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Calculus of Variations

Choquard equations with saturable reaction

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Abstract

We investigate normalized solutions of the following Choquard equation perturbed by saturable nonlinearity

$$\begin{cases} -\Delta u + \lambda u = (I_\alpha * |u|^p)\,|u|^{p-2}u + \mu \frac{g(x)+u^2}{1+g(x)+u^2}u & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c > 0, \end{cases}$$

where $2_\alpha := \frac{N+\alpha}{N} \le p \le 2^*_\alpha := \frac{N+\alpha}{N-2}$, $\mu \in \mathbb{R}\backslash\{0\}$, and $g(x)$ is a bounded intensity function on $\mathbb{R}^N$. Under different assumptions on p, μ and $g(x)$, we prove several existence and nonexistence results. We also describe some properties on the associated Lagrange multipliers λ, including the asymptotic behavior as $c \to 0$ and the relationship with the distribution potential $g(x)$.

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Contents

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1 Introduction

In this paper, we consider the Choquard equations with a saturable perturbation

$$i\partial_t \Phi - \Delta\Phi = \left(I_\alpha * |\Phi|^p\right)|\Phi|^{p-2}\Phi + \mu\frac{g(x)+|\Phi|^2}{1+g(x)+|\Phi|^2}\Phi,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{1}$$

where $N \ge 2$, $2_\alpha \le p \le 2_\alpha^*$ ($2_\alpha = \frac{N+\alpha}{N}$, $2_\alpha^* = \frac{N+\alpha}{N-2}$ if $N \ge 3$ and $2_\alpha^* = \infty$ if $N = 2$), the parameter $\mu \in \mathbb{R}\backslash\{0\}$ and I_α is the Riesz potential of order $\alpha \in (0, N)$ defined by

$$I_\alpha = \frac{A(N,\alpha)}{|x|^{N-\alpha}} \quad \text{with} \quad A(N,\alpha) = \frac{\Gamma(\frac{N-\alpha}{2})}{\pi^{N/2}2^\alpha\Gamma(\frac{\alpha}{2})} \text{ for each } x \in \mathbb{R}^N\backslash\{0\}, \tag{2}$$

and $*$ is the convolution product on $\mathbb{R}^N$. The constant $2_\alpha = \frac{N+\alpha}{N}$ is the lower critical exponent and $2_\alpha^* = \frac{N+\alpha}{N-2}$ is the upper critical exponent in the sense of Hardy-Littlewood-Sobolev inequality. $g(x) \in C(\mathbb{R}^N, \mathbb{R})$ is a bounded function, which is usually called the intensity (distribution) function.

In the case $\mu = 0$, Eq. (1) becomes the well-known Choquard–Pekar equation. When $N = 3$ and $\alpha = p = 2$, this equation has several physical origins, such as the description by Pekar of the quantum physics of a polaron at rest [26], and the model by Choquard of an electron trapped in its own hole as a certain approximation to Hartree–Fock theory of one component plasma [15].

An important topic on Eq. (1) is to study their standing wave solutions. A standing wave solution of Eq. (1) is a solution of the form $\Phi(t,x) = e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ and u satisfies the stationary equation

$$-\Delta u + \lambda u = \left(I_\alpha * |u|^p\right)|u|^{p-2}u + \mu\frac{g(x)+|u|^2}{1+g(x)+|u|^2}u \text{ in } \mathbb{R}^N. \tag{3}$$

A possible choice is then to fix $\lambda \in \mathbb{R}$, and to search for solutions to Eq. (3) as critical points of the energy functional

$$\begin{aligned} E(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + \lambda|u|^2)dx - \frac{1}{2p}\int_{\mathbb{R}^N}\left(I_\alpha * |u|^p\right)|u|^p dx \\ - \frac{\mu}{2}\int_{\mathbb{R}^N}\left[|u|^2 - \ln\left(1 + \frac{|u|^2}{1+g(x)}\right)\right]dx. \end{aligned}$$

Alternatively, one can search for solutions to Eq. (3) with the frequency λ unknown. In this case $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier and L^2-norms of solutions are prescribed, which are usually called normalized solutions. This study seems particularly meaningful from the physical point of view, since solutions of Eq. (1) conserve their mass along time.

In this paper we are concerned with this issue. For $c > 0$ given, we are interested in finding solutions to

$$\begin{cases} -\Delta u + \lambda u = (I_\alpha * |u|^p)\,|u|^{p-2}u + \mu \frac{g(x)+|u|^2}{1+g(x)+|u|^2} u \ \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c. \end{cases} \tag{P_c}$$

Solutions of problem (P_c) can be obtained as critical points of the energy functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^p dx - \frac{\mu}{2}\int_{\mathbb{R}^N}\left[|u|^2 - \ln\left(1 + \frac{|u|^2}{1+g(x)}\right)\right] dx \tag{4}$$

on the constraint

$$S(c) := \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |u|^2 dx = c\}. \tag{5}$$

Note that J is a well-defined and C^1 functional on $S(c)$ with Fré chet derivative

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v dx - \int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^{p-2} u v dx - \mu \int_{\mathbb{R}^N} \frac{g(x) + |u|^2}{1 + g(x) + |u|^2} u v dx$$

for any $v \in H^1(\mathbb{R}^N)$.

In recent years, there has been much attention on normalized solutions to the Choquard equation

$$-\Delta u + \lambda u = \left(I_\alpha * |u|^p\right)|u|^{p-2}u \ \text{ in } \mathbb{R}^N. \tag{6}$$

When $N = 3$ and $\alpha = p = 2$, the existence and uniqueness of normalized solutions for Eq. (6) was proved by Lieb [15], and the orbital stability of the normalized ground states set was studied by Lions [22] . Recently, the existence of normalized solutions for Eq. (6) was established in [35], depending on the exponent $2_\alpha < p < 2^*_\alpha$. By considering the minimizer of constrained on the Pohozaev manifold, Luo [24] obtained the existence and instability of normalized ground state for Eq. (6) with $\bar{p} := \frac{N+\alpha+2}{N} < p < 2^*_\alpha$. It is remarkable that $\bar{p}$ is called the L^2-critical exponent for Hartree type nonlinearity, which is the threshold exponent for many dynamical properties such as global existence vs. blow-up, and the stability or instability of ground states. For the generalized Choquard equation, we refer the reader to [2, 14].

Very recently, for Choquard equation with a power perturbation, there are some results on normalized solutions, see for example, [3, 7, 12, 33, 34]. In particular, Li [12] considered the upper critical Choquard equation with a power perturbation

$$-\Delta u + \lambda u = \left(I_\alpha * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha - 2}u + \mu|u|^{q-2}u \text{ in } \mathbb{R}^N, \tag{7}$$

where $\mu > 0$ and $2 < q < 2 + \frac{4}{N}$. He proved the existence and orbital stability of the normalized ground states for Eq. (7). Moreover, the second normalized solution was found as well, which is positive, radial symmetric, exponential decay and orbital instable.

We note that in the existing literature there seems no result concerned on the Choquard equations with a saturable perturbation, i.e. Eq. (3), whether λ is fixed or unknown. Inspired by this fact, in this paper we will exhaustively study the nonexistence and existence of normalized solutions for this type of equations when the perturbation is focusing or defocusing, i.e. problem (P_c) with $\mu > 0$ or $\mu < 0$. We will examine how the presence of a saturable perturbation influences the situation in our context, particularly, when the exponent p is the lower or upper critical exponent, in view of the fact that Eq. (6) with $\lambda = 1$ has no nontrivial smooth H^1 solution [9, 25]. Moreover, some properties on the associated Lagrange multipliers λ, including the asymptotic behavior as $c \to 0$ and the relationship with the function $g(x)$, are described.

We wish to point out that the saturable nonlinearity is used to describe photorefractive media [5, 6]. From the mathematical point of view, it is a kind of asymptotically linear term at infinity. Lin et al. [18] firstly studied normalized solutions for the Schrödinger equation with saturable nonlinearity

$$-\Delta u + \lambda u = \mu \frac{g(x) + |u|^2}{1 + g(x) + |u|^2} u \text{ in } \mathbb{R}^N, \tag{8}$$

where $\mu > 0$. It is true that the functional I corresponding to Eq. (8) is bounded from below on $S(c)$. Thus, one may consider the following minimization problem

$$\sigma(c) := \inf_{u \in S(c)} I(u) \tag{9}$$

to get normalized ground states of Eq. (8). When $g(x) \equiv 0$, $N = 2$ and $\mu > \Gamma$ for some $\Gamma > 0$, the existence of minimizer of problem (9) can be proved by Lin et al. [18] via the energy estimate method. Moreover, Lin et al. [19] got the estimate of λ and the minimum (ground state) energy $\sigma(c)$ by developing a virial theorem. When $g(x)$ becomes nonzero, Lin et al. [20] employed a convexity argument to obtain the existence of minimizer of problem (9) when $\mu > 0$ is sufficiently large.

Let us get back to the problem what we would like to study. Compared with the study of the Choquard equation with or without a power perturbation, there seems to be more challenging for problem (P_c). Firstly, when $p = 2_\alpha$ and $\mu > 0$, we find that the usual method can not be used to rule out vanishing of the minimizing sequence when the concentration–compactness principle is applied, due to the special structure of the nonlocal term. Secondly, a convexity method by Lin et al. [20] can be used to rule out the dichotomy of the minimizing sequence in studying normalized ground states of Eq. (8). However, for problem (P_c), when combined nonlinearities appear, particularly when $\mu < 0$, such an argument is not applicable directly. Thirdly, as we will see, the functional J will no longer be bounded from below on $S(c)$ when $\overline{p} < p < 2_\alpha^*$. When the Pohozaev mainfold approach is used, the fibering map related to the Pohozaev mainfold has an extremely complicated form arising from saturable nonlinearity, which seems to have never been concerned before. In order to overcome these considerable difficulties, new ideas and techniques have been explored. More details will be discussed in the next sections.

Before stating our main results, we agree that when $p = 2_\alpha^*$ is involved, we always assume that $N \geq 3$. For the other cases, we require $N \geq 2$. Next, we give the definition of ground state in the following sense.

Definition 1.1 We say that u is a ground state of problem (P_c) if it is a solution to problem (P_c) having minimal energy among all the solutions:

$$J|'_{S(c)}(u) = 0 \text{ and } J(u) = \inf\{J(v) \mid J|'_{S(c)}(v) = 0 \text{ and } v \in S(c)\}.$$

We assume that the intensity function $g(x)$ satisfies the following conditions:

$(D1)$ The function $g(x) \equiv g_1 > -1$ is a constant function;
$(D2)$ The function $g(x) = g(x_1, x_2, \cdots, x_N)$ is periodic with period 1 with respect to variables x_1 to x_N respectively, and satisfies $-1 < g_1 \leq g(x) \leq g_2$ for $x \in \mathbb{R}^N$, where g_1 and g_2 are constants;

($D3$) The function $g(x)$ satisfies $-1 < g(x) \leq \lim_{|x|\to\infty} g(x) = g_1$ for $x \in \mathbb{R}^N$, where g_1 is a constant;

($D4$) The function $g(x)$ satisfies $g(x) \geq \lim_{|x|\to\infty} g(x) = g_1 > -1$ for $x \in \mathbb{R}^N$, where g_1 is a constant.

Theorem 1.2 *Let $\mu > 0$. Then the following statements are true.*

(i) Assume that $p = 2_\alpha$ and one of conditions ($D1$) − ($D4$) holds. Then there exists $\mu_0 > 0$ such that for every $\mu > \mu_0$, the infimum

$$\sigma(c) < -\frac{1}{22_\alpha \|Q_{2_\alpha}\|_2^{22_\alpha}} c^{2_\alpha} - \frac{\mu g_1 c}{2(1+g_1)}$$

is achieved by $w \in S(c)$, which is a ground state of problem (P_c) with some

$$\bar{\lambda} > \frac{1}{2_\alpha \|Q_{2_\alpha}\|_2^{22_\alpha}} c^{2_\alpha - 1} + \frac{\mu g_1}{1+g_1},$$

where Q_{2_α} is given in (13) below;

(ii) Assume that $2_\alpha < p \leq \overline{p}$ and one of conditions ($D1$), ($D3$) holds. In addition, we assume that if $p = \bar{p}$, then $c < \|Q_{\bar{p}}\|_2^{4(\bar{p}-1)/(N+\alpha-\bar{p}(N-2))}$, where $Q_{\bar{p}}$ is given in (14) below. Then the infimum

$$\sigma(c) < -\frac{\mu g_1 c}{2(1+g_1)}$$

is achieved by $w \in S(c)$, which is a ground state of problem (P_c) with some $\bar{\lambda} > \frac{\mu g_1}{1+g_1}$.

(iii) Assume that $\overline{p} < p < 2_\alpha^*$ and condition ($D1$) with $-1 < g_1 \leq 0$ holds. Then there exists $\bar{c} > 0$ such that for every $c < \bar{c}$, problem (P_c) has a solution $(w, \bar{\lambda}) \in H_r^1(\mathbb{R}^N) \times \mathbb{R}^+$. In particular, we have

$$J(w) > K_1 c^{-\frac{N+\alpha-p(N-2)}{Np-N-\alpha-2}} - \frac{\mu c}{2}$$

and

$$\bar{\lambda} > K_2 c^{-\frac{2p-2}{Np-N-\alpha-2}} - \frac{\mu}{1+g_1}\left(\frac{Np}{Np-N-\alpha} - g_1\right),$$

where $K_1, K_2 > 0$ are two constants;

Theorem 1.3 *Let $\mu < 0$. Then the following statements are true.*

(i) Assume that $p = 2_\alpha$ and condition ($D1$) holds. Then the functional J has no critical point on $S(c)$. In other words, there is no solution for problem (P_c) for all $\lambda \in \mathbb{R}$;

(ii) Assume that $2_\alpha < p \leq \overline{p}$ and one of conditions ($D1$), ($D4$) with $g_1 \geq \frac{2c}{\xi} - 1$ holds, where $\xi > 0$ is given in (49) below. In addition, we assume that $c < \|Q_{\bar{p}}\|_2^{4(\bar{p}-1)/(N+\alpha-\bar{p}(N-2))}$ if $p = \bar{p}$. Then the infimum

$$\sigma(c) \leq \frac{|\mu| g_1 c}{2(1+g_1)}$$

is achieved by $w \in S(c)$, which is a ground state of problem (P_c) with some $\bar{\lambda} \geq -|\mu|$.

(iii) Assume that $\overline{p} < p < 2_\alpha^*$ and condition ($D1$) with $-1 < g_1 \leq 0$ holds. Then there exists $\tilde{c} > 0$ such that for all $c < \tilde{c}$, problem (P_c) has a solution $(w, \bar{\lambda}) \in H_r^1(\mathbb{R}^N) \times \mathbb{R}^+$. Moreover, we have

$$J(w) > K_3 c^{-\frac{N+\alpha-p(N-2)}{Np-N-\alpha-2}} - \frac{|\mu| c}{2(1+g_1)}\left(\frac{Np-\alpha}{Np-N-\alpha} - g_1\right),$$

and

$$\bar{\lambda} > K_4 c^{-\frac{2p-2}{Np-N-\alpha-2}} - |\mu|,$$

where $K_3, K_4 > 0$ *are two constants;*

(iv) *Assume that* $p = 2_\alpha^*$ *and condition* $(D1)$ *holds. Then there is no solution for problem* (P_c) *for all* $\lambda \geq \frac{|\mu|(N-2-2g_1)}{2(1+g_1)}$.

Remark 1.1 (I) In Theorem 1.2 (i), we require the parameter $\mu > 0$ large enough when $p = 2_\alpha$ due to the feature of the lower critical nonlocal term. However, for other cases of p, i.e. Theorems 1.2 $(ii) - (iii)$ and 1.3 $(ii) - (iii)$, we do not need such assumption by using the new estimate trick.

(II) From the above two theorems, we find that when $p = 2_\alpha$, there are opposite results if the saturable perturbation is focuing or defocusing;

(III) In Theorems 1.2 (iii) and 1.3 (iii), we only give the existence results of problem (P_c) when the intensity function $g(x)$ is a constant. When $g(x)$ is not a constant function, such as $g(x)$ satisfies any of conditions $(D2) - (D4)$, finding normalized solutions of problem (P_c) would be an interesting issue.

