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POSITIVE SOLUTIONS FOR RESONANT SINGULAR NON-AUTONOMOUS (p,q)-EQUATIONS

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Abstract. We consider a singular elliptic equation, driven by the non-autonomous (p,q)-operator and with a resonant perturbation. Using variational tools together with truncation and comparison techniques, we show that if the L^∞-norm of the coefficient of the singular term is small enough, then the problem has at least two positive smooth solutions.

1. Introduction.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear, non-autonomous singular Dirichlet (p,q)-equation

$$\left\{ \begin{array}{l} -\Delta_p^{\alpha_1} u(z) - \Delta_q^{\alpha_2} u(z) = \xi(z) u(z)^{-\eta} + f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ 0 < \eta < 1 < q < p,\ u > 0. \end{array} \right\} \tag{1}$$

For $\alpha \in C^{0,1}(\bar{\Omega})$ with $0 < \hat{c} \leq \alpha(z)$ for all $z \in \bar{\Omega}$ and for $s \in (1, \infty)$, by Δ_s^α we denote the non-autonomous (weighted) s-Laplace differential operator defined by

$$\Delta_s^\alpha u = \operatorname{div}(\alpha(z)|Du|^{s-2}Du) \text{ for all } u \in W_0^{1,s}(\Omega).$$

Problem (1) is driven by the sum of two such operators with different exponents (we have $1 < q < p$) and in general with distinct weights α_1, α_2. The differential operator governing (1) is non-autonomous and non-homogeneous. In the reaction (right-hand side) of (1), we have the competing effects of a singular term $u \to$

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$\xi(z)u^{-\eta}$ (with $\xi \in L^\infty(\Omega) \setminus \{0\}$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$ and $0 < \eta < 1$) and of a Carathéodory perturbation $f(z,x)$ (that is, for all $x \geq 0$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega, x \to f(z,x)$ is continuous). We assume that $f(z,\cdot)$ is $(p-1)$-linear as $x \to +\infty$ and we can have resonance with respect to the first eigenvalue of $(-\Delta_p^{\alpha_1}, W_0^{1,p}(\Omega))$. Moreover, $f(z,\cdot)$ can change sign as we move on $\mathbb{R}_+ = [0,+\infty)$.

In the past most works on nonlinear singular elliptic problems, required that the perturbation of the singular term is positive and $(p-1)$-superlinear satisfying the well-known Ambrosetti-Rabinowitz condition (the AR-condition for short). In fact in many papers this perturbation is simply of the power type. We mention the works of Giacomoni-Schindler-Takáč [6], Giacomoni-Kumar-Sreenadh [5], Irving-Koch [7], Leonardi-Papageorgiou [8], Papageorgiou-Rădulescu-Repovš [11], Papageorgiou-Smyrlis [14], Papageorgiou-Winkert [15], Perera-Zhang [16] and the references therein.

In the recent paper [3], Bobkov-Tanaka considered autonomous (p,q)-equation with Dirichlet boundary condition and employed several minimax variational methods to determine three generally different ranges of parameters such that the problem admits a given number of distinct pairs of solutions with a prescribed sign of energy. For a biharmonic problem with two weights, we refer readers to [18] where existence and multiplicity of solutions were obtained via an alternative Ricceri's result. Recently, Arruda-Nascimento [1] and Bień-Majdak-Papageorgiou [2], considered nonlinear autonomous singular equations with a $(p-1)$-superlinear perturbation which can be sign changing (indefinite). Our work here complements the two aforementioned papers, by considering non-autonomous equations with $(p-1)$-linear, resonant perturbation. Using variational tools, we prove two multiplicity theorems for the non-coercive resonant problem.

2. **Mathematical background and hypothesis.** In the study of problem (1), the main function spaces are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$. On account of the Poincaré inequality, on $W_0^{1,p}(\Omega)$ we can consider the norm

$$\|u\| = \|Du\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

The space $C_0^1(\bar{\Omega})$ is an ordered Banach space with positive (order) cone $C_+ = \{u \in C_0^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : \ u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0 \right\},$$

where $\dfrac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Given $\alpha \in C^{0,1}(\bar{\Omega})$ with $0 < \hat{c} \leq \alpha(z)$ for all $z \in \bar{\Omega}$, we consider the following nonlinear eigenvalue problem

$$-\Delta_p^\alpha u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) \text{ in } \Omega, \ u|_{\partial\Omega} = 0. \tag{2}$$

From the Appendix of Liu-Papageorgiou [10], we know that problem (2) has a smallest eigenvalue $\hat{\lambda}_1^\alpha(p) > 0$ which is isolated, simple and admits the following variational characterization,

$$0 < \hat{\lambda}_1^\alpha(p) = \inf\left\{ \frac{\int_\Omega \alpha(z)|Du|^p \mathrm{d}z}{\|u\|_p^p} : \ u \in W_0^{1,p}(\Omega), \ u \neq 0 \right\}. \tag{3}$$

The nonlinear regularity theory of Lieberman [9], implies that if $u \in W_0^{1,p}(\Omega)$ is an eigenfunction of (2), then $u \in C_0^1(\bar{\Omega})$. In particular, if u is an eigenfunction corresponding to $\hat{\lambda}_1^{\alpha}(p) > 0$, then u has fixed sign and the nonlinear maximum principle of Pucci-Serrin [17] implies that $u \in \operatorname{int} C_+$ or $u \in -\operatorname{int} C_+$.

Let $\hat{u}_1$ denote the positive L^p normalized (that is, $\|\hat{u}_1\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1^{\alpha}(p) > 0$. Then $\hat{u}_1 \in \operatorname{int} C_+$ and realizes the infimum in (3).

We will also consider the following weighted version of (2),

$$-\Delta_p^{\alpha} u(z) = \tilde{\lambda} m(z)|u(z)|^{p-2}u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0, \tag{4}$$

with $m \in L^{\infty}(\Omega)\backslash\{0\}$, $m(z) \geq 0$ for a.a. $z \in \Omega$. We have the same spectral analysis for (4) as for (3). So, (4) has a smallest eigenvalue $\tilde{\lambda}_1^{\alpha}(p, m) > 0$, which is isolated, simple and has the following variational characterization,

$$0 < \tilde{\lambda}_1^{\alpha}(p, m) = \inf\left\{ \frac{\int_{\Omega} \alpha(z)|Du|^p \mathrm{d}z}{\int_{\Omega} m(z)|u|^p \mathrm{d}z} :\ u \in W_0^{1,p}(\Omega),\ u \neq 0 \right\}. \tag{5}$$

Again, the corresponding eigenfunctions have fixed sign and belong in $\pm\operatorname{int} C_+$. We point out that for both problems (2) and (4), the principal eigenvalue is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have nodal (sign-changing) eigenfunctions. Using (5), we obtain easily the following monotonicity property for the map $m \to \tilde{\lambda}_1^{\alpha}(p, m)$.

