi_1 \geqslant \frac{u_\lambda(x)}{h(d(x))} \geqslant \xi_2$, for all $x \in \Omega_\delta$. Letting $\varepsilon \searrow 0$ in the above inequality, we obtain $\lim_{d(x)\searrow 0} \frac{u_\lambda(x)}{h(d(x))} = \xi_0$. The proof of Theorem 1.1 is now complete. $\square$

Note added in proof

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References

[1] F.-C. Cîrstea, V. Rădulescu, Existence and uniqueness of blow-up solutions for a class of logistic equations, Comm. Contemp. Math. 4 (2002) 559–586.
[2] F.-C. Cîrstea, V. Rădulescu, Uniqueness of the blow-up boundary solution of logistic equations with absorption, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 447–452.
[3] M. Ghergu, V. Rădulescu, Sublinear singular elliptic problems with two parameters, J. Differential Equations, in press.
[4] C. Gui, F.H. Lin, Regularity of an elliptic problem with a singular nonlinearity, Proc. Roy. Soc. Edinburgh Sect. A 123 (1993) 1021–1029.
[5] L. Hörmander, The Analysis of Linear Partial Differential Operators I, Springer-Verlag, Berlin, 1983.
[6] A.C. Lazer, P.J. McKenna, On a singular nonlinear elliptic boundary value problem, Proc. Amer. Math. Soc. 3 (1991) 720–730.
[7] P. Mironescu, V. Rădulescu, The study of a bifurcation problem associated to an asymptotically linear function, Nonlinear Anal. 26 (1996) 857–875.
[8] J. Shi, M. Yao, On a singular nonlinear semilinear elliptic problem, Proc. Roy. Soc. Edinburgh Sect. A 128 (1998) 1389–1401.