# Planar Schrödinger equations with critical exponential growth

Sitong Chen[a], Vicenţiu D. Rădulescu[b,c,d,e,f], Xianhua Tang[a], Lixi Wen[g,c,*]

[a] *School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan 410083, People's Republic of China*

[b] *Faculty of Applied Mathematics, AGH University of Kraków, 30-059 Kraków, Poland*

[c] *Department of Mathematics, University of Craiova, 200585 Craiova, Romania*

[d] *Brno University of Technology, Faculty of Electrical Engineering and Communication, Technická 3058/10, Brno, 61600, Czech Republic*

[e] *Simion Stoilow Institute of Mathematics of the Romanian Academy, 010702 Bucharest, Romania*

[f] *School of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, People's Republic of China*

[g] *Changsha University of Science and Technology, School of Mathematics and Statistics, Changsha, Hunan 410114, People's Republic of China*

**Abstract**

In this paper, we study the following quasilinear Schrödinger equation:

$$-\varepsilon^2\Delta u+V(x)u-\varepsilon^2\Delta(u^2)u=g(u),\quad x\in\mathbb{R}^2,$$

where $\varepsilon>0$ is a small parameter, $V\in\mathcal{C}(\mathbb{R}^2,\mathbb{R})$ is uniformly positive and allowed to be unbounded from above, and $g\in\mathcal{C}(\mathbb{R},\mathbb{R})$ has a critical exponential growth at infinity. In the autonomous case, when $\varepsilon>0$ is fixed and $V(x)\equiv V_0\in\mathbb{R}^+$, we first present a remarkable relationship between the existence of least energy solutions and the range of $V_0$ without any monotonicity conditions on $g$. Based on some new strategies, we establish the existence and concentration of positive solutions for the above singularly perturbed problem. In particular, our approach not only permits to extend the previous results to a wider class of potentials $V$ and source terms $g$, but also allows a uniform treatment of two kinds of representative nonlinearities that $g$ has extra restrictions at infinity or near the origin, namely $\liminf_{|t|\to+\infty}\frac{tg(t)}{e^{\alpha_0 t^4}}$ or $g(u)\ge C_{q,V}u^{q-1}$ with $q>4$ and $C_{q,V}>0$ is an implicit value depending on $q,V$ and the best constant of the embedding $H^1(\mathbb{R}^2)\subset L^q(\mathbb{R}^2)$, considered in the existing literature. To the best of our knowledge, there have not been established any similar results, even for simpler semilinear Schrödinger equations. We believe that our approach could be adopted and modified to treat more general elliptic partial differential equations involving critical exponential growth.

**Keywords:** quasilinear Schrödinger equation; critical exponential growth; Trudinger-Moser inequality; semi-classical state; ground state.

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*This paper is dedicated to the memory of Professor Haim Brezis*

## 1 Introduction

This paper is concerned with the following quasilinear Schrödinger equation

$$-\varepsilon^2\Delta u+V(x)u-\varepsilon^2\Delta(u^2)u=g(u),\quad x\in\mathbb{R}^2, \qquad (Q)_\varepsilon$$

*Corresponding Author. E-mail address: mathsitongchen@mail.csu.edu.cn (S.T. Chen), radulescu@inf.ucv.ro (V. D. Rădulescu), tangxh@mail.csu.edu.cn (X.H. Tang), wlxcsust@csust.edu.cn (L.X. Wen).

where $\varepsilon > 0$ is a small parameter, $V$ satisfies the following assumptions:
(V1) $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^2} V(x) > 0$;

(V2) there is a bounded domain $\Lambda \subset \mathbb{R}^2$ such that

$$\min_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x), \tag{1.1}$$

and $g$ has the following critical exponential growth at infinity:

(G1) $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and there exists $\alpha_0 > 0$ such that

$$\lim_{|t| \to +\infty} \frac{|g(t)|}{e^{\alpha t^4}} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ +\infty, & \text{for all } \alpha < \alpha_0. \end{cases} \tag{1.2}$$

This kind of nonlinearity has the maximal growth which allows us to treat $(Q)_\varepsilon$ variationally in a suitable function space, as the counterpart to the higher dimensions $N \geq 3$ in which the critical exponent is $2(2^*) = 4N/(N-2)$, see below for more details. On the potential $V$, besides the local condition (V2), we do not require any global condition other than (V1), which is even allowed to be unbounded from above. We are interested in the so called semi-classical states for $(Q)_\varepsilon$, which are families of solutions $u_\varepsilon$ developing a spike shape around one or more distinguished points of the space, while vanishing asymptotically elsewhere as $\varepsilon \to 0$.

Quasilinear equations like $(Q)_\varepsilon$ appear naturally in mathematical physics and have been derived as models of several physical phenomena, such as in the theory of superfluid film, Heisenberg ferromagnets and magnons, in dissipative quantum mechanics, and in condensed matter theory, see [6, 29, 40].

In recent years, the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \kappa\Delta(u^2)u = g(u), \quad x \in \mathbb{R}^N \tag{1.3}$$

with $\kappa > 0$, $N \geq 1$, $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, has attracted a lot of attention and many existence results have been obtained under variant assumptions on $V$ and $g$ by using variational methods. For example, when $g(u) = |u|^{q-2}u$ with $4 < q < 2(2^*)$ ($2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$), the existence of a positive ground state solution for (1.3) was proved by Poppenberg-Schmitt-Wang [41] and Liu-Wang [30] by using a constrained minimization argument, which gives a solution of (1.3) with an unknown Lagrange multiplier $\lambda$ in front of $g(u)$. A new variable replacement $v = f^{-1}(u)$ was introduced by Colin-Jeanjean [13] and Liu-Wang-Wang [32], where $f$ is defined by

$$f'(t) = \frac{1}{\sqrt{1 + 2|f(t)|^2}} \quad \text{on} \quad [0, +\infty), \qquad f(-t) = -f(t) \quad \text{on} \quad (-\infty, 0]. \tag{1.4}$$

With this change of variable, the quasilinear problem can be transformed to a semilinear problem, and some effective methods for semilinear problems can be applied to treat the resulting equation. These arguments can also be extended to the more general subcritical case in the sense that $|g(u)| \leq C(u^2 + |u|^{q-1})$ with $C > 0$ and $4 < q < 2(2^*)$. As observed by Liu-Wang-Wang [32], the number $2(2^*)$ behaves like a critical exponent for (1.3) if $N \geq 3$. When $N \geq 3$ and $g(u) = |u|^{q-2}u + |u|^{2(2^*)-2}u$ with $4 < q < 2(2^*)$, motivated by the celebrated work of Brezis-Nirenberg [5] on critical Sobolev exponent problems for semilinear elliptic equations, the authors in [21, 44] got the existence of nontrivial solutions for (1.3), see also [2, 12, 17, 23, 31, 34, 35] for more results. These results in the critical growth case were also extended to the singularly perturbed problem of the form:

$$-\varepsilon^2\Delta u + V(x)u - \kappa\varepsilon^2\Delta(u^2)u = g(u), \quad x \in \mathbb{R}^N, \tag{1.5}$$

with $N \geq 3$, see [9, 25, 26, 46] and references therein.

Note that although there have been many works on the existence of nontrivial solutions for (1.3) and (1.5) with $N \geq 3$ in the critical growth case, rather less has been done when $N = 2$. In fact, the dimension $N = 2$ is very special, as the corresponding Sobolev embedding yields $H^1(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$ for all $q \geq 2$, but $H^1(\mathbb{R}^2) \not\subset L^\infty(\mathbb{R}^2)$. In this case, the Trudinger-Moser inequality in $\mathbb{R}^2$ below can be treated as a substitute of the Sobolev inequality in the higher dimensions $N \geq 3$, as it establishes the sharp maximal exponential integrability for functions in $H^1(\mathbb{R}^2)$.

**Lemma 1.1.** (Trudinger-Moser inequality [1,7,8]) i) *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x < \infty;$$

ii) *if $u \in H^1(\mathbb{R}^2), \|\nabla u\|_2^2 \leq 1, \|u\|_2 \leq M < \infty$, and $\alpha < 4\pi$, then there exists a constant $\mathcal{C}(M, \alpha)$, which depends only on $M$ and $\alpha$, such that*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \leq \mathcal{C}(M, \alpha). \tag{1.6}$$

In particular, the threshold $\alpha = 4\pi$ in Lemma 1.1 plays an analogous role of the critical Sobolev exponent $2^* = 2N/(N-2)$. As we know, for the semilinear Schrödinger equation:

$$-\Delta u + V(x)u = g(u), \qquad x \in \mathbb{R}^2, \tag{1.7}$$

the function $g(t)$ is said to have critical exponential growth if $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and there exists $\alpha_0 > 0$ such that

$$\lim_{|t| \to +\infty} \frac{|g(t)|}{e^{\alpha t^2}} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ +\infty, & \text{for all } \alpha < \alpha_0. \end{cases} \tag{1.8}$$

It is interesting to note that for quasilinear Schrödinger equation (1.3) with $\kappa > 0$, the above definition of critical exponential growth changes into that $g$ satisfies (G1) because of the fact:

$$u \in H := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} u^2 |\nabla u|^2 \mathrm{d}x < \infty \right\} \quad \Rightarrow \quad u^2 \in H^1(\mathbb{R}^2). \tag{1.9}$$

From now on, we will focus our attention on quasilinear Schrödinger equations with the critical exponential growth. Since we look for positive solutions, as usual, we always assume that $g(t) = 0$ for all $t \in (-\infty, 0]$. Let us describe the relevant works below. Before this, we first introduce the following assumptions used in the references:

(V1′) $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ and $0 < \inf_{x \in \mathbb{R}^2} V(x) \leq V(x) \leq \liminf_{|y| \to +\infty} V(y) < +\infty$ for all $x \in \mathbb{R}^2$;

(V2′) $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ is a 1-periodic positive function;

(G2) $g(t) = o(t)$ as $t \to 0$;

(AR) there exists $\mu_1 > 4$ such that $g(t)t \geq \mu_1 G(t) \geq 0$ for all $t \geq 0$, where $G(t) = \int_0^t g(s)\mathrm{d}s$;

(MN) $\frac{g(t)}{t^3}$ is increasing on $t \in (0, +\infty)$;

(M1) $\lim_{|t| \to +\infty} \frac{tg(t)}{e^{\alpha_0 t^4}} = +\infty$;

(M2) there exists a constant $q > 2$ such that for all $t \geq 0$,

$$g(t) \geq C_q t^{q-1} \text{ with } C_q > \left[\frac{\mu_1(q-2)}{q(\mu_1 - 4)}\right]^{(q-2)/2} \left(\frac{\alpha_0}{\pi}\right)^{(q-2)/2} S_q^q(\theta), \tag{1.10}$$

where

$$S_q(\theta) := \inf_{u \in H_r^1(\mathbb{R}^2) \setminus \{0\}} \frac{\left[\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \theta u^2\right) \mathrm{d}x + \left(\int_{\mathbb{R}^2} u^2 |\nabla u|^2 \mathrm{d}x\right)^{1/2}\right]^{1/2}}{\left(\int_{\mathbb{R}^2} |u|^q \mathrm{d}x\right)^{1/q}}. \tag{1.11}$$

As far as we know, the study on $(Q)_\varepsilon$ involving critical exponential growth started with two papers [20] and [37] in 2007, in which $\varepsilon = 1$ and two types of linear potentials were considered, that $V$ satisfies the asymptotic condition (V1′) and the periodic condition (V2′), respectively. Precisely, by the change of variable and the Mountain Pass theorem, the existence of a positive solution for $(Q)_\varepsilon$ with $\varepsilon = 1$ was proved by do Ó-Miyagaki-Soares [20] under assumptions (V1′), (G1), (G2), (AR) and (M1), and by Moameni [37] under assumptions (V2′), (G1), (G2), (AR) and (M2) with $\theta = \max_{x\in\mathbb{R}^2} V(x)$. Later, the results obtained in [20] and [37] were extended to the singularly perturbed equation $(Q)_\varepsilon$ with a small parameter $\varepsilon > 0$ and a more general class of potentials $V$ requiring only (V1) and (V2) by do Ó-Moameni-Soares [18], and by do Ó-Soares [19], respectively. In particular, based on the results obtained in [20] and [37], with a penalization technique and Mountain Pass arguments in a nonstandard Orlicz space, the authors in [18] and [19] obtained a parameter family of positive solutions which concentrates, as $\varepsilon \to 0$, near a local minimum of the potential $V$, if $g$ satisfies (G1), (G2), (AR), (MN) and (M1), and $g$ satisfies (G1), (G2), (AR), (MN) and (M2) with $\theta = V_0$, respectively.

We would like to emphasize that a key tool in [18, 20] and [19, 37] is conditions (M1) and (M2), respectively, to overcome the loss of compactness due to the critical behavior of the nonlinearity, each of which can help to show that Mountain Pass level is in the range of compactness of the associated functional. In fact, the analogous conditions as (M1) and (M2) have appeared in most of the studies for elliptic problems with a nonlinear term of exponential growth. For example, for semilinear Schrödinger equation (1.7), the following two conditions:

(M3) $\lim_{|t|\to+\infty} \frac{tg(t)}{e^{\alpha_0 t^2}} = \gamma_0 > \frac{e}{\alpha_0} \max_{x\in\mathbb{R}^2} V(x)$;

(M4) there exists a constant $q_0 > 2$ such that for all $t \geq 0$,

$$g(t) \geq C_{q_0} t^{q-1} \text{ with } C_{q_0} > \left(\frac{q_0 - 2}{q_0}\right)^{(q_0-2)/2} \left(\frac{\alpha_0}{4\pi}\right)^{(q_0-2)/2} \gamma_{q_0}^{q_0/2},$$

where

$$\gamma_{q_0} := \inf_{u\in H^1(\mathbb{R}^2), \|u\|_{q_0}=1} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \max_{x\in\mathbb{R}^2} V(x)u^2\right) \mathrm{d}x \tag{1.12}$$

were assumed by de Figueiredo-Miyagaki-Ruf [14, 15] and Alves-do Ó-Marcos [3], respectively, see also Miyagaki-Alves-Souto-Montenegro [4], Masmoudi-Sani [36, (8.4)], Ruf-Sani [42], Chen-Tang-Wei [11] and Chen-Qin-Rădulescu-Tang [10] for more progresses in this direction. Note that with these two types of representative conditions, one can obtain the desired upper bound for the Mountain Pass level in two different ways: 1) by means of an estimate involving Moser's sequence of functions; 2) by choosing the appropriately large number $C_q$ in (1.11) (or $C_{q_0}$ in (1.12)) relying on some minimizing problems related to the embedding $H^1(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$.

As pointed out by Masmoudi-Sani [36, Remark 8.2], it seems to be difficult to compare the growth conditions (M1) with (M2) as they prescribe the growth of $g$ at infinity and near the origin respectively. In addition, we also note the following unpleasant facts on (M1) and (M2):

I) It is still unknown whether condition (M1) can be weakened in the sense of finding an exact lower bound of $\liminf_{|t|\to+\infty} \frac{tg(t)}{e^{\alpha_0 t^4}}$ like (M3) due to the competing effect of the second order nonhomogeneous term $\Delta(u^2)u$.

II) Condition (M2) used in [19,37] involves the implicit value of the best constant of the embedding $H^1(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$, which is so far unknown and still an open challenging problem, moreover, (M2) also relies on the parameter $\mu_1 > 4$ appearing in (AR).

Now a natural question arises:

*Can we find a unified condition involving both* (M1) *and* (M2)
*to achieve the desired estimation for the Mountain Pass level related with* $(Q)_\varepsilon$*?*

In the present paper, we shall introduce some new strategies and techniques, and give an affirmative answer to the above question, which permit to not only extend the results of [18–20, 37] to a wider class on nonlinearities, but also unify the existing results on two types of nonlinearities satisfying (M1) or (M2) in this subject. More precisely, the purpose of this paper is two-fold:

- When $\varepsilon = 1$ and $V(x) \equiv V_0 \in \mathbb{R}^+$ in $(Q)_\varepsilon$, we shall present a remarkable relationship between the existence of least energy solutions and the range of $V_0$ without any monotonicity conditions on $g$, and give a new existence criterion, both fully covering and weakening those required in the existing literature.
- When $\varepsilon > 0$ is a small parameter and $V$ satisfies (V1) and (V2) in $(Q)_\varepsilon$, based on the new necessary conditions, we establish the existence of a family of positive solutions for $(Q)_\varepsilon$ concentrating around local minima of the potential $V$, as $\varepsilon \to 0$, where $V$ just satisfy (V1) and (V2), and is allowed to be unbounded from above.

For the first purpose, let us consider the following quasilinear autonomous Schrödinger equation with constant potential

$$-\Delta u + V_0 u - \Delta(u^2)u = g(u), \tag*{$(Q)_0$}$$

where $V_0 > 0$, and $g$ satisfies (G1), (G2) and the following condition:

(G3) $\lim_{|t|\to\infty} \frac{G(t)}{t^2} = +\infty$ and $g(t)t \geq 2G(t) \geq 0$ for all $t \geq 0$,

which is much weaker than the condition of Ambrosetti-Rabinowitz type:

(AR′) $g(t)t \geq 4G(t) \geq 0$ for all $t \geq 0$,

used in the previous works. By searching for the range of $V_0$, we establish the existence of a *least energy solution* for $(Q)_0$, and also give a fine maximum characterization of the least energy solution. We recall that a solution $u \in H \setminus \{0\}$ of $(Q)_0$ is said to be a least energy solution if and only if $\Phi_0(u)$ equals the least energy

$$c_0^* := \inf\left\{\Phi_0(u) \;\middle|\; u \in H \setminus \{0\}, \Phi_0'(u) = 0\right\}, \tag{1.13}$$

where $H$ is the working space defined by (1.9), and $\Phi_0 : H \to \mathbb{R}$ is the energy functional corresponding to $(Q)_0$ defined by

$$\Phi_0(u) = \frac{1}{2}\int_{\mathbb{R}^2}(1+2u^2)|\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} V_0 u^2 \mathrm{d}x - \int_{\mathbb{R}^2} G(u)\mathrm{d}x, \tag{1.14}$$

see Section 2 for more details on $H$ and $\Phi_0$ (see also [33, 43]).

It is worth pointing out that all literatures dealing with quasilinear Schrödinger problems involving critical exponential growth used a change of variable, which reduces a quasilinear problem to a semilinear problem. In this way, some classical arguments developed by Brezis and Nirenberg [5] can be adopted and modified to treat the reduced equation to restore the compactness. Although this transformation approach is quite effective to find nontrivial solutions, it seems not to be applicable to find least energy solutions for the original problem, since *it is unknown whether a least energy solution of the reduced semilinear problem is the one of the original quasilinear problem after a change of variable, which is the main reason why there have been no related existence results of least energy solutions in this topic up to date.* This forces the implementation of new ideas to search for a least energy solution.

To state the results in this direction, we define the set

$$\Gamma_0 := \left\{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^2)) : \gamma(0) = 0, \Phi_0(f(\gamma(1))) < 0\right\}, \tag{1.15}$$

where $f$ is defined by (1.4). Inspired by Ibrahim-Masmoudi-Nakanishi [27] and Masmoudi-Sani [36], we also define the Trudinger-Moser ratio

$$C_{\mathrm{TM}}^*(G) := \sup\left\{\frac{2}{\|u\|_2^2}\int_{\mathbb{R}^2} G(u)\mathrm{d}x \;\middle|\; u \in H \setminus \{0\}, 2\|\nabla u\|_2^2 + \|\nabla(u^2)\|_2^2 \leq \frac{4\pi}{\alpha_0}\right\}, \tag{1.16}$$

see more details in (2.56)-(2.58) below. Our first result is as follows.

**Theorem 1.2.** *Assume that $g$ satisfies* (G1)-(G3)*. Then for any $V_0 \in (0, C^*_{\rm TM}(G))$, equation $(Q)_0$ admits a positive least energy solution $w$ having the maximum characterization:*

$$\Phi_0(w) = \max_{t\in[0,1]} \Phi_0(f(\tilde{\gamma}(t))) \quad \textit{for some function } \tilde{\gamma} \in \Gamma_0. \tag{1.17}$$

In particular, based on the result of Theorem 1.2, with a little extra work we can also prove the following existence result for $(Q)_\varepsilon$ with $\varepsilon = 1$ when $V$ satisfies either (V1′) or (V2′) considered in [20] or [37].

**Theorem 1.3.** *Assume that $V$ satisfies either* (V1′) *or* (V2′)*, and $g$ satisfies* (G1)-(G3) *and* (AR′)*. Let $\overline{V} := \sup_{x\in\mathbb{R}^2} V(x) < C^*_{\rm TM}(G)$. Then $(Q)_\varepsilon$ with $\varepsilon = 1$ admits a positive solution.*

For the study of singularly perturbed problem $(Q)_\varepsilon$ with a small parameter $\varepsilon > 0$, besides (G1)-(G3), we introduce the following assumptions on $g$:

(G4) $\lim_{t\to+\infty} \frac{G(t)}{g(t)t} = 0$;

(G5) $\frac{g(t)}{t^3}$ is non-decreasing on $t \in (0, +\infty)$.

Note that (G4) is weaker than the condition widely used in the literature below

(G4′) there exist $M_0 > 0$ and $t_0 > 0$ such that $G(t) \le M_0 g(t)$ for all $t \ge t_0$,

which is reasonable for functions $f(t)$ behaving as $e^{\alpha_0 t^2}$. The result in this direction is stated as follows.

**Theorem 1.4.** *Assume that $V$ satisfies* (V1) *and* (V2) *with $\min_{x\in\Lambda} V(x) < C^*_{\rm TM}(G)$, and $g$ satisfies* (G1)-(G5)*. Then there exists $\varepsilon_0^* > 0$ such that for all $0 < \varepsilon \le \varepsilon_0^*$, $(Q)_\varepsilon$ possesses a positive solution $u_\varepsilon \in \mathcal{C}^{2,\alpha}_{\rm loc}(\mathbb{R}^2)$ for some $\alpha \in (0,1)$ with the following properties:*

(i) *$u_\varepsilon(z)$ has a unique local maximum (hence global) $z_\varepsilon \in \mathbb{R}^2$ and $z_\varepsilon \in \Lambda$;*

(ii) *$\lim_{\varepsilon\to 0} V(\varepsilon y_\varepsilon) = \min_{x\in\Lambda} V(x)$;*

(iii) *there exist positive constants $\Pi_0$ and $\kappa_0$, independent on $\varepsilon$, such that*

$$u_\varepsilon(z) \le \Pi_0 \exp\left(-\frac{\kappa_0}{\varepsilon}|z|\right), \qquad \forall\ z \in \mathbb{R}^2,\ \varepsilon \in (0, \varepsilon_0^*].$$

Some remarks on Theorems 1.2-1.4 are in order.

