UNIQUENESS OF POSITIVE SOLUTIONS TO FRACTIONAL NONLINEAR ELLIPTIC EQUATIONS WITH HARMONIC POTENTIAL

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ABSTRACT. In this paper, we establish the uniqueness of positive solutions to the following fractional nonlinear elliptic equation with harmonic potential

$$(-\Delta)^{s} u + \left(\omega + |x|^2\right) u = |u|^{p-2} u \quad \text{in } \mathbb{R}^n,$$

where $n \geq 1$, $0 < s < 1$, $\omega > -\lambda_{1,s}$, $2 < p < \frac{2n}{(n-2s)^+}$, and $\lambda_{1,s} > 0$ is the lowest eigenvalue of the operator $(-\Delta)^{s} + |x|^2$. This solves an open question raised in [15] concerning the uniqueness of solutions to the equation.

1. INTRODUCTION

In this paper, we are concerned with the uniqueness of positive solutions to the following fractional nonlinear elliptic equation with harmonic potential,

$$(-\Delta)^{s} u + \left(\omega + |x|^2\right) u = |u|^{p-2} u \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $n \geq 1$, $0 < s < 1$, $\omega > -\lambda_{1,s}$, $2 < p < 2_s^* := \frac{2n}{(n-2s)^+}$ and $\lambda_{1,s} > 0$ is the lowest eigenvalue of $(-\Delta)^{s} + |x|^2$ given by

$$\lambda_{1,s} := \inf \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx + |x|^2 |u|^2 \, dx : u \in \Sigma_s, \int_{\mathbb{R}^n} |u|^2 \, dx = 1 \right\}, \tag{1.2}$$

$$\Sigma_s := H^s(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, |x|^2 \, dx).$$

Here the fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) := C(n,s) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy, \quad C(n,s) := \frac{s 2^{2s} \Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}} \gamma(1-s)}.$$

The Sobolev space $H^s(\mathbb{R}^n)$ is defined by the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|u\| := \|u\|_2 + [u]_s,$$

where

$$[u]_s := \left(\frac{C(n,s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx dy \right)^{\frac{1}{2}}.$$

Equation (1.1) appears in the study of standing waves to the following fractional nonlinear Schrödinger equation with harmonic potential,

$$i\partial_t \psi + (-\Delta)^s \psi + |x|^2 \psi = |\psi|^{p-2} \psi \quad \text{in } \mathbb{R} \times \mathbb{R}^n, \tag{1.3}$$

where a standing wave to (1.3) is a solution of the form

$$\psi(t, x) = e^{-i\omega t} u(x), \quad \omega \in \mathbb{R}.$$

Equation (1.3) is of particular interest in fractional quantum mechanics, where ψ denotes the probability density function of a particle trapped inside a trapping potential well modeled by $|x|^2$.

Following the early work due to Laskin [13, 14], the study of the following nonlinear elliptic equation driven by the fractional Laplacian has attracted much attention in recent decades,

$$(-\Delta)^s u + V(x) u = |u|^{p-2} u \quad \text{in } \mathbb{R}^n, \tag{1.4}$$

where $n \geq 1$, $0 < s < 1$ and $2 < p < 2_s^*$. The existence of solutions to (1.4) can be straightforwardly achieved with the help of minimax arguments, see for example [16]. Mathematically, it is important and fundamental to investigate the uniqueness of solutions to (1.4). Let $V(x) = \omega > 0$. When $s = 1$, by applying shooting method, the uniqueness of positive solutions to (1.4) was established by Kwong [12]. While $0 < s < 1$, the consideration of the uniqueness of solutions to (1.1) is challengeable, due to the fractional Laplacian operator is nonlocal and ODE techniques adapted to deal with the case $s = 1$ are no longer available. It was first proved by Amick and Toland in [1] that (1.4) has a unique positive solution for $n = 1$, $s = 1/2$ and $p = 3$. Later, Fall and Valdinoci in [4] established the uniqueness of positive solutions to (1.4) for s close to 1. The result has been extended by Frank, Lenzmann and Silvestre [7, 8], who obtained the uniqueness of positive solutions with Morse index one to (1.4) for any $0 < s < 1$. Very recently, Fall and Weth in [5] further showed that (1.4) admits a unique positive solution.

Observe that Σ_s is compactly embedded into $L^q(\mathbb{R}^n)$ for any $2 \leq q < 2_s^*$. Then, by using minimax arguments in [16], one can simply derive the existence of solutions to (1.1). However, the uniqueness of solutions to (1.1) is a delicate issue, which is left as open problem in [15]. When $s = 1$, by applying ODE techniques due to Yanagida and Yotsutani [17], the uniqueness of positive solutions to (1.1) was established in [10, 11]. While $0 < s < 1$, the uniqueness of positive solutions to (1.1) is unknown for a long time. Recently, by employing the approach developed in [7, 8], Gou in [9] successfully derived that (1.1) has a unique ground state. However, the uniqueness of positive solutions to (1.1) is unknown so far. As an extension of the result in [9], the aim of the present paper is to establish the uniqueness of positive solutions to (1.1).