2 Preliminaries

For convenience, we set

$$A(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx \text{ and } B(u) = \int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right) |u|^p dx.$$

Then the functional J defined in (4) can be reformulated as

$$J(u) = \frac{1}{2}A(u) - \frac{1}{2p}B(u) - \frac{\mu}{2}\int_{\mathbb{R}^N} \left[u^2 - \ln\left(1 + \frac{|u|^2}{1+g(x)}\right)\right] dx.$$

In what follows, we recall several important inequalities which will be often used in the paper.

(1) Hardy-Littlewood-Sobolev inequality ([17]): Let $t, r > 1$ and $0 < \alpha < N$ with $1/t + (N-\alpha)/N + 1/r = 2$. For $\bar{f} \in L^t(\mathbb{R}^N)$ and $\bar{h} \in L^r(\mathbb{R}^N)$, there exists a sharp constant $C(t, N, \alpha, r)$ independent of u and v, such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\bar{f}(x)\bar{h}(y)}{|x-y|^{N-\alpha}} dxdy \leq C(t, N, \alpha, r)\|\bar{f}\|_t \|\bar{h}\|_r. \tag{10}$$

(2) Gagliardo-Nirenberg inequality ([32]): For every $N \geq 1$ and $s \in (2, 2^*)$, there exists a constant $C_{N,s}$ depending on N and on s such that

$$\|u\|_s \leq C_{N,s}\|\nabla u\|_2^{\frac{N(s-2)}{2s}} \|u\|_2^{1-\frac{N(s-2)}{2s}}, \ \forall u \in H^1(\mathbb{R}^N). \tag{11}$$

(3) Gagliardo-Nirenberg inequality of Hartree type ([16, 35]): Let $N \geq 1$. For $p = 2_\alpha$, it holds

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^{2_\alpha})|u|^{2_\alpha} dx \leq \frac{1}{\|Q_{2_\alpha}\|_2^{22_\alpha}} \|u\|_2^{22_\alpha}, \tag{12}$$

where

$$Q_{2_\alpha} = C\left(\frac{\hat{b}}{\hat{b}^2 + |x-\hat{a}|^2}\right)^{N/2}, \tag{13}$$

with $C > 0$ is a fixed constant, $\hat{a} \in \mathbb{R}^N$ and $\hat{b} > 0$ are parameters. For $2_\alpha < p < 2^*_\alpha$, it holds

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \le \frac{p}{\|Q_p\|_2^{2p-2}} \|\nabla u\|_2^{Np-N-\alpha} \|u\|_2^{N+\alpha-p(N-2)}, \tag{14}$$

where Q_p is a positive ground state solution of the following equation

$$-\frac{N(p-1)-\alpha}{2}\Delta u + \frac{-(N-2)p+N+\alpha}{2} u = (I_\alpha * |u|^p)|u|^{p-2}u \text{ in } \mathbb{R}^N.$$

We now recall two known estimates on the saturable nonlinearity.

Lemma 2.1 *[21, Lemma 2.2] For each $2 < q \le \min\{4, 2^*\}$ ($2^* = \infty$ if $N = 1, 2$; $2^* = \frac{2N}{N-2}$ if $N \ge 3$), there exists a constant*

$$\mathbf{A}_q = \begin{cases} 1/2, & \text{if } q = 4, \\ \frac{q^{(q-2)/2}(4-q)^{(4-q)/2}}{2q}, & \text{if } 2 < q \le \min\{4, 2^*\} \text{ and } q \ne 4, \end{cases}$$

such that

$$s^2 - \ln\left(1 + \frac{s^2}{1+g(x)}\right) \le \frac{g(x)}{1+g(x)} s^2 + \frac{\mathbf{A}_q}{(1+g(x))^{q/2}} s^q \text{ for all } s \ge 0.$$

Lemma 2.2 *[21, Lemma 2.3] For each $2 < q \le \min\{4, 2^*\}$ ($2^* = \infty$ if $N = 1, 2$; $2^* = \frac{2N}{N-2}$ if $N \ge 3$), there exists a constant*

$$\mathbf{B}_q = \begin{cases} 1, & \text{if } q = 4; \\ \frac{32^{(q+4)/2}(q-2)^{(5-q)/2}(\sqrt{q+14}-3\sqrt{q-2})^{(4-q)/2}}{q(\sqrt{q+14}-\sqrt{q-2})^3} & \text{if } 2 < q \le \min\{4, 2^*\} \text{ and } q \ne 4, \end{cases}$$

such that

$$\frac{g(x)+s^2}{1+g(x)+s^2} s^2 \le \frac{g(x)}{1+g(x)} s^2 + \frac{\mathbf{B}_q}{(1+g(x))^{q/2}} s^q \text{ for all } s \ge 0.$$

Next, we show the variant of the classical Brezis-Lieb lemma for Riesz potential as follows.

Lemma 2.3 *[25, Lemma 2.4] Let $\alpha \in (0, N)$, $p \in [1, \infty)$ and $\{u_n\}$ be a bounded sequence in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. If $u_n \to u$ a.e. on $\mathbb{R}^N$ as $n \to \infty$, then $\lim_{n\to\infty} B(u_n) - B(u_n - u) = B(u)$.*

Lemma 2.4 *Assume that the function $g(x)$ is weakly differentiable on $\mathbb{R}^N$. Let $u \in H^1(\mathbb{R}^N)$ be a weak solution to the equation:*

$$-\Delta u + \lambda u = \left(I_\alpha * |u|^p\right)|u|^{p-2}u + \mu \frac{g(x)+|u|^2}{1+g(x)+|u|^2} u. \tag{15}$$

Then u satisfies the Pohozaev identity

$$\frac{N-2}{2}A(u) + \frac{N(\lambda-\mu)}{2}\int_{\mathbb{R}^N}|u|^2 dx = \frac{N+\alpha}{2p}B(u) - \frac{\mu N}{2}\int_{\mathbb{R}^N}\ln\left(1+\frac{|u|^2}{1+g(x)}\right)dx + \frac{\mu}{2}\int_{\mathbb{R}^N}\frac{|u|^2\nabla g(x)\cdot x}{(1+g(x))(1+g(x)+|u|^2)}dx.$$

Furthermore, it holds

$$A(u) - \frac{Np-N-\alpha}{2p}B(u) + \frac{\mu}{2}\int_{\mathbb{R}^N}\frac{|u|^2\nabla g(x)\cdot x}{(1+g(x))(1+g(x)+|u|^2)}dx = \frac{\mu N}{2}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{|u|^2}{1+g(x)}\right) - \frac{|u|^2}{1+g(x)+|u|^2}\right]dx. \tag{16}$$

Proof We follow the argument of Lehrer and Maia [11, Proposition 2.1]. By multiplying both sides of Eq. (15) by $x \cdot \nabla u$ and integrating on $\mathbb{R}^N$, we easily get the Pohozaev identity

$$\frac{N-2}{2}A(u)+\frac{N(\lambda-\mu)}{2}\int_{\mathbb{R}^N}|u|^2dx=\frac{N+\alpha}{2p}B(u)-\frac{\mu N}{2}\int_{\mathbb{R}^N}\ln\left(1+\frac{|u|^2}{1+g(x)}\right)dx$$
$$+\frac{\mu}{2}\int_{\mathbb{R}^N}\frac{|u|^2\nabla g(x)\cdot x}{(1+g(x))(1+g(x)+|u|^2)}dx. \quad (17)$$

Moreover, by multiplying both sides of Eq. (15) by u and integrating on $\mathbb{R}^N$, we have

$$A(u)+\lambda\int_{\mathbb{R}^N}|u|^2dx-B(u)-\mu\int_{\mathbb{R}^N}\frac{g(x)+|u|^2}{1+g(x)+|u|^2}|u|^2dx=0. \quad (18)$$

Combining (17) and (18), it follows that

$$A(u)-\frac{Np-N-\alpha}{2p}B(u)+\frac{\mu}{2}\int_{\mathbb{R}^N}\frac{|u|^2\nabla g(x)\cdot x}{(1+g(x))(1+g(x)+|u|^2)}dx$$
$$=\frac{\mu N}{2}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{|u|^2}{1+g(x)}\right)-\frac{|u|^2}{1+g(x)+|u|^2}\right]dx.$$

We complete the proof. □

Following the idea of Soave [27] and Cingolani and JeanJean [4], we will introduce a natural constraint manifold $\mathcal{M}(c)$ that contains all the critical points of the functional J restricted to $S(c)$. For each $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ and $t>0$, we consider the dilations

$$u^t(x):=t^{\frac{N}{2}}u(tx) \text{ for all } x\in\mathbb{R}^N.$$

Then a direct calculation shows that $\|u^t\|_2^2=\|u\|_2^2$, $A(u^t)=t^2A(u)$, $B(u^t)=t^{Np-N-\alpha}B(u)$, and

$$\int_{\mathbb{R}^N}\ln\left(1+\frac{|u^t|^2}{1+g(x)}\right)dx=\frac{1}{t^N}\int_{\mathbb{R}^N}\ln\left(1+\frac{t^N|u|^2}{1+g(x/t)}\right)dx.$$

Define the fibering map $t\in(0,\infty)\to f_u(t):=J(u^t)$ given by

$$f_u(t)=\frac{t^2}{2}A(u)-\frac{t^{Np-N-\alpha}}{2p}B(u)-\frac{\mu}{2}\int_{\mathbb{R}^N}|u|^2dx+\frac{\mu}{2t^N}\int_{\mathbb{R}^N}\ln\left(1+\frac{t^N|u|^2}{1+g(x/t)}\right)dx. \quad (19)$$

By calculating the first and second derivatives of $f_u(t)$, we have

$$f_u'(t)=tA(u)-\frac{(Np-N-\alpha)t^{Np-N-\alpha-1}}{2p}B(u)-\frac{\mu N}{2t^{N+1}}\int_{\mathbb{R}^N}\ln\left(1+\frac{t^N|u|^2}{1+g(x/t)}\right)dx$$
$$+\frac{\mu N}{2t}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g(x/t)+t^N|u|^2}dx+\frac{\mu}{2t}\int_{\mathbb{R}^N}\frac{|u|^2\nabla g(x/t)\cdot x}{(1+g(x/t))(1+g(x/t)+t^N|u|^2)}dx$$

and

$$f_u''(t)=A(u)-\frac{(Np-N-\alpha)(Np-N-\alpha-1)t^{Np-N-\alpha-2}}{2p}B(u)$$
$$+\frac{\mu N(N+1)}{2t^{N+2}}\int_{\mathbb{R}^N}\ln\left(1+\frac{t^N|u|^2}{1+g(x/t)}\right)dx-\frac{\mu N^2}{2t^2}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g(x/t)+t^N|u|^2}dx$$

$$
\begin{aligned}
&+\frac{\mu N}{2t^2}\int_{\mathbb{R}^N}\frac{|u|^2\nabla g(x/t)\cdot x}{(1+g(x/t))(1+g(x/t)+t^N|u|^2)}dx\\
&-\frac{\mu N}{2t^2}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g(x/t)+t^N|u|^2}dx+\frac{\mu N}{2t^2}\int_{\mathbb{R}^N}\frac{u^2\nabla g(x/t)\cdot x}{(1+g(x/t)+t^N|u|^2)^2}dx\\
&-\frac{\mu N^2t^{N-2}}{2}\int_{\mathbb{R}^N}\frac{|u|^4}{(1+g(x/t)+t^N|u|^2)^2}dx\\
&-\frac{\mu}{2t^2}\int_{\mathbb{R}^N}\frac{u^2\nabla g(x/t)\cdot x}{(1+g(x/t))(1+g(x/t)+t^N|u|^2)}dx\\
&+\frac{\mu}{2t}\int_{\mathbb{R}^N}\frac{u^2(\nabla g(x/t)\cdot x)_t}{(1+g(x/t))(1+g(x/t)+t^N|u|^2)}dx\\
&+\frac{\mu}{2t^2}\int_{\mathbb{R}^N}\frac{u^2(\nabla g(x/t)\cdot x)^2}{(1+g(x/t))^2(1+g(x/t)+t^N|u|^2)}dx\\
&+\frac{\mu}{2t^2}\int_{\mathbb{R}^N}\frac{u^2(\nabla g(x/t)\cdot x)^2}{(1+g(x/t))(1+g(x/t)+t^N|u|^2)^2}dx\\
&-\frac{\mu N t^{N-2}}{2}\int_{\mathbb{R}^N}\frac{u^4\nabla g(x/t)\cdot x}{(1+g(x/t))(1+g(x/t)+t^N|u|^2)^2}dx.
\end{aligned}
$$

Notice that $\frac{d}{dt}J(u^t)=f_u'(t)=\frac{Q(u^t)}{t}$, where

$$
\begin{aligned}
Q(u):=A(u)-\frac{Np-N-\alpha}{2p}B(u)+\frac{\mu}{2}\int_{\mathbb{R}^N}\frac{|u|^2\nabla g(x)\cdot x}{(1+g(x))(1+g(x)+|u|^2)}dx\\
-\frac{\mu N}{2}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{|u|^2}{1+g(x)}\right)-\frac{|u|^2}{1+g(x)+|u|^2}\right]dx.
\end{aligned}
$$

Actually $Q(u)=0$ corresponds to the Pohozaev identity (16). Then we define

$$
\mathcal{M}(c):=\{u\in S(c)\mid Q(u)=0\}=\{u\in S(c)\mid f_u'(1)=0\},
$$

which appears as a natural constraint. We also recognize that for any $u\in S(c)$, the function $u^t=t^{N/2}u(tx)$ belongs to $\mathcal{M}(c)$ if and only if $t\in\mathbb{R}^+$ is a critical point of the fibering map $f_u(t)$, namely $f_u'(t)=0$. In particular, $u\in\mathcal{M}(c)$ if and only if $f_u'(1)=0$. Thus, it is natural to split $\mathcal{M}(c)$ into three parts corresponding to local maxima, local minima and points of inflection. Following [29], we define

$$
\begin{aligned}
\mathcal{M}^+(c)&:=\left\{u\in S(c)\mid f_u'(1)=0,\ f_u''(1)>0\right\},\\
\mathcal{M}^0(c)&:=\left\{u\in S(c)\mid f_u'(1)=0,\ f_u''(1)=0\right\},\\
\mathcal{M}^-(c)&:=\left\{u\in S(c)\mid f_u'(1)=0,\ f_u''(1)<0\right\}.
\end{aligned}
$$

If we assume that $g(x)\equiv g_1>-1$ is a constant, then for each $u\in\mathcal{M}(c)$, we have

$$
\begin{aligned}
f_u''(1)&=A(u)-\frac{(Np-N-\alpha)(Np-N-\alpha-1)}{2p}B(u)+\frac{\mu N(N+1)}{2}\int_{\mathbb{R}^N}\ln\left(1+\frac{|u|^2}{1+g_1}\right)dx\\
&\quad-\frac{\mu N(N+1)}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g_1+|u|^2}dx-\frac{\mu N^2}{2}\int_{\mathbb{R}^N}\frac{|u|^4}{(1+g_1+|u|^2)^2}dx\\
&=-\frac{(Np-N-\alpha)(Np-N-\alpha-2)}{2p}B(u)+\frac{\mu N(N+2)}{2}\int_{\mathbb{R}^N}\ln\left(1+\frac{|u|^2}{1+g_1}\right)dx
\end{aligned}
$$

$$-\frac{\mu N(N+2)}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g_1+|u|^2}dx-\frac{\mu N^2}{2}\int_{\mathbb{R}^N}\frac{|u|^4}{\left(1+g_1+|u|^2\right)^2}dx \tag{20}$$

$$=-(Np-N-\alpha-2)A(u)+\frac{\mu N(Np-\alpha)}{2}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{|u|^2}{1+g_1}\right)-\frac{|u|^2}{1+g_1+|u|^2}\right]dx$$

$$-\frac{\mu N^2}{2}\int_{\mathbb{R}^N}\frac{|u|^4}{\left(1+g_1+|u|^2\right)^2}dx \tag{21}$$

$$=(N+2)A(u)-\frac{(Np-\alpha)(Np-N-\alpha)}{2p}B(u)-\frac{\mu N^2}{2}\int_{\mathbb{R}^N}\frac{|u|^4}{\left(1+g_1+|u|^2\right)^2}dx. \tag{22}$$

Furthermore, following the argument of Soave [27], we have the following lemma.

Lemma 2.5 *If $\mathcal{M}^0(c)=\emptyset$, then $\mathcal{M}(c)$ is a submanifold of codimension 2 of $H^1(\mathbb{R}^N)$ and a submanifold of codimension 1 in $S(c)$.*

Next, we shall give a general minimax theorem to establish the existence of a Palais-Smale sequence.