**Remark 1.5.** *In many non-autonomous elliptic problems, it turns out that information on the least energy level of an associated autonomous problem is crucial if there is no extra compactness condition, since the least energy level often appears as the first level of possible loss of compactness. Theorem* 1.2 *appears to be the first result on the existence of least energy solutions for $(Q)_0$ involving the critical exponential growth without the additional monotonicity assumption on $g(t)/t^3$. We believe that the result of Theorem* 1.2 *could be useful for the study of other non-autonomous quasilinear problems and its singular perturbation forms involving critical exponential growth.*

**Remark 1.6.** *The condition $\overline{V} < C^*_{\rm TM}(G)$ in Theorem* 1.3 *can be derived from either the asymptotic condition* (M1′) *or the global growth condition* (M2′) *as below:*

(M1′) *$\lim_{t\to+\infty} \frac{t^4 G(t)}{e^{\alpha_0 t^4}} = +\infty$;*

(M2′) *there exists a constant $q > 2$ such that for all $t \ge 0$,*

$$g(t) \ge \tilde{C}_q t^{q-1} \text{ with } \tilde{C}_q > \left(\frac{q-2}{2q}\right)^{(q-2)/2} \left(\frac{\alpha_0}{\pi}\right)^{(q-2)/2} S_q^q(\overline{V}),$$

*(see Remark 2.10 below for more details). Note that* (M1′) *and* (M2′) *are weaker than* (M1) *and* (M2) *used in* [20] *and* [37]*, respectively, since*

$$\lim_{t\to+\infty}\frac{t^4G(t)}{e^{\alpha_0 t^4}}=\lim_{t\to+\infty}\frac{4G(t)+tg(t)}{4\alpha_0 e^{\alpha_0 t^4}}\quad\textit{and}\quad C_q>\tilde{C}_q,$$

*where $C_q$ and $\tilde{C}_q$ appear in* (M2) *and* (M2′)*. Besides,* (AR′) *is obviously weaker than* (AR) *required in* [20,37]*. In this sense, Theorem* 1.3 *can be regarded as a unified improvement of the results of* [20] *and* [37]*.*

**Remark 1.7.** (i) *We believe that the ideas and techniques for the proofs of Theorem* 1.4 *could be adopted and modified to treat more elliptic partial differential equations involving critical exponential growth.*

Indeed, on the one hand, our working space is the Sobolev space

$$E:=\left\{u\in H^1(\mathbb{R}^2):\int_{\mathbb{R}^2}V(x)u^2\mathrm{d}x<\infty\right\},\tag{1.18}$$

which is more general than the Orlicz space used in [20,37]. On the other hand, our proofs do not rely on the Schwarz symmetrization principle which may fail for some equations, such as bi-harmonic equations, or the concentration-compactness type argument which is not available if the nonlinearity has critical exponential growth.

(ii) *To our knowledge, there have not been any similar results as Theorem* 1.4 *in the literature when the nonlinearity has critical exponential growth, even for simpler semilinear Schrödinger equations, namely $(Q)_\varepsilon$ in the absence of the nonhomogeneous term $\Delta(u^2)u$.*

Let us point out the main difficulties and highlights for the proofs of Theorem 1.2 and Theorem 1.4, respectively, before ending this section.

The proof of Theorem 1.2 is based on the constrained minimization argument:

$$A_0=\inf_{u\in\mathcal{P}_0}\Phi_0(u)=\inf_{u\in\mathcal{P}_0}\frac{1}{2}\int_{\mathbb{R}^2}(1+2u^2)|\nabla u|^2\mathrm{d}x\quad\text{with}\quad\mathcal{P}_0=\{u\in H\setminus\{0\}:P_0(u)=0\},$$

where $H$ and $\Phi_0$ are defined by (1.9) and (1.14), and $P_0:H\to\mathbb{R}$ is the Pohozaev functional related to the Pohozaev identity for $(Q)_0$, see (2.5) below, which differs considerably from previous works relying on a variable replacement. In particular, we need to implement new estimates on the minimum $A_0$. Moreover, special care is needed to gain more information on the least energy level. Precisely, the principal difficulties lie in two aspects:

(I) The existing arguments estimating the (Mountain Pass) Minimax level for semilinear problems do not work without the change of variable, so we have to search for other tools to obtain a desired upper bound for the minimization problem $A_0$, in order to resolve the loss of compactness, which may be produced not only by the concentration phenomena but also by the vanishing phenomena.

(II) It is more involved to construct the maximum characterization of least energy solutions for $(Q)_0$ without any monotonicity assumptions on $g$, even in the absence of $\Delta(u^2)u$, since $\Phi_0$ has no saddle point structure with regard to the fibres $\{tu:t>0\}\subset H$.

To overcome the two difficulties, we employ some new strategies and delicate analyses, summarized as follows.

- To restore the compactness, we propose **a new necessary and sufficient condition** for the boundedness and the compactness of general nonlinear functionals in $H$, in terms of the growth and decay of the nonlinear function, not only among exponent and power functionals. With this condition, we can treat uniformly two types of nonlinearities studied in the existing literature, that is (M1) or (M2) holds.

- To find the maximum characterization of least energy solutions for $(Q)_0$, we construct **a good sample path** having some special minimax properties. Note that, in the dimension $N \geq 3$, such a path is easy to be constructed by means of the fibres $\{u(\cdot/t) : t > 0\} \subset H^1(\mathbb{R}^N)$. Unfortunately, in the dimension $N = 2$, the path only relying on the dilation $u(x/t)$ does not belong to the class of admissible paths, and one can not find an analogous saddle point structure, even in the absence of $\Delta(u^2)u$. This, together with the competing effect of $\Delta(u^2)u$, enforces us to develop a different technical construction.

As a by-product of Theorem 1.2, we first establish the Pohozaev identity for quasilinear problems in $\mathbb{R}^2$ by using a method different from those in the higher dimensions, which especially can be adopted and modified to nonautonomous situations.

Our proof of Theorem 1.4 relies on the combination of a change of variable and a penalization technique, which is motivated by the arguments of [20,37], (see also [9,25,26,46] for the dimension $N \geq 3$). The main ingredient of the penalization technique lies in a reduction of the nonlinearity $g$ outside $\Lambda$ (see (3.6) below) in such a way that the modified energy functional $I_\varepsilon$, defined by (3.9) below, will satisfy a (local) Cerami condition at certain levels for any fixed small $\varepsilon > 0$, see Lemma 3.13 below. Along this line, for any fixed small $\varepsilon > 0$, we can find a one parameter family of critical points $\{v_\varepsilon\}$ for $I_\varepsilon$. To ensure that the parameter family of critical points of the modified functionals satisfies, after a rescaling and a change of variable, the original problem, what is most challenging is to obtain the following uniform decay on the family $\tilde{v}_\varepsilon(x) := v_\varepsilon(\varepsilon x)$:

$$\tilde{v}_\varepsilon(x) \to 0 \ \text{ as } |x| \to \infty \ \text{ uniformly in } \ \varepsilon \in (0, \varepsilon_0^*] \tag{1.19}$$

for some small constant $\varepsilon_0^* > 0$. In particular, compared with the previous works, **some new difficulties occur** in the proof procedures:

(I) The lack of conditions (M1) and (M2) prevents us from using the existing methods for controlling the Mountain Pass level by a fine threshold, which is the essential step to restore the compactness for any fixed small $\varepsilon > 0$.

(II) Without the condition (AR) required in [20,37], it is more complicated to derive two types of boundedness results and convergence results; one when $\varepsilon$ is fixed, especially the other one to obtain uniform conclusions when $\varepsilon \to 0$;

(III) To obtain the convergence of the parameter family of critical points $\{v_\varepsilon\}$ for $I_\varepsilon$ mentioned above, the concentration-compactness type argument dealing with the higher dimensions does not work, since there is no BL-splitting property caused by critical exponential growth.

(IV) The proof of the uniform decay (1.19) in [20,37] depends strongly on the Schwarz symmetrization principle, especially the following equalities

$$\int_{|x|\geq R} e^{2\alpha\left(\tilde{v}_\varepsilon^2 - 1\right)} \tilde{v}_\varepsilon^2 \mathrm{d}x = \int_{|x|\geq R} e^{2\alpha\left[(\tilde{v}_\varepsilon^*)^2 - 1\right]} (\tilde{v}_\varepsilon^*)^2 \mathrm{d}x = \sum_{k=1}^{\infty} \int_{|x|\geq R} (\tilde{v}_\varepsilon^*)^{2k+2} \mathrm{d}x, \tag{1.20}$$

where $\tilde{v}_\varepsilon^*$ denotes the Schwarz symmetrization of $\tilde{v}_\varepsilon$. In this way, (1.19) can be derived from the Radial Lemma and standard elliptic estimates. But, it seems to be rather difficult to get the local equality between the function $\tilde{v}_\varepsilon$ and its symmetric decreasing rearrangement $\tilde{v}_\varepsilon^*$ in (1.20). As far as we know, it remains open whether this uniform decay holds without the help of the local equality (1.20).

These difficulties enforce the implementation of new ideas and strategies for the proof of Theorem 1.4. For example,

- to conquer the difficulty (I), instead of estimating directly the Mountain Pass level, we control successfully the Mountain Pass level by the least energy $c_0^*$, defined by (1.13), with stretched variables and new inequalities;

- to conquer the difficulties (II) and (III), in contrast to previous works, we take full advantage of the information on $\langle I'_\varepsilon(v_\varepsilon), f(v_\varepsilon)/f'(v_\varepsilon)\rangle$, and some new estimates and delicate analyses are employed particularly when it comes to the uniform conclusions as $\varepsilon \to 0$;

- to conquer the difficulty (IV), different from the existing literature, we establish the convergence of $\tilde{v}_\varepsilon$, after suitable translation, in $H^1(\mathbb{R}^2)$ by some subtle arguments, and then indirectly prove the uniform decay (1.19) by using some new analytical techniques, where the Schwarz symmetrization principle is not required.

The paper is organized as follows. In Section 2, we study the existence of least energy solutions for $(Q)_0$, and establish its maximum characterization, whereby Theorem 1.2 is proved. In Section 3, we introduce the modified problem with penalized nonlinearity, and obtain a one parameter family of mountain-pass critical points for modified energy functionals. Section 4 is devoted to the study of $L^\infty$-estimate and behavior of mountain-pass critical points after the stretched variables as $\varepsilon \to 0$. In Section 5, we prove that the parameter family of critical points of the modified functionals satisfy, after a rescaling and a change of variable, the original problem $(Q)_\varepsilon$, and complete the proof of Theorem 1.4.

Throughout the paper, we make use of the following notations:

- $H^1(\mathbb{R}^2)$ denotes the Sobolev space equipped with the norm $\|u\| = [\int_{\mathbb{R}^2}(|\nabla u|^2 + u^2)\mathrm{d}x]^{1/2}$;
- $L^s(\mathbb{R}^2)(1 \le s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^2} |u|^s \mathrm{d}x\right)^{1/s}$;
- For any $x \in \mathbb{R}^2$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^2 : |y - x| < r\}$ and $B_r = B_r(0)$;
- $C_1, C_2 \cdots$ denote positive (possibly different) constants, possibly dependent on $\varepsilon$.

## 2 Least energy solutions for $(Q)_0$

In this section, we study the existence of least energy solutions for $(Q)_0$, and establish its maximum characterization, which completes the proof of Theorem 1.2. For this, we first introduce the variational setting for $(Q)_0$. Note that $H$, defined by (1.9), is not a vector space (it is not closed under the sum), nevertheless it is a complete metric space with distance

$$d_H(u, v) = \|u - v\| + \|\nabla(u^2) - \nabla(v^2)\|_2.$$

From (G1)-(G3), it follows that

$$\lim_{t\to 0} \frac{G(t)}{t^2} = 0 \tag{2.1}$$

and

$$\lim_{t\to+\infty} \frac{t^4 G(t)}{e^{\alpha t^4}} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ +\infty, & \text{for all } \alpha < \alpha_0. \end{cases} \tag{2.2}$$

Then we have for any $\epsilon > 0, \alpha > \alpha_0$ and $q > 0$, there exists $C = C(\epsilon, \alpha, q) > 0$ such that

$$2G(t) \le g(t)t \le \epsilon t^2 + C|t|^q \left(e^{\alpha t^4} - 1\right), \quad \forall\, t \in \mathbb{R}. \tag{2.3}$$

Using (2.3) and Lemma 1.1, it is easy to check that $\Phi_0$, defined by (1.14), is continuous on $H$. Formally, our problem has a variational structure. For any $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and $u \in H$, $u + \phi \in H$, and we can compute the Gateaux derivative:

$$\langle \Phi_0'(u), \phi\rangle = \int_{\mathbb{R}^2} \left[(1 + 2u^2)\nabla u \cdot \nabla\phi + 2|\nabla u|^2 u\phi + V_0 u\phi\right] \mathrm{d}x - \int_{\mathbb{R}^2} g(u)\phi \mathrm{d}x. \tag{2.4}$$

Therefore, $u \in H$ is a solution of $(Q)_0$ if and only if this derivative vanishes along any direction in $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, see [33] for more details.

## 2.1 Existence of least energy solutions

First, we provide a Pohozaev type identity for $(Q)_0$. The strategy of the proof is motivated by a truncation argument due to Kavian (see [47, Appendix B]), but some differences occur due to the presence of $\Delta(u^2)u$. For this, let us define the Pohozaev functional:

$$P_0(u) = V_0\|u\|_2^2 - 2\int_{\mathbb{R}^2} G(u)\mathrm{d}x. \tag{2.5}$$

**Lemma 2.1.** *Assume that $g$ satisfies* (G1)-(G3). *If $u \in H$ is a weak solution of $(Q)_0$, then we have the Pohozaev identity $P_0(u) = 0$.*

*Proof.* Let $\psi \in \mathcal{C}^\infty([0,+\infty),[0,1])$ such that $\psi(r) = 1$ for $r \in [0,1]$ and $\psi(r) = 0$ for $r \in [2,+\infty)$. Define $\psi_n(x) := \psi(|x|^2/n^2)$ on $\mathbb{R}^2$ for $n \in \mathbb{N}$. Then there exists $C_1 > 0$ such that

$$0 \le \psi_n(x) \le C_1, \quad |x||\nabla\psi_n(x)| \le C_1 \qquad \forall\, x \in \mathbb{R}^2. \tag{2.6}$$

Since $u$ is a weak solution of $(Q)_0$, by a standard regularity argument (see the appendix of [33]), we can show that $u, u^2 \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$. By Lemma 1.1 and (2.3), we have $\int_{\mathbb{R}^2} G(u)\mathrm{d}x < \infty$. Multiplying $(Q)_0$ by $\psi_n(x\cdot\nabla u)$, we have for every $n \in \mathbb{N}$,

$$0 = \left[-\Delta u + V_0 u - \Delta(u^2)u\right]\psi_n(x\cdot\nabla u). \tag{2.7}$$

It is clear that, for every $n \in \mathbb{N}$,

$$-\psi_n g(u)(x\cdot\nabla u) = -\mathrm{div}(x\psi_n G(u)) + 2\psi_n G(u) + G(u)(x\cdot\nabla\psi_n), \tag{2.8}$$

$$\begin{aligned}-\psi_n\Delta u(x\cdot\nabla u) = &-\mathrm{div}\left\{\left[\nabla u(x\cdot\nabla u) - x\frac{|\nabla u|^2}{2}\right]\psi_n\right\}\\ &-\frac{|\nabla u|^2}{2}(x\cdot\nabla\psi_n) + (x\cdot\nabla u)(\nabla\psi_n\cdot\nabla u),\end{aligned} \tag{2.9}$$

$$\psi_n u(x\cdot\nabla u) = \frac{1}{2}\mathrm{div}\left(u^2\psi_n x\right) - u^2\psi_n - \frac{1}{2}u^2(x\cdot\nabla\psi_n) - \frac{1}{2}u^2\psi_n \tag{2.10}$$

and

$$\begin{aligned}-\psi_n\Delta(u^2)u(x\cdot\nabla u) = &-\mathrm{div}\left[2\psi_n u^2(x\cdot\nabla u)\cdot\nabla u - \psi_n u^2|\nabla u|^2\cdot x\right]\\ &-(x\cdot\nabla\psi_n)u^2|\nabla u|^2 + 2(x\cdot\nabla u)(\nabla\psi_n\cdot\nabla u)u^2.\end{aligned} \tag{2.11}$$

Hence, for every $n \in \mathbb{N}$, it follows from (2.7), (2.8), (2.9), (2.10), (2.11) and the divergence theorem that

$$\begin{aligned}&\int_{\partial B_{2n}}\left\{\frac{1}{2n}|x\cdot\nabla u|^2(1+2u^2) - n|\nabla u|^2(1+2u^2) - nV_0u^2 + 2nG(u)\right\}\psi_n\mathrm{d}\sigma\\ &= -\int_{B_{2n}}\left[V_0u^2 - 2G(u)\right]\psi_n\mathrm{d}x - \frac{1}{2}\int_{B_{2n}}\left\{|\nabla u|^2(1+2u^2) + V_0u^2 - 2G(u)\right\}(x\cdot\nabla\psi_n)\mathrm{d}x\\ &\quad+\int_{B_{2n}}(x\cdot\nabla u)(\nabla\psi_n\cdot\nabla u)(1+2u^2)\mathrm{d}x,\end{aligned} \tag{2.12}$$

which, together with the fact that $\psi_n|_{\partial B_{2n}} = 0$, implies

$$\begin{aligned}\int_{B_{2n}}\left[V_0u^2 - 2G(u)\right]\psi_n\mathrm{d}x = &-\frac{1}{2}\int_{B_{2n}}\left\{|\nabla u|^2(1+2u^2) + V_0u^2 - 2G(u)\right\}(x\cdot\nabla\psi_n)\mathrm{d}x\\ &+\int_{B_{2n}}(x\cdot\nabla u)(\nabla\psi_n\cdot\nabla u)(1+2u^2)\mathrm{d}x\end{aligned}$$

$$= -\frac{1}{2}\int_{B_{\sqrt{2}n}\setminus B_n} \left[|\nabla u|^2(1+2u^2) + V_0u^2 - 2G(u)\right](x\cdot\nabla\psi_n)\mathrm{d}x$$
$$+ \int_{B_{\sqrt{2}n}\setminus B_n} (x\cdot\nabla u)(\nabla\psi_n\cdot\nabla u)(1+2u^2)\mathrm{d}x. \tag{2.13}$$

From (2.6), (2.13) and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\left[V_0u^2 - 2G(u)\right]\mathrm{d}x\right| &= \left|\lim_{n\to\infty}\int_{B_{2n}}\left[V_0u^2 - 2G(u)\right]\psi_n\mathrm{d}x\right| \\ &\le \frac{1}{2}\lim_{n\to\infty}\int_{B_{\sqrt{2}n}\setminus B_n}\left[3|\nabla u|^2(1+2u^2) + V_0u^2 + 2G(u)\right]|x||\nabla\psi_n|\mathrm{d}x \\ &\le \frac{C_1}{2}\lim_{n\to\infty}\int_{B_{\sqrt{2}n}\setminus B_n}\left[3|\nabla u|^2(1+2u^2) + V_0u^2 + 2G(u)\right]\mathrm{d}x = 0.\end{aligned}$$

This, together with (2.5), shows that $P_0(u) = 0$, as desired. □

In the following, we will solve the constrained minimization problem:

$$A_0 = \inf_{u\in\mathcal{P}_0}\Phi_0(u) \quad \text{with} \quad \mathcal{P}_0 = \{u \in H\setminus\{0\} : P_0(u) = 0\}. \tag{2.14}$$

**Lemma 2.2.** *Assume that $g$ satisfies* (G1)-(G3). *Then there exists a minimizing sequence $\{u_n\}\subset\mathcal{P}_0$ satisfying $\|u_n\|_2 = 1$ for $A_0$. In particular,*

$$A_0 = \inf_{u\in\mathcal{P}_0}\frac{1}{2}\int_{\mathbb{R}^2}(1+2u^2)|\nabla u|^2\mathrm{d}x = \inf_{u\in\mathcal{P}_0^1}\frac{1}{2}\int_{\mathbb{R}^2}(1+2u^2)|\nabla u|^2\mathrm{d}x, \tag{2.15}$$

*where $\mathcal{P}_0$ is given by* (2.5), *and $\mathcal{P}_0^1 = \mathcal{P}_0\cap\{u\in H : \|u\|_2 = 1\}$.*

*Proof.* First, we verify that $\mathcal{P}_0 \neq \emptyset$. Let $u \in H\setminus\{0\}$ be fixed and define a function $\zeta(t) := P_0(tu)$ on $(0,\infty)$. Using (G1)-(G3), it is easy to check that $\zeta(t) > 0$ for small $t > 0$ and $\zeta(t) < 0$ for large $t > 0$. Then there exists $t_u > 0$ such that $\zeta(t_u) = P_0(t_uu) = 0$, and so $\mathcal{P}_0 \neq \emptyset$. Note that

$$\Phi_0(v) = \Phi_0(v) - \frac{1}{2}P_0(v) = \frac{1}{2}\int_{\mathbb{R}^2}(1+2v^2)|\nabla v|^2\mathrm{d}x, \quad \forall\ v\in\mathcal{P}_0. \tag{2.16}$$

Thus we can assume that there exists a minimizing sequence $\{u_n\}\subset\mathcal{P}_0$ satisfying

$$\frac{1}{2}\int_{\mathbb{R}^2}(1+2u_n^2)|\nabla u_n|^2\mathrm{d}x \to A_0.$$

Let $\tilde{u}_n = u_n(\|u_n\|_2^{1/2}x)$. Then a simple computation leads to $\tilde{u}_n\in\mathcal{P}_0, \|\tilde{u}_n\|_2 = 1$ and $\|\nabla\tilde{u}_n\|_2 = \|\nabla u_n\|_2$. This shows that $\tilde{u}_n\in\mathcal{P}_0^1$. From this and the fact that $\mathcal{P}_0^1\subset\mathcal{P}_0$, (2.15) follows directly. The proof is completed. □

To prove Theorem 1.2, we also need to show that the minimizer of $A_0$ is indeed a least energy solution of $(Q)_0$. For this, we have the following important result.

**Lemma 2.3.** *Assume that $g$ satisfies* (G1)-(G3).

(i) *If $u\in H$ is a critical point of $\Phi_0$ on the set $\mathcal{P}_0$, then it is a nontrivial solution of $(Q)_0$ under a suitable change of scale;*

(ii) *If the infimum $A_0$ is attained, then $A_0 = c_0^*$, where the definition of $c_0^*$ is given by* (1.13).

