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Claude Vallée, Vicențiu D. Rădulescu & Kossi Atchonouglo

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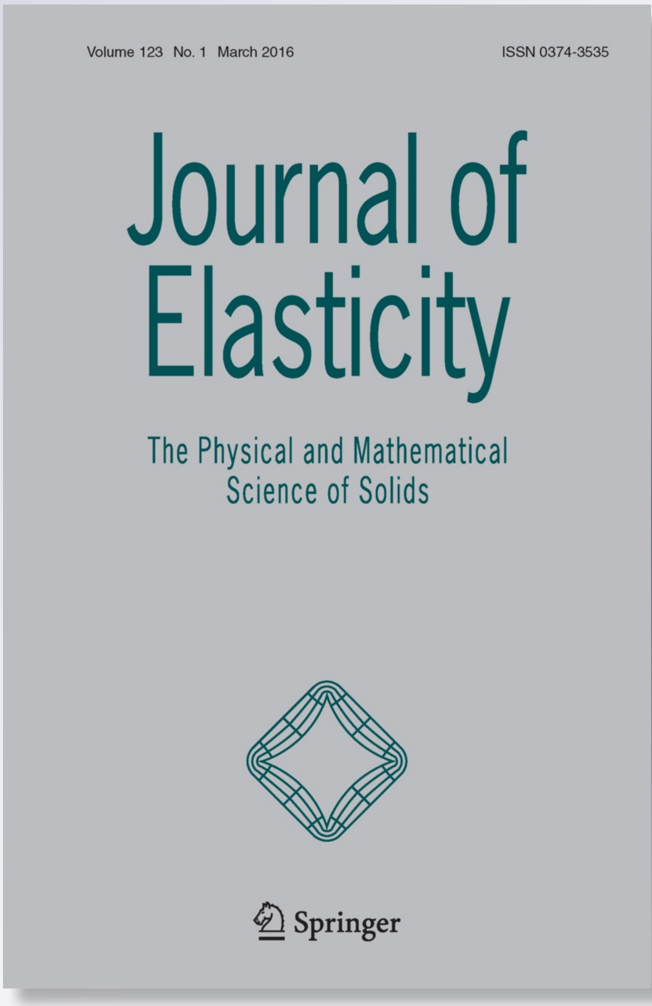

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New Variational Principles for Solving Extended Dirichlet-Neumann Problems

Claude Vallée[1] · **Vicenţiu D. Rădulescu**[2,3] · **Kossi Atchonouglo**[4]

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Abstract We extend in this paper the classical variational methods devoted to solve the Dirichlet-Neumann problems. We assume that the intensive and extensive parameters are related by a maximal monotone multifunction. The Fitzpatrick's method allows us to elaborate new variational principles.

Keywords Dirichlet-Neumann problems · Primal-dual variational problems · Fitzpatrick functions · Fitzpatrick sequences · Uzawa-type algorithm · Heat conduction · Nonlinear elasticity

Mathematics Subject Classification Primary 30E25 · 90C46 · 90C25 · Secondary 80A20 · 74B20

This paper is dedicated to the memory of the distinguished mechanician and dear friend, Professor Claude Vallée. We learned a lot from Claude's original mechanical ideas and his large scientific knowledge was very useful for us. He lost the battle with a serious illness in November 2014. Professor Claude Vallée will remain for ever in our souls and hearts.

✉ V.D. Rădulescu
vicentiu.radulescu@imar.ro

C. Vallée
claude.vallee@univ-poitiers.fr

K. Atchonouglo
kossi.atchonouglo@gmail.com

1 Institut Pprime, Université de Poitiers, UPR CNRS 3346, 86034 Poitiers, France

2 Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

3 Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 014700 Bucharest, Romania

4 Faculté des Sciences, Département de Physique, Université de Lomé, BP 1515 Lomé, Togo

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1 Introduction

1.1 Functional Analysis Preliminaries

Let Ω be a bounded, connected, open subset of $\mathbb{R}^N$ whose boundary Γ is Lipschitz continuous, the set Ω being locally on a single side of Γ. Let $\boldsymbol{n}$ denote the outer unit normal to Γ. The boundary Γ is composed of two disjoints subsets Γ_0 and Γ_1, the $d\Gamma$-measure of Γ_0 being positive.