Proposition 1. *If $m_1, m_2 \in L^{\infty}(\Omega)\backslash\{0\}$, $0 \leq m_1(z) \leq m_2(z)$ for a.a. $z \in \Omega$ and $m_1 \neq m_2$, then*

$$\tilde{\lambda}_1^{\alpha}(p, m_2) < \tilde{\lambda}_1^{\alpha}(p, m_1).$$

If X is a reflexive Banach space and $V : X \to X^*$ is a continuous and monotone map, then $V(\cdot)$ is maximal monotone (see [12, p. 117]. We say that $V(\cdot)$ is "coercive", if

$$\|u\|_X \to \infty \Longrightarrow \|V(u)\|_{X^*} \to +\infty.$$

We have the following surjectivity result (see [12, p. 135]).

Proposition 2. *If X is a reflexive Banach space and $V : X \to X^*$ is continuous, monotone and coercive, then $V(\cdot)$ is surjective.*

A useful tool in the study of singular problems, is the so called "Hardy's inequality" (see Papageorgiou-Rădulescu-Repovš [12, p. 66])

Proposition 3. *If $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \bar{\Omega}$, then*

$$\left\| \frac{u}{\hat{d}} \right\|_p \leq c^* \|Du\|_p \text{ for some } c^* > 0 \text{ and all } u \in W_0^{1,p}(\Omega),\ 1 < p < \infty.$$

For $u \in L^1(\Omega)$, we write $0 \prec u$, if for all $K \subset \Omega$ compact, $0 < c_K \leq u(z)$ for a.a. $z \in K$.

Our hypotheses on the weights α_1, α_2 and the coefficient $\xi(\cdot)$ are the following:

H_0 : $\alpha_1, \alpha_2 \in C^{0,1}(\bar{\Omega}), 0 < \hat{c} \leq \alpha_1(z), \alpha_2(z)$ for all $z \in \bar{\Omega}$ and $\xi \in L^{\infty}(\Omega)$ such that $0 \prec \xi$.

Let $V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega)\right)^*, \left(\frac{1}{p}+\frac{1}{p'}=1\right)$ be the nonlinear map defined by

$$\langle V(u), h\rangle = \int_\Omega \left[\alpha_1(z)|Du|^{p-2} + \alpha_2(z)|Du|^{q-2}\right](Du, Dh)\mathrm{d}z, \quad \forall\, u, h \in W_0^{1,p}(\Omega).$$

This map has the following properties (see Gasinski-Papageorgiou [4, p. 279]).

Proposition 4. *If hypotheses H_0 hold, then $V(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus, maximal monotone too) and of type $(S)_+$, that is,*

$$\text{“}if\ u_n \xrightarrow{w} u\ in\ W_0^{1,p}(\Omega)\ and\ \limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0,$$

$$then\ u_n \to u\ in\ W_0^{1,p}(\Omega).\text{”}$$

Let X be a Banach space and $\varphi \in C^1(X)$. We say that $\varphi(\cdot)$ satisfies the “C-condition”, if it has the following property:

$$\text{“Every sequence } \{u_n\}_{n\in\mathbb{N}} \subset X \text{ such that } \{\varphi(u_n)\} \subset \mathbb{R} \text{ is bounded}$$

$$\text{and } (1 + \|u_n\|_X)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

$$\text{admits a strongly convergent subsequence.”}$$

Also, we set

$$\mathcal{K}_\varphi = \{u \in X : \ \varphi'(u) = 0\} \text{ (the critical set of } \varphi(\cdot)).$$

Our hypotheses on the perturbation $f(z, x)$ are the following:

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $\hat{\alpha}_\rho \in L^\infty(\Omega)$ such that

$$|f(z, x)| \le \hat{\alpha}_\rho(z) \text{ for a.a. } z \in \Omega, \text{all } 0 \le x \le \rho;$$

(ii) there exist $l \in L^\infty(\Omega)$ and $\tau \in (q, p)$ such that

$$\hat{\lambda}_1^{\alpha_1}(p) \le \liminf_{x\to+\infty} \frac{f(z, x)}{x^{p-1}} \le \limsup_{x\to+\infty} \frac{f(z, x)}{x^{p-1}} \le l(z) \text{ uniformly for a.a. } z \in \Omega,$$

and if $F(z, x) = \int_0^x f(z, s)\mathrm{d}s$, then

$$0 < \hat{\beta} \le \liminf_{x\to+\infty} \frac{pF(z, x) - f(z, x)x}{x^\tau} \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exists $\delta > 0$ such that

$$-\xi(z) \le f(z, x) \le \gamma_\theta < 0 \text{ for a.a. } z \in \Omega, \text{ all } 0 < \vartheta \le x \le \delta;$$

(iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$, such that for a.a. $z \in \Omega$, the function

$$x \to f(z, x) + \hat{\xi}_\rho x^{p-1}$$

is nondecreasing on $[0, \rho]$.

Remark 1. Since we look for positive solutions of (1) and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$. Hypotheses H_1 (i), (ii) imply that $f(z,\cdot)$ is $(p-1)$-linear as $x \to +\infty$, and we can have resonance with respect to the principal eigenvalue $\hat{\lambda}_1^{\alpha_1}(p) > 0$ of $(-\Delta_p^{\alpha_1}, W_0^{1,p}(\Omega))$. As we will see in the process of the proof (see the proof of Proposition 10), the resonance occurs from the right of $\hat{\lambda}_1^{\alpha_1}(p)$ in the sense that

$$\hat{\lambda}_1^{\alpha_1}(p)x^p - pF(z,x) \to -\infty \text{ uniformly for a.a. } z \in \Omega, \text{ as } x \to +\infty.$$

and this makes the relevant energy functional noncoercive.

3. **Auxiliary problems.** In this section, we consider two auxiliary problems, the solutions of which will provide an ordered pair of upper and lower solutions for problem (1). Then in section 4 using these solutions and truncation and comparison techniques, we will show the existence and multiplicity of positive solutions for problem (1).

First we consider the following auxiliary Dirichlet problem:

$$-\Delta_p^{\alpha_1} u(z) - \Delta_q^{\alpha_2} u(z) = \xi(z) \text{ in } \Omega, \; u|_{\partial\Omega} = 0. \tag{6}$$

Using Proposition 2 (see also Papageorgiou-Rădulescu-Repovš [11, Proposition 10]), we have the following result concerning problem (6).