Theorem 1.1. *Let $n \geq 1$, $0 < s < 1$, $\omega > -\lambda_{1,s}$ and $2 < p < 2_s^*$. Then problem* (1.1) *admits a unique positive solution in Σ_s.*

To prove Theorem 1.1, relying on the elements attained in [9], it suffices to show that any positive solution to (1.1) is non-degenerate. The non-degeneracy of ground states to (1.1) was derived in [15], which has been used to establish the uniqueness of ground states to (1.1) in [9]. Nevertheless, arguments proposed in [15] are not available to verify the non-degeneracy of positive solutions to (1.1). To gain the non-degeneracy of positive solutions to (1.1), we are inspired by ingredients developed in [2, 5]. Let $u \in \Sigma_s$ be a positive solution to (1.1). The aim is to demonstrate that u is non-degenerate, i.e. $Ker[\mathcal{L}_{+,s}] = 0$, where

$$\mathcal{L}_{+,s} = (-\Delta)^s + \left(\omega + |x|^2\right) - (p-1)|u|^{p-2},$$

see Theorem 2.2. To attain the desired conclusion, we begin with proving that any positive solution to (1.1) is radially symmetric and decreasing, see Theorem 2.1, the proof of which is based on moving planes method. However, due to the presence of the harmonic potential, the procedure carried out in the current context is different from the one conducted in [6], where the radial symmetry of

positive solutions to (1.4) with $V(x) = \omega > 0$ was revealed. Next we need to introduce the following eigenvalue problem,

$$(-\Delta)^s v + \left(\omega + |x|^2\right) v = \Lambda |u|^{p-2} v, \quad v \in \Sigma_s. \tag{1.5}$$

It is not difficult to find that (1.5) possesses a sequence of eigenvalues $0 < \Lambda_1 < \Lambda_2 \leq \Lambda_3 \leq \cdots$, because Σ_s is compactly embedded into $L^q(\mathbb{R}^n)$ for any $2 \leq q < 2_s^*$. Observe that Λ_1 is simple and any eigenfunction corresponding to Λ_1 is positive. And eigenfunctions corresponding to other eigenvalues are orthogonal to the eigenfunctions corresponding to Λ_1 with respect to the scalar product in $L^2(\mathbb{R}^n, |u|^{p-2} dx)$, which are indeed sign-changing. Since $u \in \Sigma_s$ is a positive solution to (1.1), then we can conclude that $\Lambda_1 = 1$. At this point, the key element is to check that $\Lambda_2 > p - 1$. If this is true, then we know that $Ker[\mathscr{L}_{+,s}] = 0$. If $Ker[\mathscr{L}_{+,s}] \neq 0$, then we would get that $p - 1$ is an eigenvalue to (1.5). However, this is impossible, because Λ_1 is the first eigenvalue and Λ_2 is the second one to (1.5). And there exists no eigenvalues between Λ_1 and Λ_2. To deduce that $\Lambda_2 > p - 1$, we shall argue by contradiction that $\Lambda_2 \leq p - 1$. In order to reach a contradiction, we first need to infer that any eigenfunction corresponding to Λ_2 is non-radial, the proof of which is beneficial from the polarization arguments, see for example [2]. Then, making use of the Picone type result established in [5], we are able to reach a contradiction. This completes the proof.

2. PROOF OF THEOREM 1.1

In this section, we are going to establish Theorem 1.1. To do this, we first prove the radial symmetry and decrease of any positive solution to (1.1). In what follows, we shall use the symbol $X \lesssim Y$ to denote that $X \leq CY$ for some proper constant $C > 0$.

Theorem 2.1. *Let $n \geq 1$, $0 < s < 1$, $\omega > -\lambda_{1,s}$ and $2 < p < 2_s^*$. Then any positive solution to (1.1) is radially symmetric and decreasing.*

Proof. Let $u \in \Sigma_s$ be a positive solution to (1.1). In order to prove that u is radially symmetric and decreasing, we shall make use of moving planes method. Define

$$\Sigma_\lambda := \left\{x \in \mathbb{R}^n : x_1 > \lambda\right\}, \quad T_\lambda := \left\{x \in \mathbb{R}^n : x_1 = \lambda\right\}, \quad u_\lambda(x) := u(x_\lambda),$$

where $x_\lambda := (2\lambda - x_1, x_2, \cdots, x_n)$. By proceeding bootstrap procedure, we can derive that $u \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $u(x) \to 0$ as $|x| \to \infty$. It then follows that there exists $R > 0$ such that

$$(-\Delta)^s u + \frac{\alpha}{2} u \leq 0, \quad |x| > R,$$

where $\alpha > 0$. This along with [6, Lemma 4.3] and maximum principle gives that

$$0 < u(x) \lesssim \frac{1}{|x|^{n+2s}}, \quad x \in \mathbb{R}^n. \tag{2.1}$$

Now define

$$\lambda_0 := \sup_{\lambda \leq 0} A_\lambda, \quad A_\lambda := \{\lambda \leq 0 : u_\lambda \leq u \text{ in } \Sigma_\lambda\}. \tag{2.2}$$

First we verify that the set A_λ is not empty. For convenience, we define

$$w_\lambda(x) := \begin{cases} (u_\lambda - u)^+ (x), & x \in \Sigma_\lambda, \\ (u_\lambda - u)^- (x), & x \in \mathbb{R}^n \backslash \Sigma_\lambda, \end{cases}$$

where

$$(u_\lambda - u)^+ (x) := \max\{(u_\lambda - u)(x), 0\}, \quad (u_\lambda - u)^- (x) := \min\{(u_\lambda - u)(x), 0\}.$$