*Proof.* (i) Let $u \in H$ be a critical point of $\Phi_0$ on the set $\mathcal{P}_0$. Then there is a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$-\Delta u - u\Delta(u^2) + V_0 u - g(u) = 2\lambda[V_0 u - g(u)],$$

namely,

$$-\Delta u - u\Delta(u^2) = (2\lambda - 1)[V_0 u - g(u)]. \tag{2.17}$$

Since $u \neq 0$, we deduce from (2.17) that

$$2\lambda - 1 \neq 0 \quad \text{and} \quad V_0 u - g(u) \neq 0. \tag{2.18}$$

For any $T > 0$, by (G1), (G2) and (G3), there exist $0 < t_1 < t_2 < T$ and $-T < t_3 < t_4 < 0$ such that

$$2G(t) - g(t)t \leq 0, \quad \forall\, t \in [-T, T], \quad \text{and} \quad 2G(t) - g(t)t < 0, \quad \forall\, t \in [t_3, t_4] \cup [t_1, t_2]. \tag{2.19}$$

Hence, it follows from (2.19) and the definition of $\mathcal{P}_0$ that

$$\int_{\mathbb{R}^2} [V_0 u - g(u)]u \mathrm{d}x = \int_{\mathbb{R}^2} [2G(u) - g(u)u] \mathrm{d}x < 0. \tag{2.20}$$

This implies that there exists $w \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that

$$\langle P_0'(u), w\rangle = \int_{\mathbb{R}^2} [V_0 u - g(u)] w \mathrm{d}x < 0. \tag{2.21}$$

By multiplying (2.17) by $w$ and integrating, we have

$$\int_{\mathbb{R}^2} \left[(1 + 2u^2)\nabla u \cdot \nabla w + 2|\nabla u|^2 u w\right] \mathrm{d}x = (2\lambda - 1) \int_{\mathbb{R}^2} [V_0 u - g(u)] w \mathrm{d}x. \tag{2.22}$$

Using the fact $P_0(u) = 0$ and (2.21), it is easy to see that for small enough $\epsilon > 0$,

$$P_0(u + \epsilon w) < P_0(u) = 0. \tag{2.23}$$

Let

$$A(\varphi) := \frac{1}{2} \int_{\mathbb{R}^2} (1 + 2\varphi^2)|\nabla \varphi|^2 \mathrm{d}x, \qquad \forall\, \varphi \in H.$$

Noting that $A(u) = A_0$, by (2.22) and (2.23), we have

$$\begin{aligned} A(u + \epsilon w) &= \frac{1}{2} \int_{\mathbb{R}^2} \left[1 + 2\left(u^2 + 2\epsilon u w + \epsilon^2 w^2\right)\right] \left(|\nabla u|^2 + 2\epsilon \nabla u \cdot \nabla w + \epsilon^2 |\nabla w|^2\right) \mathrm{d}x \\ &= A_0 + \epsilon(2\lambda - 1) \int_{\mathbb{R}^2} [V_0 u - g(u)] w \mathrm{d}x + O(\epsilon^2). \end{aligned} \tag{2.24}$$

We claim that $2\lambda - 1 < 0$. Otherwise, if $2\lambda - 1 > 0$, then there exists $\epsilon_0 > 0$ small enough such that

$$P_0(u + \epsilon_0 w) < 0 \quad \text{and} \quad A(u + \epsilon_0 w) < A_0 \tag{2.25}$$

due to (2.23) and (2.24). Let $u_0 = u + \epsilon_0 w$. Then (2.25) yields $P_0(u_0) < 0$ and $P_0(s u_0) > 0$ for $s > 0$ small enough as a consequence of (G2). Therefore, there exists $s_0 \in (0, 1)$ such that $P_0(s_0 u_0) = 0$, and so (2.16) and (2.25) yield

$$A(s_0 u_0) = \frac{s_0^2}{2} \int_{\mathbb{R}^2} (1 + 2 s_0^2 u_0^2)|\nabla u_0|^2 \mathrm{d}x < s_0^2 A(u_0) < A_0. \tag{2.26}$$

This shows that $s_0 u_0 \in \mathcal{P}_0$ and $\Phi_0(s_0 u_0) < A_0$, which contradicts to the definition of $A_0$. Hence, we have $2\lambda - 1 < 0$ as claimed. Thus,

$$\tilde{u}(x) := u\left(\frac{x}{\sqrt{1 - 2\lambda}}\right) \quad \text{for a.e. } x \in \mathbb{R}^2 \tag{2.27}$$

is a nontrivial solution of $(Q)_0$.

(ii) If the infimum $A_0$ is attained by $u$, then $u$ is a critical point of $\Phi_0$ on the set $\mathcal{P}_0$. Applying the above Conclusion (i), we have $\Phi_0'(\tilde{u}) = 0$ and $A_0 = \Phi_0(\tilde{u}) \geq c_0^*$, where $\tilde{u}$ is defined by (2.27). To prove $A_0 = \Phi_0(\tilde{u}) = c_0^*$, it remains to show that $A_0 \leq c_0^*$. Note that Lemma 2.1 shows that if $\Phi_0'(v) = 0$ for $v \in H$, then $v$ satisfies the Pohozaev identity $P_0(v) = 0$, namely,

$$\Big\{ u \in H \setminus \{0\}, \ \Big| \ \Phi_0'(u) = 0 \Big\} \subset \mathcal{P}_0.$$

This implies that $A_0 \leq c_0^*$. The proof is completed. □

Before studying the attainability of $A_0$, we give a necessary and sufficient condition for the boundedness and the compactness of general nonlinear functionals in $H$, motivated by Ibrahim-Masmoudi-Nakanishi [27] and Masmoudi-Sani [36].

**Lemma 2.4.** *Suppose that $l : \mathbb{R} \to [0, +\infty)$ is a Borel function and define a functional $H$ by $L(u) := \int_{\mathbb{R}^2} l(u(x))\mathrm{d}x$. Then for any $K > 0$ we have the following properties* (B) *and* (C)*:*

(B) *Boundedness: The following* (i) *and* (ii) *are equivalent.*

(i) $\limsup_{|t|\to+\infty} e^{-2|t|^4/K}|t|^4 l(t) < \infty$ *and* $\limsup_{|t|\to 0} |t|^{-2} l(t) < \infty$.

(ii) *There exists a constant $C_{l,K} > 0$ such that*

$$u \in H, \ \|\nabla(u^2)\|_2 \leq 2\pi K \Rightarrow L(u) \leq C_{l,K}\left(\|u\|_2^2 + \|u\|_4^4\right). \tag{2.28}$$

(C) *Compactness: The following* (iii) *and* (iv) *are equivalent.*

(iii) $\limsup_{|t|\to+\infty} e^{-2|t|^4/K}|t|^4 l(t) = 0$ *and* $\lim_{|t|\to 0} |t|^{-2} l(t) = 0$.

(iv) *For any radially symmetric sequence $\{u_n\} \subset H$ satisfying $\|\nabla(u_n^2)\|_2^2 \leq 2\pi K$, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ and $u_n^2 \rightharpoonup u^2$ in $H^1(\mathbb{R}^2)$, there holds $L(u_n) \to L(u)$.*

*Proof. Necessity of* (i) *and* (iii)*:*

To prove the necessity of (i) and (iii), we first consider the much easier case with the condition as $u \to 0$. Let $\varphi_n(x)$ be a sequence of radial functions in $H$ defined by

$$\varphi_n(x) = \begin{cases} a_n & \text{if } 0 \leq |x| < R_n, \\ a_n(1 - |x| + R_n) & \text{if } R_n \leq |x| < R_n + 1, \\ 0 & \text{if } |x| \geq R_n + 1, \end{cases} \tag{2.29}$$

for some sequences $a_n \to 0$ and $R_n \to \infty$ chosen later. We have

$$\|\nabla(\varphi_n^2)\|_2^2 = O(a_n^4 R_n), \qquad \|\nabla \varphi_n\|_2^2 = O(a_n^2 R_n), \tag{2.30}$$

$$\|\varphi_n^2\|_2^2 = O(a_n^4 R_n^2), \qquad \|\varphi_n\|_2^2 = O(a_n^2 R_n^2) \tag{2.31}$$

and

$$L(\varphi_n) \geq \pi R_n^2 l(a_n). \tag{2.32}$$

If (i) is violated by $\limsup_{t\to 0} |t|^{-2} l(t) = \infty$, then we can find a sequence $a_n \searrow 0$ such that $l(a_n) \geq n|a_n|^2$. Let $R_n = a_n^{-1/2} + a_n^{-1} n^{-1/4}$. Then $R_n \to +\infty$ and $a_n R_n \to 0$, so (2.30), (2.31) and (2.32) yield $\|\nabla(\varphi_n^2)\|_2^2 \to 0$, $\|\nabla \varphi_n\|_2^2 \to 0$, $\|\varphi_n^2\|_2^2 \to 0$, $\|\varphi_n\|_2^2 \to 0$ and $L(\varphi_n) \geq \pi n a_n^2 R_n^2 \to \infty$.

If (iii) is violated by $\limsup_{t\to 0} |t|^{-2} l(t) > 0$, then we can find a sequence $a_n \searrow 0$ and $\delta > 0$ such that $l(a_n) \geq \delta |a_n|^2$. Let $R_n = 1/a_n$. Then $R_n \to \infty$, $a_n R_n = 1$, $a_n^2 R_n \to 0$ and $L(\varphi_n) \geq \pi \delta a_n^2 R_n^2 \geq \pi \delta$.

It remains to treat the case where the condition for $|t| \to +\infty$ fails. Choose sequences $1 \ll b_n \nearrow \infty$ and $K_n \nearrow K$ such that

$$2\pi(K_n - K) + O\left(\frac{\log b_n}{b_n^2}\right) \nearrow 0, \qquad c_n := e^{-2b_n^4/K_n} b_n^4 l(b_n) \to \limsup_{s\to\infty} e^{-2s^4/K} s^4 l(s) \tag{2.33}$$

and let $R_n = e^{-b_n^4/K_n} b_n^2$. We define a radial function $\psi_n \in H^1(\mathbb{R}^2)$ by

$$\psi_n(x) = \begin{cases} b_n, & \text{if } 0 \le |x| < R_n, \\ b_n \left| \frac{\log |x|}{\log R_n} \right|^{1/2}, & \text{if } R_n \le |x| < \frac{1}{2}, \\ 2b_n(1-|x|) \left| \frac{\log 2}{\log R_n} \right|^{1/2}, & \text{if } \frac{1}{2} \le |x| < 1, \\ 0, & \text{if } |x| \ge 1. \end{cases} \tag{2.34}$$

Noting that

$$\log R_n = -\frac{b_n^4}{K_n} + 2\log b_n, \tag{2.35}$$

by straightforward computations, we have for large $n \in \mathbb{N}$,

$$\begin{aligned} \|\nabla(\psi_n^2)\|_2^2 &= 4\int_{\mathbb{R}^2} \psi_n^2 |\nabla \psi_n|^2 \mathrm{d}x \\ &= \frac{2\pi b_n^4}{|\log R_n|^2} \int_{R_n}^{\frac{1}{2}} \frac{1}{r} \mathrm{d}r + \frac{128\pi b_n^4 |\log 2|^2}{|\log R_n|^2} \int_{\frac{1}{2}}^{1} r(r-1)^2 \mathrm{d}r \\ &= \frac{2\pi b_n^4}{|\log R_n|} + O\left( \frac{b_n^4}{|\log R_n|^2} \right) \\ &= 2\pi K_n + \frac{2\pi K_n^2 \log b_n}{b_n^4 - 2K_n \log b_n} + O\left( \frac{b_n^4}{(b_n^4 - 2K_n \log b_n)^2} \right) \\ &\le 2\pi K_n + O\left( \frac{\log b_n}{b_n^4} \right), \end{aligned} \tag{2.36}$$

$$\begin{aligned} \|\nabla \psi_n\|_2^2 &= \frac{\pi b_n^2}{2|\log R_n|} \int_{R_n}^{\frac{1}{2}} \frac{1}{r|\log r|} \mathrm{d}r + \frac{\pi b_n^2 \log 2}{2|\log R_n|} \\ &= \frac{\pi b_n^2 \log|\log R_n|}{2|\log R_n|} + O\left( \frac{b_n^2}{|\log R_n|} \right) \\ &= O\left( \frac{\log b_n}{b_n^2} \right), \end{aligned} \tag{2.37}$$

$$\begin{aligned} \|\psi_n\|_4^4 &= \pi b_n^4 R_n^2 + \frac{2\pi b_n^4}{|\log R_n|^2} \int_{R_n}^{\frac{1}{2}} r|\log r|^2 \mathrm{d}r + \frac{32\pi b_n^4 |\log 2|^2}{|\log R_n|^2} \int_{\frac{1}{2}}^{1} r(r-1)^4 \mathrm{d}r \\ &= O\left( \frac{b_n^4}{|\log R_n|^2} \right) = O\left( \frac{1}{b_n^4} \right), \end{aligned} \tag{2.38}$$

$$\begin{aligned} \|\psi_n\|_2^2 &= \pi b_n^2 R_n^2 + \frac{2\pi b_n^2}{|\log R_n|} \int_{R_n}^{\frac{1}{2}} r|\log r| \mathrm{d}r + \frac{8\pi b_n^2 \log 2}{|\log R_n|} \int_{\frac{1}{2}}^{1} r(r-1)^2 \mathrm{d}r \\ &= O\left( \frac{b_n^2}{|\log R_n|} \right) = O\left( \frac{1}{b_n^2} \right) \end{aligned} \tag{2.39}$$

and

$$L(\psi_n) \ge \pi R_n^2 l(b_n) = \frac{\pi R_n^2 c_n e^{2b_n^4/K_n}}{b_n^4} = \pi c_n. \tag{2.40}$$

Then (2.33), (2.36), (2.37), (2.38) and (2.39) imply that $\{\psi_n\}$ and $\{\psi_n^2\}$ are bounded in $H^1(\mathbb{R}^2)$ and

$$\|\nabla(\psi_n^2)\|_2^2 \le 2\pi K \quad \text{for large } n \in \mathbb{N}. \tag{2.41}$$

If the condition (i) fails at infinity, namely $c_n \to \infty$, then it follows from (2.38), (2.39) and (2.40) that

$$\|\psi_n\|_4^4 + \|\psi_n\|_2^2 \to 0 \ \text{ and } \ L(\psi_n) \to \infty,$$

which, together with (2.41), implies that the condition (ii) does not hold. Note that $\psi_n(x) \to 0$ for a.e. $x \in \mathbb{R}^2$, because $|\psi_n(x)| \le \epsilon$ if $|x| \ge e^{-\epsilon b_n} = o(1)$ for any $\epsilon > 0$. Jointly with the boundedness of $\{\|\psi_n\|\}$ and $\{\|\psi_n^2\|\}$, we get $\psi_n \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$ and $\psi_n^2 \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$. If the condition (iii) fails at infinity, namely $c_n > 0$ for large $n$, then it follows from (2.40) that

$$\liminf_{n\to\infty} L(\phi_n) > 0,$$

which, together with (2.41), implies that the condition (iv) does not hold. This ends the proof for the necessity of (i) and (iii).

*Necessity of* (ii) *and* (iv)*:*

We now prove the necessity of (ii), that is (i) implies (ii). Assume that the condition (i) holds. Let us define a new Borel measurable function $\tilde{l}(t)$ by

$$\tilde{l}(t) = l((2\pi K)^{1/4} t), \quad \forall\, t \ge 0. \tag{2.42}$$

It is easy to check that

$$\lim_{t\to+\infty} \frac{(1+t^2)^2 \tilde{l}(t)}{e^{4\pi t^4} - 1} = \lim_{t\to+\infty} \frac{t^4 \tilde{l}(t)}{e^{4\pi t^4}} < +\infty \tag{2.43}$$

and

$$\lim_{t\to 0} \frac{\tilde{l}(t)}{t^2} < +\infty. \tag{2.44}$$

From (2.43), (2.44) and the condition (i), we deduce that there exist constants $K_1, K_2 > 0$, dependent on $l$ and $K$, such that

$$\tilde{l}(t) \le K_1 t^2 + K_2 \frac{e^{4\pi t^4} - 1}{(1+t^2)^2}, \quad \forall\, t \ge 0. \tag{2.45}$$

By the Trudinger-Moser inequality with the exact growth, we have

$$\int_{\mathbb{R}^2} \frac{e^{4\pi|v|^4} - 1}{(1+|v|^2)^2} \mathrm{d}x \le \mathcal{C}\|v^2\|_2^2 = \mathcal{C}\|v\|_4^4, \quad \forall\, v^2 \in H^1(\mathbb{R}^2) \text{ with } \|\nabla(v^2)\|_2 \le 1. \tag{2.46}$$

From (2.45) and (2.46), it follows that for any $v \in H$ with $\|\nabla(v^2)\|_2 \le 1$, there holds

$$\begin{aligned} \int_{\mathbb{R}^2} \tilde{l}(v)\mathrm{d}x &\le K_1\|v\|_2^2 + K_2 \int_{\mathbb{R}^2} \frac{e^{4\pi|v|^4} - 1}{(1+|v|^2)^2} \mathrm{d}x \\ &\le K_1\|v\|_2^2 + K_2\mathcal{C}\|v\|_4^4. \end{aligned} \tag{2.47}$$

Let $v = (2\pi K)^{-1/4} u$ for $u \in H$. Then (2.42) and (2.47) imply that for any $u \in H$ satisfying $\|\nabla(u^2)\|_2^2 \le 2\pi K$, there holds

$$L(u) = \int_{\mathbb{R}^2} l(u)\mathrm{d}x = \int_{\mathbb{R}^2} \tilde{l}(v)\mathrm{d}x \le C_{l,K}\left(\|u\|_2^2 + \|u\|_4^4\right). \tag{2.48}$$

This shows that the condition (ii) holds.

Next, we turn to prove the necessity of (iv), that is (iii) implies (iv). Assume that the condition (iii) holds. Obviously, the condition (ii) is true because (iii) yields (i). For any radially symmetric sequence $\{u_n\} \subset H$ satisfying $\|\nabla(u_n^2)\| \le 2\pi K$ and $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$, we will verify that

$$\lim_{n\to\infty} [L(u_n) - L(u)] = \lim_{n\to\infty} \int_{\mathbb{R}^2} [l(u_n) - l(u)]\mathrm{d}x = 0. \tag{2.49}$$

By the radial Sobolev inequality, we have

$$|u_n(r)|^2 \leq \mathcal{C}\frac{\|u_n\|_2\|\nabla u_n\|_2}{r}, \tag{2.50}$$

and hence, $u_n(r) \to 0$ as $r \to \infty$ uniformly in $n$. This, together with (2.50) and the fact that $l(t) = o(t^2)$ as $t \to 0$, implies that for any $\varepsilon > 0$ there is $R > 0$ independent of $n$ such that

$$\int_{\mathbb{R}^2 \setminus B_R} l(u_n)\mathrm{d}x = 2\pi \int_R^{\infty} l(u_n) r\mathrm{d}r \leq 2\pi\varepsilon \int_R^{\infty} |u_n|^2 r\mathrm{d}r \leq \varepsilon\|u_n\|_2^2 \tag{2.51}$$

and

$$\int_{\mathbb{R}^2 \setminus B_R} l(u)\mathrm{d}x \leq \varepsilon. \tag{2.52}$$

Moreover, using the condition (ii) and the fact that $l(s) = o(e^{2s^4/K}|s|^{-4})$ as $s \to \infty$, there is $L > 1$ independent of $n$ such that

$$\int_{|u_n|>L} l(u_n)\mathrm{d}x \leq \varepsilon \int_{|u_n|>L} e^{2u_n^4/K}|u_n|^{-4}\mathrm{d}x \leq \varepsilon C_1\left(\|u_n\|_2^2 + \|u_n\|_4^4\right) \tag{2.53}$$

and

$$\int_{|u|>L} l(u)\mathrm{d}x \leq \varepsilon. \tag{2.54}$$

Combining (2.51), (2.52), (2.53) and (2.54), we get

$$\begin{aligned}
\lim_{n\to\infty}[L(u_n) - L(u)] &\leq \lim_{n\to\infty}\left[\int_{\mathbb{R}^2\setminus B_R}[l(u_n) - l(u)]\mathrm{d}x + \int_{B_R}[l(u_n) - l(u)]\mathrm{d}x\right] \\
&\leq C_2\varepsilon + \limsup_{n\to\infty}\left[\int_{|u_n|>L} l(u_n)\mathrm{d}x + \int_{|u|>L} l(u)\mathrm{d}x\right] \\
&\quad + \limsup_{n\to\infty}\left[\int_{|u_n|\leq L, |x|\leq R} l(u_n)\mathrm{d}x - \int_{|u|\leq L, |x|\leq R} l(u)\mathrm{d}x\right] \\
&\leq C_3\varepsilon,
\end{aligned}$$

where we have used the Lebesgue dominated convergence theorem in the last step. This accomplishes the proof for the necessity of (ii) and (iv). □

Now, we establish a relation between the attainability of $A_0$ and the Trudinger-Moser inequality with the exact growth. From (2.1), (2.2), (2.28), (2.46) and the Gagliardo-Nirenberg inequality, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^2} G(u)\mathrm{d}x \leq \mathcal{C}_1\left(\|u\|_2^2 + \|u\|_4^4\right) \leq \mathcal{C}_2\|u\|_2^2\left(1 + \|\nabla u\|_2^2\right), \\
&\qquad \forall\, u \in H \text{ with } 2\|\nabla u\|_2^2 + \|\nabla(u^2)\|_2^2 \leq L < \frac{4\pi}{\alpha_0}.
\end{aligned} \tag{2.55}$$

For this purpose, inspired by Ibrahim-Masmoudi-Nakanishi [27] and Masmoudi-Sani [36], we introduce the Trudinger-Moser ratio

$$C_{\mathrm{TM}}^L(G) = \sup\left\{\frac{2}{\|u\|_2^2}\int_{\mathbb{R}^2} G(u)\mathrm{d}x \;\middle|\; u \in H\setminus\{0\}, 2\|\nabla u\|_2^2 + \|\nabla(u^2)\|_2^2 \leq L\right\}, \tag{2.56}$$

the Trudinger-Moser threshold:

$$R(G) := \sup\left\{L > 0 \;\middle|\; C_{\mathrm{TM}}^L(G) < +\infty\right\} \tag{2.57}$$

and we denote by $C^*_{\mathrm{TM}}(G)$ the ratio at the threshold, i.e.

$$C^*_{\mathrm{TM}}(G) = C^{R(G)}_{\mathrm{TM}}(G). \tag{2.58}$$

By (2.55) and Lemma 2.4, we have $R(G) = 4\pi/\alpha_0$.

In this section, to apply Schwarz symmetrization to $(Q)_0$, as usual we let

$$\tilde{g}(t) = \begin{cases} g(t), & \text{for all } t > 0, \\ -g(-t), & \text{for all } t \leq 0. \end{cases} \tag{2.59}$$

Observe that $\tilde{g}$ satisfies the same conditions as $g$. Furthermore, by the maximum principle, solutions of $(Q)_0$ with $\tilde{g}$ are also solutions of $(Q)_0$ with $g$. Hence there is no loss in generality in replacing $g$ by $\tilde{g}$, and we will always adopt the convention that $g$ has been replaced by $\tilde{g}$; we keep however the same notation $g$ in the following discussion of this section.