Proposition 5. *If hypotheses H_0 hold, then problem (6) has a unique solution $\underline{u}_\xi \in \operatorname{int} C_+$ and*

$$\underline{u}_\xi \to 0 \textit{ in } C^{0,1}(\bar{\Omega}) \textit{ as } \|\xi\|_\infty \to 0.$$

On account of this proposition, we can find $\gamma_1 > 0$ such that

$$\|\xi\|_\infty < \gamma_1 \Longrightarrow \xi(z) < \xi(z)\underline{u}_\xi^{-\eta}(z) \text{ for a.a. } z \in \Omega. \tag{7}$$

We consider a second auxiliary Dirichlet problem

$$-\Delta_p^{\alpha_1} u(z) - \Delta_q^{\alpha_2} u(z) = \xi(z)u^{-\eta}(z) \text{ in } \Omega, \; u|_{\partial\Omega} = 0. \tag{8}$$

For this problem, we have a similar result.

Proposition 6. *If hypotheses H_0 hold, then problem (8) has a unique solution $\bar{u}_\xi \in \operatorname{int} C_+$ and*

$$\bar{u}_\xi \to 0 \textit{ in } C^{0,1}(\bar{\Omega}) \textit{ as } \|\xi\|_\infty \to 0.$$

Proof. We know that $\underline{u}_\xi \in \operatorname{int} C_+$ (see Proposition 5). So, using [12, Proposition 4.1.22, p. 274], we can find $c_1 > 0$ such that

$$\hat{d} \leq c_1 \underline{u}_\xi \text{ (recall that } \hat{d}(z) = d(z, \partial\Omega) \text{ for all } z \in \bar{\Omega}). \tag{9}$$

Then for every $h \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
&\left|\int_\Omega \xi(z)\underline{u}_\xi^{-\eta} h \mathrm{d}z\right| \\
&\leq \|\xi\|_\infty \int_\Omega \underline{u}_\xi^{1-\eta}\frac{h}{\underline{u}_\xi}\mathrm{d}z \\
&\leq c_2\|\xi\|_\infty \int_\Omega \frac{h}{\hat{d}}\mathrm{d}z \text{ for some } c_2>0 \text{ (see (9) and recall that } \underline{u}_\xi \in \operatorname{int} C_+) \\
&\leq c_3\|\xi\|_\infty\|h\| \text{ (see Proposition 3)}, \\
\Rightarrow &\xi(z)\underline{u}_\xi^{-\eta} \in W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega)\right)^*.
\end{aligned} \tag{10}$$

We write problem (8) in the following form

$$V(u) = \xi(\cdot)u^{-\eta} \text{ in } W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega)\right)^*. \tag{11}$$

Invoking Propositions 2 and 4, we can find $\bar{u}_\xi \in W_0^{1,p}(\Omega)$ solution of (1). Moreover, the strict monotonicity of $V(\cdot)$ implies that this solution is unique. Using (9), we have

$$0 \leq \xi(z)\underline{u}_\xi^{-\eta}(z) \leq c_4\hat{d}^{-\eta} \text{ for some } c_4>0.$$

Since $\xi(\cdot)\underline{u}_\xi^{-\eta}(\cdot) \in L^\infty_{\text{loc}}(\Omega)$, we can use Theorem B1 of Giacomoni-Schindler-Takáč [6] (see also Giacomoni-Kumar-Sreenadh [5], Theorem 1.7) and obtain that $\bar{u}_\xi \in C_+ \setminus \{0\}$. We have

$$\begin{aligned}
&-\Delta_p^{\alpha_1}\bar{u}_\xi - \Delta_q^{\alpha_2}\bar{u}_\xi \leq 0 \text{ in } \Omega \\
\Rightarrow &\bar{u}_\xi \in \operatorname{int} C_+ \text{ (see Pucci-Serrin [17, pp. 111, 120])}.
\end{aligned}$$

Next we show the last assertion of the proposition. We have

$$\langle V(\bar{u}_\xi), h\rangle = \int_\Omega \xi(z)\underline{u}_\xi^{-\eta}(z) h \mathrm{d}z \text{ for all } h \in W_0^{1,p}(\Omega). \tag{12}$$

In (12) we use the test function $h = \bar{u}_\xi \in W_0^{1,p}(\Omega)$. We obtain

$$\begin{aligned}
&\hat{c}\,\|\bar{u}_\xi\|^p \leq \int_\Omega \alpha(z)\,|D\bar{u}_\xi|^p\,\mathrm{d}z \leqslant c_3\|\xi\|_\infty\|\bar{u}_\xi\| \text{ (see (10) and hypotheses } H_0), \\
\Rightarrow &\|\bar{u}_\xi\| \to 0 \text{ as } \|\xi\|_\infty \to 0.
\end{aligned} \tag{13}$$

Then as before, invoking Theorem B.1 of [6] (see also Theorem 1.7 of [5]), we can find $\alpha \in (0,1)$ and $c_5>0$ such that

$$\bar{u}_\xi \in C_0^{1,\alpha}(\bar{\Omega}), \quad \|\bar{u}_\xi\|_{C_0^{1,\alpha}(\bar{\Omega})} \leq c_5. \tag{14}$$

We know that $C_0^{1,\alpha}(\bar{\Omega}) \hookrightarrow C_0^1(\bar{\Omega})$ compactly (Arzela-Ascoli theorem). Then from (13) and (14), we infer that

$$\bar{u}_\xi \to 0 \text{ in } C_0^1(\bar{\Omega}) \text{ as } \|\xi\|_\infty \to 0.$$

This completes the proof. ∎

Propositions 5 and 6 imply that we can find $0<\gamma_2\leq\gamma_1$ (see (7)) such that

$$\|\xi\|_\infty<\gamma_2 \Longrightarrow \|\bar{u}_\xi\|_\infty<\left(\frac{1}{2}\right)^{\frac{1}{\eta}},\ 0\leq \bar{u}_\xi\leq\delta \text{ for all } z\in\bar{\Omega}, \tag{15}$$

with $\delta>0$ as postulated by hypothesis H_1(iii).