Since $u \in \Sigma_s$ is a solution to (1.1), then

$$\int_{\Sigma_\lambda} (-\Delta)^s (u_\lambda - u)(u_\lambda - u)^+ \, dx + \int_{\Sigma_\lambda} \omega(u_\lambda - u)(u_\lambda - u)^+ + \left(|x_\lambda|^2 u_\lambda - |x|^2 u\right)(u_\lambda - u)^+ \, dx$$
$$= \int_{\Sigma_\lambda} (|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u)(u_\lambda - u)^+ \, dx. \tag{2.3}$$

Observe that

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} w_\lambda|^2 \, dx = \int_{\mathbb{R}^n} (-\Delta)^s w_\lambda w_\lambda \, dx = \int_{\Sigma_\lambda} (-\Delta)^s w_\lambda w_\lambda \, dx + \int_{\mathbb{R}^n \setminus \Sigma_\lambda} (-\Delta)^s w_\lambda w_\lambda \, dx$$
$$= 2 \int_{\Sigma_\lambda} (-\Delta)^s w_\lambda w_\lambda \, dx \leq 2 \int_{\Sigma_\lambda} (-\Delta)^s (u_\lambda - u)(u_\lambda - u)^+ \, dx. \tag{2.4}$$

Since $\lambda \leq 0$, then $|x| \leq |x_\lambda|$ for $x \in \Sigma_\lambda$ and

$$\int_{\Sigma_\lambda} \left(|x_\lambda|^2 u_\lambda - |x|^2 u\right)(u_\lambda - u)^+ \, dx \geq \int_{\Sigma_\lambda} |x|^2 (u_\lambda - u)(u_\lambda - u)^+ \, dx = \int_{\Sigma_\lambda} |x|^2 |w_\lambda|^2 \, dx. \tag{2.5}$$

Moreover, using the mean value theorem and (2.1), we can derive that

$$\int_{\Sigma_\lambda} (|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u)(u_\lambda - u)^+ \, dx = \int_{\Sigma_\lambda} \frac{|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u}{(u_\lambda - u)^+} |(u_\lambda - u)^+|^2 \, dx$$
$$\lesssim \int_{\Sigma_\lambda} |u_\lambda|^{p-2} |(u_\lambda - u)^+|^2 \, dx$$
$$\lesssim \int_{\Sigma_\lambda} \frac{1}{|x_\lambda|^{(n+2s)(p-2)}} |w_\lambda|^2 \, dx \tag{2.6}$$
$$\leq \left(\int_{\Sigma_\lambda} \frac{1}{|x_\lambda|^{(n+2s)p}} \, dx\right)^{\frac{p-2}{p}} \left(\int_{\Sigma_\lambda} |w_\lambda|^p \, dx\right)^{\frac{2}{p}},$$

where

$$\int_{\Sigma_\lambda} \frac{1}{|x_\lambda|^{(n+2s)p}} \, dx = \int_{\mathbb{R}^n \setminus \Sigma_\lambda} \frac{1}{|x|^{(n+2s)p}} \, dx \leq \int_{\mathbb{R}^n \setminus B_{|\lambda|}(0)} \frac{1}{|x|^{(n+2s)p}} \, dx = \frac{C}{|\lambda|^{(n+2s)p-n}},$$

where $C = C(n, s, p) > 0$. Consequently, we conclude from (2.6) that

$$\int_{\Sigma_\lambda} (|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u)(u_\lambda - u)^+ \, dx \lesssim \frac{1}{|\lambda|^{(n+2s)p-n}} \left(\int_{\Sigma_\lambda} |w_\lambda|^p \, dx\right)^{\frac{2}{p}}. \tag{2.7}$$

Combining (2.3), (2.4), (2.5) and (2.7) and applying the embedding inequality in Σ_s, we then obtain that

$$\left(\int_{\Sigma_\lambda} |w|^p \, dx\right)^{\frac{2}{p}} \lesssim \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} w_\lambda|^2 \, dx + \int_{\mathbb{R}^n} \left(\omega + |x|^2\right) |w_\lambda|^2 \, dx$$
$$\lesssim \int_{\Sigma_\lambda} (|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u)(u_\lambda - u)^+ \, dx \lesssim \left(\frac{1}{|\lambda|^{(n+2s)p-n}}\right)^{\frac{p-2}{p}} \left(\int_{\Sigma_\lambda} |w_\lambda|^p \, dx\right)^{\frac{2}{p}}.$$

Now choosing $|\lambda| > 0$ large enough, we then reach a contradiction from above. This in turn infers that $w_\lambda = 0$ in Σ_λ for $|\lambda| > 0$ large enough. Therefore, we conclude that $u_\lambda \leq u$ in Σ_λ for $|\lambda| > 0$ large enough. As a consequence, there holds that the set A_λ is not empty and λ_0 is finite defined by (2.2).

Next we are going to demonstrate that $u = u_{\lambda_0}$ in Σ_{λ_0}. Suppose by contradiction that $u \neq u_{\lambda_0}$ and $u_{\lambda_0} \leq u$ in Σ_{λ_0}. If there exists $x_0 \in \Sigma_{\lambda_0}$ such that $u(x_0) = u_{\lambda_0}(x_0)$, by applying the fact that u is a solution to (1.1), then

$$(-\Delta)^s u_{\lambda_0}(x_0) - (-\Delta)^s u(x_0) \leq 0, \tag{2.8}$$

because of $|x| \leq |x_{\lambda_0}|$. However, we find that

$$
\begin{aligned}
(-\Delta)^s u_{\lambda_0}(x_0) - (-\Delta)^s u(x_0) &= C(n,s) \int_{\mathbb{R}^n} \frac{u(y) - u_{\lambda_0}(y)}{|x-y|^{n+2s}} \, dy \\
&= C(n,s) \left(\int_{\Sigma_{\lambda_0}} \frac{u(y) - u_{\lambda_0}(y)}{|x_0 - y|^{n+2s}} \, dy + \int_{\mathbb{R}^n \setminus \Sigma_{\lambda_0}} \frac{u(y) - u_{\lambda_0}(y)}{|x_0 - y|^{n+2s}} \, dy \right) \\
&= C(n,s) \int_{\Sigma_{\lambda_0}} \left(u(y) - u_{\lambda_0}(y) \right) \left(\frac{1}{|x_0 - y|^{n+2s}} - \frac{1}{|x_0 - y_{\lambda_0}|^{n+2s}} \right) dy > 0.
\end{aligned}
\tag{2.9}
$$