Definition 2.6 [8, Definition 3.1] Let Θ be a closed subset of a metric space $X\subset H^1(\mathbb{R}^N)$. We say that a class $\mathcal{F}$ of compact subsets of X is a homotopy-stable family with closed boundary Θ provided that

(a) every set in $\mathcal{F}$ contains Θ;
(b) for any set $H\in\mathcal{F}$ and any $\eta\in C([0,1]\times X,X)$ satisfying $\eta(s,x)=x$ for all $(s,x)\in(\{0\}\times X)\cup([0,1]\times\Theta)$, we have that $\eta(\{1\}\times H)\in\mathcal{F}$.

Lemma 2.7 *[8, Theorem 3.2] Let φ be a C^1-functional on a complete connected C^1-Finsler manifold X (without boundary) and consider a homotopy stable family $\mathcal{F}$ of compact subsets of X with a closed boundary Θ. Set*

$$\theta=\theta(\varphi,\mathcal{F})=\inf_{H\in\mathcal{F}}\max_{u\in H}\varphi(u)$$

and suppose that $\sup\varphi(\Theta)<\theta$. Then for any sequence of sets $\{H_n\}$ in $\mathcal{F}$ such that $\lim_{n\to\infty}\sup_{H_n}\varphi=\theta$, there exists a sequence $\{u_n\}$ in X such that

(i) $\lim_{n\to\infty}\varphi(u_n)=\theta$; ($ii$) $\lim_{n\to\infty}\left\|\varphi'(u_n)\right\|=0$; ($iii$) $\lim_{n\to\infty}dist(u_n,H_n)=0$.
Furthermore, if φ' is uniformly continuous, then u_n can be chosen to be in H_n for each n.

Lemma 2.8 *Assume that $\mu\in\mathbb{R}\backslash\{0\}$, $\bar{p}<p<2^*_\alpha$ and condition $(D1)$ with $1<g_1\leq 0$ holds. Let $\{u_n\}\subset\mathcal{M}^-(c)\cap H_r^1(\mathbb{R}^N)$ be a bounded Palais-Smale sequence for J restricted to $S(c)$ at level β. In addition, we assume that one of the two following conditions holds:*

(i) $\beta>\frac{\mu|g_1|c}{2(1+g_1)}$ and $c<c_0^*$ for some $c_0^*>0$ if $\mu>0$;
(ii) $\beta>\frac{|\mu|c}{2}$ and $c<c_1$ for some $c_1>0$ if $\mu<0$.

Then up to a subsequence, $u_n\to u_0$ strongly in $H^1(\mathbb{R}^N)$ and $u_0\in S(c)$ is a solution of problem (P_c) for some $\bar{\lambda}>0$.

Proof Since $\{u_n\}\subset\mathcal{M}^-(c)$ is bounded and the embedding $H_r^1(\mathbb{R}^N)\hookrightarrow L^s(\mathbb{R}^N)$ is compact for $s\in(2,2^*)$, there exists $u_0\in H_r^1(\mathbb{R}^N)$ such that $u_n\rightharpoonup u_0$ weakly in $H_r^1(\mathbb{R}^N)$, $u_n\to u_0$ strongly in $L^s(\mathbb{R}^N)$ for $s\in(2,2^*)$, and a.e. in $\mathbb{R}^N$. By the Lagrange multipliers rule, there exists $\lambda_n\in\mathbb{R}$ such that for every $\varphi\in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}(\nabla u_n\nabla\varphi+\lambda_n u_n\varphi)dx-\int_{\mathbb{R}^N}\left(I_\alpha*|u_n|^p\right)|u_n|^{p-2}u_n\varphi dx$$

$$-\mu \int_{\mathbb{R}^N} \frac{g_1+|u_n|^2}{1+g_1+|u_n|^2} u_n \varphi dx = o(1)\|\varphi\|, \tag{23}$$

where $o(1) \to 0$ as $n \to \infty$. In other words, u_n solves

$$-\Delta u_n + \lambda_n u_n = \left(I_\alpha * |u_n|^p\right)|u_n|^{p-2}u_n + \mu \frac{g_1+|u_n|^2}{1+g_1+|u_n|^2} u_n \text{ in } \mathbb{R}^N. \tag{24}$$

In particular, we have

$$\lambda_n c = -A(u_n) + B(u_n) + \mu \int_{\mathbb{R}^N} \frac{g_1+|u_n|^2}{1+g_1+|u_n|^2}|u_n|^2 dx + o(1). \tag{25}$$

We note that

$$\mu \int_{\mathbb{R}^N} \frac{g_1+|u_n|^2}{1+g_1+|u_n|^2}|u_n|^2 dx < \mu \int_{\mathbb{R}^N} |u_n|^2 dx = \mu c \text{ if } \mu > 0, \tag{26}$$

and

$$\begin{aligned}\mu \int_{\mathbb{R}^N} \frac{g_1+|u_n|^2}{1+g_1+|u_n|^2}|u_n|^2 dx &= \mu c - \mu \int_{\mathbb{R}^N} \frac{1}{1+g_1+|u_n|^2}|u_n|^2 dx \\ &\le -\frac{g_1}{1+g_1}|\mu| c \text{ if } \mu < 0.\end{aligned} \tag{27}$$

Then it follows from (14) and (25–27) that $\{\lambda_n\}$ is bounded, since $\{u_n\} \subset \mathcal{M}^-(c)$ is bounded. So we can assume that $\lambda_n \to \bar{\lambda} \in \mathbb{R}$ as $n \to \infty$. In the following we shall determine the sign of $\bar{\lambda}$ by considering two seperate cases.

Case (I) : $\mu > 0$. By Lemma 2.2 one has

$$\begin{aligned}\frac{s^4}{\left(1+g_1+s^2\right)^2} &\le \frac{s^4}{\left(1+g_1+s^2\right)(1+g_1)} \\ &= \frac{g_1+s^2}{1+g_1+s^2}s^2 - \frac{g_1}{1+g_1}s^2 \\ &\le \frac{\mathbf{B}_q}{(1+g_1)^{q/2}}s^q \text{ for } s \ge 0,\end{aligned} \tag{28}$$

where $2 < q \le \min\{4, 2^*\}$. For $\{u_n\} \subset \mathcal{M}^-(c) \cap H_r^1(\mathbb{R}^N)$, by (14), (22) and (28), we have

$$\begin{aligned}A(u_n) &< \frac{(Np-\alpha)(Np-N-\alpha)}{2p(N+2)} B(u_n) + \frac{\mu N^2}{2(N+2)} \int_{\mathbb{R}^N} \frac{|u_n|^4}{\left(1+g_1+|u_n|^2\right)^2} dx \\ &\le \frac{c^{\frac{N+\alpha-p(N-2)}{2}}(Np-\alpha)(Np-N-\alpha)}{2(N+2)\|Q_p\|_2^{2p-2}} A(u_n)^{\frac{Np-N-\alpha}{2}} \\ &\quad + \frac{\mu \mathbf{B}_q N^2 C_{N,q}^q c^{\frac{2N-q(N-2)}{4}}}{2(N+2)(1+g_1)^{q/2}} A(u_n)^{\frac{N(q-2)}{4}}.\end{aligned}$$

For the convenience of calculation, we choose $q = \bar{q} = 2 + \frac{4}{N}$. Then the above inequality becomes

$$A(u_n) < \frac{c^{\frac{N+\alpha-p(N-2)}{2}}(Np-\alpha)(Np-N-\alpha)}{2(N+2)\|Q_p\|_2^{2p-2}} A(u_n)^{\frac{Np-N-\alpha}{2}} + \frac{\mu \mathbf{B}_{\bar{q}} N^2 C_{N,\bar{q}}^{\bar{q}} c^{2/N}}{2(N+2)(1+g_1)^{\bar{q}/2}} A(u_n),$$

which implies that

$$A(u_n) > \Lambda_c := \left[\frac{\|Q_p\|_2^{2p-2}\left(2(N+2)(1+g_1)^{\bar{q}/2} - \mu \mathbf{B}_{\bar{q}} N^2 C_{N,\bar{q}}^{\bar{q}} c^{2/N}\right)}{c^{\frac{N+\alpha-p(N-2)}{2}}(1+g_1)^{\bar{q}/2}(Np-\alpha)\,(Np-N-\alpha)} \right]^{\frac{2}{Np-N-\alpha-2}} > 0, \tag{29}$$

since

$$c < c_0 := \left(\frac{2(N+2)(1+g_1)^{\bar{q}/2}}{\mu \mathbf{B}_{\bar{q}} N^2 C_{N,\bar{q}}^{\bar{q}}} \right)^{N/2}. \tag{30}$$

It is clear that $\Lambda_c \to \infty$ as $c \to 0$. By the facts of $Q(u_n) = o(1)$ and of $\ln(1+x) < x$ for $x > 0$ and (29), we have

$$\begin{aligned}
\lambda_n c &= -A(u_n) + B(u_n) + \mu \int_{\mathbb{R}^N} \frac{g_1 + |u_n|^2}{1+g_1+|u_n|^2}|u_n|^2 dx + o(1) \\
&= -A(u_n) + \frac{2p}{Np-N-\alpha} A(u_n) + \mu \int_{\mathbb{R}^N} \frac{g_1+|u_n|^2}{1+g_1+|u_n|^2}|u_n|^2 dx \\
&\quad - \frac{\mu Np}{Np-N-\alpha} \int_{\mathbb{R}^N} \left[\ln\left(1 + \frac{|u_n|^2}{1+g_1}\right) - \frac{|u_n|^2}{1+g_1+|u_n|^2} \right] + o(1) \\
&> \frac{N+\alpha-p(N-2)}{Np-N-\alpha} A(u_n) - \frac{\mu Np}{Np-N-\alpha} \int_{\mathbb{R}^N} \frac{|u_n|^4}{(1+g_1)(1+g_1+|u_n|^2)} dx \\
&\quad + \mu \int_{\mathbb{R}^N} \frac{g_1+|u_n|^2}{1+g_1+|u_n|^2}|u_n|^2 dx + o(1) \\
&> \frac{N+\alpha-p(N-2)}{Np-N-\alpha} A(u_n) - \frac{\mu Np}{Np-N-\alpha} \int_{\mathbb{R}^N} \frac{|u_n|^4}{(1+g_1)(1+g_1+|u_n|^2)} dx \\
&\quad + \mu \int_{\mathbb{R}^N} \frac{g_1|u_n|^2 + |u_n|^4 + g_1^2|u_n|^2 + g_1|u_n|^4}{(1+g_1)(1+g_1+|u_n|^2)} dx + o(1) \\
&= \frac{N+\alpha-p(N-2)}{Np-N-\alpha} A(u_n) + \frac{\mu g_1}{1+g_1} \int_{\mathbb{R}^N} |u_n|^2 dx \\
&\quad - \frac{\mu(N+\alpha)}{(1+g_1)(Np-N-\alpha)} \int_{\mathbb{R}^N} \frac{|u_n|^4}{1+g_1+|u_n|^2} dx + o(1) \\
&\geq \frac{N+\alpha-p(N-2)}{Np-N-\alpha} \Lambda_c + \frac{\mu g_1}{1+g_1} c - \frac{\mu(N+\alpha)}{(1+g_1)(Np-N-\alpha)} c + o(1),
\end{aligned}$$

which implies that there exists a positive constant $c_0^* \leq c_0$ such that

$$\bar{\lambda} \geq \frac{N+\alpha-p(N-2)}{c(Np-N-\alpha)} \Lambda_c + \frac{\mu g_1}{1+g_1} - \frac{\mu(N+\alpha)}{(1+g_1)(Np-N-\alpha)} > 0 \text{ for } c < c_0^*.$$

Case (II) : $\mu < 0$. Since $Q(u_n) = o(1)$, by (14) and the fact of

$$\ln\left(1 + \frac{s}{a}\right) - \frac{s}{a+s} \geq 0 \text{ for } s \geq 0 \text{ and } a > 0, \tag{31}$$

we have

$$A(u_n) = \frac{Np-N-\alpha}{2p} B(u_n) + \frac{\mu N}{2} \int_{\mathbb{R}^N} \left[\ln\left(1 + \frac{|u_n|^2}{1+g_1}\right) - \frac{|u_n|^2}{1+g_1+|u_n|^2} \right] dx$$

$$\leq \frac{Np-N-\alpha}{2p}B(u_n)$$
$$\leq \frac{Np-N-\alpha}{2\|Q_p\|_2^{2p-2}}A(u_n)^{\frac{N(p-1)-\alpha}{2}}c^{\frac{N+\alpha-p(N-2)}{2}},$$

which implies that

$$A(u_n) \geq \left(\frac{2\|Q_p\|_2^{2p-2}}{Np-N-\alpha}c^{-\frac{N+\alpha-p(N-2)}{2}}\right)^{\frac{2}{N(p-1)-\alpha-2}}. \tag{32}$$

Then it follows from (25), (31), (32) and the fact of $Q(u_n)=o(1)$ that

$$\begin{aligned}
\lambda_n c &= -A(u_n)+B(u_n)+\mu\int_{\mathbb{R}^N}\frac{g_1+|u_n|^2}{1+g_1+|u_n|^2}|u_n|^2dx+o(1)\\
&= \frac{N+\alpha-p(N-2)}{Np-N-\alpha}A(u_n)+\mu\int_{\mathbb{R}^N}\frac{g_1+|u_n|^2}{1+g_1+|u_n|^2}|u_n|^2dx\\
&\quad -\frac{\mu Np}{Np-N-\alpha}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{|u_n|^2}{1+g_1}\right)-\frac{|u_n|^2}{1+g_1+|u_n|^2}\right]+o(1)\\
&\geq \frac{N+\alpha-p(N-2)}{Np-N-\alpha}A(u_n)-|\mu|c+o(1)\\
&\geq \frac{N+\alpha-p(N-2)}{Np-N-\alpha}\left[\frac{2\|Q_p\|_2^{2p-2}}{Np-N-\alpha}c^{-\frac{-(N-2)p+N+\alpha}{2}}\right]^{\frac{2}{N(p-1)-\alpha-2}}-|\mu|c+o(1),
\end{aligned}$$

which implies that $\bar{\lambda}>0$ for

$$c<c_1:=\left[\frac{N+\alpha-p(N-2)}{|\mu|(Np-N-\alpha)}\right]^{\frac{N(p-1)-\alpha-2}{2p-2}}\left[\frac{2\|Q_p\|_2^{2p-2}}{Np-N-\alpha}\right]^{\frac{1}{p-1}}.$$

Next, we claim that $u_0\not\equiv 0$. Assume on the contrary. Then by (10), we have $B(u_n)=o(1)$. Next we consider two separate cases depending on μ.

Case (i) : $\mu>0$. By $Q(u_n)=o(1)$ and the fact that $\ln(1+s)<s$ for all $s>0$, we deduce that

$$\begin{aligned}
\beta+o(1) &= J(u_n)\\
&= \frac{1}{2}A(u_n)-\frac{1}{2p}B(u_n)-\frac{\mu}{2}\int_{\mathbb{R}^N}\left[|u_n|^2-\ln\left(1+\frac{|u_n|^2}{1+g_1}\right)\right]dx\\
&= \frac{Np-N-\alpha-2}{4p}B(u_n)-\frac{\mu}{2}\int_{\mathbb{R}^N}\left[|u_n|^2-\ln\left(1+\frac{|u_n|^2}{1+g_1}\right)\right]dx\\
&\quad +\frac{\mu N}{4}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{|u_n|^2}{1+g_1}\right)-\frac{|u_n|^2}{1+g_1+|u_n|^2}\right]dx\\
&< -\frac{\mu}{2}\int_{\mathbb{R}^N}\left[|u_n|^2-\frac{|u_n|^2}{1+g_1}\right]dx+\frac{\mu N}{4}\int_{\mathbb{R}^N}\left(\frac{1}{1+g_1}-\frac{1}{1+g_1+|u_n|^2}\right)|u_n|^2dx+o(1)\\
&= -\frac{\mu g_1 c}{2(1+g_1)}+\frac{\mu N}{4}\int_{\mathbb{R}^N}\left(\frac{g_1+|u_n|^2}{1+g_1+|u_n|^2}-\frac{g_1}{1+g_1}\right)|u_n|^2dx+o(1)\\
&\leq -\frac{\mu g_1 c}{2(1+g_1)}+\frac{\mu N\mathbf{B}_q}{4(1+g_1)^{q/2}}\int_{\mathbb{R}^N}|u_n|^q dx+o(1)
\end{aligned}$$

$$\leq \frac{\mu|g_1|c}{2(1+g_1)} + o(1),$$

where we have used Lemma 2.2 with $2 < q < \min\{4, 2^*\}$. Clearly, this is a contradiction with $\beta > \frac{\mu|g_1|c}{2(1+g_1)}$, and so $u_0 \not\equiv 0$.