Let

$$H_r := H \cap \{u \in H \mid u(x) = u(|x|) \text{ a.e. in } \mathbb{R}^2\}.$$

In the following, we will solve the constrained minimization problem $A_0$, given by (2.14).

**Lemma 2.5.** *Assume that $g$ satisfies* (G1)-(G3). *If $A_0 < \pi/\alpha_0$, then $A_0$ is attained and $A_0 = \Phi_0(u)$, where $u \in H_r$ is, under a suitable change of scale, a positive least energy solution of equation $(Q)_0$.*

*Proof.* We may always assume that there exists a sequence $\{u_n\} \subset \mathcal{P}_0 \cap H_r$ satisfying

$$\frac{1}{2}\int_{\mathbb{R}^2} (1 + 2u_n^2)|\nabla u_n|^2 \mathrm{d}x \to A_0 < \frac{\pi}{\alpha_0} \text{ and } \|u_n\|_2 = 1 \tag{2.60}$$

by Schwarz symmetrization and Lemma 2.2. Then there exists some function $u \in H_r$ such that $u_n \rightharpoonup u$ and $u_n^2 \rightharpoonup u^2$ in $H^1(\mathbb{R}^2)$.

Picking up $\frac{2}{K} > \alpha_0$ satisfying $\lim_{n\to\infty} \|\nabla(u_n^2)\|_2^2 \leq 2\pi K$, then (2.2) yields

$$\lim_{|t|\to+\infty} \frac{|t|^4 G(t)}{e^{2|t|^4/K}} = 0. \tag{2.61}$$

From (2.1), (2.61) and (C) of Lemma 2.4, we derive that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} G(u_n)\mathrm{d}x = \int_{\mathbb{R}^2} G(u)\mathrm{d}x. \tag{2.62}$$

Since $P_0(u_n) = 0$ and $\|u_n\|_2 = 1$, by (2.62), we have

$$0 < V_0 = \lim_{n\to\infty} V_0\|u_n\|_2^2 = 2\lim_{n\to\infty} \int_{\mathbb{R}^2} G(u_n)\mathrm{d}x = 2\int_{\mathbb{R}^2} G(u)\mathrm{d}x, \tag{2.63}$$

which implies that $u \neq 0$. Now, we prove that the infimum $A_0$ is attained by $u$. By the weak lower semicontinuity of the norm and (2.62), we have

$$P_0(u) = V_0\|u\|_2^2 - 2\int_{\mathbb{R}^2} G(u)\mathrm{d}x \leq \lim_{n\to\infty} \left(V_0\|u_n\|_2^2 - 2\int_{\mathbb{R}^2} G(u_n)\mathrm{d}x\right) = 0 \tag{2.64}$$

and

$$0 < \frac{1}{2}\int_{\mathbb{R}^2} (1 + 2u^2)|\nabla u|^2 \mathrm{d}x \leq \lim_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^2} (1 + 2u_n^2)|\nabla u_n|^2 \mathrm{d}x = A_0. \tag{2.65}$$

Next, it remains only to show that $u \in \mathcal{P}_0$, namely $P_0(u) = 0$. Set

$$h(t) = P_0(tu) = t^2 V_0\|u\|_2^2 - 2\int_{\mathbb{R}^2} G(tu)\mathrm{d}x.$$

Then $h(1) \leq 0$ by (2.64), and from (2.3), we can deduce that $h(t) > 0$ for $t > 0$ small enough. Consequently, there exists $t_0 \in (0, 1]$ such that $P_0(t_0 u) = 0$, namely $t_0 u \in \mathcal{P}_0$. This together with (2.65) leads to

$$A_0 \leq \frac{t_0^2}{2} \int_{\mathbb{R}^2} (1 + 2t_0^2 u^2)|\nabla u|^2 \mathrm{d}x \leq t_0^2 A_0.$$

The above inequality and (2.65) show that $t_0 = 1$ and $\frac{1}{2}\int_{\mathbb{R}^2}(1 + 2u^2)|\nabla u|^2 \mathrm{d}x = A_0$. Combining (2.63) with the fact that $P_0(u) = 0$, we have $\|u\|_2 = 1$. Applying Lemma 2.3, we have that this $u$ is a least energy solution of $(Q)_0$ under a suitable change of scale. Noting that $\langle \Phi_0'(u), -u^- \rangle = 0$, where $u^\pm = \max\{\pm u, 0\}$, it follows that $u^- = 0$ and so $u = u^+ \geq 0$. Arguing as in the proof of [22, Page 3368], we can derive that $u > 0$ in $\mathbb{R}^2$. The proof is completed. □

**Lemma 2.6.** *Assume that $g$ satisfies* (G1)-(G3). *Then $A_0 < \pi/\alpha_0$ if and only if $V_0 < C^*_{\mathrm{TM}}(G)$, where $C^*_{\mathrm{TM}}(G)$ is given by* (2.58).

*Proof.* First, we verify that $V_0 < C^*_{\mathrm{TM}}(G)$ yields $A_0 < \pi/\alpha_0$. We distinguish two cases: $C^*_{\mathrm{TM}}(G) < +\infty$ and $C^*_{\mathrm{TM}}(G) = +\infty$. In the case $C^*_{\mathrm{TM}}(G) < +\infty$, since $V_0 < C^*_{\mathrm{TM}}(G)$, then $V_0 < C^*_{\mathrm{TM}}(G) - \epsilon_0$ for some $\epsilon_0 > 0$. By the definition of $C^*_{\mathrm{TM}}(G)$, there exists some $u_0 \in H \setminus \{0\}$ such that

$$2\|\nabla u_0\|_2^2 + \|\nabla(u_0^2)\|_2^2 \leq R(G) = 4\pi/\alpha_0 \quad \text{and} \quad V_0 < C^*_{\mathrm{TM}}(G) - \epsilon_0 < \frac{2}{\|u_0\|_2^2} \int_{\mathbb{R}^2} G(u_0)\mathrm{d}x. \tag{2.66}$$

Then

$$P_0(u_0) = V_0\|u_0\|_2^2 - 2\int_{\mathbb{R}^2} G(u_0)\mathrm{d}x < 0. \tag{2.67}$$

Let $h_0(t) = P_0(tu_0)$ for $t > 0$. Since $h_0(1) < 0$ by (2.67), and $h_0(t) > 0$ for $t > 0$ small enough by (2.3), there exists $t_0 \in (0, 1)$ such that $h_0(t_0) = P_0(t_0 u_0) = 0$, namely $t_0 u_0 \in \mathcal{P}_0$. Therefore, we have

$$A_0 \leq \frac{t_0^2}{2} \int_{\mathbb{R}^2} (1 + 2t_0^2 u_0^2)|\nabla u_0|^2 \mathrm{d}x < \frac{1}{2} \int_{\mathbb{R}^2} (1 + 2u_0^2)|\nabla u_0|^2 \mathrm{d}x \leq \frac{1}{4} R(G) = \frac{\pi}{\alpha_0},$$

which shows that $A_0 < \pi/\alpha_0$ in the case $C^*_{\mathrm{TM}}(G) < +\infty$. In the case $C^*_{\mathrm{TM}}(G) = +\infty$, for any $V_0 > 0$, there exists some $u_0 \in H \setminus \{0\}$ such that

$$2\|\nabla u_0\|_2^2 + \|\nabla(u_0^2)\|_2^2 \leq R(G) \quad \text{and} \quad V_0\|u_0\|_2^2 < 2\int_{\mathbb{R}^2} G(u_0)\mathrm{d}x.$$

Hence we can repeat the same arguments as above to get the desired conclusion.

Now, we prove that $A_0 < \frac{\pi}{\alpha_0}$ yields $V_0 < C^*_{\mathrm{TM}}(G)$. Clearly, if $C^*_{\mathrm{TM}}(G) = +\infty$, then $V_0 < C^*_{\mathrm{TM}}(G)$ and the proof is completed. Therefore, without loss of generality, we may assume that $C^*_{\mathrm{TM}}(G) < +\infty$. Applying Lemma 2.5 we know that $A_0$ is achieved by some function $u \in H_r$, that is

$$P_0(u) = V_0\|u\|_2^2 - 2\int_{\mathbb{R}^2} G(u)\mathrm{d}x = 0 \tag{2.68}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^2} (1 + 2u^2)|\nabla u|^2 \mathrm{d}x = A_0 < \frac{\pi}{\alpha_0}. \tag{2.69}$$

Define the function

$$\psi(t) = \frac{2}{t^2\|u\|_2^2} \int_{\mathbb{R}^2} G(tu)\mathrm{d}x, \qquad \forall\, t > 0.$$

Then (2.68) yields $\psi(1) = V_0$. Note that $\psi(t)$ is monotone increasing by (G3). Define the function

$$\phi(t) = t^2 \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + t^4 \int_{\mathbb{R}^2} 2u^2|\nabla u|^2 \mathrm{d}x, \qquad \forall\, t > 0.$$

By (2.69) and the continuity of $\phi$, we know that there exists $t_0 > 1$ such that

$$\phi(t_0) = \int_{\mathbb{R}^2} \left[1 + 2(t_0 u)^2\right] |\nabla(t_0 u)|^2 \mathrm{d}x = \frac{2\pi}{\alpha_0}. \tag{2.70}$$

Set $v = t_0 u$. Then we have

$$2\|\nabla v\|_2^2 + \|\nabla(v^2)\|_2^2 = \frac{4\pi}{\alpha_0},$$

and so

$$C^*_{\mathrm{TM}}(G) \geq \frac{2}{\|v\|_2^2} \int_{\mathbb{R}^2} G(v) \mathrm{d}x = \psi(t_0) > \psi(1) = V_0.$$

This completes the proof. □

## 2.2 Maximum characterization of the least energy solution

In order to finish the proof of Theorem 1.2, in addition to proving the existence of a least energy solution, we also need to establish the maximum characterization. For this purpose, we make the change of variable by $v = f^{-1}(u)$, where $f$ is defined by (1.4). After the change of variable, we obtain the following functional:

$$I_0(v) = \Phi_0(u) = \Phi_0(f(v)) = \frac{1}{2} \int_{\mathbb{R}^2} \left[|\nabla v|^2 + V_0 f^2(v)\right] \mathrm{d}x - \int_{\mathbb{R}^2} G(f(v)) \mathrm{d}x. \tag{2.71}$$

About the change of variable $f(t)$, we have the following lemma, see [13, 18, 32].

**Lemma 2.7.** *The following properties involving $f(t)$ and its derivative hold:*

(f1) *$f$ is uniquely defined, $\mathcal{C}^\infty$ and invertible;*

(f2) $0 < f'(t) \leq 1$ *for all* $t \in \mathbb{R}$*;*

(f3) $|f(t)| \leq |t|$ *for all* $t \in \mathbb{R}$*;*

(f4) $f(t)/t \to 1$ *as* $t \to 0$*;*

(f5) $f(t)/\sqrt{t} \to 2^{1/4}$ *as* $t \to +\infty$*;*

(f6) $f(t)/2 \leq tf'(t) \leq f(t)$ *for all* $t > 0$ *and* $f(t) \leq tf'(t) \leq f(t)/2$ *for all* $t \leq 0$*;*

(f7) $|f(t)| \leq 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$*;*

(f8) $|f(t)f'(t)| \leq 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$*;*

(f9) *there exists a positive constant $\theta_0$ such that*

$$|f(t)| \geq \begin{cases} \theta_0 |t|, & |t| \leq 1, \\ \theta_0 |t|^{1/2}, & |t| > 1; \end{cases}$$

(f10) $t \mapsto f(t)f'(t)/|t|$ *is strictly decreasing on* $(-\infty, 0) \cup (0, +\infty)$*;*

(f11) $t \mapsto f^3(t)f'(t)/|t|$ *is strictly increasing on* $(-\infty, 0) \cup (0, +\infty)$.

By (2.3) and Lemma 2.7, we have for any $\epsilon > 0, \alpha > \alpha_0$ and $q > 0$, there exists $C = C(\epsilon, \alpha, q) > 0$ such that

$$\begin{aligned} 2G(f(t)) &\leq g(f(t))f(t) \leq \epsilon f^2(t) + C|f(t)|^q \left(e^{\alpha f^4(t)} - 1\right) \\ &\leq \epsilon t^2 + C|t|^q \left(e^{2\alpha t^2} - 1\right), \quad \forall\, t \in \mathbb{R}. \end{aligned} \tag{2.72}$$

Using (2.72), Lemmas 1.1 and 2.7, one can check that $I_0 \in \mathcal{C}^1(H^1(\mathbb{R}^2),\mathbb{R})$, moreover,

$$\langle I_0'(v), v\rangle = \int_{\mathbb{R}^2} \left[|\nabla v|^2 + V_0 f(v) f'(v) v\right] \mathrm{d}x - \int_{\mathbb{R}^2} g(f(v)) f'(v) v \mathrm{d}x, \quad \forall\ v \in H^1(\mathbb{R}^2) \tag{2.73}$$

and

$$\begin{aligned}\langle I_0'(v), f(v)/f'(v)\rangle = &\int_{\mathbb{R}^2} \left(1 + \frac{2f^2(v)}{1+2f^2(v)}\right) |\nabla v|^2 \mathrm{d}x + \int_{\mathbb{R}^2} V_0 f^2(v) \mathrm{d}x \\ &- \int_{\mathbb{R}^2} g(f(v)) f(v) \mathrm{d}x, \quad \forall\ v \in H^1(\mathbb{R}^2).\end{aligned} \tag{2.74}$$

As in [13], critical points of $I_0$ are solutions of the semilinear equation

$$-\Delta v + V_0 f(v) f'(v) = g(f(v)) f'(v). \tag{$\mathcal{S}$)$_0$.$$

Then $v$ is a solution of $(\mathcal{S})_0$ if and only if $u = f(v)$ solves $(Q)_0$, see [13,32].

We define the following Mountain Pass level for $I_0$:

$$c_0 = \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} I_0(\gamma(t)) \ \text{ with } \ \Gamma_0 = \left\{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^2) : \gamma(0) = 0, I_0(\gamma(1)) < 0\right\}. \tag{2.75}$$

**Remark 2.8.** *As in* [20, Proposition 3.1]*, we can get the geometric hypotheses of the Mountain Pass theorem for $I_0$. Then the Mountain Pass level $c_0$ in* (2.75) *is well-defined. Moreover, the following proof will yield $c_0 \le c_0^*$, where $c_0^*$ is the least energy for $\Phi_0$ defined by* (1.13)*.*

*Proof of Theorem* 1.2. If $V_0 < C^*_{\mathrm{TM}}(G)$, then Lemma 2.6 leads to $A_0 < \pi/\alpha_0$. Hence the assumptions of Lemma 2.5 are fulfilled. This, jointly with (ii) of Lemma 2.3, shows that $(Q)_0$ has a positive least energy solution $u_0 \in H$ satisfying

$$A_0 = c_0^* = \frac{1}{2}\int_{\mathbb{R}^2} (1 + 2u_0^2)|\nabla u_0|^2 \mathrm{d}x < \pi/\alpha_0. \tag{2.76}$$

Next, we give the maximum characterization of the least energy solution. Let $w = f^{-1}(u_0)$. Then

$$c_0^* = \Phi_0(u_0) = \Phi_0(f(w)) = I_0(w), \ \ \Phi_0'(u_0) = 0 \ \text{ and } \ I_0'(w) = 0. \tag{2.77}$$

We define a curve $\gamma$, constituted of the three pieces given by:

$$\gamma(\theta) = \begin{cases} \frac{\theta}{t_1} w_{t_1}, & \text{if } \theta \in [0, t_1], \\ w_{[t_3(\theta - t_1) + (t_2 - \theta)t_1]/(t_2 - t_1)}, & \text{if } \theta \in [t_1, t_2], \\ \frac{t_2(\theta - t_2) + t_3 - \theta}{t_3 - t_2} w_{t_3}, & \text{if } \theta \in [t_2, t_3], \end{cases} \tag{2.78}$$

where $w_t(x) = w(x/t)$ and $0 < t_1 < 1 < t_2 < t_3$ are determined later. It is easy to check that $\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^2))$. Since $\langle I_0'(w), w\rangle = 0$ by (2.77), then (2.73) yields

$$\int_{\mathbb{R}^2} \left[g(f(w)) - V_0 f(w)\right] f'(w) w \mathrm{d}x = \int_{\mathbb{R}^2} |\nabla w|^2 \mathrm{d}x > 0.$$

Then we can find $t_2 > 1$ such that

$$\int_{\mathbb{R}^2} \left[g(f(\xi w)) - V_0 f(\xi w)\right] f'(\xi w) w \mathrm{d}x > 0, \quad \forall\ \xi \in [1, t_2]. \tag{2.79}$$

Note that for any fixed $t > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\xi} I_0(\xi w_t) = \langle I_0'(\xi w_t), w_t\rangle$$

$$= \xi \left\{ \|\nabla w_t\|_2^2 - \int_{\mathbb{R}^2} [g(f(\xi w_t)) - V_0 f(\xi w_t)] \frac{f'(\xi w_t) w_t}{\xi} \mathrm{d}x \right\}$$

$$= \xi \left\{ \|\nabla w\|_2^2 - t^2 \int_{\mathbb{R}^2} [g(f(\xi w)) - V_0 f(\xi w)] \frac{f'(\xi w) w}{\xi} \mathrm{d}x \right\}. \tag{2.80}$$

Choosing $t_1 \in (0,1)$, we have

$$\|\nabla w\|_2^2 - t_1^2 \int_{\mathbb{R}^2} [g(f(\xi w)) - V_0 f(\xi w)] \frac{f'(\xi w) w}{\xi} \mathrm{d}x > 0, \quad \forall\, \xi \in [0,1]. \tag{2.81}$$

By (2.79), we can also choose $t_3 > t_2$ such that

$$\|\nabla w\|_2^2 - t_3^2 \int_{\mathbb{R}^2} [g(f(\xi w)) - V_0 f(\xi w)] \frac{f'(\xi w) w}{\xi} \mathrm{d}x \le -\frac{2}{t_2^2 - 1} \|\nabla w\|_2^2, \quad \forall\, \xi \in [1, t_2]. \tag{2.82}$$

Thus we can see by (2.81) that the function $I_0\left(\frac{\theta}{t_1} w_{t_1}\right)$ is increasing on $\theta \in [0, t_1]$ and takes its maximal at $\theta = t_1$, namely

$$I_0(\gamma(\theta)) = I_0\left(\frac{\theta}{t_1} w_{t_1}\right) \le I_0(w_{t_1}), \quad \forall\, \theta \in [0, t_1]. \tag{2.83}$$

Since $\Phi_0'(u_0) = 0$ and $u_0 = f(w)$, then Lemma 2.1 gives

$$P_0(u_0) = \int_{\mathbb{R}^2} \left[V_0 u_0^2 - 2G(u_0)\right] \mathrm{d}x = \int_{\mathbb{R}^2} \left[V_0 f^2(w) - 2G(f(w))\right] \mathrm{d}x = 0. \tag{2.84}$$

From (1.4), (2.76) and (2.84), we derive that

$$\begin{aligned} I_0(w_t) &= \frac{1}{2}\|\nabla w\|_2^2 + \frac{t^2}{2} \int_{\mathbb{R}^2} \left[V_0 f^2(w) - 2G(f(w))\right] \mathrm{d}x \\ &= \frac{1}{2}\|\nabla w\|_2^2 = \frac{1}{2} \int_{\mathbb{R}^2} (1 + 2u_0^2) |\nabla u_0|^2 \mathrm{d}x \\ &= \Phi_0(u_0) = c_0^*, \quad \forall\, t > 0, \end{aligned} \tag{2.85}$$

which implies

$$I_0(w_{[t_3(\theta - t_1) + (t_2 - \theta) t_1]/(t_2 - t_1)}) = I_0(w) = \Phi_0(u_0) = c_0^*, \quad \forall\, \theta \in [t_1, t_2]. \tag{2.86}$$

Next by (2.80) and (2.82), we have $I_0(\xi w_{t_3})$ is decreasing on $\xi \in [1, t_2]$. Noting that

$$\frac{t_2(\theta - t_2) + t_3 - \theta}{t_3 - t_2} \in [1, t_2] \;\Leftrightarrow\; \theta \in [t_2, t_3],$$

we know that $I_0\left(\frac{t_2(\theta - t_2) + t_3 - \theta}{t_3 - t_2} w_{t_3}\right)$ is decreasing on $\theta \in [t_2, t_3]$. Therefore,

$$I_0(\gamma(\theta)) \le I_0(\gamma(t_2)) = I_0(t_2 w_{t_3}), \quad \forall\, \theta \in [t_2, t_3]. \tag{2.87}$$

Moreover, (2.82) yields

$$\begin{aligned} I_0(\gamma(t_3)) = I_0(t_2 w_{t_3}) &= I_0(w_{t_3}) + \int_1^{t_2} \frac{\mathrm{d}}{\mathrm{d}\xi} I_0(\xi w_{t_3}) \mathrm{d}\xi \\ &\le \frac{1}{2}\|\nabla w\|_2^2 - \int_1^{t_2} \frac{2\xi}{t_2^2 - 1} \|\nabla w\|_2^2 \mathrm{d}\xi \\ &= -\frac{1}{2}\|\nabla w\|_2^2 < 0. \end{aligned} \tag{2.88}$$

Combining (2.77), (2.83), (2.86) and (2.87), we have

$$I_0(\gamma(\theta)) \le I_0(w) = c_0^*, \qquad \forall\ \theta \in [0, t_3]. \tag{2.89}$$

Let $\gamma_0(\theta) = \gamma(t_3\theta)$ for all $\theta \in [0, 1]$. Then $\gamma_0 \in \Gamma_0$ by (2.88), where the definition of $\Gamma_0$ is given in (2.75). From this, (2.75), (2.77) and (2.89), we derive

$$c_0 \le \max_{t\in[0,1]} I_0(\gamma_0(t)) = I_0(w) = c_0^*. \tag{2.90}$$

Hence, (1.17) follows from (2.77), (2.90) and Lemma 2.7, and so Theorem 1.2 is proved. □

**Lemma 2.9.** *Assume that $g$ satisfies* (G1)-(G3) *and* (M2′). *Then $A_0 \le c_0 < \pi/\alpha_0$ where $A_0$ and $c_0$ are given by* (2.14) *and* (2.75), *respectively.*

*Proof.* Arguing as in the proof of [37, Lemma 3.5], we can get the following estimate on the Mountain Pass level:

$$c_0 \le \frac{q-2}{2q\tilde{C}_q^{(q-2)/2}} S_q(V_0)^q, \tag{2.91}$$

replacing (M2) used in [37] by (M2′). By (1.10) and (2.91), we have $c_0 < \pi/\alpha_0$. Based on the general minimax principle [30, Proposition 2.8] (see also [28]), we can construct a Cerami sequence $\{v_n\}$ with $I_0(v_n) \to c_0$ and with the extra property that $\mathcal{P}_0(v_n) \to 0$. By modifying the proof of [20, 37], we can deduce that there exists $v_0 \in H^1(\mathbb{R}^2) \setminus \{0\}$ such that $I_0'(v_0) = 0$ and $I_0(v_0) = c_0$. In particular, we can take advantage of the additional information $\mathcal{P}_0(v_n) \to 0$ to get the boundedness of $\{\|v_n\|\}$ without the condition (AR) required in [20, 37], which is the main difference from those. Hence, $A_0 \le c_0 < \pi/\alpha_0$ follows from the definition of $A_0$, since $\mathcal{P}_0(v_0) = 0$. The proof is completed. □

**Remark 2.10.** (i) *Recalling* (2.1) *and in light of (B) of Lemma* 2.4, *we can easily derive that $C^*_{\mathrm{TM}}(G) = +\infty$ if and only if $\lim_{t\to+\infty} \frac{t^2G(t)}{e^{\alpha_0 t^4}} = +\infty$. Hence, if $g$ satisfies* (M1′), *then $V_0 < C^*_{\mathrm{TM}}(G) = +\infty$ is obvious.*

(ii) *From Lemmas* 2.6 *and* 2.9, *we can easily derive that* (M2′) *implies the inequality $V_0 < C^*_{\mathrm{TM}}(G)$.*

# 3 Modified problem

Under (V1), we know that $E$, defined by (1.18), is a Hilbert space with the inner product

$$(u, v) = \int_{\mathbb{R}^2} [\nabla u \cdot \nabla v + V(x)uv]\,\mathrm{d}x, \qquad \forall\ u, v \in E$$

and the induced norm denoted by $\|u\| = (u, u)^{1/2}$. Then $E \hookrightarrow H^1(\mathbb{R}^2)$, and so for $s \in [2, \infty)$, there exists $\gamma_s > 0$ such that

$$\|v\|_s \le \gamma_s\|v\|, \qquad \forall\ v \in E. \tag{3.1}$$

Observe that formally $(Q)_\varepsilon$ is the Euler-Lagrange equation associated to the following functional

$$\Phi_\varepsilon(u) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}(1 + 2u^2)|\nabla u|^2\mathrm{d}x + \int_{\mathbb{R}^2} V(x)u^2\mathrm{d}x - \int_{\mathbb{R}^2} G(u)\mathrm{d}x. \tag{3.2}$$

As in Section 2.2, we make the change of variable by $v = f^{-1}(u)$, and get the functional:

$$J_\varepsilon(v) = \Phi_\varepsilon(u) = \Phi_\varepsilon(f(v)) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}^2} |\nabla v|^2\mathrm{d}x + \int_{\mathbb{R}^2} V(x)f^2(v)\mathrm{d}x - \int_{\mathbb{R}^2} G(f(v))\mathrm{d}x, \tag{3.3}$$

where $f$ is defined by (1.4). Using (2.72), Lemmas 1.1 and 2.7, one can check that $J_\varepsilon \in \mathcal{C}^1(E, \mathbb{R})$. Then $v_\varepsilon$ is a critical point of $J_\varepsilon(v)$ if and only if $u_\varepsilon = f(v_\varepsilon)$ is a solution of $(Q)_\varepsilon$, see [13, 32].