Consider the function space

$$H(\mathrm{div};\Omega) := \left\{\mathbf{y} \in \left(L^2(\Omega)\right)^N \middle| \ \mathrm{div}\,\mathbf{y} \in L^2(\Omega)\right\}. \tag{1}$$

Then $H(\mathrm{div};\Omega)$ is a Hilbert space when endowed with the norm

$$\|\mathbf{y}\|_{H(\mathrm{div};\Omega)} := \left(|\mathbf{y}|^2_{L^2} + |\mathrm{div}\,\mathbf{y}|^2_{L^2}\right)^{1/2}.$$

As argued in Lions & Magenes [28] for smooth bodies and in Girault & Raviart [24] for Lipschitz continuous domains (see also Ciarlet, Geymonat & Krasucki [20, 21]), for all $\mathbf{y} \in H(\mathrm{div};\Omega)$ we can define its "outer normal component" $\langle \mathbf{y}, \mathbf{n}\rangle_\Gamma$ along Γ as an element of $H^{-1/2}(\Gamma)$ such that the following Green formula holds:

$$\int_\Omega \mathrm{grad}\,\theta \cdot \mathbf{y}\, d\Omega + \int_\Omega \theta \ \mathrm{div}\ \mathbf{y}\, d\Omega = \langle \theta, \mathbf{y}\mathbf{n}\rangle_\Gamma, \quad \forall(\theta, \mathbf{y}) \in H^1(\Omega) \times H(\mathrm{div};\Omega). \tag{2}$$

We have denoted by $\mathrm{grad}\,\theta \cdot \mathbf{y}$ the scalar product in $\mathbb{R}^N$ between $\mathrm{grad}\,\theta$ and $\mathbf{y}$. The duality pairing between $H^{-1/2}(\Omega)$ and $H^{1/2}(\Omega)$ is denoted $\langle\cdot,\cdot\rangle_\Gamma$. The above definition makes sense because $\langle \mathbf{y}, \mathbf{n}\rangle_\Gamma \in L^2(\Gamma) \subset H^{-1/2}(\Gamma)$. We refer to Allaire [1, Sect. 4.4.2] for related properties of the function space $H(\mathrm{div};\Omega)$.

The definition of the function space $H(\mathrm{div};\Omega)$ in relation (1) and the statement of the Green formula in (2) are appropriate for modelling the heat conduction phenomena with $N = 2$ or 3. In elasticity, $H(\mathrm{div};\Omega)$ is generalized as

$$H(\mathrm{div};\Omega) := \left\{\mathbf{y} \in \left(L^2(\Omega)\right)^d;\ \mathrm{div}\ \mathbf{y} \in \left(L^2(\Omega)\right)^N\right\}, \tag{3}$$

where $d = N(N+1)/2$. Accordingly, the Green formula reads

$$\int_\Omega \mathbf{y} : \nabla u\, d\Omega + \int_\Omega u \cdot (\mathrm{div}\ \mathbf{y})\, d\Omega = \langle \mathbf{y}\,\mathbf{n}, u\rangle_\Gamma$$
$$\text{for all } (u, \mathbf{y}) \in \left(H^1(\Omega)\right)^N \times H(\mathrm{div};\Omega), \tag{4}$$

where $\mathbf{y} : \nabla u$ denotes the double contracted product of the stress tensor $\mathbf{y}$ and the displacement gradient tensor ∇u. We refer to Amrouche, Ciarlet, Gratie & Kesavan [3, 4] for more details and related properties.

1.2 Dirichlet-Neumann Problem

Let $\Phi : \mathbb{R}^N \to \mathbb{R}$ be a convex and differentiable potential and let Φ^* be its Legendre-Fenchel transform [31], namely

$$\Phi^*(\mathbf{y}) = \sup_{\mathbf{x}\in\mathbb{R}^N} \left\{\mathbf{x}\cdot\mathbf{y} - \Phi(\mathbf{y})\right\}.$$

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Remark 1 In the Green formula (4) the tensor $\mathbf{y}$ is symmetric and the double contracted product $\mathbf{y} : \nabla u$ can be replaced by $\mathbf{y} : \nabla_s u$, where $\nabla_s u$ is the symmetric part of ∇u.

Consider the following Dirichlet-Neumann problems.