Hence we reach a contradiction from (2.8) and (2.9). This obviously shows that $u_{\lambda_0} < u$ in Σ_{λ_0}. To derive the desired conclusion, it suffices to verify that there exists $\epsilon > 0$ small enough such that $u_\lambda < u$ in Σ_λ for any $\lambda_0 < \lambda < \lambda_0 + \epsilon$. Let us denote $P = (\lambda, 0, \cdots, 0) \in \mathbb{R}^n$. In view of (2.3), (2.4) and (2.5) and the embedding inequality in Σ_s, we know that there exists $C > 0$ such that

$$
\left(\int_{\Sigma_\lambda} |w|^p \, dx \right)^{\frac{2}{p}} \leq C \int_{\Sigma_\lambda} (|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u)(u_\lambda - u)^+ \, dx.
\tag{2.10}
$$

It is simple to compute that

$$
\begin{aligned}
\int_{\Sigma_\lambda \cap B_R(P)} (|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u)(u_\lambda - u)^+ \, dx &= \int_{\Sigma_\lambda \cap B_R(P)} \frac{|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u}{(u_\lambda - u)^+} (u_\lambda - u)^+ \, dx \\
&\lesssim \int_{\Sigma_\lambda \cap B_R(P) \cap supp(u_\lambda - u)} |w_\lambda|^2 \, dx \\
&\leq \left| \Sigma_\lambda \cap B_R(P) \cap supp(u_\lambda - u) \right|^{\frac{p-2}{p}} \left(\int_{\Sigma_\lambda} |w_\lambda|^p \, dx \right)^{\frac{2}{p}},
\end{aligned}
\tag{2.11}
$$

where we used the mean value theorem and the fact that $u \in L^\infty(\mathbb{R}^n)$. Let $R > 0$ and $R_0 > 0$ be such that $\Sigma_\lambda \setminus B_R(P) \subset \mathbb{R}^n \setminus B_{R_0}(0)$, which are determined later. Therefore, by invoking the mean value theorem and (2.1) as previously, we are able to obtain that

$$
\begin{aligned}
\int_{\Sigma_\lambda \setminus B_R(P)} (|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u)(u_\lambda - u)^+ \, dx &\lesssim \int_{\Sigma_\lambda \setminus B_R(P)} |u_\lambda|^{p-2} |w_\lambda|^2 \, dx \\
&\lesssim \left(\int_{\mathbb{R}^n \setminus B_{R_0}(0)} \frac{1}{|x_\lambda|^{(n+2s)p}} \, dx \right)^{\frac{p-2}{p}} \left(\int_{\Sigma_\lambda} |w_\lambda|^p \, dx \right)^{\frac{2}{p}} \\
&\lesssim \left(\frac{1}{R_0^{(n+2s)p-n}} \right)^{\frac{p-2}{p}} \left(\int_{\Sigma_\lambda} |w_\lambda|^p \, dx \right)^{\frac{2}{p}}.
\end{aligned}
$$

Now we choose $R_0 > 0$ large enough such that

$$
\int_{\Sigma_\lambda \setminus B_R(P)} (|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u)(u_\lambda - u)^+ \, dx < \frac{1}{4C} \left(\int_{\Sigma_\lambda} |w_\lambda|^p \, dx \right)^{\frac{2}{p}}.
\tag{2.12}
$$

Then we choose $R > 0$ such that $\Sigma_\lambda \setminus B_R(P) \subset \mathbb{R}^n \setminus B_{R_0}(0)$. Moreover, applying (2.11), we choose $\epsilon > 0$ small enough such that

$$
\int_{\Sigma_\lambda \cap B_R(P)} (|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u)(u_\lambda - u)^+ \, dx < \frac{1}{4C} \left(\int_{\Sigma_\lambda} |w|^p \, dx \right)^{\frac{2}{p}}.
\tag{2.13}
$$

Accordingly, from (2.12) and (2.13), there holds that

$$
\int_{\Sigma_\lambda} (|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u)(u_\lambda - u)^+ \, dx < \frac{1}{2C} \left(\int_{\Sigma_\lambda} |w|^p \, dx \right)^{\frac{2}{p}}.
$$

Coming back to (2.10), we then have that $w_\lambda = 0$ in Σ_λ. This readily indicates that $u = u_{\lambda_0}$ in Σ_{λ_0}. Further, by (2.8) and (2.9), we can get that $\lambda_0 = 0$. Indeed, if $\lambda_0 < 0$, then

$$(-\Delta)^s u_{\lambda_0}(x_0) - (-\Delta)^s u(x_0) < 0.$$

This clearly contradicts (2.9). Let $0 \le x_1 < x_1'$, then $-x_1' < -x_1 < 0$. Define

$$\lambda := \frac{-x_1 - x_1'}{2} < 0.$$

Since $u_\lambda < u$ in Σ_λ, then

$$u(-x_1', 0, \cdots, 0) = u_\lambda((-x_1', 0, \cdots, 0)_\lambda) < u((-x_1', 0, \cdots, 0)_\lambda) = u_\lambda(-x_1, 0, \cdots, 0) < u(-x_1, 0, \cdots, 0).$$