### 3.1 Penalized nonlinearity

To find a critical point of $J_\varepsilon(v)$, defined by (3.3), we introduce the penalized nonlinearity, following the idea of del Pino-Felmer [16]. We may suppose, without loss of generality, that the boundary $\partial\Lambda$ is smooth and $0 \in \Lambda$ and $V(0) = \inf_{x\in\Lambda} V(x)$ by the translation invariance of the problem. To simplify the notation, in what follows, we let $\min_{x\in\Lambda} V(x) = V_0$. Using (V1), (V2), (G2) and (G3), we can choose numbers $k > 2$ and $\beta_0 > 0$ such that

$$g(\beta_0) = \beta_0 V_0 k^{-1}, \qquad \inf_{x\in\mathbb{R}^2} V(x) > 2V_0 k^{-1}, \tag{3.4}$$

and set

$$\bar{g}(t) := \begin{cases} g(t), & 0 \le t \le \beta_0, \\ V_0 k^{-1} t, & t > \beta_0. \end{cases} \tag{3.5}$$

We consider the modified nonlinearity that is the Carathéodory function

$$g(x,t) := \begin{cases} \chi_\Lambda(x) g(t) + (1 - \chi_\Lambda(x))\, \bar{g}(t), & t \ge 0, \\ 0, & t < 0, \end{cases} \tag{3.6}$$

where $\chi_\Lambda$ is the characteristic function on $\Lambda$ defined by

$$\chi_\Lambda(x) = \begin{cases} 1, & x \in \Lambda, \\ 0, & x \in \mathbb{R}^2 \setminus \Lambda. \end{cases} \tag{3.7}$$

Let $\bar{G}(t) := \int_0^t \bar{g}(s)\mathrm{d}s$ and $G(x,t) := \int_0^t g(x,s)\mathrm{d}s$. We have the following properties on $g$.

**Proposition 3.1.** *Assume that $g$ satisfies* (G1)-(G5). *Then*

(g1) *$g(x,t) = o(t)$ uniformly in $x$ as $t \to 0$ and $g(x,t) \le g(t)$ for all $x \in \mathbb{R}^2$ and $t \ge 0$;*

(g2) *$0 \le 4G(x,t) \le g(x,t)t$ for all $x \in \Lambda$ and $t \ge 0$, or $x \in \mathbb{R}^2 \setminus \Lambda$ and $0 \le t \le \beta_0$;*

(g3) *$0 \le 2G(x,t) \le g(x,t)t \le V_0 k^{-1} t^2$ and $0 \le g(x, f(t)) f'(t) t \le V_0 k^{-1} f(t) f'(t) t$ for all $x \in \mathbb{R}^2 \setminus \Lambda$ and $t \ge 0$.*

This proposition and (2.72) imply that for any $\epsilon > 0, \alpha > \alpha_0$ and $q > 0$, there exists $C = C(\epsilon, \alpha, q) > 0$ such that

$$\begin{aligned} 2G(x, f(t)) &\le g(x, f(t)) f(t) \le \epsilon f^2(t) + C|f(t)|^q \left(e^{\alpha f^4(t)} - 1\right) \\ &\le \epsilon t^2 + C|t|^q \left(e^{2\alpha t^2} - 1\right), \quad \forall\ (x,t) \in \mathbb{R}^2 \times \mathbb{R}. \end{aligned} \tag{3.8}$$

For every $\varepsilon \in (0,1]$, we introduce the penalized functional $I_\varepsilon : E \to \mathbb{R}$ as follows:

$$I_\varepsilon(v) = \frac{1}{2} \int_{\mathbb{R}^2} \left[\varepsilon^2 |\nabla v|^2 + V(x) f^2(v)\right] \mathrm{d}x - \int_{\mathbb{R}^2} G(x, f(v))\mathrm{d}x. \tag{3.9}$$

Using (3.8), Lemmas 1.1 and 2.7, one can check that $I_\varepsilon \in \mathcal{C}^1(E, \mathbb{R})$, and

$$\begin{aligned} \langle I'_\varepsilon(v), \phi \rangle = &\int_{\mathbb{R}^2} \left[\varepsilon^2 \nabla v \cdot \nabla \phi + V(x) f(v) f'(v) \phi\right] \mathrm{d}x \\ &- \int_{\mathbb{R}^2} g(x, f(v)) f'(v) \phi \mathrm{d}x, \quad \forall\ \phi \in E. \end{aligned} \tag{3.10}$$

Moreover, the critical points of $I_\varepsilon$ are solutions of the modified problem:

$$-\varepsilon^2 \Delta v = f'(v)\left[g(x, f(v)) - V(x) f(v)\right], \quad x \in \mathbb{R}^2. \tag{$\mathcal{S}$}_\varepsilon$$

In this section, we try to find a positive ground state solution for modified problem $(\mathcal{S})_\varepsilon$. Precisely, we are going to prove the following theorem.

**Theorem 3.2.** *Assume that* (V1), (V2) *and* (G1)-(G5) *hold. Let* $V_0 = \min_{x\in\Lambda} V(x) < C^*_{\rm TM}(G)$, *where* $C^*_{\rm TM}(G)$ *is given by* (1.16). *Then there exists* $\varepsilon_0 > 0$ *such that* $(\mathcal{S})_\varepsilon$ *possesses a positive ground state solution for any* $\varepsilon \in (0, \varepsilon_0)$.

In this paper, we say that a solution of $(\mathcal{S})_\varepsilon$ is a *ground state solution* if it has the least energy on the Nehari manifold defined by

$$\mathcal{N}_\varepsilon := \{v \in E \setminus \{0\} : \langle I'_\varepsilon(v), v\rangle = 0\}. \tag{3.11}$$

## 3.2 Mountain pass geometry

In this subsection, we verify that $I_\varepsilon(u)$ has a mountain pass geometry, and then obtain a Cerami sequence of $I_\varepsilon(u)$ for every fixed $\varepsilon \in (0, 1]$. To this end, for $\rho > 0$, we define

$$A(v) := \int_{\mathbb{R}^2} \left[|\nabla v|^2 + f^2(v)\right] \mathrm{d}x \quad \text{and} \quad S_\rho := \{u \in E : A(v) = \rho\}. \tag{3.12}$$

Clearly, $S_\rho$ is a closed subset and disconnects the space $E$. We have the following properties:
**Proposition 3.3.** *Assume that* (V1), (V2) *and* (G1)-(G4) *hold. Then for any* $\varepsilon \in (0, 1]$,

(i) *there exist* $\rho_\varepsilon, \delta_\varepsilon > 0$ *such that* $I_\varepsilon(v) \geq \delta_\varepsilon$ *for all* $v \in S_{\rho_\varepsilon}$, *where* $S_{\rho_\varepsilon}$ *is given by* (3.12);

(ii) *there exists* $v_0 \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ *with* $A(v_0) > \rho_\varepsilon$ *such that* $I_\varepsilon(v_0) < 0$.

*Proof.* (i) From (3.8), we know that for given $\alpha > \alpha_0$, there exists $C_1 > 0$ such that

$$\begin{aligned} G(x, f(t)) &\leq \frac{V_0}{4k} f^2(t) + C_1 \left(e^{\alpha f^4(t)} - 1\right) f^3(t) \\ &\leq \frac{V_0}{4k} f^2(t) + C_1 \left(e^{2\alpha t^2} - 1\right) f^3(t), \quad \forall\ (x, t) \in \mathbb{R}^2 \times \mathbb{R}. \end{aligned} \tag{3.13}$$

In view of Lemma 1.1, we have

$$\int_{\mathbb{R}^2} \left(e^{4\alpha v^2} - 1\right) \mathrm{d}x = \int_{\mathbb{R}^2} \left(e^{4\alpha \|\nabla v\|^2 (v/\|\nabla v\|)^2} - 1\right) \mathrm{d}x \leq \mathcal{C}_1, \quad \forall\ v \in E, A(v) \leq \pi/2\alpha. \tag{3.14}$$

Note that $\|\nabla f(v)\|_2^2 \leq 2\|\nabla v\|_2^2$ for all $v \in E$. Then (3.1) (3.13), (3.14) and the Hölder inequality give

$$\begin{aligned} \int_{\mathbb{R}^2} G(x, f(v))\mathrm{d}x &\leq \frac{V_0}{4k} \|f(v)\|_2^2 + C_1 \int_{\mathbb{R}^2} \left(e^{2\alpha v^2} - 1\right) |f(v)|^3 \mathrm{d}x \\ &\leq \frac{V_0}{4k} \|f(v)\|_2^2 + C_1 \left[\int_{\mathbb{R}^2} \left(e^{4\alpha v^2} - 1\right) \mathrm{d}x\right]^{1/2} \|f(v)\|_6^3 \\ &\leq \frac{V_0}{4k} \|f(v)\|_2^2 + C_2 [A(v)]^{3/2}, \quad \forall\ v \in E, A(v) \leq \pi/2\alpha. \end{aligned} \tag{3.15}$$

Hence, it follows from (3.4), (3.9) and (3.15) that

$$\begin{aligned} I_\varepsilon(v) &= \frac{\varepsilon^2}{2} \int_{\mathbb{R}^2} |\nabla v|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^2} V(x) f^2(v) \mathrm{d}x - \int_{\mathbb{R}^2} G(x, f(v))\mathrm{d}x \\ &\geq \frac{\varepsilon^2}{4k} \min\{1, V_0\} A(v) - C_2 [A(v)]^{3/2}, \quad \forall\ v \in E, A(v) \leq \pi/2\alpha. \end{aligned} \tag{3.16}$$

Therefore, there exists $\delta_\varepsilon > 0$ and $0 < \rho_\varepsilon < \pi/2\alpha$ such that $I_\varepsilon(v) \geq \delta_\varepsilon$ for all $v \in S_{\rho_\varepsilon}$.

(ii) By (G1)-(G4), there exist $K_1, K_2 > 0$ and $\mu_0 > 4$ such that

$$G(t) \geq K_1 t^{\mu_0} - K_2 t^2, \quad \forall\ t \geq 0. \tag{3.17}$$

Using (3.17), a standard argument shows the desired conclusion. □

Using Proposition 3.3 and applying the Mountain Pass Theorem, we know that for any $\varepsilon \in (0,1]$, there exists a Cerami sequence, reads as follows.

**Lemma 3.4.** *Assume that* (V1), (V2) *and* (G1)-(G4) *hold. Then for any* $\varepsilon \in (0,1]$, *there exists a sequence* $\{v_{\varepsilon,n}\} \subset E$ *such that*

$$I_\varepsilon(v_{\varepsilon,n}) \to c_\varepsilon, \qquad \|I'_\varepsilon(v_{\varepsilon,n})\|(1+\|v_{\varepsilon,n}\|) \to 0, \tag{3.18}$$

*where*

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} I_\varepsilon(\gamma(t)) > \delta_\varepsilon \quad \textit{with} \quad \Gamma_\varepsilon = \{\gamma \in \mathcal{C}([0,1],E) : \gamma(0)=0, I_\varepsilon(\gamma(1))<0\}. \tag{3.19}$$

## 3.3 Characterization of the mountain-pass level

In this subsection, we establish the characterization of the mountain-pass level $c_\varepsilon$ for any $\varepsilon \in (0,1]$, where $c_\varepsilon$ is defined by (3.19).

Arguing as in [18, Lemma 3.7], we can easily show the following lemma.

**Lemma 3.5.** *Assume that* (V1), (V2) *and* (G1)-(G4) *hold. Let* $\varepsilon \in (0,1]$. *Then for any* $v \in E\setminus\{0\}$, *there exists* $t_v > 0$ *such that* $t_v v \in \mathcal{N}_\varepsilon$, *where* $\mathcal{N}_\varepsilon$ *is defined by* (3.11).

**Lemma 3.6.** *Assume that* (V1), (G1), (G2), (G4) *and* (G5) *hold. Let* $\varepsilon \in (0,1]$. *Then*

$$I_\varepsilon(v) \geq I_\varepsilon(tv) + \frac{1-t^2}{2}\langle I'_\varepsilon(v), v\rangle, \qquad \forall\ v \in E,\ t \geq 0. \tag{3.20}$$

*Proof.* For any $v \neq 0$, (f10) of Lemma 2.7 yields

$$\frac{1}{2}\left[f^2(v) - f^2(tv)\right] - \frac{1-t^2}{2} f(v)f'(v)v = \int_1^t \left[\frac{f(v)f'(v)}{v} - \frac{f(sv)f'(sv)}{sv}\right] sv^2 \mathrm{d}s \geq 0, \tag{3.21}$$

moreover, (G5) and (f11) of Lemma 2.7 imply

$$\begin{aligned} &G(f(tv)) - G(f(v)) + \frac{1-t^2}{2} g(f(v))f'(v)v = \int_1^t \left[g(f(sv))f'(sv)v - g(f(v))f'(v)sv\right] \mathrm{d}s \\ &= \int_1^t \left[\frac{g(f(sv))}{f^3(sv)} \cdot \frac{f^3(sv)f'(sv)}{sv} - \frac{g(f(v))}{f^3(v)} \cdot \frac{f^3(v)f'(v)}{v}\right] sv^2 \mathrm{d}s \geq 0. \end{aligned} \tag{3.22}$$

By (3.4), Proposition 3.1, (f2) and (f11) of Lemma 2.7, we have

$$\begin{aligned} &\int_{\mathbb{R}^2\setminus\Lambda} \left\{ V(x)\left[\frac{1}{2}f^2(v) - \frac{1}{2}f^2(tv) - \frac{1-t^2}{2} f(v)f'(v)v\right] \right. \\ &\left. + G(x,f(tv)) - G(x,f(v)) + \frac{1-t^2}{2} g(x,f(v))f'(v)v \right\} \mathrm{d}x \\ &= \int_{\mathbb{R}^2\setminus\Lambda} \left\{ \left[G(x,f(tv)) - \frac{V(x)}{2} f^2(tv)\right] - \left[G(x,f(v)) - \frac{V(x)}{2} f^2(v)\right] \right. \\ &\quad \left. + \frac{1-t^2}{2} \left[g(x,f(v)) - V(x)f(v)\right] f'(v)v \right\} \mathrm{d}x \\ &= -\int_{\mathbb{R}^2\setminus\Lambda} \int_1^t \left[ \frac{V(x)f(sv) - g(x,f(sv))}{f(sv)} \cdot \frac{f(sv)f'(sv)}{sv} \right. \\ &\quad \left. - \frac{V(x)f(v) - g(x,f(v))}{f(v)} \cdot \frac{f(v)f'(v)}{v} \right] sv^2 \mathrm{d}s\mathrm{d}x \geq 0. \end{aligned} \tag{3.23}$$

Hence, it follows from (3.9), (3.10), (3.21), (3.22) and (3.23) that

$$I_\varepsilon(v) - I_\varepsilon(tv)$$

$$
\begin{aligned}
&= \frac{1-t^2}{2}\int_{\mathbb{R}^2}|\nabla v|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2}V(x)\left[f^2(v)-f^2(tv)\right]\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2}\left[G(x,f(tv))-G(x,f(v))\right]\mathrm{d}x \\
&= \frac{1-t^2}{2}\langle I'_\varepsilon(v),v\rangle + \int_{\mathbb{R}^2}V(x)\left[\frac{1}{2}f^2(v)-\frac{1}{2}f^2(tv)-\frac{1-t^2}{2}f(v)f'(v)v\right]\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2}\left[G(x,f(tv))-G(x,f(v))+\frac{1-t^2}{2}g(x,f(v))f'(v)v\right]\mathrm{d}x \\
&= \frac{1-t^2}{2}\langle I'_\varepsilon(v),v\rangle + \int_{\Lambda}V(x)\left[\frac{1}{2}f^2(v)-\frac{1}{2}f^2(tv)-\frac{1-t^2}{2}f(v)f'(v)v\right]\mathrm{d}x \\
&\quad + \int_{\Lambda}\left[G(f(tv))-G(f(v))+\frac{1-t^2}{2}g(f(v))f'(v)v\right]\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2\setminus\Lambda}\left\{V(x)\left[\frac{1}{2}f^2(v)-\frac{1}{2}f^2(tv)-\frac{1-t^2}{2}f(v)f'(v)v\right]\right. \\
&\quad \left. +G(x,f(tv))-G(x,f(v))+\frac{1-t^2}{2}g(x,f(v))f'(v)v\right\}\mathrm{d}x \\
&\geq \frac{1-t^2}{2}\langle I'_\varepsilon(v),v\rangle, \quad \forall\, t\geq 0.
\end{aligned}
$$

This shows that (3.20) holds. □

**Corollary 3.7.** *Assume that* (V1), (V2) *and* (G1)-(G5) *hold. Let* $\varepsilon\in(0,1]$. *Then*

$$
m_\varepsilon := \inf_{v\in\mathcal{N}_\varepsilon} I_\varepsilon(v) = \inf_{v\in E\setminus\{0\}}\sup_{t>0} I_\varepsilon(tv). \tag{3.24}
$$

**Lemma 3.8.** *Assume that* (V1), (V2) *and* (G1)-(G5) *hold. Let* $\varepsilon\in(0,1]$. *Then* $m_\varepsilon = c_\varepsilon$, *where* $c_\varepsilon$ *is given by* (3.19).

Let

$$
m_0 := \inf_{v\in\mathcal{N}_0} I_0(v) \quad \text{with} \quad \mathcal{N}_0 = \left\{v\in H^1(\mathbb{R}^2)\setminus\{0\} : \langle I'_0(v),v\rangle = 0\right\}. \tag{3.25}
$$

Clearly, the results in the above lemmas on modified problem $(\mathcal{S})_\varepsilon$ still work for the autonomous problem $(\mathcal{S})_0$. Combining with the result obtained in Section 2, we have the following theorem.

**Theorem 3.9.** *Assume that* (G1)-(G5) *hold. Let* $0 < V_0 < C^*_{\mathrm{TM}}(G)$, *where* $C^*_{\mathrm{TM}}(G)$ *is given by* (1.16). *Then* $(\mathcal{S})_0$ *has a positive solution* $v_0\in H^1(\mathbb{R}^2)$ *such that*

$$
I_0(v_0) = c_0 = m_0 = \inf_{v\in\mathcal{N}_0} I_0(v) = \inf_{v\in H^1(\mathbb{R}^2)\setminus\{0\}}\sup_{t>0} I_0(tv) < \frac{\pi}{\alpha_0}. \tag{3.26}
$$

### 3.4 Local Cerami condition

In this subsection, we will prove that $I_\varepsilon$ satisfies the Cerami condition in a certain level. For simplicity, we denote the Cerami sequence $\{v_{\varepsilon,n}\}$ given by Lemma 3.4 by $\{v_n\}$ in this subsection.

**Lemma 3.10.** *Assume that* (V1), (V2) *and* (G1)-(G5) *hold. Let* $\varepsilon\in(0,1]$. *Then any sequence* $\{v_n\}$ *satisfying* (3.18) *is bounded in* $E$.