1. *Heat conduction case*. Assume that there are given three functions: $Q \in L^2(\Omega)$, $q \in L^2(\Gamma_1)$, and $\theta_0 \in L^2(\Gamma_0)$. Find $(\mathbf{y}, \theta) \in H(\text{div}; \Omega) \times H^1(\Omega)$ satisfying the partial differential equation

$$\text{div}\, \mathbf{y} + Q = 0 \quad \text{in } \Omega \tag{5}$$

subjected to the boundary conditions

$$\mathbf{y} \cdot \boldsymbol{n} = q \quad \text{on } \Gamma_1 \tag{6a}$$

$$\text{tr}\,\theta = \theta_0 \quad \text{on } \Gamma_0 \tag{6b}$$

and the constitutive law

$$\mathbf{y} = D\Phi(\text{grad}\,\theta), \tag{7}$$

where tr denotes the trace operator in $L^2(\Gamma_0)$.

2. *Elasticity case*. Assume that there are given three functions: $f \in L^2(\Omega)$, $F \in L^2(\Gamma_1)$, and $u_0 \in L^2(\Gamma_0)$. Find $(\mathbf{y}, \mathbf{x}) \in H(\text{div}; \Omega) \times H^1(\Omega)$ satisfying the partial differential equation

$$\text{div}\,\mathbf{y} + f = 0 \quad \text{in } \Omega \tag{8}$$

subjected to the boundary conditions

$$\mathbf{y} \cdot \boldsymbol{n} = F \quad \text{on } \Gamma_1 \tag{9a}$$

$$u = u_0 \quad \text{on } \Gamma_0 \tag{9b}$$

and the constitutive law

$$\mathbf{y} = D\Phi\,(\mathbf{x}). \tag{10}$$

1.3 Examples of Linear Fourier's Heat Conduction Law

1.3.1 Isotropic Case

In the particular case where $N = 3$ and θ is the temperature, the potential is

$$\Phi(\text{grad}\,\theta) = \frac{1}{2}\lambda|\,\text{grad}\,\theta|^2, \tag{11}$$

where $\lambda > 0$ is the Fourier heat condition coefficient. The vector $\mathbf{y} = \lambda\,\text{grad}\,\theta$ is the opposite of the heat flux density vector, the normal component $\langle \mathbf{y}, \mathbf{n} \rangle = \mathbf{n} \cdot (\lambda\,\text{grad}\,\theta)$ of $\mathbf{y}$ is given ($=q$) on a part Γ_1 of the boundary, and θ is given ($= \theta_0$) on the complementary part Γ_0. The heat equation with source term Q reads

$$\text{div}(\lambda\,\text{grad}\,\theta) + Q = 0 \quad \text{in } \Omega.$$

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1.3.2 Anisotropic Case

In the anisotropic linear case, the potential equation (11) is replaced by

$$\Phi(\operatorname{grad}\theta) = \frac{1}{2}(S\operatorname{grad}\theta)\cdot\operatorname{grad}\theta,$$

where S stands for the heat conduction tensor, which is either homogeneous or non-homogeneous, but it is symmetric and positive definite. In both isotropic and anisotropic cases, the vector $\mathbf{y}$ is linked to the gradient of the temperature by a constitutive law of the type Eq. (7).

1.4 Primal Method

A *primal method* to find θ consists in minimizing the energy functional

$$G(\theta) = \int_{\Omega} \Phi(\operatorname{grad}\theta)\, d\Omega - \int_{\Omega} Q\theta\, d\Omega - \int_{\Gamma_1} q\theta\, d\Gamma$$

in the class of θ satisfying the condition (6b).

The stationarity conditions of the functional G are equivalent to (5), (6a) and (7).

1.5 Dual Method

A *dual method* consists in finding directly $\mathbf{y}$ by minimizing the functional

$$H(\mathbf{y}) = \int_{\Omega} \Phi^*(\mathbf{y})\, d\Omega - \int_{\Gamma_0} \theta_0 \langle \mathbf{y}, \mathbf{n}\rangle\, d\Gamma$$

for all $\mathbf{y}$ satisfying (5) and (6a).

The stationarity conditions of the functional H assert the existence of a scalar field θ satisfying both equation

$$\operatorname{grad}\theta = D\Phi^*(\mathbf{y}) \tag{12}$$

and relation (6b).

Hence θ is viewed as the solution of the primal problem. As one determines numerically $\mathbf{y}$, the field $\operatorname{grad}\theta$ follows from Eq. (12). Next, since θ is given on Γ_0, then θ is determined.