Case (ii) : $\mu < 0$. Using the fact of $Q(u_n) = o(1)$ and (31) gives

$$\begin{aligned}
\beta + o(1) &= J(u_n) \\
&= \frac{Np - N - \alpha - 2}{4p} B(u_n) - \frac{\mu}{2} \int_{\mathbb{R}^N} \left[|u_n|^2 - \ln\left(1 + \frac{|u_n|^2}{1+g_1}\right)\right] dx \\
&\quad + \frac{\mu N}{4} \int_{\mathbb{R}^N} \left[\ln\left(1 + \frac{|u_n|^2}{1+g_1}\right) - \frac{|u_n|^2}{1+g_1+|u_n|^2}\right] dx \\
&< \frac{|\mu|}{2} \int_{\mathbb{R}^N} |u_n|^2 dx + o(1) \\
&= \frac{|\mu| c}{2} + o(1),
\end{aligned}$$

which contradicts with $\beta > \frac{|\mu|c}{2}$, and so $u_0 \not\equiv 0$.

Finally, let us prove that $u_n \to u_0$ in $H^1(\mathbb{R}^N)$. Since $u_n \rightharpoonup u_0$ in $H_r^1(\mathbb{R}^N)$ and $\lambda_n \to \bar{\lambda} \in \mathbb{R}$ as $n \to \infty$, by (23) one has

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\nabla u_0 \nabla \varphi + \bar{\lambda} u_0 \varphi) dx - \int_{\mathbb{R}^N} \left(I_\alpha * |u_0|^p\right) |u_0|^{p-2} u_0 \varphi dx \\
&-\mu \int_{\mathbb{R}^N} \frac{g_1 + |u_0|^2}{1+g_1+|u_0|^2} u_0 \varphi dx = o(1),
\end{aligned} \tag{33}$$

for every $\varphi \in H^1(\mathbb{R}^N)$. Taking $\varphi = u_n - u_0$ in (23) and (33), and subtracting, we get

$$\begin{aligned}
o(1) = &\int_{\mathbb{R}^N} (|\nabla(u_n - u_0)|^2 + \bar{\lambda}|u_n - u_0|^2) dx - (B'(u_n) - B'(u_0))(u_n - u_0) \\
&-\mu \int_{\mathbb{R}^N} \left(\frac{g_1 + |u_n|^2}{1+g_1+|u_n|^2} u_n - \frac{g_1 + |u_0|^2}{1+g_1+|u_0|^2} u_0 \right)(u_n - u_0) dx.
\end{aligned} \tag{34}$$

By [13, Lemma 2.4], we have

$$(B'(u_n) - B'(u_0))(u_n - u_0) = o(1). \tag{35}$$

Next, we claim that

$$\int_{\mathbb{R}^N} \left(\frac{g_1 + |u_n|^2}{1+g_1+|u_n|^2} u_n - \frac{g_1 + |u_0|^2}{1+g_1+|u_0|^2} u_0 \right)(u_n - u_0) dx = o(1). \tag{36}$$

We observe that if $-1 < g_1 \leq 0$, then there exists $\bar{C} > 0$ such that

$$\frac{g_1 + s^2}{1+g_1+s^2} \leq \bar{C} s^2 \text{ for } s > 0. \tag{37}$$

Then by (37) and the Hölder inequality, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} \frac{g_1 + |u_n|^2}{1+g_1+|u_n|^2} u_n (u_n - u_0) dx \right| \\
&\leq \left(\int_{\mathbb{R}^3} |u_n - u_0|^p \, dx \right)^{1/p} \left(\int_{\mathbb{R}^3} \left| \frac{g_1 + |u_n|^2}{1+g_1+|u_n|^2} u_n \right|^q dx \right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\int_{\mathbb{R}^3} |u_n - u_0|^p\,dx\right)^{1/p} \left(\int_{\mathbb{R}^3} \left|\frac{g_1 + |u_n|^2}{1 + g_1 + |u_n|^2}\right|^q |u_n|^q dx\right)^{1/q} \\
&\le \bar{C} \left(\int_{\mathbb{R}^3} |u_n - u_0|^p\,dx\right)^{1/p} \left(\int_{\mathbb{R}^3} |u_n|^{2q} |u_n|^q dx\right)^{1/q} \\
&= \bar{C} \left(\int_{\mathbb{R}^3} |u_n - u_0|^p\,dx\right)^{1/p} \left(\int_{\mathbb{R}^3} |u_n|^{3q} dx\right)^{1/q} \\
&\le \tilde{C} \left(\int_{\mathbb{R}^3} |u_n - u_0|^p\,dx\right)^{1/p} \\
&< \varepsilon,
\end{aligned} \tag{38}
$$

where $p \in (2, 6)$ and $q = \frac{p}{p-1} \in (\frac{6}{5}, 2)$. Similarly, we also have

$$
\left|\int_{\mathbb{R}^3} \frac{g_1 + |u_0|^2}{1 + g_1 + |u_0|^2} u_0 (u_n - u_0) dx\right| < \varepsilon. \tag{39}
$$

Thus, using (38) and (39) leads to

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3} \left(\frac{g_1 + |u_n|^2}{1 + g_1 + |u_n|^2} u_n - \frac{g_1 + |u_0|^2}{1 + g_1 + |u_0|^2} u\right) (u_n - u_0) dx\right| \\
&\le \left|\int_{\mathbb{R}^3} \frac{g_1 + |u_n|^2}{1 + g_1 + |u_n|^2} u_n (u_n - u_0) dx\right| + \left|\int_{\mathbb{R}^3} \frac{g_1 + |u_0|^2}{1 + g_1 + |u_0|^2} u_0 (u_n - u_0) dx\right| \\
&< 2\varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we complete the claim of (36). Hence, it follows from (34–36) that

$$
\int_{\mathbb{R}^N} (|\nabla (u_n - u_0)|^2 + \bar{\lambda} |u_n - u_0|^2) dx = o(1),
$$

which implies that $u_n \to u_0$ in $H^1(\mathbb{R}^N)$, since $\bar{\lambda} > 0$. We complete the proof. □

3 The case $\mu > 0$

3.1 The subcase $2_\alpha \le p \le \bar{p}$

In this subsection, we consider the case of $2_\alpha \le p \le \bar{p}$. As we will see, the functional J is bounded below on $S(c)$. We have the following result.

Lemma 3.1 *Assume that $\mu > 0$, $2_\alpha \le p \le \bar{p}$ and one of conditions $(D1) - (D4)$ holds. In addition, we assume that $c < \| Q_{\bar{p}} \|_2^{4(\bar{p}-1)/(N+\alpha-\bar{p}(N-2))}$ if $p = \bar{p}$. Then the functional J is bounded from below and coercive on $S(c)$.*

Proof For $u \in S(c)$, it follows from (12) and (14) that

$$
\begin{aligned}
J(u) &= \frac{1}{2} A(u) - \frac{1}{2p} B(u) - \frac{\mu}{2} \int_{\mathbb{R}^N} \left[|u|^2 - \ln\left(1 + \frac{|u|^2}{1 + g(x)}\right)\right] dx \\
&\ge \begin{cases} \frac{1}{2} A(u) - \frac{1}{2 2_\alpha \| Q_{2_\alpha} \|_2^{2 2_\alpha}} c^{2_\alpha} - \frac{\mu c}{2}, & \text{if } p = 2_\alpha, \\ \frac{1}{2} A(u) - \frac{c^{\frac{N+\alpha-p(N-2)}{2}}}{2 \| Q_p \|_2^{2p-2}} A(u)^{\frac{Np-N-\alpha}{2}} - \frac{\mu c}{2}, & \text{if } 2_\alpha < p \le \bar{p}, \end{cases}
\end{aligned}
$$

which implies that J is coercive and bounded from below on $S(c)$. We complete the proof. □

Now we are ready to prove Theorem 1.2 $(i)-(ii)$. In the following, we only give the proof when $g(x)$ satisfies condition $(D1)$, since the other cases are similar.

Let $\{u_n\} \subset S(c)$ be a minimizing sequence for $\sigma(c)$ on $H^1(\mathbb{R}^N)$. Then $\{u_n\}$ is bounded on $H^1(\mathbb{R}^N)$ by Lemma 3.1. First of all, we claim that

$$\eta := \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 dx > 0. \tag{40}$$

We consider two seperate cases.

Case (i) : $p = 2_\alpha$. There exists a constant $\mu_0 > 0$ such that for all $\mu > \mu_0$,

$$\sigma(c) < -\frac{1}{22_\alpha \| Q_{2_\alpha} \|_2^{22_\alpha}} c^{2_\alpha} - \frac{\mu g_1 c}{2(1+g_1)}. \tag{41}$$

Indeed, we can fix some $u \in S(c)$ and choose a constant $\mu_0 > 0$ such that

$$\frac{1}{2}A(u) - \frac{1}{22_\alpha}\left(B(u) - \frac{1}{\| Q_{2_\alpha} \|_2^{22_\alpha}} c^{2_\alpha}\right) - \frac{\mu_0}{2}\int_{\mathbb{R}^N}\left[\frac{|u|^2}{1+g_1} - \ln\left(1+\frac{|u|^2}{1+g_1}\right)\right] dx < 0,$$

where we have used the fact of $\ln(1+x) < x$ for all $x > 0$. Using the above inequality, together with (12) gives

$$\begin{aligned}
\sigma(c) &\le \frac{1}{2}A(u) - \frac{1}{22_\alpha}B(u) - \frac{\mu}{2}\int_{\mathbb{R}^N}\left[|u|^2 - \ln\left(1+\frac{|u|^2}{1+g_1}\right)\right] dx \\
&< \frac{1}{2}A(u) - \frac{1}{22_\alpha}\left(B(u) - \frac{1}{\| Q_{2_\alpha} \|_2^{22_\alpha}} c^{2_\alpha}\right) - \frac{1}{22_\alpha \| Q_{2_\alpha} \|_2^{22_\alpha}} c^{2_\alpha} \\
&\quad -\frac{\mu_0}{2}\int_{\mathbb{R}^N}\left[\frac{|u|^2}{1+g_1} - \ln\left(1+\frac{|u|^2}{1+g_1}\right)\right] dx - \frac{\mu g_1}{2(1+g_1)}\int_{\mathbb{R}^N}|u|^2 dx \\
&< -\frac{1}{22_\alpha \| Q_{2_\alpha} \|_2^{22_\alpha}} c^{2_\alpha} - \frac{\mu g_1 c}{2(1+g_1)} \text{ for all } \mu > \mu_0.
\end{aligned} \tag{42}$$

For $\mu > \mu_0$ fixed, we assume on the contrary that $\eta = 0$. By Lions's lemma [31], one has $\|u_n\|_s \to 0$ as $n \to \infty$ for any $2 < s < 2^*$. Then it follows from (12) and Lemma 2.1 with $2 < q < \min\{4, 2^*\}$ that

$$\begin{aligned}
\sigma(c) + o(1) &= J(u_n) \\
&= \frac{1}{2}A(u_n) - \frac{1}{22_\alpha}B(u_n) - \frac{\mu}{2}\int_{\mathbb{R}^N}\left[|u_n|^2 - \ln\left(1+\frac{|u_n|^2}{1+g_1}\right)\right] dx \\
&\ge \frac{1}{2}A(u_n) - \frac{1}{22_\alpha \| Q_{2_\alpha} \|_2^{22_\alpha}} c^{2_\alpha} - \frac{\mu g_1}{2(1+g_1)}\int_{\mathbb{R}^N}|u_n|^2 dx - \frac{\mu \mathbf{A}_q}{2(1+g_1)^{q/2}}\int_{\mathbb{R}^N}|u_n|^q dx \\
&\ge \frac{1}{2}A(u_n) - \frac{1}{22_\alpha \| Q_{2_\alpha} \|_2^{22_\alpha}} c^{2_\alpha} - \frac{\mu g_1}{2(1+g_1)}\int_{\mathbb{R}^N}|u_n|^2 dx + o(1) \\
&\ge -\frac{1}{22_\alpha \| Q_{2_\alpha} \|_2^{22_\alpha}} c^{2_\alpha} - \frac{\mu g_1 c}{2(1+g_1)} + o(1),
\end{aligned} \tag{43}$$

which contradicts with (42). Thus, (40) holds.

Case (ii) : $2_\alpha < p \le \bar{p}$. Fix $u \in S(c)$, by the fact of $\ln(1+x) < x$ for $x > 0$ we have

$$\sigma(c) \le J(u^t) = \frac{t^2}{2}A(u) - \frac{t^{Np-N-\alpha}}{2p}B(u) - \frac{\mu c}{2} + \frac{\mu}{2t^N}\int_{\mathbb{R}^N}\ln\left(1+\frac{t^N|u|^2}{1+g_1}\right) dx$$

$$
\begin{aligned}
&< \frac{t^2}{2}A(u) - \frac{t^{Np-N-\alpha}}{2p}B(u) - \frac{\mu c}{2} + \frac{\mu}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g_1}dx \\
&\to -\frac{\mu g_1 c}{2(1+g_1)} \text{ as } t \to 0,
\end{aligned}
$$

which implies that

$$
\sigma(c) < -\frac{\mu g_1 c}{2(1+g_1)}. \tag{44}
$$

Assume on the contrary that $\eta = 0$. By Lions's lemma [31], one has $\|u_n\|_s \to 0$ as $n \to \infty$ for $2 < s < 2^*$, which implies that $B(u_n) = o(1)$ by (10). Using this, together with Lemma 2.1 with $2 < q < \min\{4, 2^*\}$, yields

$$
\begin{aligned}
\sigma(c) + o(1) &= J(u_n) \\
&= \frac{1}{2}A(u_n) - \frac{1}{2p}B(u_n) - \frac{\mu}{2}\int_{\mathbb{R}^N}\left[|u_n|^2 - \ln\left(1 + \frac{|u_n|^2}{1+g_1}\right)\right]dx \\
&= \frac{1}{2}A(u_n) - \frac{\mu g_1}{2(1+g_1)}\int_{\mathbb{R}^N}|u_n|^2dx - \frac{\mu \mathbf{A}_q}{2(1+g_1)^{q/2}}\int_{\mathbb{R}^N}|u_n|^q dx + o(1) \\
&\geq -\frac{\mu g_1 c}{2(1+g_1)} + o(1),
\end{aligned}
$$

which contradicts with (44). Thus, (40) holds.

According to (40), there exists $y_n \in \mathbb{R}^N$ such that

$$
\limsup_{n\to\infty}\int_{B_1(y_n)}|u_n|^2dx \geq \frac{\eta}{2}. \tag{45}
$$

Let $w_n(x) := u_n(x + y_n)$. Then it holds $A(w_n) = A(u_n),\ B(w_n) = B(u_n)$, and

$$
\int_{\mathbb{R}^N}\left[|w_n|^2 - \ln\left(1 + \frac{|w_n|^2}{1+g_1}\right)\right]dx = \int_{\mathbb{R}^N}\left[|u_n|^2 - \ln\left(1 + \frac{|u_n|^2}{1+g_1}\right)\right]dx.
$$

Moreover, $\{w_n\}$ is also a bounded minimizing sequence for $\sigma(c)$ on $S(c)$, and

$$
\limsup_{n\to\infty}\int_{B_1(0)}|w_n|^2dx \geq \frac{\eta}{2}.
$$

Then, we can assume that $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$, $w_n \to w \neq 0$ in $L^2(B_1(0))$ and $w_n(x) \to w(x) \neq 0$ a.e. on $B_1(0)$. By Egoroff's theorem we can find a constant $\delta > 0$ such that

$$
w_n(x) \to w(x) \text{ uniformly in } E, \text{ and } meas(E) > 0, \tag{46}
$$

where $E \subset \{x \mid |w(x)| \geq \delta,\ x \in B_1(0)\} \subset B_1(0)$.