Proposition 7. *If hypotheses H_0 hold and $\|\xi\|_\infty<\gamma_2$, then $\underline{u}_\xi(z)\leq\bar{u}_\xi(z)$ for all $z\in\bar{\Omega}$.*

Proof. On account of (7), we have

$$V(\underline{u}_\xi)=\xi(\cdot)\leq\xi(\cdot)\bar{u}_\xi^{-\eta}=V(\bar{u}_\xi) \text{ in } W^{-1,p'}(\Omega).$$

By the weak comparison principle (see Pucci-Serrin [17, Theorem 3.4.1, p. 61]), we conclude that $\underline{u}_\xi\leq\bar{u}_\xi$. ☐

4. **Positive solutions.** In this section, using the results from section 3, we prove existence and multiplicity of positive solutions for problem (1). In what follows

$$\left[\underline{u}_\xi,\bar{u}_\xi\right]=\left\{u\in W_0^{1,p}(\Omega):\ \underline{u}_\xi(z)\leq u(z)\leq\bar{u}_\xi(z) \text{ for a.a. } z\in\Omega\right\},$$
$$\text{and } \operatorname{int}_{C_0^1(\bar{\Omega})}\left[\underline{u}_\xi,\bar{u}_\xi\right]=\text{interior in } C_0^1(\bar{\Omega}) \text{ of } \left[\underline{u}_\xi,\bar{u}_\xi\right]\cap C_0^1(\bar{\Omega}).$$

Proposition 8. *If hypotheses H_0, H_1 hold and $\|\xi\|_\infty<\gamma_2$, then problem (1) has a solution $u_0\in\operatorname{int}_{C_0^1(\bar{\Omega})}\left[\underline{u}_\xi,\bar{u}_\xi\right]$.*

Proof. From Proposition 7, we know that

$$\underline{u}_\xi\leq\bar{u}_\xi \ \text{ (recall } 0<\gamma_2\leq\gamma_1\text{)}. \tag{16}$$

So, we can introduce the Carathéodory function $\hat{g}(z,x)$ defined by

$$\hat{g}(z,x)=\begin{cases}\xi(z)\underline{u}_\xi^{-\eta}(z)+f(z,\underline{u}_\xi(z)) & \text{if } x<\underline{u}_\xi(z),\\ \xi(z)x^{-\eta}+f(z,x) & \text{if } \underline{u}_\xi(z)\leq x\leq\bar{u}_\xi(z),\\ \xi(z)\bar{u}_\xi^{-\eta}(z)+f(z,\bar{u}_\xi(z)) & \text{if } \bar{u}_\xi(z)<x.\end{cases} \tag{17}$$

We set $\hat{G}(z,x)=\int_0^x\hat{g}(z,s)\mathrm{d}s$ and consider the functional $\hat{\varphi}:W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\hat{\varphi}(u)=\frac{1}{p}\int_\Omega\alpha_1(z)|Du|^p\mathrm{d}z+\frac{1}{q}\int_\Omega\alpha_2(z)|Du|^q\mathrm{d}z-\int_\Omega\hat{G}(z,u)\mathrm{d}z \text{ for all } u\in W_0^{1,p}(\Omega).$$

From Papageorgiou-Smyrlis [14, Proposition 3], we know that $\hat{\varphi}\in C^1\left(W_0^{1,p}(\Omega)\right)$. From (17) and hypotheses H_0, we see that $\hat{\varphi}(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we see that $\hat{\varphi}(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0\in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}&\hat{\varphi}(u_0)=\inf\left\{\hat{\varphi}(u):\ u\in W_0^{1,p}(\Omega)\right\},\\ &\Rightarrow\left\langle\hat{\varphi}'(u_0),h\right\rangle=0 \text{ for all } h\in W_0^{1,p}(\Omega),\\ &\Rightarrow\langle V(u_0),h\rangle=\int_\Omega\hat{g}(z,u_0)h\mathrm{d}z \text{ for all } h\in W_0^{1,p}(\Omega).\end{aligned} \tag{18}$$

In (18) first we use the test function $h = (u_0 - \bar{u}_\xi)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\left\langle V(u_0), (u_0 - \bar{u}_\xi)^+ \right\rangle \\
&= \int_\Omega \left[\xi(z)\bar{u}_\xi^{-\eta} + f(z, \bar{u}_\xi)\right](u_0 - \bar{u}_\xi)^+ \mathrm{d}z \text{ (see (17))} \\
&\leq \int_\Omega \xi(z)\bar{u}_\xi^{-\eta}(u_0 - \bar{u}_\xi)^+ \mathrm{d}z \text{ (see (15) and hypothesis } H_1\text{(iii))} \\
&= \left\langle V(\bar{u}_\xi), (u_0 - \bar{u}_\xi)^+ \right\rangle \text{ (see Proposition 6)} \\
\Rightarrow & u_0 \leq \bar{u}_\xi \text{ (see Proposition 4).}
\end{aligned}$$

Next we test (18) with $h = (\underline{u}_\xi - u_0)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\left\langle V(u_0), (\underline{u}_\xi - u_0)^+ \right\rangle \\
&= \int_\Omega \left[\xi(z)\underline{u}_\xi^{-\eta} + f(z, \underline{u}_\xi)\right](\underline{u}_\xi - u_0)^+ \mathrm{d}z \text{ (see (17))} \\
&\geq \int_\Omega \xi(z)\left[\underline{u}_\xi^{-\eta} - 1\right](\underline{u}_\xi - u_0)^+ \mathrm{d}z \text{ (see (15) and hypothesis } H_1\text{(iii))} \\
&\geq \int_\Omega \xi(z)(\underline{u}_\xi - u_0)^+ \mathrm{d}z \text{ (since } \underline{u}_\xi^{-\eta} - 1 > 1, \text{ see (15))} \\
&= \left\langle V(\underline{u}_\xi), (\underline{u}_\xi - u_0)^+ \right\rangle \text{ (see Proposition 5)} \\
\Rightarrow & \underline{u}_\xi \leq u_0 \text{ (see Proposition 4).}
\end{aligned}$$

We have proved that

$$u_0 \in \left[\underline{u}_\xi, \bar{u}_\xi\right].$$

The nonlinear regularity theory of Lieberman [9], implies that

$$u_0 \in \left[\underline{u}_\xi, \bar{u}_\xi\right] \cap C_0^1(\bar{\Omega}). \tag{19}$$

Let $\rho = \|\bar{u}_\xi\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_1(iv). We have

$$\begin{aligned}
&-\Delta_p^{\alpha_1} u_0 - \Delta_q^{\alpha_2} u_0 + \hat{\xi}_\rho u_0^{p-1} - \xi(z)u_0^{-\eta} \\
&= f(z, u_0) + \hat{\xi}_\rho u_0^{p-1} \\
&\leq f(z, \bar{u}_\xi) + \hat{\xi}_\rho \bar{u}_\xi^{p-1} \text{ (see (19) and hypothesis } H_1\text{(iv))} \\
&\leq \hat{\xi}_\rho \bar{u}_\xi^{p-1} \text{ (see (15) and hypothesis } H_1\text{(iii))} \\
&= -\Delta_p^{\alpha_1} \bar{u}_\xi - \Delta_q^{\alpha_2} \bar{u}_\xi + \hat{\xi}_\rho \bar{u}_\xi^{p-1} - \xi(z)\bar{u}_\xi^{-\eta} \text{(see Proposition 6).}
\end{aligned}$$