It then follows that $u(x_1', 0, \cdots, 0) < u(x_1, 0, \cdots, 0)$, due to $u_{\lambda_0} = u$ in Σ_{λ_0} and $\lambda_0 = 0$. At this moment, repeating the arguments above in any direction, we then get that u is radially symmetric and decreasing. Thus the proof is completed. $\qquad\square$

Theorem 2.2. *Let $n \ge 1$, $0 < s < 1$, $\omega > -\lambda_{1,s}$ and $2 < p < 2_s^*$. Let $u \in \Sigma_s$ be a positive solution to* (1.1). *Then there holds that $Ker[\mathcal{L}_{+,s}] = 0$, where*

$$\mathcal{L}_{+,s} = (-\Delta)^s + (\omega + |x|^2) - (p-1)|u|^{p-2}.$$

Proof. Let $u \in \Sigma_s$ be a positive solution to (1.1). It follows from Theorem 2.1 that u is radially symmetric and decreasing. First we introduce the following eigenvalue problem,

$$(-\Delta)^s v + (\omega + |x|^2) v = \Lambda |u|^{p-2} v, \quad v \in \Sigma_s. \tag{2.14}$$

It is obvious that there exists a sequence of eigenvalues $0 < \Lambda_1 < \Lambda_2 \le \Lambda_3 \le \cdots$, where Λ_1 is given by

$$\Lambda_1 = \inf \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2 \, dx + \int_{\mathbb{R}^n} (\omega + |x|^2)|v|^2 \, dx : v \in \Sigma_s, \int_{\mathbb{R}^n} |v|^2 |u|^{p-2} \, dx = 1 \right\}.$$

It is well-known that Λ_1 is the first eigenvalue, which is simple and possesses positive eigenfunctions. Moreover, eigenfunctions corresponding to the eigenvalues Λ_i for $i \ge 2$ are orthogonal to eigenfunctions corresponding to Λ_1 with respect to the scalar product in $L^2(\mathbb{R}^n, |u|^{p-2} \, dx)$, which indeed changes sign. Since $u \in \Sigma_s$ solves (1.1) and it is positive, then it is an eigenfunction corresponding to Λ_1. This leads to $\Lambda_1 = 1$. Therefore, the second eigenvalue can be denoted by

$$\Lambda_2 = \inf \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2 \, dx + \int_{\mathbb{R}^n} (\omega + |x|^2)|v|^2 \, dx : v \in \Sigma_s, \int_{\mathbb{R}^n} |v|^2 |u|^{p-2} \, dx = 1, \int_{\mathbb{R}^n} v|u|^{p-1} \, dx = 0 \right\}.$$

In the following, we shall prove that any eigenfunction corresponding to Λ_2 is not radially symmetric. Let $v \in \Sigma_s$ be an eigenfunction corresponding to Λ_2. On the contrary, we shall assume that v is radially symmetric, i.e. there exists a function $\rho : [0, \infty) \to \mathbb{R}$ such that $\rho(|x|) = v(x)$ for $x \in \mathbb{R}^n$. It follows from [15, Proposition 7] that v changes sign only once. Let $r > 0$ be such that $\rho(r) = 0$ and $\rho(t) > 0$ for $t > r$. Let $\lambda > 0$ and $P_\lambda v$ be the polarization of v with respect to the half space H_λ, where

$$P_\lambda v(x) := \begin{cases} \max\{v(x), v(\sigma_\lambda(x))\}, & x \in H_\lambda, \\ \min\{v(x), v(\sigma_\lambda(x))\}, & x \notin H_\lambda, \end{cases} \tag{2.15}$$

$$H_\lambda := \{x \in \mathbb{R}^n : x_1 < \lambda\}, \quad \sigma_\lambda(x) := (2\lambda - x_1, x_2, \cdots, x_n).$$

Since v changes sign, then $P_\lambda v$ changes sign. From the definition of the polarization, we can see that

$$(P_\lambda v)^\pm = P_\lambda v^\pm, \tag{2.16}$$

where the positive and negative parts of a function $\phi : \mathbb{R}^n \to \mathbb{R}$ are respectively defined by

$$\phi^+ := \max\{\phi(x), 0\}, \quad \phi^- := \min\{\phi(x), 0\}, \quad \phi = \phi^+ + \phi^-.$$

In view of the definition of P_λ, we then write that

$$P_\lambda v(x) := \begin{cases} \dfrac{1}{2}\left(v(x) + v(\sigma_\lambda(x))\right) + \dfrac{1}{2}|v(x) - v(\sigma_\lambda(x))|, & x \in H_\lambda, \\[2mm] \dfrac{1}{2}\left(v(x) + v(\sigma_\lambda(x))\right) - \dfrac{1}{2}|v(x) - v(\sigma_\lambda(x))|, & x \notin H_\lambda. \end{cases}$$