Next, we prove that $\|w\|_2^2 = c$. Assume on the contrary that $\rho := \|w\|_2^2 \in (0, c)$. Let

$$
\tilde{w} = \frac{w}{\sqrt{1+g_1}} \text{ and } \tilde{v}_n = \frac{w_n - w}{\sqrt{1+g_1}}.
$$

From (46) it follows that

$$
\tilde{w}^2 = \frac{w^2}{1+g_1} \geq \frac{\delta^2}{1+g_1} > 0 \ \text{ in } E \tag{47}
$$

and

$$
\tilde{v}_n^2 = \frac{(w_n - w)^2}{1+g_1} \to 0 \text{ in } E. \tag{48}
$$

Let $h(s)=s-\ln(1+s)$ for $s\geq 0$. By (46–48), applying [20, Lemma 5.2], we can find a constant $\xi>0$ such that

$$\begin{aligned}&\int_E h\left(\frac{\rho}{c}\left(\frac{(\sqrt{c}\tilde{w})^2}{\|w\|_2^2}\right)+\frac{c-\rho}{c}\frac{(\sqrt{c}\tilde{v}_n)^2}{\|w_n-w\|_2^2}\right)dx\\&\leq -\xi+\frac{\rho}{c}\int_E h\left(\frac{(\sqrt{c}\tilde{w})^2}{\|w\|_2^2}\right)dx+\frac{c-\rho}{c}\int_E h\left(\frac{(\sqrt{c}\tilde{v}_n)^2}{\|w_n-w\|_2^2}\right)dx,\end{aligned}\tag{49}$$

as $n\to\infty$. Using this, together with Brizes-Lieb lemma and Lemma 2.3, one has

$$\begin{aligned}\sigma(c)&=J(w_n)+o(1)\\&=\frac{1}{2}A(w_n)-\frac{1}{2p}B(w_n)-\frac{\mu}{2}\int_{\mathbb{R}^N}\left[|w_n|^2-\ln\left(1+\frac{|w_n|^2}{1+g_1}\right)\right]dx+o(1)\\&=\frac{\rho}{2c}A\left(\frac{\sqrt{c}w}{\|w\|_2}\right)+\frac{c-\rho}{2c}A\left(\frac{\sqrt{c}(w_n-w)}{\|w_n-w\|_2}\right)\\&\quad-\frac{1}{2p}\left(\frac{\rho}{c}\right)^p B\left(\frac{\sqrt{c}w}{\|w\|_2}\right)-\frac{1}{2p}\left(\frac{c-\rho}{c}\right)^p B\left(\frac{\sqrt{c}(w_n-w)}{\|w_n-w\|_2}\right)\\&\quad-\frac{\mu}{2}\int_{\mathbb{R}^N}h\left(\frac{\rho}{c}\left(\frac{(\sqrt{c}|\tilde{w}|)^2}{\|w\|_2^2}\right)+\left(\frac{c-\rho}{c}\right)\frac{(\sqrt{c}|\tilde{v}_n|)^2}{\|w_n-w\|_2^2}\right)dx\\&\quad-\frac{\mu I_1}{2}\int_{\mathbb{R}^N}(|\tilde{w}|^2+|\tilde{v}_n|^2)dx+o(1)\\&\geq\frac{\rho}{2c}A\left(\frac{\sqrt{c}w}{\|w\|_2}\right)+\frac{c-\rho}{2c}A\left(\frac{\sqrt{c}(w_n-w)}{\|w_n-w\|_2}\right)\\&\quad-\frac{\rho}{2pc}B\left(\frac{\sqrt{c}w}{\|w\|_2}\right)-\frac{1}{2p}\left(\frac{c-\rho}{c}\right)B\left(\frac{\sqrt{c}(w_n-w)}{\|w_n-w\|_2}\right)\\&\quad-\frac{\mu}{2}\int_{\mathbb{R}^N}h\left(\frac{\rho}{c}\left(\frac{(\sqrt{c}\tilde{w})^2}{\|w\|_2^2}\right)+\left(\frac{c-\rho}{c}\right)\frac{(\sqrt{c}\tilde{v}_n)^2}{\|w_n-w\|_2^2}\right)dx\\&\quad-\frac{\mu I_1}{2}\int_{\mathbb{R}^N}(|\tilde{w}|^2+|\tilde{v}_n|^2)dx+o(1)\\&\geq\frac{\rho}{c}J\left(\frac{\sqrt{c}w}{\|w\|_2}\right)+\frac{c-\rho}{c}J\left(\frac{\sqrt{c}(w_n-w)}{\|w_n-w\|_2}\right)+\frac{\mu\xi}{2}+o(1)\\&\geq\sigma(c)+\frac{\mu\xi}{2}+o(1),\end{aligned}$$

which is a contradiction. Thus, we have $w_n\to w$ in $L^2(\mathbb{R}^N)$. Using this, combining with the generalized Lebesgue dominated convergence theorem [30, Lemma 2.22] and (14), we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left[|w_n|^2-\ln\left(1+\frac{|w_n|^2}{1+g_1}\right)\right]dx=\int_{\mathbb{R}^N}\left[|w|^2-\ln\left(1+\frac{|w|^2}{1+g_1}\right)\right]dx.\tag{50}$$

and $\lim_{n\to\infty}B(w_n-w)=0$ for $2_\alpha\leq p\leq\bar{p}$, which implies that

$$\lim_{n\to\infty}B(w_n)=B(w)\tag{51}$$

via Lemma 2.3. Moreover, since $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$, we have

$$A(w) \le \liminf_{n\to\infty} A(w_n). \tag{52}$$

Hence, it follows from (50–52) that

$$\begin{aligned}
\sigma(c) &= \lim_{n\to\infty} J(w_n) \\
&= \lim_{n\to\infty} \left(\frac{1}{2}A(w_n) - \frac{1}{2p}B(w_n) - \frac{\mu}{2}\int_{\mathbb{R}^N} \left[|w_n|^2 - \ln\left(1 + \frac{|w_n|^2}{1+g_1}\right)\right] dx \right) \\
&\ge \frac{1}{2}A(w) - \frac{1}{2p}B(w) - \frac{\mu}{2}\int_{\mathbb{R}^N} \left[|w|^2 - \ln\left(1 + \frac{|w|^2}{1+g_1}\right)\right] dx \\
&\ge \sigma(c),
\end{aligned}$$

which indicates that $\sigma(c)$ is achieved at $w \ne 0$ and $\|w_n - w\|_{H^1} \to 0$ as $n \to \infty$.

Since w is a critical point of J restricted to $S(c)$, there exists a Lagrange multiplier $\bar{\lambda} \in \mathbb{R}$ such that $J'(w) + \bar{\lambda} w = 0$. In particular, we have

$$\begin{aligned}
\bar{\lambda} c &= -A(w) + B(w) + \mu \int_{\mathbb{R}^N} \frac{g_1 + |w|^2}{1 + g_1 + |w|^2}|w|^2 dx \\
&= -2\sigma(c) + \frac{p-1}{p}B(w) + \mu \int_{\mathbb{R}^N} \left[\ln\left(1 + \frac{|w|^2}{1+g_1}\right) - \frac{|w|^2}{1+g_1+|w|^2}\right] dx \\
&> -2\sigma(c),
\end{aligned}$$

where we have used (31). This indicates that

$$\bar{\lambda} > \frac{1}{2_\alpha \|Q_{2_\alpha}\|_2^{22_\alpha}} c^{2_\alpha - 1} + \frac{\mu g_1}{1+g_1}$$

by (42) when $p = 2_\alpha$, and

$$\bar{\lambda} > \frac{\mu g_1}{1+g_1}$$

by (44) when $2_\alpha < p \le \bar{p}$. We complete the proof.

3.2 The subcase $\bar{p} < p < 2_\alpha^*$

In this subsection, we consider the case of $\bar{p} < p < 2_\alpha^*$. For this case, the functional J is unbounded from below on $S(c)$, and it is not possible to look for a global minimizer on $S(c)$. So we shall use the Pohozaev manifold $\mathcal{M}(c)$ defined in Section 2 to find critical points of J.

Lemma 3.2 *Assume that $\mu > 0$, $\bar{p} < p < 2_\alpha^*$ and condition $(D1)$ holds. Then the functional J is coercive and bounded from below on $\mathcal{M}(c)$ for all $c > 0$. Furthermore, there exists a constant $c_2 > 0$ such that for $0 < c < c_2$, J is bounded from below by a positive constant on $\mathcal{M}^-(c)$.*

Proof For each $u \in \mathcal{M}(c)$, we have

$$A(u) - \frac{Np - N - \alpha}{2p}B(u) - \frac{\mu N}{2}\int_{\mathbb{R}^N} \left[\ln\left(1 + \frac{|u|^2}{1+g_1}\right) - \frac{|u|^2}{1+g_1+|u|^2}\right] dx = 0.$$

Using this, together with (31), leads to

$$\begin{aligned}J(u) &= \frac{1}{2}A(u) - \frac{1}{2p}B(u) - \frac{\mu}{2}\int_{\mathbb{R}^N}\left[|u|^2 - \ln\left(1+\frac{|u|^2}{1+g_1}\right)\right]dx\\ &= \left(\frac{1}{2}-\frac{1}{Np-N-\alpha}\right)A(u) - \frac{\mu}{2}\int_{\mathbb{R}^N}\frac{g_1+|u|^2}{1+g_1+|u|^2}|u|^2dx\\ &\quad+\frac{\mu}{2}\left(1+\frac{N}{Np-N-\alpha}\right)\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{|u|^2}{1+g_1}\right)-\frac{|u|^2}{1+g_1+|u|^2}\right]dx\\ &\geq \left(\frac{1}{2}-\frac{1}{Np-N-\alpha}\right)A(u) - \frac{\mu}{2}\int_{\mathbb{R}^N}|u|^2dx\\ &= \left(\frac{1}{2}-\frac{1}{Np-N-\alpha}\right)A(u) - \frac{\mu c}{2},\end{aligned} \tag{53}$$

which implies that J is bounded from below and coercive on $\mathcal{M}(c)$.

For $u \in \mathcal{M}^-(c)$, similar to the argument in Lemma 2.8, we have

$$A(u) > \Lambda_c > 0 \text{ for } c < c_0, \tag{54}$$

where Λ_c and c_0 are as (29) and (30), respectively. Note that $A(u) \to +\infty$ as $c \to 0$. Then it follows from (53) that there exist two constants $c_2 < c_0$ and $D_0 := D_0(\mu) > \frac{\mu|g_1|c}{2(1+g_1)}$ such that $J(u) > D_0$ for all $c < c_2$. □

Lemma 3.3 *Assume that $\mu > 0$, $\bar{p} < p < 2^*_\alpha$ and condition $(D1)$ holds. Then we have $\mathcal{M}^0(c) = \emptyset$ for $c < c_0$.*

Proof Suppose on the contrary. Let $u \in \mathcal{M}^0(c)$. Similar to the argument of Lemma 3.2, we deduce that for $c < c_0$,

$$\begin{aligned}A(u) &\geq \left[\frac{\|Q_p\|_2^{2p-2}\left(2(N+2)(1+g_1)^{\bar{q}/2} - \mu\mathbf{B}_{\bar{q}}N^2C_{N,\bar{q}}^{\bar{q}}c^{2/N}\right)}{c^{\frac{N+\alpha-p(N-2)}{2}}(1+g_1)^{\bar{q}/2}(Np-\alpha)(Np-N-\alpha)}\right]^{\frac{2}{Np-N-\alpha-2}}\\ &\to +\infty \text{ as } c \to 0 \text{ if } \bar{p} < p < 2^*_\alpha.\end{aligned} \tag{55}$$

On the other hand, by (21) and the fact of $\ln(1+x) < x$ for all $x > 0$, we have

$$\begin{aligned}(Np-N-\alpha-2)A(u) &= \frac{\mu N(Np-\alpha)}{2}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{|u|^2}{1+g_1}\right)-\frac{|u|^2}{1+g_1+|u|^2}\right]dx\\ &\quad-\frac{\mu N^2}{2}\int_{\mathbb{R}^N}\frac{|u|^4}{\left(1+g_1+|u|^2\right)^2}dx\\ &< \frac{\mu N(Np-\alpha)}{2(1+g_1)}\int_{\mathbb{R}^N}\frac{|u|^4}{1+g_1+|u|^2}dx\\ &< \frac{\mu N(Np-\alpha)c}{2(1+g_1)},\end{aligned}$$

which implies that

$$A(u) < \frac{\mu N(Np-\alpha)c}{2(Np-N-\alpha-2)(1+g_1)} \to 0 \text{ as } c \to 0. \tag{56}$$

Thus, from (55–56) we arrive at a contradiction on $A(u)$. We complete the proof. □

According to Lemma 3.3, it holds $\mathcal{M}(c) = \mathcal{M}^+(c) \cup \mathcal{M}^-(c)$, which is a natural constraint manifold. Next, let us prove that the submanifold $\mathcal{M}^-(c)$ is nonempty. Set

$$c_2^* := \frac{(1+g_1)^2}{\mu C_{2,4}^4},$$

where $C_{2,4}$ is the best constant in (11) with $N = 2$ and $s = 4$.

Lemma 3.4 *Assume that $\mu > 0$, $\bar{p} < p < 2_\alpha^*$ and condition $(D1)$ holds. In addtion, we further assume that $c < c_2^*$ if $N = 2$. Then for any $u \in S(c)$, there exists a constant $t_u^- > 0$ such that $u^{t_u^-} \in \mathcal{M}^-(c)$. In particular, t_u^- is a local maximum point of $f_u(t)$.*

Proof Note that for $u \in S(c)$ and $t > 0$, $u^t \in \mathcal{M}(c)$ if and only if $f_u'(t) = 0$. By the fact of $\ln(1+x) < x$ for all $x > 0$, a direct calculation shows that

$$\begin{aligned}
f_u'(t) &= tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u) - \frac{\mu N}{2t^{N+1}} \int_{\mathbb{R}^N} \ln\left(1 + \frac{t^N|u|^2}{1+g_1}\right) dx \\
&\quad + \frac{\mu N}{2t} \int_{\mathbb{R}^N} \frac{|u|^2}{1+g_1+t^N|u|^2} dx \\
&\geq tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u) - \frac{\mu N}{2t} \int_{\mathbb{R}^N} \frac{|u|^2}{1+g_1} dx \\
&\quad + \frac{\mu N}{2t} \int_{\mathbb{R}^N} \frac{|u|^2}{1+g_1+t^N|u|^2} dx \\
&= tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u) - \frac{\mu N}{2t} \int_{\mathbb{R}^N} \left[\frac{|u|^2}{1+g_1} - \frac{|u|^2}{1+g_1+t^N|u|^2}\right] dx \\
&= tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u) - \frac{\mu N}{2} \int_{\mathbb{R}^N} \frac{t^{N-1}|u|^4}{(1+g_1)(1+g_1+t^N u^2)} dx \\
&\geq tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u) - \frac{\mu N t^{N-1}}{2(1+g_1)^2} \int_{\mathbb{R}^N} |u|^4 dx. \qquad (57)
\end{aligned}$$

If $N \geq 3$, then it is clear that $f_u'(t) > 0$ for $t > 0$ small enough by (57). If $N = 2$, then from (11) and (57) we have

$$f_u'(t) \geq \left[1 - \frac{c\mu C_{2,4}^4}{(1+g_1)^2}\right] tA(u) - \frac{Np-N-\alpha}{2p} t^{N(p-1)-\alpha-1} B(u),$$

which implies that $f_u'(t) > 0$ for $t > 0$ small enough, since $c < c_2^*$.

On the other hand, it follows from (31) that

$$\begin{aligned}
f_u'(t) &= tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u) - \frac{\mu N}{2t^{N+1}} \int_{\mathbb{R}^N} \ln\left(1 + \frac{t^N|u|^2}{1+g_1}\right) dx \\
&\quad + \frac{\mu N}{2t} \int_{\mathbb{R}^N} \frac{|u|^2}{1+g_1+t^N|u|^2} dx \\
&= tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u) \\
&\quad - \frac{\mu N}{2t^{N+1}} \int_{\mathbb{R}^N} \left[\ln\left(1 + \frac{t^N|u|^2}{1+g_1}\right) - \frac{t^N|u|^2}{1+g_1+t^N|u|^2}\right] dx
\end{aligned}$$

$$\leq tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u),$$

which implies that $f_u'(t) < 0$ for $t > 0$ large enough, since $p > \bar{p}$. Therefore, there exists a constant $t_u^- > 0$ such that $f_u'(t_u^-) = 0$ and $f_u''(t_u^-) < 0$, which means that $u^{t_u^-} \in \mathcal{M}^-(c)$ and t_u^- is a local maximum point of $f_u(t)$. We complete the proof. □

Remark 3.1 From Lemma 3.4 one can see that it is difficult for us to prove the uniqueness of t_u^-, due to the complex form of fibering map arising from saturable nonlinearity. Moreover, we even can not prove that the submanifold $\mathcal{M}^+(c)$ is nonempty.