It follows from H_1(iii) and (15) that $0 \prec -f(z, \bar{u}_\xi)$, then from Proposition 7 of Papageorgiou-Rădulescu-Repovš [11], we obtain

$$\bar{u}_\xi - u_0 \in \operatorname{int} C_+. \tag{20}$$

Also we have

$$\begin{aligned}
& -\Delta_p^{\alpha_1} u_0 - \Delta_q^{\alpha_2} u_0 + \hat{\xi}_\rho u_0^{p-1} - \xi(z) u_0^{-\eta} \\
&= f(z, u_0) + \hat{\xi}_\rho u_0^{p-1} \\
&\geq f(z, \underline{u}_\xi) + \hat{\xi}_\rho \underline{u}_\xi^{p-1} \text{ (see (15), (16) and hypothesis } H_1\text{(iv))} \\
&\geq -\xi(z) + \hat{\xi}_\rho \underline{u}_\xi^{p-1} \text{ (see (15) and hypothesis } H_1\text{(iii))} \\
&\geq \xi(z)\left[1 - \underline{u}_\xi^{-\eta}\right] + \hat{\xi}_\rho \underline{u}_\xi^{p-1} \text{ (see (15))} \\
&= -\Delta_p^{\alpha_1} \underline{u}_\xi - \Delta_q^{\alpha_2} \underline{u}_\xi + \hat{\xi}_\rho \underline{u}_\xi^{p-1} - \xi(z)\underline{u}_\xi^{-\eta}.
\end{aligned}$$

Note that $-\xi(z) - \xi(z)\left[1 - \underline{u}_\xi^{-\eta}\right] = \xi(z)\left[\underline{u}_\xi^{-\eta} - 2\right] \succ 0$ and so Proposition 7 of [11] implies that

$$u_0 - \underline{u}_\xi \in \operatorname{int} C_+. \tag{21}$$

From (20) and (21), we conclude that

$$u_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})} \left[\underline{u}_\xi, \bar{u}_\xi\right]. \tag{22}$$

The proof is now complete. ☐

We will use u_0 to produce a second positive smooth solution of (1).

We introduce the Carathéodory function $g(z,x)$ defined by

$$g(z,x) = \begin{cases} \xi(z)\underline{u}_\xi^{-\eta}(z) + f(z, \underline{u}_\xi(z)) & \text{if } x \leq \underline{u}_\xi(z), \\ \xi(z)x^{-\eta} + f(z,x) & \text{if } \underline{u}_\xi(z) < x. \end{cases} \tag{23}$$

We set $G(z,x) = \int_0^x g(z,s)\mathrm{d}s$ and consider the functional $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{p}\int_\Omega \alpha_1(z)|Du|^p \mathrm{d}z + \frac{1}{q}\int_\Omega \alpha_2(z)|Du|^q \mathrm{d}z - \int_\Omega G(z,u)\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega).$$

We have $\varphi \in C^1\left(W_0^{1,p}(\Omega)\right)$ (see [14]). In what follows we define

$$\left[\underline{u}_\xi\right) = \left\{u \in W_0^{1,p}(\Omega) : \ \underline{u}_\xi(z) \leq u(z) \text{ for a.a. } z \in \Omega\right\},$$

Proposition 9. *If hypotheses H_0, H_1 hold, $\|\xi\|_\infty < \gamma_2$ and $u_0 \in \operatorname{int} C_+$ is the solution of (1) from Proposition 8, then*

$$\mathcal{K}_{\hat{\varphi}} \subset \left[\underline{u}_\xi, \bar{u}_\xi\right] \cap C_0^1(\bar{\Omega}),\ \mathcal{K}_\varphi \subset \left[\underline{u}_\xi\right) \cap C_0^1(\bar{\Omega}) \textit{ and } u_0 \textit{ is a local minimizer of } \varphi(\cdot).$$

Proof. That $\mathcal{K}_{\hat{\varphi}} \subset \left[\underline{u}_\xi, \bar{u}_\xi\right] \cap C_0^1(\bar{\Omega})$ and $\mathcal{K}_\varphi \subset \left[\underline{u}_\xi\right) \cap C_0^1(\bar{\Omega})$, follow from (17) and (23) as in the proof of Proposition 8. Note that

$$\hat{\varphi}\big|_{\left[\underline{u}_\xi, \bar{u}_\xi\right]} = \varphi\big|_{\left[\underline{u}_\xi, \bar{u}_\xi\right]}. \tag{24}$$

From the proof of Proposition 8, we know that u_0 is a minimizer of the functional $\hat{\varphi}$. From (22) and (24), we infer that

$$\begin{aligned} & u_0 \text{ is a local } C_0^1(\bar{\Omega})\text{-minimizer of } \varphi(\cdot), \\ \Rightarrow & u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi(\cdot), \\ & \text{(see Papageorgiou-Rădulescu-Zhang [13, Proposition A3]).} \end{aligned} \tag{25}$$

This completes the proof. □

On account of Proposition 9, we see that we may assume that

$$\mathcal{K}_\varphi \cap \left[\underline{u}_\xi, \bar{u}_\xi\right] = \{u_0\} \text{ and } \mathcal{K}_\varphi \text{ is finite.} \tag{26}$$

Otherwise it is clear from (23) that we already have at least one more nontrivial positive smooth solution. Moreover, from (25), (26) and Papageorgiou-Rădulescu-Repovš [12, Theorem 5.7.6, p. 449], we see that we can find $\rho \in (0,1)$ small such that

$$\varphi(u_0) < \inf\{\varphi(u) : \ \|u - u_0\| = \rho\} = m_0. \tag{27}$$

Proposition 10. *If hypotheses H_0, H_1 hold and $\|\xi\|_\infty < \gamma_2$, then*

$$\varphi(t\hat{u}_1) \to -\infty \text{ as } t \to +\infty.$$

Proof. Hypothesis H_1(ii) implies that given $\hat{\beta}_0 \in (0, \hat{\beta})$, we can find $M = M(\hat{\beta}_0) > 0$ such that

$$\hat{\beta}_0 x^\tau \le pF(z,x) - f(z,x)x \ \text{ for a.a. } x \in \Omega, \ \text{all } x \ge M. \tag{28}$$