*Proof.* Note that (3.10) yields

$$
\begin{aligned}
\langle I'_\varepsilon(v), f(v)/f'(v)\rangle &= \int_{\mathbb{R}^2}\varepsilon^2\left(1+\frac{2f^2(v)}{1+2f^2(v)}\right)|\nabla v|^2\mathrm{d}x + \int_{\mathbb{R}^2}V(x)f^2(v)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^2}g(x,f(v))f(v)\mathrm{d}x, \quad \forall\, v\in E.
\end{aligned} \tag{3.27}
$$

Moreover, by Lemma 2.7, we have

$$\begin{aligned}\|f(v_n)/f'(v_n)\|^2 &= \int_{\mathbb{R}^2} \left\{ \left[1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right] |\nabla v_n|^2 + \left[1 + 2f^2(v_n)\right] f^2(v_n) \right\} \\ &\leq 5\|v_n\|^2. \end{aligned} \tag{3.28}$$

Using (G5), it is easy to see that

$$\frac{1}{4} g(t)t - G(t) \geq 0, \quad \forall\, t \geq 0. \tag{3.29}$$

Using (G3) and (G4), it is easy to check that for each $\delta > 0$, there exists $R_\delta > 0$ satisfying

$$g(t)t \geq \delta G(t), \quad \forall\, |t| \geq R_\delta. \tag{3.30}$$

Then it follows from (3.4), (3.9), (3.18), (3.27), (3.28), (3.29) and (3.30) with $\delta > 4$ that

$$\begin{aligned} c_\varepsilon + o(1) &= I_\varepsilon(v_n) - \frac{1}{4}\langle I'_\varepsilon(v_n), f(v_n)/f'(v_n)\rangle \\ &= \frac{1}{4}\int_{\mathbb{R}^2} \left[\varepsilon^2 |\nabla f(v_n)|^2 + V(x) f^2(v_n)\right] \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^2\setminus\Lambda} \frac{V_0}{k} f^2(v_n) \mathrm{d}x \\ &\quad + \int_{\{x\in\Lambda : |f(v_n)|\leq R_\delta\}} \left[\frac{1}{4} g(f(v_n)) f(v_n) - G(f(v_n))\right] \mathrm{d}x \\ &\quad + \int_{\{x\in\Lambda : |f(v_n)|> R_\delta\}} \left[\frac{1}{4} g(f(v_n)) f(v_n) - G(f(v_n))\right] \mathrm{d}x \\ &\geq \frac{1}{4k}\int_{\mathbb{R}^2} \left[\varepsilon^2 |\nabla f(v_n)|^2 + V_0 f^2(v_n)\right] \mathrm{d}x + \frac{\delta - 4}{4\delta}\int_{\{x\in\Lambda : |f(v_n)|> R_\delta\}} g(f(v_n)) f(v_n) \mathrm{d}x. \end{aligned}$$

The above inequality implies that

$$\|f(v_n)\| \leq C_1, \quad \int_{\{x\in\Lambda : |f(v_n)|> R_\delta\}} g(f(v_n)) f(v_n) \mathrm{d}x \leq C_2. \tag{3.31}$$

Since $\langle I'_\varepsilon(v_n), v_n\rangle = o(1)$, it follows from (3.31), (f6) of Lemma 2.7 and Proposition 3.1 that

$$\begin{aligned} \varepsilon^2 \|\nabla v_n\|_2^2 &\leq \varepsilon^2 \|\nabla v_n\|_2^2 + \int_{\mathbb{R}^2} V(x) f(v_n) f'(v_n) v_n \mathrm{d}x = \int_{\mathbb{R}^2} g(x, f(v_n)) f'(v_n) v_n \mathrm{d}x + o(1) \\ &\leq C_3 \|f(v_n)\|_2^2 + \int_{\{x\in\Lambda : |f(v_n)|> R_\delta\}} g(f(v_n)) f(v_n) \mathrm{d}x + o(1) \leq C_4. \end{aligned} \tag{3.32}$$

Moreover, by (3.31), (f9) of Lemma 2.7 and the Sobolev embedding theorem, we have

$$\int_{\mathbb{R}^2} v_n^2 \mathrm{d}x = \int_{\{|v_n|\leq 1\}} v_n^2 \mathrm{d}x + \int_{\{|v_n|>1\}} v_n^2 \mathrm{d}x \leq \frac{1}{\theta_0^2}\int_{\mathbb{R}^2} f^2(v_n) \mathrm{d}x + \frac{1}{\theta_0^4}\int_{\mathbb{R}^2} f^4(v_n) \mathrm{d}x \leq C_5. \tag{3.33}$$

Combining (3.32) with (3.33), we get the boundedness of $\{\|v_n\|\}$, and the lemma is proved. □

**Lemma 3.11.** *Assume that* (V1), (V2) *and* (G1)-(G5) *hold. Let* $\varepsilon \in (0, 1]$ *and* $\{v_n\}$ *be a Cerami sequence satisfying* (3.18)*. Then for given* $\epsilon > 0$ *there exists* $R_\epsilon > 0$ *such that*

$$\limsup_{n\to\infty} \int_{|x|\geq R_\epsilon} \left[\varepsilon^2 |\nabla v_n|^2 + V(x) f(v_n) f'(v_n) v_n\right] \mathrm{d}x \leq \epsilon. \tag{3.34}$$

*Proof.* We choose $R > 0$ suitably large such that

$$\Lambda \subset \overline{B}_{R/2}(0), \tag{3.35}$$

and take a cut-off function $\eta_R \in \mathcal{C}^\infty(\mathbb{R}^2, [0,1])$ such that $\eta_R = 0$ on $B_{R/2}(0)$, $\eta_R = 1$ on $\mathbb{R}^2 \setminus B_R(0)$ and $|\nabla \eta_R| \le 3/R$. Then $\eta_R = 0$ on $\Lambda$. By (3.10), (3.18), (g3) of Proposition 3.3 and Lemma 3.6, we have

$$\begin{aligned} o(1) &= \langle I'_\varepsilon(v_n), \eta_R v_n \rangle \\ &= \int_{\mathbb{R}^2} \left[\varepsilon^2 |\nabla v_n|^2 + V(x) f(v_n) f'(v_n) v_n\right] \eta_R \mathrm{d}x + \varepsilon^2 \int_{\mathbb{R}^2} (\nabla v_n \cdot \nabla \eta_R)\, v_n \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^2} g(x, f(v_n)) f'(v_n) v_n \eta_R \mathrm{d}x \\ &\ge \frac{1}{2} \int_{\mathbb{R}^2} \left[\varepsilon^2 |\nabla v_n|^2 + V(x) f(v_n) f'(v_n) v_n\right] \eta_R \mathrm{d}x - \frac{3\varepsilon^2}{R} \int_{\mathbb{R}^2} |\nabla v_n||v_n| \mathrm{d}x \\ &\ge \frac{1}{2} \int_{|x|\ge R} \left[\varepsilon^2 |\nabla v_n|^2 + V(x) f(v_n) f'(v_n) v_n\right] \mathrm{d}x - \frac{C_6 \varepsilon^2}{R}, \end{aligned}$$

which implies

$$\int_{|x|\ge R} \left[\varepsilon^2 |\nabla v_n|^2 + V(x) f(v_n) f'(v_n) v_n\right] \mathrm{d}x \le \frac{2C_6 \varepsilon^2}{R} + o(1). \tag{3.36}$$

Hence, for given $\epsilon > 0$, there exists $R_\epsilon > 0$ such that (3.34) holds . □

From [14, Lemma 2.1] and Lemma 2.7, we can get the following lemma.

**Lemma 3.12.** *Assume that* (G1) *and* (G2) *hold. Let $v_n \rightharpoonup \bar{v}$ in $H^1(\mathbb{R}^2)$.*

(i) *If $\int_{\mathbb{R}^2} |g(x, v_n) v_n| \mathrm{d}x \le K_0$ for some constant $K_0 > 0$, then $\lim_{n\to\infty} \int_{\mathbb{R}^2} g(x, v_n)\phi \mathrm{d}x = \int_{\mathbb{R}^2} g(x, \bar{v})\phi \mathrm{d}x$ for any $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$.*

(ii) *If $\int_{\mathbb{R}^2} |g(x, f(v_n)) f(v_n)| \mathrm{d}x \le K'_0$ for some constant $K'_0 > 0$, then $\lim_{n\to\infty} \int_{\mathbb{R}^2} g(x, f(v_n)) f'(v_n)\phi \mathrm{d}x = \int_{\mathbb{R}^2} g(x, f(\bar{v})) f'(\bar{v})\phi \mathrm{d}x$ for any $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, if further* (G4) *holds, $\lim_{n\to\infty} \int_\Omega G(x, f(v_n)) \mathrm{d}x = \int_\Omega G(x, f(\bar{v})) \mathrm{d}x$ for any compact set $\Omega \subset \mathbb{R}^2$.*

**Lemma 3.13.** *Assume that* (V1), (V2) *and* (G1)-(G5) *hold. Let $\varepsilon \in (0,1]$. If $c_\varepsilon < \varepsilon^2 \pi/\alpha_0$, then there exists $v_\varepsilon > 0$ such that $I_\varepsilon(v_\varepsilon) = c_\varepsilon$ and $I'_\varepsilon(v_\varepsilon) = 0$.*

*Proof.* Applying Lemmas 3.4 and 3.10, for any $\varepsilon \in (0,1]$, there exists a bounded sequence $\{v_n\} \subset E$ satisfying (3.18). We may thus assume, passing to a subsequence if necessary, that $v_n \rightharpoonup v_\varepsilon$ in $E$, $v_n \to v_\varepsilon$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$ for $s \in [1, \infty)$ and $v_n \to v_\varepsilon$ a.e. in $\mathbb{R}^2$. Then (3.31) gives

$$\int_{\mathbb{R}^2} |g(x, f(v_n)) f(v_n)| \mathrm{d}x = \int_{\mathbb{R}^2} g(x, f(v_n)) f(v_n) \mathrm{d}x \le C_7. \tag{3.37}$$

From (3.37) and (ii) of Lemma 3.12, we can deduce that $I'_\varepsilon(v_\varepsilon) = 0$. Let $c_\varepsilon < \varepsilon^2\pi/\alpha_0$. The rest of the proof of Lemma 3.13 consists of several steps.

**Step 1:** We prove that $v_\varepsilon > 0$.
First, we claim that $v_\varepsilon \ne 0$. For this, we suppose by contradiction that $v_\varepsilon = 0$. Then $v_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$ for $s \in [1, \infty)$ and $v_n \to 0$ a.e. in $\mathbb{R}^2$. From (f3) and (f6) of Lemma 2.7 and Lemma 3.11, we then deduce that

$$\int_{\mathbb{R}^2} f(v_n) f'(v_n) v_n \mathrm{d}x = o(1), \qquad \int_{\mathbb{R}^2} f^2(v_n) \mathrm{d}x = o(1). \tag{3.38}$$

Noting that $\mathbb{R}^2 \setminus B_R(0) \subset \mathbb{R}^2 \setminus \Lambda$ by (3.35), it follows from (3.38) and (g3) of Proposition 3.1 that

$$\int_{|x|>R} G(x, f(v_n))\mathrm{d}x \leq \frac{V_0}{2k}\|f(v_n)\|_2^2 = o(1). \tag{3.39}$$

Moreover, (ii) of Lemma 3.12 yields

$$\int_{|x|\leq R} G(x, f(v_n))\mathrm{d}x = o(1). \tag{3.40}$$

Combining (3.39) with (3.40), we have

$$\int_{\mathbb{R}^2} G(x, f(v_n))\mathrm{d}x = o(1). \tag{3.41}$$

Since $c_\varepsilon < \varepsilon^2\pi/\alpha_0$, it follows from (3.9), (3.18), (3.38) and (3.41) that

$$\varepsilon^2\|\nabla v_n\|_2^2 \leq 2c_\varepsilon + 2\int_{\mathbb{R}^2} G(x, f(v_n))\mathrm{d}x + o(1) := \varepsilon^2\frac{2\pi}{\alpha_0}(1-3\bar{\epsilon}) + o(1) \tag{3.42}$$

for some $\bar{\epsilon} > 0$. Let us choose $q \in (1, 2)$ such that

$$\frac{(1+\bar{\epsilon})\,(1-2\bar{\epsilon})\,q}{1-\bar{\epsilon}} < 1. \tag{3.43}$$

Then (G1) and (f7) of Lemma 2.7 yield

$$|g(x, f(t))|^q \leq C_8\left[e^{\alpha_0(1+\bar{\epsilon})qf^4(t)} - 1\right] \leq C_8\left[e^{2\alpha_0(1+\bar{\epsilon})qt^2} - 1\right], \quad \forall\ |f(t)| \geq 1. \tag{3.44}$$

By (3.42), (3.43), (3.44) and ii) of Lemma 1.1, we have

$$\begin{aligned}
&\int_{|f(v_n)|\geq 1} |g(x, f(v_n))|^q\mathrm{d}x \leq C_8\int_{\mathbb{R}^2}\left[e^{2\alpha_0(1+\bar{\epsilon})qv_n^2} - 1\right]\mathrm{d}x \\
&= C_8\int_{\mathbb{R}^2}\left[e^{2\alpha_0(1+\bar{\epsilon})q\left(\|\nabla v_n\|_2^2+2\pi\bar{\epsilon}/\alpha_0\right)v_n^2/\left(\|\nabla v_n\|_2^2+2\pi\bar{\epsilon}/\alpha_0\right)} - 1\right]\mathrm{d}x \leq C_9.
\end{aligned} \tag{3.45}$$

Note that $f(v_n) \to 0$ in $L^s(\mathbb{R}^2)$ for any $s \geq 2$ by (3.38) and the Sobolev embedding theorem. Let $q' = q/(q-1)$. Then it follows from (3.45), the Hölder inequality and (f6) of Lemma 2.7, we have

$$\begin{aligned}
&\int_{|f(v_n)|\geq 1} g(x, f(v_n))f'(v_n)v_n\mathrm{d}x \leq \int_{|f(v_n)|\geq 1} g(x, f(v_n))f(v_n)\mathrm{d}x \\
&\leq \left[\int_{|f(v_n)|\geq 1} |g(x, f(v_n))|^q\mathrm{d}x\right]^{1/q}\|f(v_n)\|_{q'} = o(1).
\end{aligned} \tag{3.46}$$

Moreover, using (G1), (G2), (f3) of Lemma 2.7 and (3.38), we can check easily that

$$\int_{|f(v_n)|<1} g(x, f(v_n))f'(v_n)v_n\mathrm{d}x \leq C_{10}\|f(v_n)\|_2^2 = o(1). \tag{3.47}$$

Combining (3.10), (3.38), (3.46) and (3.47), we derive that

$$o(1) = \langle I_\varepsilon'(v_n), v_n\rangle = \varepsilon^2\|\nabla v_n\|_2^2 + \int_{\mathbb{R}^2} V(x)f(v_n)f'(v_n)v_n\mathrm{d}x + o(1),$$

which, together with (3.9), (3.18), (3.19), (3.41) and (f6) of Lemma 2.7, leads to

$$\delta_\varepsilon \leq c_\varepsilon + o(1) = I_\varepsilon(v_n) = o(1).$$

This contradiction shows that $v_\varepsilon \neq 0$. Noting that $\langle I'_\varepsilon(v_\varepsilon), -v_\varepsilon^-\rangle = 0$, where $v_\varepsilon^\pm = \max\{\pm v_\varepsilon, 0\}$, it follows that $v_\varepsilon^- = 0$ and so $v_\varepsilon = v_\varepsilon^+ \geq 0$. Arguing as in the proof of [22, Page 3368], we can derive that $v_\varepsilon > 0$ in $\mathbb{R}^2$.

**Step 2:** We prove that $\lim_{n\to\infty} \|\nabla v_n\|_2^2 < \frac{2\pi}{\alpha_0} + \|\nabla v_\varepsilon\|_2^2$, up to a subsequence. Suppose, by contradiction, that $\limsup_{n\to\infty} \|\nabla v_n\|_2^2 \geq \frac{2\pi}{\alpha_0} + \|\nabla v_\varepsilon\|_2^2$. Note that

$$\langle I'_\varepsilon(v_n), f(v_n)/f'(v_n)\rangle = o(1) \text{ and } \langle I'_\varepsilon(v_\varepsilon), f(v_\varepsilon)/f'(v_\varepsilon)\rangle = 0. \tag{3.48}$$

Then (3.27) and (3.48) give

$$\begin{aligned}
&\int_\Lambda [g(f(v_n))f(v_n) - g(f(v_\varepsilon))f(v_\varepsilon)]\,\mathrm{d}x + \frac{V_0}{k}\int_{\mathbb{R}^2\setminus\Lambda} \left[f^2(v_n) - f^2(v_\varepsilon)\right]\mathrm{d}x \\
&= \varepsilon^2 \int_{\mathbb{R}^2} \left[\left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2 - \left(1 + \frac{2f^2(v_\varepsilon)}{1+2f^2(v_\varepsilon)}\right)|\nabla v_\varepsilon|^2\right]\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2} V(x)\left[f^2(v_n) - f^2(v_\varepsilon)\right]\mathrm{d}x + o(1).
\end{aligned} \tag{3.49}$$

Using Lemma 3.11 and (ii) of Lemma 3.12, it is easy to see that

$$\int_\Lambda [G(f(v_n)) - G(f(v_\varepsilon))]\,\mathrm{d}x = 0. \tag{3.50}$$

Then it follows from (3.9), (3.18), (3.27), (3.48), (3.49), (3.50) and Fatou's lemma that, up to a subsequence,

$$\begin{aligned}
c_\varepsilon + o(1) &= I_\varepsilon(v_n) - \frac{1}{4}\langle I'_\varepsilon(v_n), f(v_n)/f'(v_n)\rangle \\
&= \frac{1}{4}\int_{\mathbb{R}^2}\left[\frac{\varepsilon^2|\nabla v_n|^2}{1+2f^2(v_n)} + V(x)f^2(v_n)\right]\mathrm{d}x - \frac{V_0}{4k}\int_{\mathbb{R}^2\setminus\Lambda} f^2(v_n)\mathrm{d}x \\
&\quad + \frac{1}{4}\int_\Lambda [g(f(v_n))f(v_n) - g(f(v_\varepsilon))f(v_\varepsilon)]\,\mathrm{d}x \\
&\quad + \int_\Lambda \left[\frac{1}{4}g(f(v_\varepsilon))f(v_\varepsilon) - G(f(v_\varepsilon))\right]\mathrm{d}x \\
&= \frac{1}{4}\int_{\mathbb{R}^2}\left[\frac{\varepsilon^2|\nabla v_n|^2}{1+2f^2(v_n)} + V(x)f^2(v_n)\right]\mathrm{d}x - \frac{V_0}{4k}\int_{\mathbb{R}^2\setminus\Lambda} f^2(v_n)\mathrm{d}x \\
&\quad + \frac{\varepsilon^2}{4}\int_{\mathbb{R}^2}\left[\left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2 - \left(1 + \frac{2f^2(v_\varepsilon)}{1+2f^2(v_\varepsilon)}\right)|\nabla v_\varepsilon|^2\right]\mathrm{d}x \\
&\quad - \frac{V_0}{4k}\int_{\mathbb{R}^2\setminus\Lambda}\left[f^2(v_n) - f^2(v_\varepsilon)\right]\mathrm{d}x + \int_{\mathbb{R}^2} V(x)\left[f^2(v_n) - f^2(v_\varepsilon)\right]\mathrm{d}x \\
&\quad + \int_\Lambda \left[\frac{1}{4}g(f(v_\varepsilon))f(v_\varepsilon) - G(f(v_\varepsilon))\right]\mathrm{d}x + o(1) \\
&\geq \frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|\nabla v_n|^2\mathrm{d}x - \frac{\varepsilon^2}{4}\int_{\mathbb{R}^2}\left[\left(1 + \frac{2f^2(v_\varepsilon)}{1+2f^2(v_\varepsilon)}\right)|\nabla v_\varepsilon|^2\right]\mathrm{d}x + o(1) \\
&\geq \varepsilon^2\frac{\pi}{\alpha_0} + \frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|\nabla v_\varepsilon|^2\mathrm{d}x - \frac{\varepsilon^2}{4}\int_{\mathbb{R}^2}\left[\left(1 + \frac{2f^2(v_\varepsilon)}{1+2f^2(v_\varepsilon)}\right)|\nabla v_\varepsilon|^2\right]\mathrm{d}x + o(1) \\
&= \varepsilon^2\frac{\pi}{\alpha_0} + \frac{\varepsilon^2}{4}\|\nabla f(v_\varepsilon)\|_2^2 + o(1).
\end{aligned}$$

This contradicts to the assumption $c_\varepsilon < \varepsilon^2\pi/\alpha_0$. Hence, $\lim_{n\to\infty} \|\nabla v_n\|_2^2 < \frac{2\pi}{\alpha_0} + \|\nabla v_\varepsilon\|_2^2$.