We now define

$$S_r(c) := S(c) \cap H_r^1(\mathbb{R}^N),\ \mathcal{M}_r(c) := \mathcal{M}(c) \cap H_r^1(\mathbb{R}^N) \text{ and } \mathcal{M}_r^-(c) := \mathcal{M}^-(c) \cap H_r^1(\mathbb{R}^N). \tag{58}$$

By virtue of Lemmas 3.2 and 3.4 one has

$$m_r^-(c) := \inf_{u \in \mathcal{M}_r^-(c)} J(u) \geq \inf_{u \in \mathcal{M}^-(c)} J(u) > 0.$$

Next we apply Lemma 2.7 to construct a Palais–Smale sequence $\{u_n\} \subset \mathcal{M}_r^-(c)$ for J restricted to $S(c)$. Our arguments are inspired by [1, 4]. Observe that $\Theta = \emptyset$ is admissible. First of all, we introduce the following lemma.

Lemma 3.5 *The map $u \in S_r(c) \mapsto t_u^- \in \mathbb{R}$ is of class C^1.*

Proof Consider the C^1 function $\phi : \mathbb{R} \times S_r(c) \to \mathbb{R}$ defined by $\phi(t,u) = f_u'(t)$. Since $\phi(t_u^-, u) = 0$, $\partial_t \phi(t_u^-, u) = f_u''(t_u^-) < 0$ and $\mathcal{M}^0(c) = \emptyset$, the proof is complete by using the implicit function theorem. □

Now we define the functional $G^- : S_r(c) \to \mathbb{R}$ by $G^-(u) = J(u^{t_u^-})$. Clearly, it follows from Lemma 3.5 that the functional G^- is of class C^1. We also need the following result.

Lemma 3.6 *The map $\Psi : T_u S_r(c) \to T_{u^{t_u^-}} S_r(c)$ defined by $\psi \to \psi^{t_u^-}$ is isomorphism, where $T_u S_r(c)$ denotes the tangent space to $S_r(c)$ in u.*

Proof For $\psi \in T_u S_r(c)$, we have

$$\int_{\mathbb{R}^N} u^{t_u^-}(x)\psi^{t_u^-}(x)dx = \int_{\mathbb{R}^N} (t_u^-)^{N/2} u(t_u^- x)(t_u^-)^{N/2}\psi(t_u^- x)dx = \int_{\mathbb{R}^N} u(y)\psi(y)dy = 0,$$

which implies that $\psi^{t_u^-} \in T_{u^{t_u^-}} S_r(c)$, and thus the map Ψ is well defined. Moreover, for $\forall \psi_1, \psi_2 \in T_u S_r(c)$ and $\forall k \in \mathbb{R}$, it holds

$$\Psi(\psi_1+\psi_2) = (\psi_1+\psi_2)^{t_u^-} = (t_u^-)^{N/2}(\psi_1(t_u^- x)+\psi_2(t_u^- x)) = \psi_1^{t_u^-} + \psi_2^{t_u^-} = \Psi(\psi_1)+\Psi(\psi_2)$$

and $\Psi(k\psi_1) = (k\psi_1)^{t_u^-} = k\psi_1^{t_u^-} = k\Psi(\psi_1)$. This shows that the map Ψ is linear. Finally, let us claim that the map Ψ is a bijection. For $\forall \psi_1, \psi_2 \in T_u S_r(c)$ with $\psi_1 \neq \psi_2$, by the fact of $t_u^- > 0$, we have

$$\Psi(\psi_1) = (t_u^-)^{N/2}\psi_1(t_u^- x) \neq (t_u^-)^{N/2}\psi_2(t_u^- x) = \Psi(\psi_2).$$

Moreover, let $\chi \in T_{u^{t_u^-}} S_r(c)$. Clearly, $\left((t_u^-)^{-N/2}\chi(\frac{x}{t_u^-})\right)^{t_u^-} = \chi(x)$ and

$$\int_{\mathbb{R}^N} (t_u^-)^{-N/2}\chi\left(\frac{x}{t_u^-}\right)u(x)dx = \int_{\mathbb{R}^N} \chi(y)(t_u^-)^{N/2}u(t_u^- y)dy = \int_{\mathbb{R}^N} \chi(y)u^{t_u^-}(y)dy = 0,$$

leading to $(t_u^-)^{-N/2}\chi\left(\frac{x}{t_u^-}\right) \in T_u S_r(c)$. So, Ψ is a bijection. We complete the proof. □

Lemma 3.7 *It holds $(G^-)'(u)[\psi] = J'(u^{t_u^-})[\psi^{t_u^-}]$ for any $u \in S_r(c)$ and $\psi \in T_u S_r(c)$.*

Proof The proof is similar to that of [4, Lemma 3.15] (or [1, Lemma 3.2]), and we omit it here. □

Lemma 3.8 *Assume that $\mu > 0$, $\bar{p} < p < 2_\alpha^*$ and condition $(D1)$ holds. Let $\mathcal{F}$ be a homotopy stable family of compact subsets of $S_r(c)$ with closed boundary Θ and let*

$$e_{\mathcal{F}}^- := \inf_{H\in\mathcal{F}} \max_{u\in H} G^-(u).$$

Suppose that Θ is contained in a connected component of $\mathcal{M}_r^-(c)$ and that $\max\{\sup G^-(\Theta), 0\} < e_{\mathcal{F}}^- < \infty$. Then there exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}_r^-(c)$ for J restricted to $S_r(c)$ at level $e_{\mathcal{F}}^-$.

Proof First of all, we take $\{D_n\} \subset \mathcal{F}$ such that $\max_{u\in D_n} G^-(u) < e_{\mathcal{F}}^- + \frac{1}{n}$ and define $\eta : [0, 1] \times S(c) \to S(c)$ by

$$\eta(s, u) = u^{1-s+st_u^-}.$$

Note that η is continuous. Since $t_u^- = 1$ for any $u \in \mathcal{M}_r^-(c)$ and $\Theta \subset \mathcal{M}_r^-(c)$, we have $\eta(s, u) = u$ for $(s, u) \in (\{0\} \times S_r(c)) \cup ([0, 1] \times \Theta)$. Then, according to the definition of $\mathcal{F}$, one has

$$\mathbf{A}_n := \eta(\{1\} \times D_n) = \{u^{t_u^-} \mid u \in D_n\} \in \mathcal{F}.$$

Clearly, $\mathbf{A}_n \subset \mathcal{M}_r^-(c)$ for all $n \in \mathbb{N}$. Let $v \in \mathbf{A}_n$, that is $v = u^{t_u^-}$ for some $u \in D_n$. Then $G^-(u) = J(u^{t_u^-}) = J(v) = G^-(v)$, which shows that $\max_{\mathbf{A}_n} G^- = \max_{D_n} G^-$. Thus, $\{\mathbf{A}_n\} \subset \mathcal{M}_r^-(c)$ is another minimizing sequence of $e_{\mathcal{F}}^-$. By Lemma 2.7, we obtain a Palais-Smale sequence $\{v_n\}$ for G^- on $S_r(c)$ at level $e_{\mathcal{F}}^-$ satisfying $dist(v_n, \mathbf{A}_n) \to 0$ as $n \to \infty$. For $v_n \in S_r(c)$, there exists a constant $t_{v_n}^- > 0$ such that $u_n := t_{v_n}^- v_n \in \mathcal{M}_r^-(c)$.

Next we claim that there exists a constant $C_0 > 0$ such that

$$\frac{1}{C_0} \le (t_{v_n}^-)^2 \le C_0 \text{ for } n \in \mathbb{N}. \tag{59}$$

Indeed, it holds

$$(t_{v_n}^-)^2 = \frac{A(v_n^{t_{v_n}^-})}{A(v_n)}.$$

Since $J(v_n^{t_{v_n}^-}) = G^-(v_n) \to e_{\mathcal{F}}^-$, it follows from Lemma 3.2 that there exists a constant $M_0 > 0$ such that

$$\frac{1}{M_0} \le A(v_n^{t_{v_n}^-}) \le M_0. \tag{60}$$

On the other hand, since $\{\mathbf{A}_n\} \subset \mathcal{M}_r^-(c)$ is a minimizing sequence for $e_{\mathcal{F}}^-$ and J is coercive on $\mathcal{M}^-(c)$, we have $\{\mathbf{A}_n\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$. Note that $dist(v_n, \mathbf{A}_n) \to 0$ as $n \to \infty$. Then $\sup_n A(v_n) < \infty$. Also, since $\mathbf{A}_n$ is compact for every $n \in \mathbb{N}$, there exists a $\bar{v}_n \in \mathbf{A}_n$ such that $dist(v_n, \mathbf{A}_n) = \|\bar{v}_n - v_n\|_{H^1}$. Then by Lemma 3.2, we obtain that for a constant $\delta > 0$,

$$A(v_n) \ge A(\bar{v}_n) - A(v_n - \bar{v}_n) \ge \frac{\delta}{2}. \tag{61}$$

Thus, by (60) and (61), we prove the claim.

Next, we show that $\{u_n\} \subset \mathcal{M}_r^-(c)$ is a Palais-Smale sequence for J on $S_r(c)$ at level $e_{\mathcal{F}}^-$. Denote the norm of space $T_{u_n}(S_r(c))$ and dual space of $T_{u_n}(S_r(c))$ by $\|\cdot\|$ and $\|\cdot\|_*$, respectively. Then we have

$$\|J'(u_n)\|_* = \sup_{\psi\in T_{u_n}S_r(c),\|\psi\|\le 1} |\langle J'(u_n),\psi\rangle| = \sup_{\psi\in T_{u_n}S_r(c),\|\psi\|\le 1} |\langle J'(u_n),(\psi^{-t_{v_n}^-})^{t_{v_n}^-}\rangle|. \tag{62}$$

By Lemma 3.6, we know that the map $\Psi : T_{v_n}S_r(c) \to T_{v_n^{t_{v_n}^-}} S_r(c)$ defined by $\psi \to \psi^{t_{v_n}^-}$ is isomorphism. Moreover, it follows from Lemma 3.7 that $\langle (G^-)'(v_n), \psi^{-t_{v_n}^-}\rangle = \langle J'(u_n),\psi\rangle$. Then by (62), we have

$$\|J'(u_n)\|_* = \sup_{\psi\in T_{u_n}S_r(c),\|\psi\|\le 1} |\langle J'(u_n),\psi\rangle| = \sup_{\psi\in T_{u_n}S_r(c),\|\psi\|\le 1} |\langle (G^-)'(v_n),\psi^{-t_{v_n}^-}\rangle|. \tag{63}$$

Note that $\|\psi^{-t_{v_n}^-}\| \le C\|\psi\| \le C$ by (59). Thus, from (63) it follows that $\{u_n\} \subset \mathcal{M}_r^-(c)$ is a Palais-Smale sequence for J on $S_r(c)$ at level $e_{\mathcal{F}}^-$. We complete the proof. □

Lemma 3.9 *Assume that $\mu > 0$, $\bar{p} < p < 2_\alpha^*$ and condition $(D1)$ holds. Then there exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}_r^-(c)$ for J restricted to $S_r(c)$ at level $m_r^-(c) > \frac{\mu|g_1|c}{2(1+g_1)}$.*

Proof By Lemma 3.8, we choose the set $\overline{\mathcal{F}}$ of all singletons belonging to $S_r(c)$ and $\Theta = \emptyset$, which is clearly a homotopy stable family of compact subsets of $S_r(c)$ (without boundary). Note that $e_{\overline{\mathcal{F}}}^- = \inf_{H\in\overline{\mathcal{F}}} \max_{u\in H} G^-(u) = \inf_{u\in S_r(c)} G^-(u) = \inf_{u\in\mathcal{M}_r^-(c)} J(u) = m_r^-(c)$. Then the lemma follows directly from Lemma 3.8. We complete the proof. □

Now we are ready to prove the Theorem 1.2 (iii). By Lemma 3.9, there exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{M}_r^-(c)$ for J restricted to $S(c)$ at level $m_r^-(c) > \frac{\mu|g_1|c}{2(1+g_1)}$, which is bounded in $H_r^1(\mathbb{R}^N)$ via Lemma 3.2. So, for $\bar{p} < p < 2_\alpha^*$, according to Lemma 2.8, for

$$c < \bar{c} := \begin{cases} \min\{c_0^*, c_2, c_2^*\} & \text{if } N = 2,\\ \min\{c_0^*, c_2\} & \text{if } N \ge 3,\end{cases}$$

Problem (P_c) admits a radially symmetric solution w satisfying $J(w) = m_r^-(c) > \frac{\mu|g_1|c}{2(1+g_1)}$ for some $\bar{\lambda} > 0$.

Next, we give the asymptotic behavior of $J(w)$ and $\bar{\lambda}$ as $c \to 0$. Since $w \in \mathcal{M}_r^-(c)$, by (54) one has

$$A(w) > \Lambda_c = \left[\frac{\|Q_p\|_2^{2p-2}\left(2(N+2)(1+g_1)^{\bar{q}/2} - \mu\mathbf{B}_{\bar{q}}N^2C_{N,\bar{q}}^{\bar{q}}c^{2/N}\right)}{c^{\frac{N+\alpha-p(N-2)}{2}}(1+g_1)^{\bar{q}/2}(Np-\alpha)(Np-N-\alpha)}\right]^{\frac{2}{Np-N-\alpha-2}}. \tag{64}$$

It follows from (53) and (64) that

$$\begin{aligned} J(w) &\ge \left(\frac{1}{2} - \frac{1}{Np-N-\alpha}\right)A(w) - \frac{\mu c}{2}\\ &> K_1 c^{-\frac{N+\alpha-p(N-2)}{Np-N-\alpha-2}} - \frac{\mu c}{2}, \end{aligned}$$

where

$$K_1 := \left[\frac{Np-N-\alpha-2}{2(Np-N-\alpha)}\right]\left[\frac{\|Q_p\|_2^{2p-2}\left(2(N+2)(1+g_1)^{\bar{q}/2} - \mu\mathbf{B}_{\bar{q}}N^2C_{N,\bar{q}}^{\bar{q}}c^{2/N}\right)}{(1+g_1)^{\bar{q}/2}(Np-\alpha)(Np-N-\alpha)}\right]^{\frac{2}{Np-N-\alpha-2}}$$

$$\to \left[\frac{Np-N-\alpha-2}{2(Np-N-\alpha)}\right]\left[\frac{2\|Q_p\|_2^{2p-2}(N+2)(1+g_1)^{\bar{q}/2}}{(1+g_1)^{\bar{q}/2}(Np-\alpha)(Np-N-\alpha)}\right]^{\frac{2}{Np-N-\alpha-2}} \quad \text{as } c\to 0.$$

Moreover, since $Q(w)=0$, by (64) and the fact of $\ln(1+x)<x$ for all $x>0$ one has

$$\begin{aligned}
\bar{\lambda}c &= \frac{N+\alpha-p(N-2)}{Np-N-\alpha}A(w)+\mu\int_{\mathbb{R}^N}\frac{g_1+|w|^2}{1+g_1+|w|^2}|w|^2dx \\
&\quad -\frac{\mu Np}{Np-N-\alpha}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{|w|^2}{1+g_1}\right)-\frac{|w|^2}{1+g_1+|w|^2}\right]dx \\
&> K_2 c^{-\frac{2p-2}{Np-N-\alpha-2}+1}-\frac{\mu c}{1+g_1}\left(\frac{Np}{Np-N-\alpha}-g_1\right),
\end{aligned} \tag{65}$$

where

$$K_2 := \left[\frac{N+\alpha-p(N-2)}{Np-N-\alpha}\right]\left[\frac{\|Q_p\|_2^{2p-2}\left(2(N+2)(1+g_1)^{\bar{q}/2}-\mu\mathbf{B}_{\bar{q}}N^2C_{N,\bar{q}}^{\bar{q}}c^{2/N}\right)}{(1+g_1)^{\bar{q}/2}(Np-\alpha)\,(Np-N-\alpha)}\right]^{\frac{2}{Np-N-\alpha-2}}$$

$$\to \left[\frac{N+\alpha-p(N-2)}{Np-N-\alpha}\right]\left[\frac{2\|Q_p\|_2^{2p-2}(N+2)(1+g_1)^{\bar{q}/2}}{(1+g_1)^{\bar{q}/2}(Np-\alpha)(Np-N-\alpha)}\right]^{\frac{2}{N(p-1)-\alpha-2}} \quad \text{as } c\to 0.$$

This indicates that

$$\bar{\lambda} > K_2 c^{-\frac{2p-2}{Np-N-\alpha-2}}-\frac{\mu}{1+g_1}\left(\frac{Np}{Np-N-\alpha}-g_1\right).$$

We complete the proof.