Note that

$$\begin{aligned} \frac{d}{dx}\frac{F(z,x)}{x^p} &= \frac{f(z,x)x^p - px^{p-1}F(z,x)}{x^{2p}} \\ &= \frac{f(z,x)x - pF(z,x)}{x^{p+1}} \\ &\le -\frac{\hat{\beta}_0}{x^{p+1-\tau}} \ \text{ for a.a. } z \in \Omega, \ \text{all } x \ge M \text{ (see (28))}, \end{aligned}$$

$$\Rightarrow \frac{F(z,x)}{v^p} - \frac{F(z,x)}{x^p} \le \frac{\hat{\beta}_0}{p-\tau}\left[\frac{1}{v^{p-\tau}} - \frac{1}{x^{p-\tau}}\right] \text{ for a.a. } z \in \Omega, \ \text{all } v \ge x \ge M. \tag{29}$$

Hypothesis H_1(ii) implies that

$$\frac{1}{p}\hat{\lambda}_1^{\alpha_1}(p) \le \liminf_{x\to+\infty}\frac{F(z,x)}{x^p} \text{ uniformly for a.a. } z \in \Omega. \tag{30}$$

So, if in (29) we pass to the limit as $v \to +\infty$ and use (30), we obtain

$$\begin{aligned} & \frac{\hat{\lambda}_1^{\alpha_1}(p)}{p} - \frac{F(z,x)}{x^p} \le -\frac{\hat{\beta}_0}{p-\tau}\frac{1}{x^{p-\tau}}, \\ \Rightarrow & \hat{\lambda}_1^{\alpha_1}(p)x^p - pF(z,x) \le -\frac{p\hat{\beta}_0}{p-\tau}x^\tau \text{ for a.a. } z \in \Omega, \ \text{all } x \ge M. \end{aligned} \tag{31}$$

Note that since $\|\hat{u}_1\|_p = 1$, we have

$$\varphi(t\hat{u}_1) = \frac{\hat{\lambda}_1^{\alpha_1}(p)t^p}{p} + \frac{t^q}{q}\int_\Omega \alpha_2(z)|D\hat{u}_1|^q dz - \int_\Omega G(z,t\hat{u}_1)dz \text{ for all } t > 0. \tag{32}$$

We estimate the integral $\int_\Omega G(z,t\hat{u}_1)\mathrm{d}z$. From (23), we have

$$\begin{aligned}
&\int_\Omega G(z,t\hat{u}_1)\mathrm{d}z \\
=&\int_{\{t\hat{u}_1\le \underline{u}_\xi\}}\left[\xi(z)\underline{u}_\xi^{-\eta}+f(z,\underline{u}_\xi)\right](t\hat{u}_1)\mathrm{d}z+\int_{\{\underline{u}_\xi\le t\hat{u}_1\}}\frac{\xi(z)}{1-\eta}\left[(t\hat{u}_1)^{1-\eta}-\underline{u}_\xi^{1-\eta}\right]\mathrm{d}z \\
&+\int_{\{\underline{u}_\xi\le t\hat{u}_1\}}\xi(z)\xi^{1-\eta}\mathrm{d}z+\int_{\{\underline{u}_\xi\le t\hat{u}_1\}}\left[F(z,t\hat{u}_1)-F(z,\underline{u}_\xi)\right]\mathrm{d}z \\
&+\int_{\{\underline{u}_\xi\le t\hat{u}_1\}}f(z,\underline{u}_\xi)\underline{u}_\xi\mathrm{d}z \\
\ge&\int_\Omega F(z,t\hat{u}_1)\mathrm{d}z-c_6(t+1)\text{ for some }c_6>0\text{ (see hypotheses }H_1(\text{i}),(\text{iii})).
\end{aligned}\tag{33}$$

Using (31), (32) and (33), we obtain

$$\varphi(t\hat{u}_1)\le\frac{1}{p}\int_\Omega\left[\hat{\lambda}_1^{\alpha_1}(p)(t\hat{u}_1)^p-pF(z,t\hat{u}_1)\right]\mathrm{d}z+c_7(t^q+1)\text{ for some }c_7>0,\text{all }t\ge 1.$$

$$\begin{aligned}
&\Rightarrow\frac{\varphi(t\hat{u}_1)}{t^\tau}\le-\frac{\hat{\beta}_0}{p-\tau}\|\hat{u}_1\|_\tau^\tau+\frac{c_8}{t^{\tau-q}}\text{ for some }c_8>0,\ \text{all }t\ge 1,\\
&\Rightarrow\limsup_{t\to+\infty}\frac{\varphi(t\hat{u}_1)}{t^\tau}=-\hat{\vartheta}<0,\\
&\Rightarrow\varphi(t\hat{u}_1)\to-\infty\text{ as }t\to+\infty,
\end{aligned}$$

which completes the proof. □

Remark 2. Included in the above proof is the fact that the resonance at $\hat{\lambda}_1^{\alpha_1}(p)$ occurs from the right of the principal eigenvalue, that is

$$\hat{\lambda}_1^{\alpha_1}(p)x^p-pF(z,x)\to-\infty\text{ uniformly for a.a. }z\in\Omega\text{ as }x\to+\infty\text{ (see (31))}.$$

This means that the functional $\varphi(\cdot)$ can not be coercive and so the second solution can not be obtained using the direct method of the calculus of variations.

Proposition 11. *If hypotheses H_0, H_1 hold and $\|\xi\|_\infty<\gamma_2$, then the functional $\varphi(\cdot)$ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\in\mathbb{N}}\subset W_0^{1,p}(\Omega)$ such that

$$|\varphi(u_n)|\le c_9\text{ for some }c_9>0,\ \text{all }n\in\mathbb{N},\tag{34}$$

$$(1+\|u_n\|)\,\varphi'(u_n)\to 0\text{ in }W^{-1,p'}(\Omega)\text{ as }n\to\infty.\tag{35}$$

From (35) we have

$$\langle V(u_n),h\rangle-\int_\Omega g(z,u_n)h\mathrm{d}z\le\frac{\varepsilon_n\|h\|}{1+\|u_n\|}\text{ for all }h\in W_0^{1,p}(\Omega),\ \text{with }\varepsilon_n\to 0^+.\tag{36}$$

In (36) we choose $h=-u_n^-\in W_0^{1p}(\Omega)$. Then

$$\begin{aligned}
\hat{c}\left\|u_n^-\right\|^p&\le\varepsilon_n-\int_\Omega\left[\xi(z)\underline{u}_\xi^{-\eta}+f\left(z,\underline{u}_\xi\right)\right]u_n^-\mathrm{d}z\\
&\leqslant\varepsilon_n+c_{10}\|u_n^-\|\text{ for some }c_{10}>0,\ \text{all }n\in\mathbb{N},
\end{aligned}$$

$$\Rightarrow \{u_n^-\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega) \text{ is bounded.} \tag{37}$$