It then follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} |P_\lambda v^+|^2\, dx &= \int_{H_\lambda} |P_\lambda v^+|^2\, dx + \int_{\mathbb{R}^n \setminus H_\lambda} |P_\lambda v^+|^2\, dx \\
&= \frac{1}{2} \int_{H_\lambda} |v^+|^2 + |v^+(\sigma_\lambda \cdot)|^2\, dx + \frac{1}{2} \int_{H_\lambda} \left||v^+|^2 - |v^+(\sigma_\lambda \cdot)|^2\right|\, dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^n \setminus H_\lambda} |v^+|^2 + |v^+(\sigma_\lambda \cdot)|^2\, dx - \frac{1}{2} \int_{\mathbb{R}^n \setminus H_\lambda} \left||v^+|^2 - |v^+(\sigma_\lambda \cdot)|^2\right|\, dx \\
&= \int_{\mathbb{R}^n} |v^+|^2\, dx.
\end{aligned} \tag{2.17}$$

Similarly, we can conclude that

$$\int_{\mathbb{R}^n} |P_\lambda v^-|^2\, dx = \int_{\mathbb{R}^n} |v^-|^2\, dx.$$

Now we assert that

$$\int_{\mathbb{R}^n} |x|^2 |P_\lambda v^+|^2\, dx \leq \int_{\mathbb{R}^n} |x|^2 |v^+|^2\, dx.$$

Let $R > 0$ and $V_R(|x|) := \min\left\{|x|^2, R^2\right\}$ for $x \in \mathbb{R}^n$. Then we get that the function $x \mapsto -V_R(|x|) + R^2$ is positive, decreasing and radially symmetric. By the definition of P_λ given by (2.15), then it is invariant under the polarization P_λ, i.e. $P_\lambda(-V_R + R^2) = -V_R + R^2$. It then follows from [3, Lemma 3] that

$$\int_{\mathbb{R}^d} \left(-V_R + R^2\right) |v^+|^2\, dx \leq \int_{\mathbb{R}^d} P_\lambda(-V_R + R^2) |P_\lambda v^+|^2\, dx = \int_{\mathbb{R}^d} \left(-V_R + R^2\right) |P_\lambda v^+|^2\, dx.$$

This along with (2.17) gives that

$$\int_{\mathbb{R}^n} V_R |P_\lambda v^+|^2\, dx \leq \int_{\mathbb{R}^n} V_R |v^+|^2\, dx.$$

Passing the limit as R goes to infinity, we then have the desired conclusion. Similarly, we can derive that

$$\int_{\mathbb{R}^n} |x|^2 |P_\lambda v^-|^2\, dx \leq \int_{\mathbb{R}^n} |x|^2 |v^-|^2\, dx.$$

Consequently, there holds that

$$\int_{\mathbb{R}^N} |P_\lambda v^\pm|^2\, dx = \int_{\mathbb{R}^N} |v^\pm|^2\, dx, \quad \int_{\mathbb{R}^N} |x|^2 |P_\lambda v^\pm|^2\, dx \leq \int_{\mathbb{R}^N} |x|^2 |v^\pm|^2\, dx.$$

This together with (2.16) then leads to

$$\int_{\mathbb{R}^n} (P_\lambda v)^\pm P_\lambda v\, dx = \int_{\mathbb{R}^n} |(P_\lambda v)^\pm|^2\, dx = \int_{\mathbb{R}^n} |P_\lambda v^\pm|^2\, dx = \int_{\mathbb{R}^n} |v^\pm|^2\, dx = \int_{\mathbb{R}^n} v v^\pm\, dx, \tag{2.18}$$

$$\begin{aligned}
\int_{\mathbb{R}^n} |x|^2 (P_\lambda v)^\pm P_\lambda v\, dx &= \int_{\mathbb{R}^n} |x|^2 |(P_\lambda v)^\pm|^2\, dx = \int_{\mathbb{R}^n} |x|^2 |P_\lambda v^\pm|^2\, dx \\
&\leq \int_{\mathbb{R}^n} |x|^2 |v^\pm|^2\, dx = \int_{\mathbb{R}^n} |x|^2 v v^\pm\, dx.
\end{aligned} \tag{2.19}$$

Since u is radially symmetric and decreasing, by the definition of P_λ given by (2.15), then it is invariant under the polarization P_λ, i.e. $P_\lambda u = u$. Applying (2.16) and [3, Lemma 2], we then get that

$$\int_{\mathbb{R}^n} |u|^{p-2} (P_\lambda v)(P_\lambda v)^\pm \, dx = \int_{\mathbb{R}^n} |u|^{p-2} |(P_\lambda v)^\pm|^2 \, dx = \int_{\mathbb{R}^n} |P_\lambda u|^{p-2} |P_\lambda v^\pm|^2 \, dx$$
$$\geq \int_{\mathbb{R}^n} |u|^{p-2} |v^\pm|^2 \, dx = \int_{\mathbb{R}^n} |u|^{p-2} v v^\pm \, dx, \qquad (2.20)$$

where $P_\lambda u > 0$, due to $u > 0$. In addition, making use of [2, Lemma 2.2], we know that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} P_\lambda v (-\Delta)^{\frac{s}{2}} (P_\lambda v)^\pm \, dx \leq \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} v (-\Delta)^{\frac{s}{2}} v^\pm \, dx. \qquad (2.21)$$

Since $v \in \Sigma_s$ solves (2.14) with $\Lambda = \Lambda_2$, then

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} v (-\Delta)^{\frac{s}{2}} v^\pm \, dx + \int_{\mathbb{R}^n} \left(\omega + |x|^2 \right) v v^\pm \, dx = \Lambda_2 \int_{\mathbb{R}^n} |u|^{p-2} v v^\pm \, dx. \qquad (2.22)$$