4 The case $\mu < 0$

4.1 The subcase $p = 2_\alpha$

Proof of Theorem 1.3 (i). Let $u\in S(c)$ and $t>0$. Since $p=2_\alpha$ and $\mu<0$, it follows from (31) that

$$\begin{aligned}
f_u'(t) &= tA(u)+\frac{|\mu|N}{2t^{N+1}}\int_{\mathbb{R}^N}\ln\left(1+\frac{t^N|u|^2}{1+g_1}\right)dx-\frac{|\mu|N}{2t}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g_1+t^N|u|^2}dx \\
&= tA(u)+\frac{|\mu|N}{2t^{N+1}}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{t^N|u|^2}{1+g_1}\right)-\frac{t^N|u|^2}{1+g_1+t^N|u|^2}\right]dx \\
&> 0,
\end{aligned}$$

which implies that the fibering map $f_u(t)=J(u^t)$ is strictly increasing on t. This means that the functional J has no critical point on $S(c)$. In other words, problem (P_c) has no solution for any $\lambda\in\mathbb{R}$. We complete the proof. □

4.2 The subcase $2_\alpha < p \le \bar{p}$

Lemma 4.1 *Assume that $\mu<0, 2_\alpha<p\le\bar{p}$ and one of conditions $(D1)$, $(D4)$ holds. In addition, we assume that $c<\|Q_{\bar{p}}\|_2^{4(\bar{p}-1)/(N+\alpha-\bar{p}(N-2))}$ if $p=\bar{p}$. Then the functional J is coercive and bounded from below on $S(c)$.*

Proof For $u \in S(c)$, by (14) and the fact of $\ln(1+x) < x$ for $x > 0$, we have

$$\begin{aligned}J(u) &= \frac{1}{2}A(u) - \frac{1}{2p}B(u) + \frac{|\mu|}{2}\int_{\mathbb{R}^N}\left[u^2 - \ln\left(1 + \frac{|u|^2}{1+g(x)}\right)\right]dx \\ &\geq \frac{1}{2}A(u) - \frac{c^{\frac{N+\alpha-p(N-2)}{2}}}{2\|Q_p\|_2^{2p-2}}A(u)^{\frac{Np-N-\alpha}{2}} - \frac{|\mu|}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g(x)}dx \\ &\geq \frac{1}{2}A(u) - \frac{c^{\frac{N+\alpha-p(N-2)}{2}}}{2\|Q_p\|_2^{2p-2}}A(u)^{\frac{Np-N-\alpha}{2}} - \frac{|\mu|c}{2(1+g_1)},\end{aligned}$$

which implies that J is coercive and bounded from below on $S(c)$. We complete the proof. □

Now we give the proof of Theorem 1.3 (ii). In the following, we proceed our argument only under condition $(D1)$, since the other case is similar. For $u \in S(c)$ fixed, by (31) and Lebesgue's dominated convergence theorem one has

$$\begin{aligned}\sigma(c) &\leq J(u^t) = \frac{t^2}{2}A(u) - \frac{t^{Np-N-\alpha}}{2p}B(u) + \frac{|\mu|c}{2} - \frac{|\mu|}{2t^N}\int_{\mathbb{R}^N}\ln\left(1 + \frac{t^N|u|^2}{1+g_1}\right)dx \\ &< \frac{t^2}{2}A(u) - \frac{t^{Np-N-\alpha}}{2p}B(u) + \frac{|\mu|c}{2} - \frac{|\mu|}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g_1+t^N|u|^2}dx \\ &\to \frac{g_1|\mu|c}{2(1+g_1)} \text{ as } t \to 0,\end{aligned}$$

which implies that

$$\sigma(c) \leq \frac{g_1|\mu|c}{2(1+g_1)}. \tag{66}$$

Let $\{u_n\} \subset S(c)$ be a minimizing sequence for $\sigma(c)$ on $H^1(\mathbb{R}^N)$. Then $\{u_n\}$ is bounded on $H^1(\mathbb{R}^N)$ by Lemma 4.1. Next we claim that

$$\eta := \lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^2dx > 0. \tag{67}$$

Assume on the contrary that $\eta = 0$. By Lions's lemma in [31], one has $\|u_n\|_s \to 0$ as $n \to \infty$ for $2 < s < 2^*$, which implies that $B(u_n) = o(1)$ by (10). Using this, together with the fact of $\ln(1+x) < x$ for all $x > 0$, leads to

$$\begin{aligned}\sigma(c) + o(1) = J(u_n) &= \frac{1}{2}A(u_n) - \frac{1}{2p}B(u_n) + \frac{|\mu|}{2}\int_{\mathbb{R}^N}\left[|u_n|^2 - \ln\left(1 + \frac{|u_n|^2}{1+g_1}\right)\right]dx \\ &= \frac{1}{2}A(u_n) + \frac{|\mu|g_1}{2(1+g_1)}\int_{\mathbb{R}^N}|u_n|^2dx \\ &\quad + \frac{|\mu|}{2}\int_{\mathbb{R}^N}\left[\frac{|u_n|^2}{1+g_1} - \ln\left(1 + \frac{|u_n|^2}{1+g_1}\right)\right]dx + o(1) \\ &> \frac{|\mu|g_1c}{2(1+g_1)} + o(1),\end{aligned}$$

which contradicts with (66). Thus, (67) holds. Now we define translations of $\{u_n\}$ by $w_n(x) = u_n(x+y_n)$. Clearly, $\{w_n\}$ is also a minimizing sequence for $\sigma(c)$ on $S(c)$ and w_n is bounded in $H^1(\mathbb{R}^N)$. By (67), we have

$$\limsup_{n\to\infty}\int_{B_1(0)}|w_n|^2dx \geq \frac{\eta}{2}.$$

Thus, we can assume that $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$, $w_n \to w \neq 0$ in $L^2(B_1(0))$ and $w_n(x) \to w(x) \neq 0$ a.e. on $B_1(0)$.

Next, we prove that $\|w\|_2^2 = c$. Otherwise, assume that $\rho = \|w\|_2^2 \in (0, c)$. Let

$$\tilde{w} = \frac{w}{\sqrt{1+g_1}} \text{ and } \tilde{v}_n = \frac{w_n - w}{\sqrt{1+g_1}}.$$

Similar to the argument of Theorem 1.2$(i)-(ii)$, it follows from (49) that

$$\begin{aligned}
\sigma(c) + o(1) &= J(w_n) \\
&= \frac{1}{2}A(w_n) - \frac{1}{2p}B(w_n) + \frac{|\mu|}{2}\int_{\mathbb{R}^N}\left[|w_n|^2 - \ln\left(1 + \frac{|w_n|^2}{1+g_1}\right)\right]dx \\
&= \frac{1}{2}A(w_n) - \frac{1}{2p}B(w_n) - \frac{|\mu|}{2}\int_{\mathbb{R}^N}\left[|w_n|^2 - \ln\left(1 + \frac{|w_n|^2}{1+g_1}\right)\right]dx \\
&\quad + |\mu|\int_{\mathbb{R}^N}\left[|w_n|^2 - \ln\left(1 + \frac{|w_n|^2}{1+g_1}\right)\right]dx \\
&= \frac{\rho}{2c}A\left(\frac{\sqrt{c}w}{\|w\|_2}\right) + \frac{c-\rho}{2c}A\left(\frac{\sqrt{c}(w_n - w)}{\|w_n - w\|_2}\right) \\
&\quad - \frac{1}{2p}\left(\frac{\rho}{c}\right)^p B\left(\frac{\sqrt{c}w}{\|w\|_2}\right) - \frac{1}{2p}\left(\frac{c-\rho}{c}\right)^p B\left(\frac{\sqrt{c}(w_n - w)}{\|w_n - w\|_2}\right) \\
&\quad - \frac{|\mu|}{2}\int_{\mathbb{R}^N} h\left(\frac{\rho}{c}\left(\frac{(\sqrt{c}\tilde{w})^2}{\|w\|_2^2}\right) + \frac{c-\rho}{c}\frac{(\sqrt{c}\tilde{v}_n)^2}{\|w_n - w\|_2^2}\right)dx - \frac{|\mu| I_1}{2}\int_{\mathbb{R}^N}(|\tilde{w}|^2 + |\tilde{v}_n|^2)dx \\
&\quad + |\mu|\int_{\mathbb{R}^N}\left[|w_n|^2 - \ln\left(1 + \frac{|w_n|^2}{1+g_1}\right)\right]dx \\
&\geq \frac{\rho}{2c}A\left(\frac{\sqrt{c}w}{\|w\|_2}\right) + \frac{c-\rho}{2c}A\left(\frac{\sqrt{c}(w_n - w)}{\|w_n - w\|_2}\right) \\
&\quad - \frac{\gamma\rho}{2pc}B\left(\frac{\sqrt{c}w}{\|w\|_2}\right) - \frac{\gamma(c-\rho)}{2pc}B\left(\frac{\sqrt{c}(w_n - w)}{\|w_n - w\|_2}\right) - \frac{|\mu| g_1}{2}\int_{\mathbb{R}^N}(|\tilde{w}|^2 + |\tilde{v}_n|^2)dx \\
&\quad - \frac{|\mu|}{2}\int_{\mathbb{R}^N}\left[\frac{\rho}{c}h\left(\frac{(\sqrt{c}\tilde{w})^2}{\|w\|_2^2}\right) + \frac{c-\rho}{c}h\left(\frac{(\sqrt{c}\tilde{v}_n)^2}{\|w_n - w\|_2^2}\right)\right]dx + \frac{|\mu|\xi}{2} \\
&\quad + |\mu|\int_{\mathbb{R}^N}\left[|w_n|^2 - \ln\left(1 + \frac{|w_n|^2}{1+g_1}\right)\right]dx + o(1) \\
&= \frac{\rho}{2c}A\left(\frac{\sqrt{c}w}{\|w\|_2}\right) + \frac{c-\rho}{2c}A\left(\frac{\sqrt{c}(w_n - w)}{\|w_n - w\|_2}\right) \\
&\quad - \frac{\gamma\rho}{2pc}B\left(\frac{\sqrt{c}w}{\|w\|_2}\right) - \frac{\gamma(c-\rho)}{2pc}B\left(\frac{\sqrt{c}(w_n - w)}{\|w_n - w\|_2}\right) \\
&\quad + \frac{|\mu|\rho}{2c}\int_{\mathbb{R}^N}\left[\left(\frac{\sqrt{c}|w|}{\|w\|_2}\right)^2 - \ln\left(1 + \frac{1}{1+g_1}\left(\frac{\sqrt{c}|w|}{\|w\|_2}\right)^2\right)\right]dx \\
&\quad + \frac{|\mu|(c-\rho)}{2c}\int_{\mathbb{R}^N}\left[\left(\frac{\sqrt{c}|w_n - w|}{\|w_n - w\|_2}\right)^2 - \ln\left(1 + \frac{1}{1+g_1}\left(\frac{\sqrt{c}|w_n - w|}{\|w_n - w\|_2}\right)^2\right)\right]dx \\
&\quad + K(w_n, w) + o(1) \\
&> \frac{\rho}{c}J\left(\frac{\sqrt{c}w}{\|w\|_2}\right) + \frac{c-\rho}{c}J\left(\frac{\sqrt{c}(w_n - w)}{\|w_n - w\|_2}\right) + K(w_n, w) + o(1) \\
&> \frac{\rho}{c}J\left(\frac{\sqrt{c}w}{\|w\|_2}\right) + \frac{c-\rho}{c}J\left(\frac{\sqrt{c}(w_n - w)}{\|w_n - w\|_2}\right) + o(1),
\end{aligned}$$

$$\geq \sigma(c) + o(1).$$

Clearly, this is a contradiction. Here note that

$$\begin{aligned}
K(w_n, w) &:= \frac{|\mu|\xi}{2} + |\mu| \int_{\mathbb{R}^N} \left[|w_n|^2 - \ln\left(1 + \frac{|w_n|^2}{1+g_1}\right) \right] \\
&\quad - \frac{|\mu|\rho}{c} \int_{\mathbb{R}^N} \left[\left(\frac{\sqrt{c}|w|}{\|w\|_2}\right)^2 - \ln\left(1 + \frac{1}{1+g_1}\left(\frac{\sqrt{c}|w|}{\|w\|_2}\right)^2\right) \right] dx \\
&\quad - \frac{|\mu|(c-\rho)}{c} \int_{\mathbb{R}^N} \left[\left(\frac{\sqrt{c}|w_n - w|}{\|w_n - w\|_2}\right)^2 - \ln\left(1 + \frac{1}{1+g_1}\left(\frac{\sqrt{c}|w_n - w|}{\|w_n - w\|_2}\right)^2\right) \right] dx \\
&> \frac{|\mu|\xi}{2} + \frac{|\mu| g_1 c}{1+g_1} - |\mu| c \\
&= \frac{|\mu|\xi}{2} - \frac{|\mu| c}{1+g_1} \\
&\geq 0 \text{ if } g_1 \geq \frac{2c}{\xi} - 1.
\end{aligned}$$

So we have $w_n \to w$ in $L^2(\mathbb{R}^N)$. Hence, it follows from (50–52) that

$$\begin{aligned}
\sigma(c) &= \lim_{n\to\infty} J(w_n) \\
&= \lim_{n\to\infty} \left(\frac{1}{2} A(w_n) - \frac{1}{2p} B(w_n) - \frac{\mu}{2} \int_{\mathbb{R}^N} \left[|w_n|^2 - \ln\left(1 + \frac{|w_n|^2}{1+g_1}\right) \right] dx \right) \\
&\geq \frac{1}{2} A(w) - \frac{1}{2p} B(w) - \frac{\mu}{2} \int_{\mathbb{R}^N} \left[|w|^2 - \ln\left(1 + \frac{|w|^2}{1+g_1}\right) \right] dx \\
&\geq \sigma(c),
\end{aligned}$$

which indicates that $\sigma(c)$ is achieved at $w \neq 0$ and $\|w_n - w\|_{H^1} \to 0$ as $n \to \infty$.

since w is a critical point of J restricted to $S(c)$, there exists a Lagrange multiplier $\bar{\lambda} \in \mathbb{R}$ such that $J'(w) + \bar{\lambda} w = 0$. In particular, by (66) and the fact of $\ln(1+x) < x$ for all $x > 0$ one has

$$\begin{aligned}
\bar{\lambda} c &= -A(w) + B(w) - |\mu| \int_{\mathbb{R}^N} \frac{g_1 + |w|^2}{1 + g_1 + |w|^2} |w|^2 dx \\
&= -2p\sigma(c) + (p-1)A(w) + |\mu|(p-1) \int_{\mathbb{R}^N} |w|^2 dx \\
&\quad - |\mu| p \int_{\mathbb{R}^N} \ln\left(1 + \frac{|w|^2}{1+g_1}\right) dx + |\mu| \int_{\mathbb{R}^N} \frac{|w|^2}{1 + g_1 + |w|^2} dx \\
&> -2p\sigma(c) + |\mu|(p-1)c - \frac{|\mu| pc}{1+g_1} \\
&\geq -|\mu| c,
\end{aligned}$$

leading to $\bar{\lambda} > -|\mu|$. We complete the proof.