**Step 3:** We prove that $v_n \to v_\varepsilon$ in $E$, up to a subsequence.

For this, we first verify that $\|v_n - v_\varepsilon\|_2 \to 0$. From Lemma 3.11 and the Gagliardo-Nirenberg inequality, we can deduce that for $\epsilon > 0$ small enough, there exists $R'_\epsilon > 0$ large enough such that

$$\int_{\mathbb{R}^2 \setminus B_{R'_\epsilon}} \left[f^2(v_n) + f^4(v_n)\right] \mathrm{d}x \le \epsilon. \tag{3.51}$$

Jointly with (f9) of Lemma 2.7, we have

$$\begin{aligned}\int_{\mathbb{R}^2 \setminus B_{R'_\epsilon}} v_n^2 \mathrm{d}x &\le \frac{1}{\theta_0^2} \int_{\{x\in\mathbb{R}^2\setminus B_{R'_\epsilon}:|v_n|\le 1\}} f^2(v_n)\mathrm{d}x + \frac{1}{\theta_0^4} \int_{\{x\in\mathbb{R}^2\setminus B_{R'_\epsilon}:|v_n|>1\}} f^4(v_n)\mathrm{d}x \\ &\le \frac{\theta_0^2+1}{\theta_0^4}\epsilon. \end{aligned} \tag{3.52}$$

Combining (3.52) and the fact that $v_n \to v_\varepsilon$ in $L^2(B_{R'_\epsilon})$, we get

$$\int_{\mathbb{R}^2} |v_n - v_\varepsilon|^2 \mathrm{d}x = \int_{B_{R'_\epsilon}} |v_n - v_\varepsilon|^2 \mathrm{d}x + \int_{\mathbb{R}^2 \setminus B_{R'_\epsilon}} |v_n - v_\varepsilon|^2 \mathrm{d}x \le o(1) + C_{11}\epsilon,$$

which, together with the arbitrariness of $\epsilon > 0$, yields $\|v_n - v_\varepsilon\|_2 \to 0$. From this, the Sobolev embedding theorem and Lemma 2.7, we can derive

$$\|v_n - v_\varepsilon\|_s \to 0 \quad \text{and} \quad \|f(v_n) - f(v_\varepsilon)\|_s \to 0, \qquad \forall\ s \ge 2. \tag{3.53}$$

By Step 2, we know that there exists $\hat{\epsilon} > 0$ such that, up to a subsequence,

$$\|\nabla(v_n - v_\varepsilon)\|_2^2 = \frac{2\pi(1-3\hat{\epsilon})}{\alpha_0} \text{ for large } n \in \mathbb{N}. \tag{3.54}$$

Let us choose $\hat{q} \in (1,2)$ such that

$$\frac{(1+\hat{\epsilon})^2(1-2\hat{\epsilon})\hat{q}^2}{1-\hat{\epsilon}} < 1. \tag{3.55}$$

Noting that $\Lambda$ is a bounded domain, it follows from (3.54), (3.55), the Young's inequality and the Trudinger-Moser inequality in bounded domains that

$$\begin{aligned}&\int_\Lambda |g(f(v_n))|^{\hat{q}} \mathrm{d}x \le C_{12} \int_\Lambda e^{\alpha_0(1+\hat{\epsilon})\hat{q} f^4(v_n)} \mathrm{d}x \\ &\le C_{12} \int_\Lambda e^{2\alpha_0(1+\hat{\epsilon})^2\hat{\epsilon}^{-1}\hat{q}v_\varepsilon^2} e^{2\alpha_0(1+\hat{\epsilon})^2\hat{q}(v_n-v_\varepsilon)^2} \mathrm{d}x \\ &\le \frac{(\hat{q}-1)C_{12}}{\hat{q}} \int_\Lambda e^{2\alpha_0(1+\hat{\epsilon})^2\hat{\epsilon}^{-1}\hat{q}^2(\hat{q}-1)^{-1}v_\varepsilon^2} \mathrm{d}x + \frac{C_{12}}{\hat{q}} \int_\Lambda e^{2\alpha_0(1+\hat{\epsilon})^2\hat{q}^2(v_n-v_\varepsilon)^2} \mathrm{d}x \\ &\le C_{13} + \frac{C_{12}}{\hat{q}} \int_\Lambda e^{2\alpha_0(1+\hat{\epsilon})^2\hat{q}^2\left(\|\nabla(v_n-v_\varepsilon)\|_2^2+2\pi\hat{\epsilon}/\alpha_0\right)(v_n-v_\varepsilon)^2/\left(\|\nabla(v_n-v_\varepsilon)\|_2^2+2\pi\hat{\epsilon}/\alpha_0\right)} \mathrm{d}x \le C_{14}. \end{aligned} \tag{3.56}$$

Let $\hat{q}' = \hat{q}/(\hat{q}-1)$. Then by Lemma 2.7, (3.56) and the Hölder inequality, we get

$$\left|\int_\Lambda g(f(v_n))[f(v_n) - f(v_\varepsilon)]\mathrm{d}x\right| \le \left[\int_\Lambda |g(f(v_n))|^{\hat{q}} \mathrm{d}x\right]^{1/\hat{q}} \left[\int_\Lambda |f(v_n) - f(v_\varepsilon)|^{\hat{q}'} \mathrm{d}x\right]^{1/\hat{q}'} = o(1). \tag{3.57}$$

Noting that $f(v_n) \rightharpoonup f(v_\varepsilon)$ in $H^1(\mathbb{R}^2)$, by (3.37) and (i) of Lemma 3.12, we have

$$\int_\Lambda [g(f(v_n)) - g(f(v_\varepsilon))] f(v_\varepsilon) \mathrm{d}x = o(1). \tag{3.58}$$

Combining (3.57) with (3.58), we get

$$\int_\Lambda [g(f(v_n))f(v_n) - g(f(v_\varepsilon))f(v_\varepsilon)]\,\mathrm{d}x$$

$$= \int_\Lambda g(f(v_n))[f(v_n) - f(v_\varepsilon)]\mathrm{d}x + \int_\Lambda [g(f(v_n)) - g(f(v_\varepsilon))]f(v_\varepsilon)\mathrm{d}x = o(1). \tag{3.59}$$

Therefore, it follows from (3.48), (3.49), (3.53) and (3.59) that

$$\begin{aligned} o(1) &= \int_\Lambda [g(f(v_n))f(v_n) - g(f(v_\varepsilon))f(v_\varepsilon)]\,\mathrm{d}x + \frac{V_0}{k}\int_{\mathbb{R}^2\setminus\Lambda}\left[f^2(v_n) - f^2(v_\varepsilon)\right]\mathrm{d}x \\ &= \varepsilon^2\int_{\mathbb{R}^2}\left[\left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2 - \left(1 + \frac{2f^2(v_\varepsilon)}{1+2f^2(v_\varepsilon)}\right)|\nabla v_\varepsilon|^2\right]\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^2} V(x)\left[f^2(v_n) - f^2(v_\varepsilon)\right]\mathrm{d}x + o(1), \end{aligned}$$

which, together with Fatou's lemma, implies that $\|\nabla(v_n - v_\varepsilon)\|_2 \to 0$. This, jointly with (3.53), shows that $v_n \to v_\varepsilon$ in $E$ up to a subsequence, and so $I_\varepsilon(v_\varepsilon) = c_\varepsilon$ and $I'_\varepsilon(v_\varepsilon) = 0$ provided $c_\varepsilon < \varepsilon^2\pi/\alpha_0$. The proof is completed. □

Now, to end the proof of Theorem 3.2, using Lemmas 3.8 and 3.13, it suffices to establish the desired estimate of the mountain-pass level. Hereafter, we always assume that (V1), (V2) and (G1)-(G5) hold, and let $V_0 = \min_{x\in\Lambda} V(x) < C^*_{\mathrm{TM}}(G)$, where $C^*_{\mathrm{TM}}(G)$ is given by (2.58).

## 3.5 Estimate of the mountain-pass level

In this subsection, we give the estimate of the mountain-pass level $c_\varepsilon$, defined by (3.19), and finish the proof of Theorem 3.2.

**Lemma 3.14.** *There exists $\varepsilon_0 > 0$ such that $c_\varepsilon < \varepsilon^2\pi/\alpha_0$ for all $\varepsilon \in (0, \varepsilon_0]$.*

For the proof of Lemma 3.14, we need to work with stretched variables, because of the presence of $\varepsilon^2$ before the component-$\|\nabla u\|_2^2$ in $I_\varepsilon(u)$. Precisely, we change the variables as $z = \varepsilon x$, and consider the following energy functional:

$$\mathcal{I}_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla v|^2 + V(\varepsilon x)f^2(v)\right]\mathrm{d}x - \int_{\mathbb{R}^2} G(\varepsilon x, f(v))\mathrm{d}x \tag{3.60}$$

associated to the equation:

$$-\Delta v = f'(v)\left[g(\varepsilon x, f(v)) - V(\varepsilon x)f(v)\right], \quad x \in \mathbb{R}^2, \tag*{$(\widetilde{\mathcal{S}})_\varepsilon$}$$

and defined on the Banach space

$$E_\varepsilon := \left\{v \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(\varepsilon x)v^2\mathrm{d}x < \infty\right\}. \tag{3.61}$$

It is easy to see that $\mathcal{I}_\varepsilon \in \mathcal{C}^1(E_\varepsilon, \mathbb{R})$, and

$$\begin{aligned} \langle \mathcal{I}'_\varepsilon(v), \phi\rangle &= \int_{\mathbb{R}^2}[\nabla v\cdot\nabla\phi + V(\varepsilon x)f(v)f'(v)\phi]\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^2} g(\varepsilon x, f(v))f'(v)\phi\mathrm{d}x, \quad \forall\ \phi \in E_\varepsilon. \end{aligned} \tag{3.62}$$

For every $\varepsilon \in (0, 1]$, we consider the Nehari manifold

$$\widetilde{\mathcal{N}}_\varepsilon = \{v \in E_\varepsilon \setminus \{0\} : \langle \mathcal{I}'_\varepsilon(v), v\rangle = 0\}. \tag{3.63}$$

Arguing as in Lemmas 3.5, 3.6 and Corollary 3.7, for any $\varepsilon \in (0, 1]$, we have

$$\widetilde{m}_\varepsilon := \inf_{v\in\widetilde{\mathcal{N}}_\varepsilon}\mathcal{I}_\varepsilon(v) = \inf_{v\in E_\varepsilon\setminus\{0\}}\sup_{t>0}\mathcal{I}_\varepsilon(tv). \tag{3.64}$$

The following lemma is crucial in the proof of Lemma 3.14.

**Lemma 3.15.** $\limsup_{\varepsilon\to 0} \widetilde{m}_\varepsilon \le m_0$.

*Proof.* Let $v_0$ be a positive ground state solution of $(\mathcal{S})_0$ involved in Theorem 3.9. Without loss of generality, we may assume that $v_0$ maximizes at zero. Consider the function $w_\varepsilon = \phi(\varepsilon x)v_0$, where $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2, [0,1])$ is defined by

$$\phi(x) = \begin{cases} 1, & x \in B_\rho, \\ 0, & x \in \mathbb{R}^2 \setminus B_{2\rho} \end{cases} \tag{3.65}$$

with $\rho > 0$ such that $\overline{B}_{2\rho} \subset \Lambda$. It is easy to see that $w_\varepsilon \to v_0$ in $H^1(\mathbb{R}^2)$ as $\varepsilon \to 0$. Furthermore,

$$\textit{supp } w_\varepsilon \subset \Lambda_\varepsilon := \{x \in \mathbb{R}^2 : \varepsilon x \in \Lambda\} \tag{3.66}$$

and

$$\int_{\mathbb{R}^2} V(\varepsilon x) w_\varepsilon^2 \mathrm{d}x \le \int_{\Lambda_\varepsilon} V(\varepsilon x) w_\varepsilon^2 \mathrm{d}x \le \sup_{x\in\Lambda} V(x) \|w_\varepsilon\|_2^2 \le \sup_{x\in\Lambda} V(x) \|v_0\|_2^2. \tag{3.67}$$

Then (3.66) and (3.67) imply that $w_\varepsilon \in E_\varepsilon$,

$$\int_{\mathbb{R}^2} G(\varepsilon x, f(w_\varepsilon)) \mathrm{d}x = \int_{\mathbb{R}^2} G(f(w_\varepsilon)) \mathrm{d}x \tag{3.68}$$

and

$$\int_{\mathbb{R}^2} g(\varepsilon x, f(t_\varepsilon w_\varepsilon)) f'(t_\varepsilon w_\varepsilon) t_\varepsilon w_\varepsilon \mathrm{d}x = \int_{\mathbb{R}^2} g(f(t_\varepsilon w_\varepsilon)) f'(t_\varepsilon w_\varepsilon) t_\varepsilon w_\varepsilon \mathrm{d}x. \tag{3.69}$$

Similarly as in Lemma 3.5, we derive that for each $\varepsilon \in (0,1]$, there exists $t_\varepsilon > 0$ such that $t_\varepsilon w_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$, i.e.,

$$\langle \mathcal{I}'_\varepsilon(t_\varepsilon w_\varepsilon), t_\varepsilon w_\varepsilon \rangle = 0, \tag{3.70}$$

and so $\mathcal{I}_\varepsilon(t_\varepsilon w_\varepsilon) \ge \widetilde{m}_\varepsilon$ by (3.64). By (G4), (3.17) and (3.70), arguing as in the proof of [18, Lemma 19], we can deduce that $\{t_\varepsilon\}$ is bounded. We claim that, up to a subsequence,

$$\int_{\mathbb{R}^2} \left[V(\varepsilon x) f^2(w_\varepsilon) - V_0 f^2(v_0)\right] \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0 \tag{3.71}$$

and

$$\int_{\mathbb{R}^2} \left[V(\varepsilon x) f'(w_\varepsilon) w_\varepsilon - V_0 f'(v_0) v_0\right] \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0. \tag{3.72}$$

Next, we just give the proof of (3.71), because the other is similar. Since

$$\sup_{x\in\Lambda_\varepsilon} V(\varepsilon x) \le \sup_{x\in\Lambda} V(x), \quad \forall\, \varepsilon \in (0,1]$$

and $w_\varepsilon \to v_0$ in $H^1(\mathbb{R}^2)$ as $\varepsilon \to 0$, we have

$$V(\varepsilon x) f^2(w_\varepsilon) - V_0 f^2(v_0) \to 0 \;\text{ for a.e. } x \in \mathbb{R}^2 \tag{3.73}$$

and there exists $h \in L^1(\mathbb{R}^2)$ such that

$$0 \le V(\varepsilon x) f^2(w_\varepsilon) \le \sup_{x\in\Lambda} V(x) w_\varepsilon^2 \le \sup_{x\in\Lambda} V(x) h(x) \;\text{ for a.e. } x \in \mathbb{R}^2. \tag{3.74}$$

Hence, (3.71) follows from (3.73), (3.74) and the Lebesgue dominated convergence theorem. As in the proof of Lemma 3.6, we have

$$\mathcal{I}_\varepsilon(v) \ge \mathcal{I}_\varepsilon(tv) + \frac{1-t^2}{2} \langle \mathcal{I}'_\varepsilon(v), v \rangle, \quad \forall\, v \in E_\varepsilon,\ t \ge 0. \tag{3.75}$$

Since $\{t_\varepsilon\}$ is bounded, it follows from (2.71), (2.73), (3.25), (3.60), (3.62), (3.66), (3.67), (3.71), (3.72) and (3.75) that, up to a subsequence,

$$\begin{aligned} m_0 + o_\varepsilon(1) = I_0(v_0) + o_\varepsilon(1) &= I_0(v_0) + \frac{1}{2}\int_{\mathbb{R}^2} \left[V(\varepsilon x) f^2(w_\varepsilon) - V_0 f^2(v_0)\right] \mathrm{d}x \\ &= \mathcal{I}_\varepsilon(w_\varepsilon) \geq \mathcal{I}_\varepsilon(t_\varepsilon w_\varepsilon) + \frac{1-t_\varepsilon^2}{2} \langle \mathcal{I}_\varepsilon'(w_\varepsilon), w_\varepsilon\rangle \\ &\geq \widetilde{m}_\varepsilon + \frac{1-t_\varepsilon^2}{2}\langle I_0'(v_0), v_0\rangle + \frac{1-t_\varepsilon^2}{2}\int_{\mathbb{R}^2} \left[V(\varepsilon x) f'(w_\varepsilon) w_\varepsilon - V_0 f'(v_0) v_0\right] \mathrm{d}x \\ &= \widetilde{m}_\varepsilon + o_\varepsilon(1), \end{aligned}$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. This completes the proof. □

*Proof of Lemma* 3.14. It is easy to see that $m_\varepsilon = \varepsilon^2 \widetilde{m}_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$. Then Lemma 3.14 follows directly from Lemmas 3.8 and 3.15 in the same way as that of [18]. □

*Proof of Theorem* 3.2. Theorem 3.2 follows directly from Lemmas 3.8, 3.13 and 3.14. □

By performing the scaling $x \mapsto \varepsilon x$, Theorem 3.2 also yields a one parameter family of critical points $\{\widetilde{v}_\varepsilon\}$ of $\mathcal{I}_\varepsilon$ for any $\varepsilon \in (0, \varepsilon_0]$, namely

$$\widetilde{v}_\varepsilon(x) := v_\varepsilon(\varepsilon x) \text{ for } x \in \mathbb{R}^2, \;\; \mathcal{I}_\varepsilon'(\widetilde{v}_\varepsilon) = 0 \;\text{ and }\; \mathcal{I}_\varepsilon(\widetilde{v}_\varepsilon) = \widetilde{m}_\varepsilon, \quad \forall\, \varepsilon \in (0, \varepsilon_0]. \tag{3.76}$$

This gives

$$-\Delta \widetilde{v}_\varepsilon = f'(\widetilde{v}_\varepsilon)\left[g(\varepsilon x, f(\widetilde{v}_\varepsilon)) - V(\varepsilon x) f(\widetilde{v}_\varepsilon)\right], \quad \forall\, x \in \mathbb{R}^2, \;\; \varepsilon \in (0, \varepsilon_0]. \tag{3.77}$$

In the next section, we will give the $L^\infty$-estimate and the behavior of $\widetilde{v}_\varepsilon$ as $\varepsilon \to 0$, to relate to critical points of $J_\varepsilon$, defined by (3.3). In what follows, we denote $o(1) \to 0$ as $\varepsilon \to 0$.

# 4 $L^\infty$-estimate and behavior of $\widetilde{v}_\varepsilon$ as $\varepsilon \to 0$

**Lemma 4.1.** *There is a constant $K > 0$, independent of $\varepsilon$, such that $\|\widetilde{v}_\varepsilon\|_\infty \leq K$ for all $\varepsilon \in (0, \varepsilon_0]$.*

*Proof.* With Lemma 3.15, using similar arguments as that of Lemma 3.10, we can prove that there exists a constant $K_1 > 0$, independent of $\varepsilon$, such that $\|\widetilde{v}_\varepsilon\| \leq K_1$ for all $\varepsilon \in (0, \varepsilon_0]$. As in the proof of [18, Proposition 22], we can conclude this lemma. □

**Lemma 4.2.** *There exist $\{y_\varepsilon\} \subset \mathbb{R}^2$ and $\widetilde{R}, \widetilde{\beta} > 0$, independent of $\varepsilon$, such that*

$$\int_{B_{\widetilde{R}}(y_\varepsilon)} f^2(\widetilde{v}_\varepsilon) \mathrm{d}x \geq \widetilde{\beta}, \quad \forall\, \varepsilon \in (0, \varepsilon_0].$$

*Proof.* Suppose by contradiction that the lemma does not hold. Using a result by Lions, we have $f(\widetilde{v}_\varepsilon) \to 0$ in $L^s(\mathbb{R}^2)$. By (3.8) and Lemma 4.1, we can derive that

$$\int_{\mathbb{R}^2} G(\varepsilon x, f(\widetilde{v}_\varepsilon)) \mathrm{d}x = \int_{\mathbb{R}^2} g(\varepsilon x, f(\widetilde{v}_\varepsilon)) f(\widetilde{v}_\varepsilon) f'(\widetilde{v}_\varepsilon) \widetilde{v}_\varepsilon \mathrm{d}x = o(1). \tag{4.1}$$

By (3.62), (4.1) and (f6) of Lemma 2.7, we derive that

$$\begin{aligned} 0 = \langle \mathcal{I}_\varepsilon'(\widetilde{v}_\varepsilon), \widetilde{v}_\varepsilon\rangle &= \|\nabla \widetilde{v}_\varepsilon\|_2^2 + \int_{\mathbb{R}^2} V(\varepsilon x) f(\widetilde{v}_\varepsilon) f'(\widetilde{v}_\varepsilon) \widetilde{v}_\varepsilon \mathrm{d}x + o(1) \\ &\geq \|\nabla \widetilde{v}_\varepsilon\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^2} V(\varepsilon x) f^2(\widetilde{v}_\varepsilon) \mathrm{d}x + o(1), \end{aligned} \tag{4.2}$$

which, jointly with (3.60), yields that $\widetilde{m}_\varepsilon = o(1)$. On the other hand, by a standard argument, we can prove that there exists $\sigma > 0$, independent of $\varepsilon$, such that $\widetilde{m}_\varepsilon \geq \sigma > 0$ for all $\varepsilon \in (0, \varepsilon_0]$, since $\inf_{x \in \mathbb{R}^2} V(x) > 0$. This a contradiction, and thus the lemma is proved. □

As in [18, Lemma 25, Remark 26], we have the following lemma and remark:

**Lemma 4.3.** *The family $\{\varepsilon y_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ has the following property dist$(\varepsilon y_\varepsilon, \Lambda) \le \varepsilon\widetilde{R}$.*

**Remark 4.4.** *The family $\{\varepsilon y_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ can be taken in such a way that $\varepsilon y_\varepsilon \in \Lambda$ for all $0<\varepsilon\le\varepsilon_0$. Indeed, since dist$(\varepsilon y_\varepsilon,\Lambda) < 2\varepsilon\widetilde{R}$ for all $0<\varepsilon\le\varepsilon_0$, there exists $x_\varepsilon\in\Lambda$ satisfying $|y_\varepsilon - \varepsilon^{-1}x_\varepsilon| < 2\widetilde{R}$. Thus,*

$$0<\widetilde{\beta}\le\int_{B_{\widetilde{R}}(y_\varepsilon)} f^2(\widetilde{v}_\varepsilon)\mathrm{d}x \le \int_{B_{3\widetilde{R}}(\varepsilon^{-1}x_\varepsilon)} f^2(\widetilde{v}_\varepsilon)\mathrm{d}x.$$

*Replacing $\widetilde{R}$ by $3\widetilde{R}$ in Lemma 4.3, we can replace $y_\varepsilon$ by $\varepsilon^{-1}x_\varepsilon$.*

For all $\varepsilon\in(0,\varepsilon_0]$, we let

$$w_\varepsilon(x) = \widetilde{v}_\varepsilon(x+\varepsilon y_\varepsilon), \quad \forall\, x\in\mathbb{R}^2. \tag{4.3}$$

Then Theorem 3.2 and (3.76) give that

$$-\Delta w_\varepsilon = f'(v)\left[g(\varepsilon x+\varepsilon y_\varepsilon, f(w_\varepsilon)) - V(\varepsilon x+\varepsilon y_\varepsilon)f(w_\varepsilon)\right], \quad x\in\mathbb{R}^2. \tag{4.4}$$

**Lemma 4.5.** $\lim_{\varepsilon\to0} V(\varepsilon y_\varepsilon) = V_0 = \min_{x\in\Lambda}V(x)$. *Moreover, $w_\varepsilon\to w$ in $H^1(\mathbb{R}^2)$ and $w_\varepsilon\to w$ in $\mathcal{C}^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\alpha\in(0,1)$, where $w\in H^1(\mathbb{R}^2)$ is a positive ground state solution of $(\mathcal{S})_0$.*

*Proof.* Let $\{\varepsilon_n\}$ be a sequence such that $\varepsilon_n\in(0,\varepsilon_0]$ verifying $\varepsilon_n y_{\varepsilon_n}\in\Lambda$ by Remark 4.4. We may assume that, up to a subsequence,

$$\varepsilon_n y_{\varepsilon_n}\to x_0\in\overline{\Lambda}, \quad V(x_0)\ge V_0. \tag{4.5}$$

To simplify the notation, set $\widetilde{v}_n=\widetilde{v}_{\varepsilon_n}$ and $w_n = w_{\varepsilon_n}$. Since $\{\|w_n\|\}$ is bounded due to $\|w_n\|=\|\widetilde{v}_n\|$, we may assume that there exists $w\in H^1(\mathbb{R}^2)$ such that

$$w_n\rightharpoonup w \text{ in } H^1(\mathbb{R}^2), \quad w_n\to w \text{ in } L^s_{\mathrm{loc}}(\mathbb{R}^2) \text{ for all } s\ge1 \text{ and } w_n\to w \text{ a.e. in } x\in\mathbb{R}^2. \tag{4.6}$$

By Lemma 4.2, we have $w\ne0$. Next, we divide the proof into the following steps.