Therefore, utilizing (2.18)-(2.21), we are able to conclude from (2.22) that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} P_\lambda v (-\Delta)^{\frac{s}{2}} (P_\lambda v)^\pm \, dx + \int_{\mathbb{R}^n} \left(\omega + |x|^2 \right) (P_\lambda v)(P_\lambda v)^\pm \, dx \leq \Lambda_2 \int_{\mathbb{R}^n} |u|^{p-2} (P_\lambda v)(P_\lambda v)^\pm \, dx.$$

Since $u > 0$, $P_\lambda v^+ \geq 0$ and $P_\lambda v^- \leq 0$ for $v^+ := \max\{v(x), 0\}$ and $v^- := \min\{v(x), 0\}$, by (2.16), then

$$\int_{\mathbb{R}^n} (P_\lambda v)^+ |u|^{p-1} \, dx = \int_{\mathbb{R}^n} (P_\lambda v^+) |u|^{p-1} \, dx > 0,$$

$$\int_{\mathbb{R}^n} (P_\lambda v)^- |u|^{p-1} \, dx = \int_{\mathbb{R}^n} (P_\lambda v^-) |u|^{p-1} \, dx < 0,$$

where v is sign-changing. As a consequence, we have that there exists $\alpha_1 > 0$ such that

$$\int_{\mathbb{R}^n} \left((P_\lambda v)^+ + \alpha_1 (P_\lambda v)^- \right) |u|^{p-1} \, dx = 0.$$

Observe that

$$\int_{\mathbb{R}^n} \left| (P_\lambda v)^+ + \alpha_1 (P_\lambda v)^- \right|^2 |u|^{p-2} \, dx = \int_{\mathbb{R}^n} \left(\left| (P_\lambda v)^+ \right|^2 + \alpha_1^2 \left| (P_\lambda v)^- \right|^2 \right) |u|^{p-2} \, dx > 0.$$

Therefore, we know that there exists $\alpha_2 > 0$ such that

$$\alpha_2^2 \int_{\mathbb{R}^n} \left| (P_\lambda v)^+ + \alpha_1 (P_\lambda v)^- \right|^2 |u|^{p-2} \, dx = 1.$$

This implies that $\alpha_2\left((P_\lambda v)^+ + \alpha_1(P_\lambda v)^-\right) \in \Sigma_s$ is a test function for the variational characterization of Λ_2. It then follows that

$$\Lambda_2 \le \alpha_2^2 \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{s}{2}}(P_\lambda v)^+\right|^2 dx + \alpha_2^2 \int_{\mathbb{R}^n} \left(\omega + |x|^2\right)\left|(P_\lambda v)^+\right|^2 dx$$

$$+ \alpha_1^2 \alpha_2^2 \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{s}{2}}(P_\lambda v)^-\right|^2 dx + \alpha_1^2 \alpha_2^2 \int_{\mathbb{R}^n} \left(\omega + |x|^2\right)|(P_\lambda v)^-|^2 dx$$

$$+ 2\alpha_1 \alpha_2^2 \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(P_\lambda v)^+ (-\Delta)^{\frac{s}{2}}(P_\lambda v)^- dx$$

$$\le \alpha_2^2 \Lambda_2 \int_{\mathbb{R}^n} |u|^{p-2}\left|(P_\lambda v)^+\right|^2 dx - \alpha_2^2 \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(P_\lambda v)^+ (-\Delta)^{\frac{s}{2}}(P_\lambda v)^- dx$$

$$+ \alpha_1^2 \alpha_2^2 \Lambda_2 \int_{\mathbb{R}^n} |u|^{p-2}|(P_\lambda v)^-|^2 dx - \alpha_1^2 \alpha_2^2 \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(P_\lambda v)^+ (-\Delta)^{\frac{s}{2}}(P_\lambda v)^- dx$$

$$+ 2\alpha_1 \alpha_2^2 \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(P_\lambda v)^+ (-\Delta)^{\frac{s}{2}}(P_\lambda v)^- dx$$

$$= \Lambda_2 \int_{\mathbb{R}^2} |u|^{p-2}\left|\alpha_2\left((P_\lambda v)^+ + \alpha_1(P_\lambda v)^-\right)\right|^2 dx - \alpha_2^2(1-\alpha_1)^2 \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(P_\lambda v)^+ (-\Delta)^{\frac{s}{2}}(P_\lambda v)^- dx$$

$$\le \Lambda_2,$$

where we used the fact that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(P_\lambda v)^+ (-\Delta)^{\frac{s}{2}}(P_\lambda v)^- dx > 0.$$

It then gives that $\alpha_1 = 1$. Therefore, we know that $P_\lambda v = (P_\lambda v)^+ + (P_\lambda v)^-$ is an eigenfunction corresponding to Λ_2. Note that $\overline{x} = (r + 2\lambda, 0, \cdots, 0) \notin H_\lambda$ and $\sigma_\lambda(\overline{x}) = (-r, 0, \cdots, 0)$. This then implies that

$$P_\lambda v(\overline{x}) \le v(\sigma_\lambda(\overline{x})) = \rho(|\sigma_\lambda(\overline{x})|) = 0.$$

On the other hand, since $-\overline{x} \in H_\lambda$ with $|\overline{x}| > r$, then

$$P_\lambda v(-\overline{x}) \ge v(-\overline{x}) = v(\overline{x}) = \rho(|\overline{x}|) > 0.$$