4.3 The subcase $\bar{p} < p < 2_\alpha^*$

Lemma 4.2 *Assume that $\mu < 0$, $\bar{p} < p < 2_\alpha^*$ and condition $(D1)$ holds. Then the functional J is coercive and bounded from below on $\mathcal{M}(c)$ for all $c > 0$. Furthermore, there exists*

$c_3 > 0$ *such that for every* $c < c_3$,

$$J(u) \geq \frac{|\mu|c}{2} \text{ on } \mathcal{M}^-(c).$$

Proof For $u \in \mathcal{M}(c)$, it holds

$$A(u) - \frac{Np - N - \alpha}{2p} B(u) + \frac{|\mu|N}{2} \int_{\mathbb{R}^N} \left[\ln\left(1 + \frac{|u|^2}{1+g_1}\right) - \frac{|u|^2}{1+g_1+|u|^2} \right] dx = 0.$$

Using this, together with the fact of $\ln(1+x) \leq x$ for $x \geq 0$, leads to

$$\begin{aligned} J(u) &= \frac{1}{2}A(u) - \frac{1}{2p}B(u) + \frac{|\mu|}{2}\int_{\mathbb{R}^N}\left[|u|^2 - \ln\left(1+\frac{|u|^2}{1+g_1}\right)\right]dx \\ &= \frac{Np-N-\alpha-2}{2(Np-N-\alpha)}A(u) + \frac{|\mu|}{2}\int_{\mathbb{R}^N}\frac{g_1+|u|^2}{1+g_1+|u|^2}|u|^2 dx \\ &\quad - \frac{|\mu|(Np-\alpha)}{2(Np-N-\alpha)}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{|u|^2}{1+g_1}\right) - \frac{|u|^2}{1+g_1+|u|^2}\right]dx \\ &> \frac{Np-N-\alpha-2}{2(Np-N-\alpha)}A(u) - \frac{|\mu|(Np-\alpha)}{2(Np-N-\alpha)}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g_1}dx \\ &\quad - \frac{|\mu|}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g_1+|u|^2}dx + \frac{|\mu|c}{2} \\ &\geq \frac{Np-N-\alpha-2}{2(Np-N-\alpha)}A(u) - \frac{|\mu|c}{2(1+g_1)}\left(\frac{Np-\alpha}{Np-N-\alpha} - g_1\right), \end{aligned} \tag{68}$$

which implies that J is bounded from below and coercive on $\mathcal{M}(c)$, since $p > \bar{p}$.

For $u \in \mathcal{M}^-(c)$, it follows from (14) and (22) that

$$\begin{aligned} A(u) &< \frac{(Np-\alpha)(Np-N-\alpha)}{2p(N+2)}B(u) + \frac{\mu N^2}{2(N+2)}\int_{\mathbb{R}^N}\frac{|u|^4}{(1+g_1+|u|^2)^2}dx \\ &\leq \frac{(Np-\alpha)(Np-N-\alpha)c^{\frac{N+\alpha-p(N-2)}{2}}}{2(N+2)\|Q_p\|_2^{2p-2}}A(u)^{\frac{Np-N-\alpha}{2}}, \end{aligned}$$

which implies that

$$A(u) > \left[\frac{2(N+2)\|Q_p\|_2^{2p-2}}{(Np-\alpha)(Np-N-\alpha)}\right]^{\frac{2}{Np-N-\alpha-2}} c^{-\frac{N+\alpha-p(N-2)}{Np-N-\alpha-2}}. \tag{69}$$

Note that $A(u) \to +\infty$ as $c \to 0$, and together with (68), there exists a constant $c_3 > 0$ such that

$$J(u) > K_3 c^{-\frac{N+\alpha-p(N-2)}{Np-N-\alpha-2}} - \frac{|\mu|c}{2(1+g_1)}\left(\frac{Np-\alpha}{Np-N-\alpha} - g_1\right) \geq \frac{|\mu|c}{2}$$

for all $c < c_3$, where

$$K_3 := \left[\frac{Np-N-\alpha-2}{2(Np-N-\alpha)}\right]\left[\frac{2(N+2)\|Q_p\|_2^{2p-2}}{(Np-\alpha)(Np-N-\alpha)}\right]^{\frac{2}{Np-N-\alpha-2}} > 0. \tag{70}$$

We complete the proof. □

Lemma 4.3 *Assume that* $\mu < 0$, $\bar{p} < p < 2_\alpha^*$ *and condition* $(D1)$ *holds. Then* $\mathcal{M}^0(c) = \emptyset$.

Proof Suppose on the contrary. Let $u \in \mathcal{M}^0(c)$. By (14) and (22), similar to the argument of Lemma 4.2, we have

$$A(u) \geq \left[\frac{2(N+2)\|Q_p\|_2^{2p-2}}{(Np-\alpha)(Np-N-\alpha)}\right]^{\frac{2}{Np-N-\alpha-2}} c^{-\frac{N+\alpha-p(N-2)}{Np-N-\alpha-2}}$$

and further

$$A(u) \to +\infty \text{ as } c \to 0. \tag{71}$$

On the other hand, using (21) and (31) gives

$$\begin{aligned}(Np-N-\alpha-2)\,A(u) &= -\frac{|\mu|N(Np-\alpha)}{2}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{|u|^2}{1+g_1}\right)-\frac{|u|^2}{1+g_1+|u|^2}\right]dx\\ &\quad+\frac{|\mu|N^2}{2}\int_{\mathbb{R}^N}\frac{|u|^4}{\left(1+g_1+|u|^2\right)^2}dx\\ &\leq \frac{|\mu|N^2}{2}\int_{\mathbb{R}^N}\frac{|u|^4}{\left(1+g_1+|u|^2\right)^2}dx\\ &\leq \frac{|\mu|N^2c}{2(1+g_1)},\end{aligned}$$

that is

$$A(u) \leq \frac{|\mu|N^2c}{2(1+g_1)\,(Np-N-\alpha-2)},$$

which implies that

$$A(u) \to 0 \text{ as } c \to 0.$$

Clearly, this contradicts with (71). We complete the proof. □

By virtue of Lemma 4.3, it holds $\mathcal{M}(c) = \mathcal{M}^+(c) \cup \mathcal{M}^-(c)$, which is a natural constraint manifold. Next, let us prove that the submanifold $\mathcal{M}^-(c)$ is nonempty.

Lemma 4.4 *Assume that* $\mu < 0$, $\bar{p} < p < 2_\alpha^*$ *and condition* $(D1)$ *holds. Then for* $u \in S(c)$, *there exists a constant* $\bar{t}_u^- > 0$ *such that* $u^{\bar{t}_u^-} \in \mathcal{M}^-(c)$. *In particular,* $\bar{t}_u^-$ *is a local maximum point of* $f_u(t)$.

Proof Note that for $u \in S(c)$ and $t > 0$, $u^t \in \mathcal{M}(c)$ if and only if $f_u'(t) = 0$. It follows from (31) that

$$\begin{aligned}f_u'(t) &= tA(u) - \frac{Np-N-\alpha}{2p}t^{Np-N-\alpha-1}B(u) + \frac{|\mu|N}{2t^{N+1}}\int_{\mathbb{R}^N}\ln\left(1+\frac{t^N|u|^2}{1+g_1}\right)dx\\ &\quad-\frac{|\mu|N}{2t}\int_{\mathbb{R}^N}\frac{u^2}{1+g_1+t^N|u|^2}dx\\ &= tA(u) - \frac{Np-N-\alpha}{2p}t^{Np-N-\alpha-1}B(u)\\ &\quad+\frac{|\mu|N}{2t^{N+1}}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{t^N|u|^2}{1+g_1}\right)-\frac{t^N|u|^2}{1+g_1+t^N|u|^2}\right]dx\end{aligned}$$

$$\geq tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u),$$

which implies that $f_u'(t) > 0$ for $t > 0$ small enough, since $p > \bar{p}$. On the other hand, by the fact of $\ln(1+s) < s$ for all $s > 0$, we deduce that

$$\begin{aligned}
f_u'(t) &= tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u) + \frac{|\mu|N}{2t^{N+1}} \int_{\mathbb{R}^N} \ln\left(1 + \frac{t^N|u|^2}{1+g_1}\right) dx \\
&\quad - \frac{|\mu|N}{2t} \int_{\mathbb{R}^N} \frac{|u|^2}{1+g_1+t^N|u|^2} dx \\
&\leq tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u) + \frac{|\mu|N}{2t} \int_{\mathbb{R}^N} \frac{|u|^2}{1+g_1} dx \\
&\quad - \frac{|\mu|N}{2t} \int_{\mathbb{R}^N} \frac{u^2}{1+g_1+t^N u^2} dx \\
&\leq tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u) + \frac{|\mu|N}{2t} \int_{\mathbb{R}^N} \left[\frac{|u|^2}{1+g_1} - \frac{|u|^2}{1+g_1+t^N|u|^2}\right] dx \\
&= tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u) + \frac{|\mu|N}{2(1+g_1)} \int_{\mathbb{R}^N} \frac{t^{N-1}|u|^4}{(1+g_1+t^N|u|^2)} dx \\
&\leq tA(u) - \frac{Np-N-\alpha}{2p} t^{Np-N-\alpha-1} B(u) + \frac{|\mu|Nc}{2(1+g_1)t},
\end{aligned}$$

which implies that $f_u'(t) < 0$ for $t > 0$ large enough, since $p > \bar{p}$. Therefore, according to the continuity of $f_u(t)$, there exists a constant $\bar{t}_u^- > 0$ such that $f_u'(\bar{t}_u^-) = 0$ and $f_u''(\bar{t}_u^-) < 0$, that is $u^{\bar{t}_u^-} \in \mathcal{M}^-(c)$. We complete the proof. □

By virtue of Lemmas 4.2 and 4.4 one has

$$m_r^-(c) := \inf_{u \in \mathcal{M}_r^-(c)} J(u) \geq \inf_{u \in \mathcal{M}^-(c)} J(u) \geq \frac{|\mu|c}{2} > 0,$$

where $\mathcal{M}_r^-(c)$ is defined as (58). Similar to the arguments in Sect. 3.2, we also apply Lemma 2.7 to construct a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}_r^-(c)$ for the functional J restricted to $S_r(c)$ defined as (58). Here we only give the conclusions without proof.

Lemma 4.5 *The map $u \in S_r(c) \mapsto \bar{t}_u^- \in \mathbb{R}$ is of class C^1.*

Lemma 4.6 *The map $T_u S_r(c) \to T_{u^{\bar{t}_u^-}} S_r(c)$ defined by $\psi \to \psi^{\bar{t}_u^-}$ is isomorphism, where $T_u S_r(c)$ denotes the tangent space to $S_r(c)$ in u.*

Lemma 4.7 *It holds $(G^-)'(u)[\psi] = J'(u^{\bar{t}_u^-})[\psi^{\bar{t}_u^-}]$ for any $u \in S_r(c)$ and $\psi \in T_u S_r(c)$, where the functional $G^- : S_r(c) \to \mathbb{R}$ is defined by $G^-(u) = J(u^{\bar{t}_u^-})$.*

Lemma 4.8 *Assume that $\mu < 0$, $\bar{p} < p < 2_\alpha^*$ and condition $(D1)$ holds. Let $\mathcal{F}$ be a homotopy stable family of compact subsets of $S_r(c)$ with closed boundary Θ and let*

$$e_{\mathcal{F}}^- := \inf_{H \in \mathcal{F}} \max_{u \in H} G^-(u).$$

Suppose that Θ is contained in a connected component of $\mathcal{M}_r^-(c)$ and that $\max\{\sup G^-(\Theta), 0\}$ *$< e_{\mathcal{F}}^- < \infty$. Then there exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}_r^-(c)$ for J restricted to $S_r(c)$ at level $e_{\mathcal{F}}^-$.*

According to Lemma 4.8, similar to the argument of Lemma 3.9, we have the following result.

Lemma 4.9 *Assume that $\mu < 0$, $\bar{p} < p < 2^*_\alpha$ and condition $(D1)$ holds. Then there exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}_r^-(c)$ for J restricted to $S_r(c)$ at level $m_r^-(c) \geq \frac{|\mu|c}{2}$.*

Now we are ready to prove Theorem 1.3 (iii). It follows from Lemma 4.9 that there exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}_r^-(c)$ for J restricted to $S_r(c)$ st level $m_r^-(c) > \frac{|\mu|c}{2}$, which is bounded in $H^1(\mathbb{R}^N)$ via Lemma 4.1. According to Lemmas 2.8 and 4.2, for $c < \tilde{c} := \min\{c_1, c_3\}$, problem (P_c) admits a radially symmetric solution w satisfying

$$J(w) = m_r^-(c) > K_3 c^{-\frac{N+\alpha-p(N-2)}{Np-N-\alpha-2}} - \frac{|\mu|c}{2(1+g_1)}\left(\frac{Np-\alpha}{Np-N-\alpha} - g_1\right) \geq \frac{|\mu|c}{2}$$

for some $\bar{\lambda} > 0$, where $K_3 > 0$ is as in (70). Moreover, since $Q(w) = 0$, by (31) and (69) one has

$$\begin{aligned}
\bar{\lambda}c &= \frac{N+\alpha-p(N-2)}{Np-N-\alpha}A(w) - |\mu|\int_{\mathbb{R}^N}\frac{g_1+|w|^2}{1+g_1+w^2}|w|^2dx \\
&\quad + \frac{|\mu|Np}{Np-N-\alpha}\int_{\mathbb{R}^N}\left[\ln\left(1+\frac{|w|^2}{1+g_1}\right) - \frac{|w|^2}{1+g_1+|w|^2}\right]dx \\
&> \frac{N+\alpha-p(N-2)}{Np-N-\alpha}A(w) - |\mu|c \\
&> K_4 c^{-\frac{2p-2}{Np-N-\alpha-2}+1} - |\mu|c,
\end{aligned}$$

leading to

$$\bar{\lambda} > K_4 c^{-\frac{2p-2}{Np-N-\alpha-2}} - |\mu|,$$

where

$$K_4 := \left(\frac{N+\alpha-p(N-2)}{Np-N-\alpha}\right)\left[\frac{2(N+2)\|Q_p\|_2^{2p-2}}{(Np-\alpha)(Np-N-\alpha)}\right]^{\frac{2}{Np-N-\alpha-2}}.$$

We complete the proof.

4.4 The subcase $p = 2^*_\alpha$

Proof of Theorem 1.3 (iv). Assume on the contrary. Let $u \in H^1(\mathbb{R}^N)$ be a nontrivial solution of Problem (P_c) for some $\bar{\lambda} \geq \frac{|\mu|(N-2-2g_1)}{2(1+g_1)}$. Then we have

$$A(\bar{u}) + \bar{\lambda}\int_{\mathbb{R}^N}|u|^2dx - B(u) + |\mu|\int_{\mathbb{R}^N}\frac{g_1+|u|^2}{1+g_1+|u|^2}|u|^2dx = 0$$

and

$$\frac{N-2}{2}A(u) + \frac{N(\bar{\lambda}+|\mu|)}{2}\int_{\mathbb{R}^N}|u|^2dx - \frac{N+\alpha}{22^*_\alpha}B(u) - \frac{|\mu|N}{2}\int_{\mathbb{R}^N}\ln\left(1+\frac{|u|^2}{1+g_1}\right)dx = 0.$$

Using the above two equalities, together with the fact of $\ln(1+x) < x$ for $x > 0$ gives

$$\begin{aligned}
\bar{\lambda}\int_{\mathbb{R}^N}u^2dx &= -\frac{|\mu|N}{2}\int_{\mathbb{R}^N}|u|^2dx + \frac{|\mu|N}{2}\int_{\mathbb{R}^N}\ln\left(1+\frac{|u|^2}{1+g_1}\right)dx \\
&\quad + \frac{|\mu|(N-2)}{2}\int_{\mathbb{R}^N}\frac{g_1+|u|^2}{1+g_1+|u|^2}|u|^2dx
\end{aligned}$$

$$
\begin{aligned}
&< -\frac{|\mu|Ng_1}{2(1+g_1)}\int_{\mathbb{R}^N}|u|^2dx+\frac{|\mu|(N-2)}{2}\int_{\mathbb{R}^N}|u|^2dx \\
&\quad -\frac{|\mu|(N-2)}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g_1+|u|^2}dx \\
&= \frac{|\mu|(N-2-2g_1)}{2(1+g_1)}\int_{\mathbb{R}^N}|u|^2dx-\frac{|\mu|(N-2)}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g_1+|u|^2}dx,
\end{aligned}
$$

which implies that

$$
\left(\bar{\lambda}-\frac{|\mu|(N-2-2g_1)}{2(1+g_1)}\right)\int_{\mathbb{R}^N}|u|^2dx < -\frac{|\mu|(N-2)}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{1+g_1+|u|^2}dx.
$$

This is a contradiction, since $\bar{\lambda} \geq \frac{|\mu|(N-2-2g_1)}{2(1+g_1)}$ and $u \in S(c)$. We complete the proof. □

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Declarations

Conflicts of Interest The authors confirm that there is no conflict of interest.

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