1.6 Primal-Dual Method

We associate to the problem governed by Eq. (7) the natural optimization problem: minimize the energy functional

$$F(\theta,\mathbf{y}) = \int_{\Omega} \left[\Phi(\operatorname{grad}\theta) + \Phi^*(\mathbf{y})\right] d\Omega - \int_{\Gamma_0} \theta_0\, \langle \mathbf{y}, \mathbf{n}\rangle\, d\Gamma - \int_{\Gamma_1} q\theta\, d\Gamma - \int_{\Omega} Q\theta\, d\Omega \tag{13}$$

with respect to all θ satisfying (6b) and all $\mathbf{y}$ satisfying (5) and (6a).

Theorem 1 *Let Φ be strictly convex and differentiable. Then the minimization of the globally convex functional F defined in (13) solves the initial Dirichlet-Neumann problem.*

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Proof Expression (13) shows clearly that F is globally convex. Next, we show that F is non-negative. For this purpose we combine the Fenchel inequality

$$\Phi\,(\operatorname{grad}\theta) + \Phi^*(\mathbf{y}) \geq \mathbf{y}\cdot\operatorname{grad}\theta,$$

the tensorial calculus formula

$$\mathbf{y}\cdot\operatorname{grad}\theta = \operatorname{div}(\theta\mathbf{y}) - \theta\,\operatorname{div}\mathbf{y}$$

and the Green formula (2). It follows that

$$F(\theta,\mathbf{y}) \geq \int_{\Gamma_1} \theta\big(\langle\mathbf{y},\mathbf{n}\rangle - q\big)\,d\Gamma + \int_{\Gamma_0} (\theta-\theta_0)\langle\mathbf{y},\mathbf{n}\rangle\,d\Gamma - \int_{\Omega} \theta(\operatorname{div}\mathbf{y} + Q)\,d\Omega = 0.$$

In the above formula the equality holds solely if the Fenchel inequality reduces to the equality

$$\Phi(\operatorname{grad}\theta) + \Phi^*(\mathbf{y}) = \mathbf{y}\cdot\operatorname{grad}\theta,$$

hence if the constitutive law is fulfilled as Eq. (7) or equivalently Eq. (12). □

Remark 2 Numerically it is appropriate to perform this minimization by an Uzawa type algorithm alternating successive refinements of the fields θ and $\mathbf{y}$.

1.7 Maximal Cyclically Monotone Constitutive Laws

When the potential Φ is not differentiable, but only convex and semi-continuous, the three variational methods (primal, dual, and primal-dual) subsist with the same energy functional (see Moreau [30]). For many materials the constitutive law (see Sect. 2.1.1) is multivalued, that is,

$$\mathbf{y} \in \mathbf{A}(\operatorname{grad}\theta) \tag{14}$$

where $\mathbf{A}$ is a set-valued function. When $\mathbf{A}$ is maximal cyclically monotone (see Sect. 2.2), there exists a proper convex and lower semi-continuous function Φ such that Eq. (14) becomes

$$\mathbf{y} \in \partial\Phi\,(\operatorname{grad}\theta). \tag{15}$$

In the above subdifferential inclusion, $\partial\Phi$ stands for the subdifferential of the function Φ.

The constitutive law (15) can be easily inverted as

$$\operatorname{grad}\theta \in \partial\Phi^*(\mathbf{y}). \tag{16}$$

Theorem 2 *Let Φ be strictly convex and differentiable. Then the minimization of the globally convex functional F defined by* (13) *solves the initial Dirichlet-Neumann problem.*

Proof The extension of the proof of Theorem 1 is based on the equivalence between the subdifferential inclusions (15) and (16) with the Fenchel scalar condition

$$\Phi(\operatorname{grad}\theta) + \Phi^*(\mathbf{y}) = \mathbf{y}\cdot\operatorname{grad}\theta, \tag{17}$$

as observed by Moreau [31]. □

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In Deformable Solid Mechanics such kind of laws apply to Generalized Standard Materials (GSM) (see Sect. 2.1.3). This concludes the formulation of the primal, dual and primal-dual methods for solving the Dirichlet-Neumann problem. We refer to the reference books by Ciarlet [18, 19], which develop several related results, including the primal-dual principle (Hellinger-Reissner energy) for the elasticity problem.

It may happen that the graph of the operator A relying the flux vector $\mathbf{y}$ with the gradient vector $\mathbf{x} = \operatorname{grad}\theta$ is only k-monotone up to a finite integer n, k does not run up to infinity as for cyclically monotone operators (see Sect. 2.1).