Hence we can conclude that $P_\lambda v$ is not radial and $P_\lambda v \ne v$, because v is assumed to be radially symmetric. Define $\psi_\lambda := v - P_\lambda v$. Clearly, $\psi_\lambda \in \Sigma_s$ is also an eigenfunction corresponding to Λ_2. Since $P_\lambda v \ge v$ on H_λ, then $\psi_\lambda \le 0$ and $\psi_\lambda \ne 0$ on H_λ. Using maximum principle, we then know that $\psi_\lambda < 0$ on H_λ. It then shows that $v < v \circ \sigma_\lambda$ on H_λ for any $\lambda > 0$. Thus we are able to conclude that v is strictly increasing with respect to the x_1 direction in the half space H_0. We then reach a contradiction from the assumption v is radial and the fact that v is sign-changing and $v(x) \to 0$ as $|x| \to \infty$.

Next we are going to prove that $\Lambda_2 > p - 1$. Suppose by contradiction that $\Lambda_2 \le p - 1$. Let $v \in \Sigma_s$ be an eigenfunction corresponding to Λ_2. Define

$$T_\mu := \left\{x \in \mathbb{R}^n : x \cdot \mu = 0\right\},$$

where $\mu \in \mathbb{R}^n$ is a unit vector. Denote σ_μ be the reflection with respect to T_μ. Define

$$\psi_\mu := \frac{v - v \circ \sigma_\mu}{2} \in \Sigma_s.$$

Obviously, we find that ψ_μ enjoys the equation

$$(-\Delta)^s \psi_\mu + \left(\omega + |x|^2\right)\psi_\mu = \Lambda_2 |u|^{p-2}\psi_\mu. \tag{2.23}$$

Define $\varphi := -\nabla u \cdot \mu$. Then $\varphi \in \Sigma_s$ satisfies the equation

$$(-\Delta)^s \varphi + \left(\omega + |x|^2\right)\varphi + 2(x \cdot \mu)u = (p-1)|u|^{p-2}\varphi.$$

Since u is radially symmetric and decreasing, then φ is odd with respect to the reflection T_μ and $\varphi \ge 0$ in $\{x \in \mathbb{R}^n : x \cdot \mu > 0\}$. It immediately follows from [5, Lemma 5.3] that $\varphi > 0$ on $\{x \in \mathbb{R}^n : x \cdot \mu > 0\}$

and $\nabla\varphi \cdot \mu > 0$ in $\{x \in \mathbb{R}^n : x \cdot \mu = 0\}$. Let $\chi : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\chi = 1$ on $(-1,1)$ and $\chi = 0$ on $(-2,2)$. Define

$$\psi_\mu^R(x) := \psi_\mu(x)\chi\left(\frac{|x|}{R}\right), \quad R > 0.$$

Applying [5, Lemma 2.1], we then get that

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}\psi_\mu^R|^2\, dx + \int_{\mathbb{R}^n} \left(\omega + |x|^2\right)|\psi_\mu^R|^2\, dx - (p-1)\int_{\mathbb{R}^n} |u|^{p-2}|\psi_\mu^R|^2\, dx \ge \int_{\{x\cdot\mu>0\}\cap\{y\cdot\mu>0\}} H^\mu_{\psi_\mu^R,\varphi}\, dx,$$

where

$$H^\mu_{\psi_\mu^R,\varphi}(x,y) := C(N,s)\varphi(x)\varphi(y)\left(\frac{\psi_\mu^R(x)}{\varphi(x)} - \frac{\psi_\mu^R(y)}{\varphi(y)}\right)^2\left(\frac{1}{|x-y|^{n+2s}} - \frac{1}{|\sigma_\mu(x)-y|^{n+2s}}\right).$$

Taking the limit as $R \to \infty$, we then know that

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}\psi_\mu|^2\, dx + \int_{\mathbb{R}^n} \left(\omega + |x|^2\right)|\psi_\mu|^2\, dx - (p-1)\int_{\mathbb{R}^n} |u|^{p-2}|\psi_\mu|^2\, dx \ge \int_{\{x\cdot\mu>0\}\cap\{y\cdot\mu>0\}} H^\mu_{\psi_\mu,\varphi}\, dx \ge 0.$$

Using (2.23) and the assumption that $\Lambda_2 \le p - 1$, we then obtain that

$$0 \ge \left(\Lambda_2 - p + 1\right)\int_{\mathbb{R}^n} |u|^{p-2}|\psi_\mu|^2\, dx \ge \int_{\{x\cdot\mu>0\}\cap\{y\cdot\mu>0\}} H^\mu_{\psi_\mu,\varphi}\, dx \ge 0.$$

As a result, we have that $\psi_\mu = 0$. Since v is nonradial, then there exists a unit vector $\overline{\mu} \in \mathbb{R}^n$ such that $\psi_{\overline{\mu}} \ne 0$. We then reach a contradiction. This implies that $\Lambda_2 > p - 1$. Since $1 = \Lambda_1 < p - 1 < \Lambda_2$, then $Ker[\mathscr{L}_{s,+}] = 0$ and the proof is completed. $\qquad\square$

Proof of Theorem 1.1. From Theorems 2.1 and 2.2, we know that the conditions of Lemma 2.4 in [9] hold true. Then, replacing the roles of ground states by positive solutions to (1.1) and closely following the discussions in [9], we are able to conclude the desired conclusion. This completes the proof. $\qquad\square$

STATEMENTS AND DECLARATIONS

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