We will show that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded, too. We argue by contradiction. So, suppose that at least for a subsequence, we have

$$\|u_n^+\| \to \infty \text{ as } n \to \infty. \tag{38}$$

We set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. Hence we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega),\ y_n \to y \text{ in } L^p(\Omega),\ y \geq 0. \tag{39}$$

From (36) and (37), we have

$$\left\langle V(u_n^+), h\right\rangle - \int_\Omega g(z, u_n^+) h \mathrm{d}z \leq c_{11}\|h\| \text{ for some } c_{11} > 0,\ \text{all } n \in \mathbb{N}. \tag{40}$$

Let $A_p^{\alpha_1} : W_0^{1,p}(\Omega) \to \mathbb{R}$ and $A_q^{\alpha_2} : W_0^{1,q}(\Omega) \to \mathbb{R}$ be the bounded, continuous, strictly monotone, $(S)_+$-type maps defined by

$$\langle A_p^{\alpha_1}(u), h\rangle = \int_\Omega \alpha_1(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} \mathrm{d}z, \quad \forall\ u, h \in W_0^{1,p}(\Omega),$$
$$\langle A_q^{\alpha_2}(u), h\rangle = \int_\Omega \alpha_2(z)|Du|^{q-2}(Du, Dh)_{\mathbb{R}^N} \mathrm{d}z, \quad \forall\ u, h \in W_0^{1,p}(\Omega).$$

Evidently, $V = A_p^{\alpha_1} + A_q^{\alpha_2}$. From (40), we obtain

$$\begin{aligned} &\left\langle A_p^{\alpha_1}(y_n), h\right\rangle + \frac{1}{\left\|u_n^+\right\|^{p-q}} \left\langle A_q^{\alpha_2}(y_n), h\right\rangle \\ \leq &\frac{c_{11}}{\left\|u_n^+\right\|^{p-1}}\|h\| + \int_\Omega \frac{g(z, u_n^+)}{\left\|u_n^+\right\|^{p-1}} h \mathrm{d}z \text{ for all } h \in W_0^{1,p}(\Omega),\ \text{all } n \in \mathbb{N}. \end{aligned} \tag{41}$$

Note that

$$\begin{aligned} &\int_\Omega \frac{g(z, u_n^+)}{\|u_n^+\|^{p-1}} h \mathrm{d}z \\ = &\frac{1}{\left\|u_n^+\right\|^{p-1}} \int_{\{u_n^+ \leq \underline{u}_\xi\}} \left[\xi(z)\underline{u}_\xi^{-\eta} + f\left(z, \underline{u}_\xi\right)\right] h \mathrm{d}z \\ &+ \frac{1}{\left\|u_n^+\right\|^{p-1}} \int_{\{\underline{u}_\xi < u_n^+\}} \xi(z)(u_n^+)^{-\eta} h \mathrm{d}z + \int_{\{\underline{u}_\xi < u_n^+\}} \frac{f(z, u_n^+)}{\left\|u_n^+\right\|^{p-1}} h \mathrm{d}z \\ &\qquad \text{for all } n \in \mathbb{N},\ \text{all } h \in W_0^{1,p}(\Omega) \text{ (see (23))}. \end{aligned} \tag{42}$$

We see that

$$\frac{1}{\left\|u_n^+\right\|^{p-1}} \int_{\{u_n^+ \leqslant \underline{u}_\xi\}} \left[\xi(z)\underline{u}_\xi^{-\eta} + f\left(z, \underline{u}_\xi\right)\right] h \mathrm{d}z \to 0 \text{ as } n \to \infty \text{ (see (10), (38))}. \tag{43}$$

Also, we have

$$\begin{aligned}
0 &\le \frac{1}{\|u_n^+\|^{p-1}} \int_{\{\underline{u}_\xi < u_n^+\}} \xi(z)\left(u_n^+\right)^{-\eta} h\mathrm{d}z \\
&\le \frac{1}{\|u_n^+\|^{p-1}} \int_{\{\underline{u}_\xi < u_n^+\}} \xi(z)\underline{u}_\xi^{-\eta} h\mathrm{d}z \\
&\le \frac{1}{\|u_n^+\|^{p-1}} \int_\Omega \xi(z)\underline{u}_\xi^{-\eta} h\mathrm{d}z \\
&\le \frac{c_3\|\xi\|_\infty}{\|u_n^+\|^{p-1}}\|h\| \text{ (see (10)) for all } n\in\mathbb{N},
\end{aligned}$$

$$\Rightarrow \frac{1}{\|u_n^+\|^{p-1}} \int_{\{\underline{u}_\xi < u_n^+\}} \xi(z)\left(u_n^+\right)^{-\eta} h\mathrm{d}z \to 0 \text{ as } n\to\infty. \tag{44}$$

Hypotheses H_1(i),(ii) imply that

$$\left\{\frac{f\left(\cdot, u_n^+(\cdot)\right)}{\|u_n^+\|^{p-1}}\right\}_{n\in\mathbb{N}} \subset L^{p'}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$\frac{f\left(\cdot, u_n^+(\cdot)\right)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \hat{l}(\cdot)y^{p-1} \text{ in } L^{p'}(\Omega) \text{ as } n\to\infty,$$

$$\text{with } \hat{\lambda}_1^{\alpha_1}(p) \le \hat{l}(z) \le l(z) \text{ for a.a. } z\in\Omega \text{ (see } H_1\text{(ii))}.$$

Recall that $y \ge 0$ (see (39)). On $\{z\in\Omega : \ y(z)>0\}$ we have $u_n^+(z)\to+\infty$ and so

$$\int_{\{\underline{u}_\xi < u_n^+\}} \frac{f\left(z, u_n^+\right)}{\|u_n^+\|^{p-1}} h\mathrm{d}z \to \int_\Omega \hat{l}(z)y^{p-1}h\mathrm{d}z \text{ as } n\to\infty. \tag{45}$$

In (41) we choose the test function $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n\to\infty$ and use (43), (44), (45), we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\left\langle A_p^{\alpha_1}\left(y_n\right), y_n - y\right\rangle = 0, \\
\Rightarrow & y_n \to y \text{ in } W_0^{1,p}(\Omega),\ \|y\| = 1,\ y\ge 0.
\end{aligned} \tag{46}$$

If in (41) we pass to the limit as $n\to\infty$ and use (43), (44), (45), (46), we obtain

$$\begin{aligned}
&\left\langle A_p^{\alpha_1}(y), h\right\rangle = \int_\Omega \hat{l}(z)y^{p-1}h\mathrm{d}z \text{ for all } h\in W_0^{1,p}(\Omega), \\
\Rightarrow & -\Delta_p^{\alpha_1} y(z) = \hat{l}(z)y(z)^{p-1} \text{ in } \Omega,\ y|_{\partial\Omega} = 0.
\end{aligned} \tag{47}$$

First suppose that $\hat{l} \not\equiv \hat{\lambda}_1^{\alpha_1}(p)$. We have

$$\tilde{\lambda}_1^{\alpha_1}(p,\hat{l}) < \tilde{\lambda}_1^{\alpha_1}(p,\hat{\lambda}_1^{\alpha_1}(p)) = 1 \text{ (see Proposition 1).}$$

Then from (47) we infer that $y=0$ or y is nodal (sign changing). Both possibilities contradict (46).