**Step 1:** We prove that $w\in H^1(\mathbb{R}^2)\setminus\{0\}$ is a ground state solution of $(\mathcal{S})_0$.

We define

$$\chi(x) = \lim_{n\to\infty}\chi_\Lambda(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n}) \text{ a.e. in } x\in\mathbb{R}^2 \tag{4.7}$$

and

$$\widetilde{g}(x,t) = \chi(x)g(t) + (1-\chi(x))\,\bar{g}(t) \quad \forall\,(x,t)\in\mathbb{R}^2\times\mathbb{R}. \tag{4.8}$$

Then we have

$$g(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n}, f(w_n))f(w_n)\to\widetilde{g}(x,f(w))f(w) \text{ a.e. in } x\in\mathbb{R}^2 \tag{4.9}$$

and

$$G(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n}, f(w_n))\to\widetilde{G}(x,f(w))f(w) \text{ a.e. in } x\in\mathbb{R}^2, \tag{4.10}$$

where $\widetilde{G}(x,t)=\int_0^t\widetilde{g}(x,s)\mathrm{d}s$. By (4.4), we have

$$\begin{aligned}&\int_{\mathbb{R}^2}\left[\nabla w_n\cdot\nabla\phi + V(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n})f(w_n)f'(w_n)\phi\right]\mathrm{d}x\\ &=\int_{\mathbb{R}^2} g(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n}, f(w_n))f'(w_n)\phi\mathrm{d}x, \quad \forall\,\phi\in\mathcal{C}_0^\infty(\mathbb{R}^2).\end{aligned} \tag{4.11}$$

Noting that Lemma 4.1 and (4.3) give

$$\|w_n\|_\infty\le C_\infty \text{ with some constant } C_\infty>0 \text{ independent of } n, \tag{4.12}$$

it follows from the Lebesgue dominated convergence theorem that

$$\int_{\mathbb{R}^2} g(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n}, f(w_n))f'(w_n)\phi \mathrm{d}x = \int_{\mathbb{R}^2} \widetilde{g}(x, f(w))f'(w)\phi \mathrm{d}x = o(1), \quad \forall\ \phi \in \mathcal{C}_0^\infty(\mathbb{R}^2). \tag{4.13}$$

Taking the limit in (4.11), using (4.5) and (4.13), we see that $w$ satisfies

$$\int_{\mathbb{R}^2} [\nabla w\cdot\nabla\phi + V(x_0)f(w)f'(w)\phi]\,\mathrm{d}x = \int_{\mathbb{R}^2} \widetilde{g}(x, f(w))f'(w)\phi \mathrm{d}x, \quad \forall\ \phi \in \mathcal{C}_0^\infty(\mathbb{R}^2). \tag{4.14}$$

Therefore, $w$ is a critical point of the functional given by

$$\widetilde{\mathcal{I}}(v) = \frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla v|^2 + V(x_0)f^2(v)\right]\mathrm{d}x - \int_{\mathbb{R}^2}\widetilde{G}(x, f(v))\mathrm{d}x, \qquad \forall\ v\in H^1(\mathbb{R}^2). \tag{4.15}$$

To end this step, it remains to show that

$$x_0\in\Lambda \ \text{ and } \ V(x_0) = V_0. \tag{4.16}$$

Indeed, if $x_0\in\Lambda$ can be proved, we then get $\varepsilon_n y_{\varepsilon_n}\in\Lambda$ for $n\in\mathbb{N}$ sufficiently large. Hence, $\chi(x)=1$ for all $x\in\mathbb{R}^2$ and $w$ is a critical point of the following functional

$$I_{x_0}(v) = \frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla v|^2 + V(x_0)f^2(v)\right]\mathrm{d}x - \int_{\mathbb{R}^2} G(f(v))\mathrm{d}x, \qquad \forall\ v\in H^1(\mathbb{R}^2), \tag{4.17}$$

and so the conclusion follows if further $V(x_0)=V_0$. We next prove that (4.16) holds. Denoting by $c_{x_0}$ the mountain-pass level associated to the functional $I_{x_0}$ and by $\widetilde{c}$ the mountain-pass level associated to the functional $\widetilde{\mathcal{I}}$, we claim that $c_{x_0}\le\widetilde{c}$. In fact, since $\widetilde{G}(x,t)\le G(t)$ for all $x\in\mathbb{R}^2$ and $t\in\mathbb{R}$, we obtain $I_{x_0}(v)\le\widetilde{\mathcal{I}}(v)$ for all $v\in H^1(\mathbb{R}^2)$, and this implies that $c_{x_0}\le\widetilde{c}$. Arguing as in Corollary 3.7 and Lemma 3.8, we can get $\widetilde{\mathcal{I}}(w)\ge\widetilde{c}$ since $\widetilde{\mathcal{I}}'(w)=0$. Moreover, using the fact $V(x_0)\ge V_0$, it is easy to check that $c_0\le c_{x_0}$. Thus, we have

$$m_0 = c_0 \le c_{x_0} \le \widetilde{c} \le \widetilde{\mathcal{I}}(w), \tag{4.18}$$

where $c_0$ and $m_0$ are given by (2.75) and (3.25). Let us define the set

$$A_n = \left\{x\in\mathbb{R}^2 : \varepsilon_n x+\varepsilon_n y_{\varepsilon_n}\in\Lambda\right\}. \tag{4.19}$$

If $x\in A_n$, then (3.29) and (4.8) imply that

$$\begin{aligned}&V(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n})f^2(w_n) + g(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n}, f(w_n))f(w_n) - 4G(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n}, f(w_n))\\ =&V(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n})f^2(w_n) + g(f(w_n))f(w_n) - 4G(f(w_n)) \ge 0.\end{aligned} \tag{4.20}$$

If $x\notin A_n$, then (3.4), (3.5) and (4.8) imply that

$$\begin{aligned}&V(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n})f^2(w_n) + g(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n}, f(w_n))f(w_n) - 4G(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n}, f(w_n))\\ \ge&V(\varepsilon_n x+\varepsilon_n y_{\varepsilon_n})f^2(w_n) - \frac{V_0}{k}f^2(w_n) \ge 0.\end{aligned} \tag{4.21}$$

Noting that

$$\varepsilon_n x+\varepsilon_n y_{\varepsilon_n}\to x_0 \ \text{ a.e. in } x\in\mathbb{R}^2,$$

then it follows from (3.76), (4.6), (4.9), (4.10), (4.14), (4.15), (4.18), (4.20), (4.21), Fatou's lemma and semicontinuity of the norm that

$$m_0 \le \widetilde{\mathcal{I}}(w) = \widetilde{\mathcal{I}}(w) - \frac{1}{4}\left\langle \widetilde{\mathcal{I}}'(w), f(w)/f'(w)\right\rangle$$

$$
\begin{aligned}
&= \frac{1}{4}\int_{\mathbb{R}^2}\left[|\nabla f(w)|^2 + V(x_0)f^2(w)\right]\mathrm{d}x \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^2}\left[\widetilde{g}(x,f(w))f(w) - 4\widetilde{G}(x,f(w))\right]\mathrm{d}x \\
&\le \frac{1}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^2}|\nabla f(w_n)|\mathrm{d}x + \frac{1}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^2}\left[V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})f^2(w_n)\right. \\
&\quad \left. + g(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n}, f(w_n))f(w_n) - 4G(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n}, f(w_n))\right]\mathrm{d}x \\
&= \frac{1}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^2}|\nabla f(\widetilde{v}_n)|\mathrm{d}x + \frac{1}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^2}\left[V(\varepsilon_n x)f^2(\widetilde{v}_n)\right. \\
&\quad \left. + g(\varepsilon_n x, f(\widetilde{v}_n))f(\widetilde{v}_n) - 4G(\varepsilon_n x, f(\widetilde{v}_n))\right]\mathrm{d}x \\
&\le \limsup_{n\to\infty}\left[\mathcal{I}_{\varepsilon_n}(\widetilde{v}_n) - \frac{1}{4}\left\langle \mathcal{I}'_{\varepsilon_n}(\widetilde{v}_n), f(\widetilde{v}_n)/f'(\widetilde{v}_n)\right\rangle\right] \\
&= \limsup_{\varepsilon\to 0}\widetilde{m}_\varepsilon \le m_0,
\end{aligned}
\tag{4.22}
$$

which, together with (4.18), implies

$$
f(w_n) \to f(w) \ \text{ in } \ H^1(\mathbb{R}^2) \ \text{ and } \ m_0 = c_0 = c_{x_0} = \widetilde{c} = \widetilde{\mathcal{I}}(w). \tag{4.23}
$$

Using (4.23) and the fact that the mountain pass level $c_0$ on the constant potential $V_0$ is continuous and increasing, we can obtain (4.16) holds. This completes this step.

**Step 2:** We prove that $w_n \to w$ in $H^1(\mathbb{R}^2)$.

By (4.23) and (f9) of Lemma 2.7, we know that there exists $h_1 \in L^1(\mathbb{R}^2)$ such that

$$
|w_n|^2 \le \frac{1}{\theta_0^2}f^2(w_n) + \frac{1}{\theta_0^4}f^4(w_n) \le h_1(x) \ \text{ for a.e. } \ x \in \mathbb{R}^2. \tag{4.24}
$$

Using (4.24) and the Lebesgue dominated convergence theorem, we have $\|w_n - w\|_2 \to 0$, which, jointly with the Sobolev embedding theorem, gives $\|w_n - w\|_s \to 0$ for all $s \ge 2$. From this, (3.8), (4.12), Hölder inequality and the Lebesgue dominated convergence theorem, we have

$$
\int_{\mathbb{R}^2} V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})f(w_n)f'(w_n)(w_n - w)\mathrm{d}x = o(1) \tag{4.25}
$$

and

$$
\int_{\mathbb{R}^2} g(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n}, f(w_n))f'(w_n)(w_n - w)\mathrm{d}x = o(1). \tag{4.26}
$$

Therefore, it follows from (4.4), (4.25) and (4.26) that

$$
\begin{aligned}
o(1) &= \int_{\mathbb{R}^2}\nabla w_n \cdot \nabla(w_n - w)\mathrm{d}x + \int_{\mathbb{R}^2} V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})f(w_n)f'(w_n)(w_n - w)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^2} g(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n}, f(w_n))f'(w_n)(w_n - w)\mathrm{d}x \\
&= \int_{\mathbb{R}^2}\nabla w_n \cdot \nabla(w_n - w)\mathrm{d}x + o(1),
\end{aligned}
$$

which implies that $\|\nabla(w_n - w)\|_2 \to 0$. This shows that $w_n \to w$ in $H^1(\mathbb{R}^2)$.

**Step 3:** We verify that $w_n \to w$ in $\mathcal{C}^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\alpha \in (0,1)$.

The previous two steps imply

$$
-\Delta(w_n - w) = H_n(x) \ \text{ in } \ \mathbb{R}^2, \tag{4.27}
$$

where

$$
H_n(x) = V_0 f(w)f'(w) - V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})f(w_n)f'(w_n)
$$

$$+ g(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n}, f(w_n))f'(w_n) - g(f(w))f'(w).$$

By (4.6) and (4.12), we have $H_n(x) \to 0$ for a.e. $x \in \mathbb{R}^2$. Note that for each compact subset $D$ of $\mathbb{R}^2$ we have $|H_n|, |w| \le C_D$ for some positive constant $C_D$ dependent on $D$ due to (4.12) and the fact that $\{|\varepsilon_n x + \varepsilon_n y_{\varepsilon_n}|\}_{n\ge 1}$ is bounded for all $x \in D$ . Thus, it follows from the Lebesgue dominated convergence theorem that $H_n \to 0$ in $L^s_{\text{loc}}(\mathbb{R}^2)$ for all $s \ge 1$. The rest of the proof is the same as the one in [18]. Indeed, using [24, Theorem 9.11], we can conclude that $w_n \to w$ in $W^{2,s}_{\text{loc}}(\mathbb{R}^2)$ for all $s \ge 1$, and so $w_n \to w$ in $\mathcal{C}^{1,\alpha}_{\text{loc}}(\mathbb{R}^2)$ for some $\alpha \in (0, 1)$. Now, by [24, Theorem 6.2], we have $w_n \to w$ in $\mathcal{C}^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)$ for some $\alpha \in (0, 1)$ and the lemma is proved. □

**Lemma 4.6.** *There is a constant $M_0 > 0$, independent of $x \in \mathbb{R}^2$ and $\varepsilon \in (0, \varepsilon_0]$, such that*

$$0 < w_\varepsilon(x) \le M_0 \int_{B_1(x)} w_\varepsilon(y)\mathrm{d}y, \qquad \forall\, \varepsilon \in (0, \varepsilon_0],\ x \in \mathbb{R}^2. \tag{4.28}$$

*Proof.* Let $\varepsilon \in (0, \varepsilon_0]$. By (G1), (G2), (4.12), Lemma 2.7 and Proposition 3.1, there exists a constant $\varrho_1 > 0$ such that

$$g(\varepsilon x + \varepsilon y_\varepsilon, f(t)) \le \varrho_1 f(t) \le \varrho_1 t, \qquad \forall\, t \in [0, C_\infty]. \tag{4.29}$$

Since $w_\varepsilon > 0$, then it follows from (4.4) and (4.29) that

$$\begin{aligned} -\Delta w_\varepsilon &= f'(w_\varepsilon)\,[g(\varepsilon x + \varepsilon y_\varepsilon, f(w_\varepsilon)) - V(\varepsilon x + \varepsilon y_\varepsilon)f(w_\varepsilon)] \\ &\le \varrho_1 w_\varepsilon, \qquad x \in \mathbb{R}^2, \end{aligned} \tag{4.30}$$

which implies that $w_\varepsilon$ is a sub-solution of the equation $(-\Delta - \varrho_1)w = 0$, and hence (4.28) follows from the sub-solution estimate (see [45, Theorem C. 1.2]). □

**Lemma 4.7.** *There exists $\varepsilon^* \in (0, \varepsilon_0]$ sufficiently small such that the family $\widetilde{v}_\varepsilon(x) \to 0$ as $|x| \to \infty$ uniformly in $\varepsilon \in (0, \varepsilon^*]$.*

*Proof.* To prove this, it suffices to show that there exists $\varepsilon^* \in (0, \varepsilon_0]$ such that $w_\varepsilon(x) \to 0$ as $|x| \to \infty$ uniformly in $\varepsilon \in (0, \varepsilon^*]$, since $\widetilde{v}_\varepsilon(x) = w_\varepsilon(x - \varepsilon y_\varepsilon)$ and $\{\varepsilon y_\varepsilon\}$ is bounded by (4.3) and Lemma 4.3. Suppose by contradiction that there exists $\delta_1 > 0$, $\varepsilon_n \in (0, \varepsilon^*]$ and $\{x_n\} \subset \mathbb{R}^2$ with $\varepsilon_n \to 0$ and $|x_n| \to \infty$ such that $w_{\varepsilon_n}(x_n) \ge \delta_1$. From Lemma 4.5, we have $w_{\varepsilon_n} \to w$ in $H^1(\mathbb{R}^2)$, where $w$ is given by Lemma 4.5. Hence, it follows from (4.28) and the Hölder inequality that

$$\begin{aligned} \delta_1 \le w_{\varepsilon_n}(x_n) &\le M_0 \int_{B_1(x_n)} w_{\varepsilon_n}(y)\mathrm{d}y \\ &\le M_0 \int_{B_1(x_n)} |w_{\varepsilon_n}(y) - w(y)|\mathrm{d}y + M_0 \int_{B_1(x_n)} |w(y)|\mathrm{d}y \\ &\le M_0\sqrt{\pi}\|w_{\varepsilon_n} - w\|_2 + M_0 \int_{B_1(x_n)} |w(y)|\mathrm{d}y = o(1), \end{aligned}$$

which is a contradiction. This completes the proof. □

**Lemma 4.8.** *There exist $\Pi_0, \kappa_0 > 0$, independent of $x \in \mathbb{R}^2$ and $\varepsilon \in (0, \varepsilon^*]$, such that*

$$0 < \widetilde{v}_\varepsilon(x) \le \Pi_0 \exp(-\kappa_0|x|), \qquad \forall\, \varepsilon \in (0, \varepsilon^*],\ x \in \mathbb{R}^2. \tag{4.31}$$

*Proof.* Using Lemma 4.8, (f2) and (f4) of Lemma 2.7 and (g1) of Proposition 3.1, we have

$$\lim_{|x|\to\infty} \frac{f(\widetilde{v}_\varepsilon(x))}{\widetilde{v}_\varepsilon(x)} = 1, \qquad \lim_{|x|\to\infty} \frac{g(\varepsilon x, f(\widetilde{v}_\varepsilon(x)))}{\widetilde{v}_\varepsilon(x)} = 0, \quad \text{uniformly in } \ \varepsilon \in (0, \varepsilon^*]. \tag{4.32}$$

Then there exists a constant $R_1 > 0$, independent of $x \in \mathbb{R}^2$ and $\varepsilon$, such that

$$0 < \frac{3}{4}\widetilde{v}_\varepsilon(x) \le f(\widetilde{v}_\varepsilon(x)) \le 1, \quad g(\varepsilon x, f(\widetilde{v}_\varepsilon(x))) \le \frac{V_0}{k}\widetilde{v}_\varepsilon(x), \quad \forall\, \varepsilon \in (0, \varepsilon^*],\ x \in \mathbb{R}^2 \text{ with } |x| \ge R_1,$$

where $k > 2$ is given by (3.4). This, together with (3.4) and (3.77), implies

$$\Delta\widetilde{v}_\varepsilon = f'(\widetilde{v}_\varepsilon)\left[V(\varepsilon x)f(\widetilde{v}_\varepsilon) - g(\varepsilon x, f(\widetilde{v}_\varepsilon))\right] \geq \frac{V_0}{4k}\widetilde{v}_\varepsilon, \quad \forall\ \varepsilon \in (0, \varepsilon^*],\ x \in \mathbb{R}^2 \text{ with } |x| \geq R_1. \tag{4.33}$$

Set $\widetilde{w}_\varepsilon(x) = \widetilde{v}_\varepsilon(x) - Ke^{-\sqrt{\frac{V_0}{4k}}(|x|-R_1)}$, where $K$ is given in Lemma 4.1. Then

$$\Delta\widetilde{w}_\varepsilon(x) \geq \frac{V_0}{4k}\widetilde{w}_\varepsilon(x), \qquad \forall\ x \in \mathbb{R}^2 \text{ with } |x| \geq R_1.$$

By the maximum principle (see [39]), we conclude that $\widetilde{w}_\varepsilon(x) \leq 0$ for $|x| \geq R_1$, i.e.,

$$|\widetilde{v}_\varepsilon(x)| \leq Ke^{-\sqrt{\frac{V_0}{4k}}(|x|-R_1)}, \qquad \forall\ x \in \mathbb{R}^2 \text{ with } |x| \geq R_1.$$

Therefore, there exist $\Pi_0, \kappa_0 > 0$, independent of $x$ and $\varepsilon$, such that (4.31) holds. □

# 5 Proof of Theorem 1.4

Let $v_\varepsilon(x) = \widetilde{v}_\varepsilon(x/\varepsilon)$ for all $\varepsilon \in (0, \varepsilon^*]$, where $\widetilde{v}_\varepsilon$ and $\varepsilon^*$ are given by (3.76) and Lemma 4.7.

**Lemma 5.1.** *There exists $\varepsilon_0^* \in (0, \varepsilon^*]$ sufficiently small such that $u_\varepsilon = f(v_\varepsilon)$ is a nontrivial solution of $(Q)_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0^*]$.*

*Proof.* In view of Lemma 4.7, there exists $R^* > 0$ such that

$$\widetilde{v}_\varepsilon(x) \leq \beta_0, \qquad \forall\ |x| \geq R^*, \tag{5.1}$$

where $\beta_0$ is given by (3.5). Since $\Lambda_\varepsilon = \{x \in \mathbb{R}^2 : \varepsilon x \in \Lambda\}$ and $\Lambda$ is bounded, we have $|\Lambda_\varepsilon|$ is large enough provided that $\varepsilon$ is small enough. Thus we can choose $\varepsilon_0^* \in (0, \varepsilon^*]$ sufficiently small such that $B_{R^*} \subset \Lambda_{\varepsilon_0^*}$. Jointly with (3.77) and (5.1), we conclude that for all $\varepsilon \in (0, \varepsilon_0^*]$, $\widetilde{v}_\varepsilon$ satisfies:

$$-\Delta\widetilde{v}_\varepsilon = f'(\widetilde{v}_\varepsilon)\left[g(f(\widetilde{v}_\varepsilon)) - V(\varepsilon x)f(\widetilde{v}_\varepsilon)\right], \qquad x \in \mathbb{R}^2.$$

which implies that $v_\varepsilon(x) = \widetilde{v}_\varepsilon(x/\varepsilon)$ satisfies

$$-\varepsilon^2\Delta v_\varepsilon = f'(v_\varepsilon)\left[g(f(v_\varepsilon)) - V(x)f(v_\varepsilon)\right], \qquad x \in \mathbb{R}^2. \tag{5.2}$$

Hence, $u_\varepsilon = f(v_\varepsilon)$ is a positive solution of $(Q)_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0^*]$, and the proof is completed. □

*Proof of Theorem* 1.4. Let $\varepsilon_0^*$ be given in Lemma 5.1. With Lemma 5.1, to end the proof, it remains to verify that $u_\varepsilon$, obtained in Lemma 5.1, satisfies (i)-(iii) of Theorem 1.4. We first prove that (i) holds. From Lemma 4.5, we know that for all $\varepsilon \in (0, \varepsilon_0^*]$, $w_\varepsilon$ possesses a global maximum point $x_\varepsilon \in B_\rho$ for some $\rho > 0$. Considering the translation $\widetilde{w}_\varepsilon = w_\varepsilon(\cdot + x_\varepsilon)$, we may assume that the function $w_\varepsilon$ achieves its global maximum at the origin of $\mathbb{R}^2$ without of loss of generality. Using the fact that $w$ is spherically symmetric, $\partial w/\partial r < 0$ for all $r > 0$ and $w_n \to w$ in $\mathcal{C}^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)$, by [38, Lemma 4.2], we can conclude that $w_\varepsilon$ possesses no critical point other than the origin for all $\varepsilon \in (0, \varepsilon_0^*]$. Notice that the maximum value of $v_\varepsilon(z) = v(\varepsilon x) = \widetilde{v}_\varepsilon(x) = w_\varepsilon(x - y_\varepsilon)$ is achieved at the point $z_\varepsilon = \varepsilon y_\varepsilon \in \Lambda$. As the function $f$ is strictly increasing, the maximum value of $u_\varepsilon(z) = f(v_\varepsilon(z))$ is also achieved in this point. As $\nabla u_\varepsilon = f'(v_\varepsilon)\nabla v_\varepsilon$, $u_\varepsilon$ possesses no critical point other than $z_\varepsilon$, and so the item (i) of Theorem 1.4 is proved. The item (ii) is a consequence of Lemma 4.5. Finally, by (f3) of Lemma 2.7 and Lemma 4.8, we have

$$\begin{aligned} 0 < u_\varepsilon(z) = f(v_\varepsilon(z)) \leq v_\varepsilon(z) &= \widetilde{v}_\varepsilon\left(\frac{z}{\varepsilon}\right) \\ &\leq \Pi_0 \exp\left(-\frac{\kappa_0}{\varepsilon}|z|\right), \qquad \forall\ z \in \mathbb{R}^2,\ \varepsilon \in (0, \varepsilon_0^*], \end{aligned} \tag{5.3}$$

and thus the item (iii) of Theorem 1.4 is proved. □

# Declarations

## Conflict of interest

The authors declare that there is no conflict of interest. We also declare that this manuscript has no associated data.

## Data availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.


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