In the next section we will ask us if there is a variational principles to solve primal, dual or primal-dual Dirichlet-Neumann problem.

2 *n*-Monotone Constitutive Laws

The constitutive laws of *Standard Materials* are described by differentiable potentials. Cyclically monotone set-valued constitutive laws characterize *Generalized Standard Materials* (GSM) and are modeled by convex lower semi-continuous potentials. However, this extension fails to describe some important models, including the Coulomb dry friction law. In 1991, considering an implicit constitutive law, de Saxcé & Feng [37] proposed a new extension. This new class, called *Implicit Standard Material* (ISM), is modeled by a bipotential. In the particular case corresponding to a Generalized Standard Material, the bipotential reduces to the sum of the potential and its Legendre-Fenchel conjugate. Independently, in order to simplify the study of monotone operators, Fitzpatrick [22] proposed to replace the n-monotone operators by point-to-point functions, nowadays called Fitzpatrick's functions. It appears that Fitzpatrick's functions are special bipotentials [39].

In this section we develop the main mathematical tools necessary for the qualitative analysis of GSM and ISM, in relationship with the theory developed by Fitzpatrick.

2.1 Generalized and Implicit Standard Materials

2.1.1 Constitutive Laws

Let X be a real Banach space and let X^* be its dual space. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X.

A *constitutive law* relating an extensive variable $x \in X$ and an intensive variable $y \in X^*$ can be regarded as a subset of $X \times X^*$. This subset is seen as the graph $G(T)$ of a multivalued operator $T : X \to 2^{X^*}$, where

$$G(T) := \{(x, y) \in X \times X^*;\ y \in Tx\}.$$

Example 1 For heat conduction x is $\operatorname{grad}\theta$, while for elasticity x is $\nabla_s u$.

2.1.2 Standard Materials

The constitutive law is of the type

$$y = \mathrm{D}\,\Phi(x) \tag{18}$$

where the potential Φ is differentiable and convex (see Eq. (7)). The inverse constitutive law reads (see Eq. (12))

$$x = \mathrm{D}\,\Phi^*(y). \tag{19}$$

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A material whose behaviour is governed by such a constitutive law is referred to as a *Standard Material*.

2.1.3 Generalized Standard Materials

Equations (18) and (19) describe single-valued relations between variables x and y. However, for many materials, such relations reveal to be multi-valued.

The constitutive law defined by the formula Eq. (15) is generalized in

$$y \in \partial \Phi(x) \tag{20}$$

where the potential is convex and l.s.c. The inverse constitutive law reads

$$x \in \partial \Phi^*(y). \tag{21}$$

As in Eq. (17) the constitutive law can be summarized by a scalar relation

$$\Phi(x) + \Phi^*(y) = \langle x, y \rangle. \tag{22}$$

2.1.4 Implicit Standard Materials

Equation (22) can be seen as an extremal case of Fenchel's inequality

$$\Phi(x) + \Phi^*(y) \geq \langle x, y \rangle, \tag{23}$$

which holds for all $x \in X$ and $\mathbf{y} \in X^*$.

In relationship with applications to the dry friction phenomenon, G. de Saxcé observed that Eq. (23) can be weakened to

$$b(x, y) \geq \langle x, y \rangle,$$

where the function $b(x, y)$ called bipotential is assumed to be

(i) convex and l.s.c. in x;
(ii) convex and l.s.c. in y;
(iii) bounded from below by the duality product, i.e., $b(x, y) \geq \langle x, y \rangle$.

In the particular case of Generalized Standard Materials the bipotential reduces to the sum of the potential and its conjugate.

A material is referred as to an *Implicit Standard Material* if it is described by one of the following equivalent implicit constitutive laws:

(i) y belongs to the subdifferential of $b(\xi, y)$ with respect to ξ at x,
(ii) x belongs to the subdifferential of $b(x, \eta)$ with respect to η at y,
(iii) $b(x, y) = \langle x, y \rangle$.

We refer to Buliga, de Saxcé & Vallée [14–16] for more details about Implicit Standard Materials. We just point out that this model is relevant to describe various phenomena, such as generalized Drücker-Prager plasticity [36], unilateral contact with Coulomb dry friction [37], modified Cam-Clay model [25, 36], non-associated plasticity of soils [10], nonlinear kinematical hardening rule for cyclic plasticity of metals [27], Lemaître's plastic-ductile damage law [26], and shakedown of non-standard elasto-plastic materials [11].