Next suppose that $\hat{l} = \hat{\lambda}_1^{\alpha_1}(p)$ for a.a. $z \in \Omega$. From (47) and (46) it follows that

$$y = \theta \hat{u}_1 \text{ with } \theta > 1,$$
$$\Rightarrow y \in \text{int } C_+.$$

We infer that $u_n^+(z) \to +\infty$ for a.a. $z \in \Omega$, as $n \to \infty$. From (34) and (37), we have for some $c_{12} > 0$,

$$-\int_\Omega \alpha_1(z)|Du_n^+|^p \mathrm{d}z - \frac{p}{q}\int_\Omega \alpha_2(z)|Du_n^+|^q \mathrm{d}z + \int_\Omega pG(z, u_n^+)\mathrm{d}z \le c_{12} \text{ for all } n \in \mathbb{N}. \tag{48}$$

Also, if in (36) we use the test function $h = u_n^+ \in W_0^{1,p}(\Omega)$, then

$$\int_\Omega \alpha_1(z)|Du_n^+|^p \mathrm{d}z + \int_\Omega \alpha_2(z)|Du_n^+|^q \mathrm{d}z - \int_\Omega g(z, u_n^+)u_n^+ \mathrm{d}z \le \varepsilon_n \text{ for all } n \in \mathbb{N}. \tag{49}$$

We add (48) and (49) and obtain

$$\int_\Omega \left[pG(z, u_n^+) - g(z, u_n^+)u_n^+\right] \mathrm{d}z \le c_{13} + \left(\frac{p}{q} - 1\right)\int_\Omega \alpha_2(z)|Du_n^+|^q \mathrm{d}z \quad \text{for some } c_{13} > 0, \text{ all } n \in \mathbb{N}. \tag{50}$$

We have

$$\begin{aligned}
&\int_\Omega \left[pG(z, u_n^+) - g(z, u_n^+)u_n^+\right] \mathrm{d}z \\
&= \int_{\{u_n^+ \le \underline{u}_\xi\}} \left[p\xi(z)\underline{u}_\xi^{-\eta}u_n^+ + f(z, u_n^+)u_n^+\right] \mathrm{d}z + \int_{\{\underline{u}_\xi < u_n^+\}} p\left[F(z, u_n^+) - F(z, \underline{u}_\xi)\right] \mathrm{d}z \\
&\quad + \int_{\{\underline{u}_\xi < u_n^+\}} pf(z, \underline{u}_\xi)\underline{u}_\xi \mathrm{d}z - \int_{\{u_n^+ \le \underline{u}_\xi\}} \left[\xi(z)\underline{u}_\xi^{-\eta}u_n^+ + f(z, \underline{u}_\xi)u_n^+\right] \mathrm{d}z \\
&\quad - \int_{\{\underline{u}_\xi < u_n^+\}} \xi(z)(u_n^+)^{1-\eta} \mathrm{d}z - \int_{\{\underline{u}_\xi < u_n^+\}} f(z, u_n^+)u_n^+ \mathrm{d}z \text{ for all } n \in \mathbb{N} \text{ (see (23))}, \\
&\ge \int_{\{\underline{u}_\xi < u_n^+\}} \left[pF(z, u_n^+) - f(z, u_n^+)u_n^+\right] \mathrm{d}z - c_{14}\left(1 + \|u_n^+\|^{1-\eta}\right) \\
&\qquad \text{for some } c_{14} > 0, \text{all } n \in \mathbb{N}, \\
&\ge \int_\Omega \left[pF(z, u_n^+) - f(z, u_n^+)u_n^+\right] \mathrm{d}z - c_{15}\left(1 + \|u_n^+\|^{1-\eta}\right) \\
&\qquad \text{for some } c_{15} > 0, \text{all } n \in \mathbb{N} \text{ (see hypothesis } H_1\text{(i))}.
\end{aligned} \tag{51}$$

Using (51) in (50), we obtain

$$\begin{aligned}
&\int_\Omega \left[pF(z, u_n^+) - f(z, u_n^+)u_n^+\right] \mathrm{d}z \le c_{16}\left(1 + \|u_n^+\|^q\right) \mathrm{d}z \text{ for some } c_{16} > 0, \text{all } n \in \mathbb{N}, \\
\Rightarrow &\int_\Omega \frac{pF(z, u_n^+) - f(z, u_n^+)u_n^+}{(u_n^+)^\tau} y_n^\tau \mathrm{d}z \le c_{16}\left(\frac{1}{\|u_n^+\|^\tau} + \frac{1}{\|u_n^+\|^{\tau - q}}\right) \text{ for all } n \in \mathbb{N}.
\end{aligned} \tag{52}$$

In (52) we pass to the limit as $n \to \infty$. Using Fatou's lemma, hypothesis H_1(ii) and (38), we obtain

$$0 < \hat{\beta}|\Omega|_N \le 0,$$

a contradiction (here $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$). Therefore $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we infer that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded (see (37)).

We may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega),\ u_n \to u \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{53}$$

In (36), we choose the test function $h = u_n - u \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \to \infty$. We obtain

$$\lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0,$$
$$\Rightarrow u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 4).}$$

This proves that the functional $\varphi(\cdot)$ satisfies the C-condition. ☐

Now we are ready for the multiplicity theorem.

Theorem 1. *If hypotheses H_0, H_1 hold and $\|\xi\|_\infty$ is small, then problem (1) has at least two positive solutions*

$$u_0,\ \hat{u} \in \operatorname{int} C_+,\ u_0 \neq \hat{u}.$$

Proof. From Proposition 8, we already have one positive solution

$$u_0 \in \operatorname{int} C_+.$$

Also (27) and Propositions 10 and 11 permit the use of the mountain pass theorem. So we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in \mathcal{K}_\varphi \subset \left[\underline{u}_\xi\right) \cap C_0^1(\bar{\Omega}),\ \varphi(u_0) < m_0 \leq \varphi(\hat{u}).$$

Therefore $\hat{u} \in \operatorname{int} C_+$ is a second positive solution of (1) (see (23)) and $u_0 \neq \hat{u}$. ☐

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