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2.2 Monotone Maximal Operators

2.2.1 Monotonicity

Let $T : X \to 2^{X^*}$ be a multi-valued operator. Then T is *monotone* if

$$\langle x_2 - x_1, y_2 - y_1 \rangle \geq 0 \quad \text{for all } x_1,\, x_2 \in X,\;\; y_1 \in T x_1,\;\; y_2 \in T x_2.$$

Example 2 Assume that T is single-valued, linear and positive. Then T is monotone.

Example 3 Let $\Phi : X \to \mathbb{R}$ be convex and lower semi-continuous. Then $T := \partial \Phi$ is monotone. We point out that a monotone multifunction is not necessarily the subdifferential of a convex lower semi-continuous potential. We refer to Moreau [31] and Rockafellar [34] for details and related properties.

2.2.2 Maximality

Assume that $T : X \to 2^{X^*}$ is a monotone operator. Then T is said to be *maximal monotone* if there is no monotone proper enlargement of T. In other words the maximal monotone operators are monotone multi-valued operators whose graphs cannot be enlarged without destroying monotonicity. In other to prove that T is maximal monotone one must establish that

$$(x, y) \notin G(T) \quad \Longrightarrow \quad \exists\, (x_1, y_1) \in G(T) \text{ such that } \langle x - x_1, y - y_1 \rangle < 0.$$

In [33, 38] it has been established that the maximality assumption is equivalent to one of the following statements:

$$\big[(x, y) \in X \times X^* \text{ and } \langle x - x_1, y - y_1 \rangle \geq 0,\ \forall\, (x_1, y_1) \in G(T)\big] \quad \Longrightarrow \quad (x, y) \in G(T);$$

$$\Big[(x, y) \in X \times X^* \text{ and } \inf_{y_1 \in T x_1} \langle x - x_1, y - y_1 \rangle \geq 0\Big] \quad \Longrightarrow \quad (x, y) \in G(T).$$

Example 4 If X is a real Hilbert space then any linear positive single-valued operator $T : X \to X$ is maximal monotone.

Example 5 (Rockafellar [34, 35]) If $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous convex function, then $T = \partial \Phi$ is a maximal monotone operator.

2.2.3 Properties

We conclude with the following useful properties for testing the maximality of monotone operators:

$$\text{(i)} \quad (x, y) \in G(T) \quad \Longrightarrow \quad \inf_{y_1 \in T x_1} \langle x - x_1, y - y_1 \rangle = 0;$$

$$\text{(ii)} \quad (x, y) \notin G(T) \quad \Longrightarrow \quad \inf_{y_1 \in T x_1} \langle x - x_1, y - y_1 \rangle < 0;$$

$$\text{(iii)} \quad \forall x \in X, \forall y \in X^*, \ \inf_{y_1 \in T x_1} \langle x - x_1, y - y_1 \rangle \leq 0.$$

For detailed proofs and related properties we refer to Phelps & Simons [33] and Simons [38].

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2.3 Finite Monotonicity

2.3.1 *n-Cyclically Monotonicity*

For an integer $n \geq 2$, a multifunction $T : X \to 2^{X^*}$ is n-cyclically monotone [7], provided that

$$\left.\begin{array}{l}(x_i, y_i) \text{ are } n \text{ pairs of } G(T)\\ (x_{n+1}, y_{n+1}) = (x_1, y_1)\end{array}\right\} \implies \sum_{i=1}^{n} \langle x_{i+1} - x_i, y_i \rangle \leq 0.$$

This definition shows that 2-cyclically monotonicity simplifies to ordinary monotonicity. We also observe that $(n+1)$-cyclically monotonicity implies n-cyclically monotonicity.

2.3.2 *Examples*

The first example expresses the finite monotonicity of positive semi-definite symmetric linear mappings [39].

Example 6 Let X be a real Hilbert space and let $S : X \to X$ be a linear symmetric single-valued operator. Assume that S is positive definite. Then the single-valued operator T defined by $Tx = \{Sx\}$ is n-cyclically monotone for all $n \geq 2$.

We have seen in Example 5 that the subdifferential of a convex lower semi-continuous function is maximal monotone. The following example establishes an important converse of this property (see Rockafellar [34]).

Example 7 Let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then $T = \partial\Phi$ is a maximal monotone operator. Conversely, assume that $T : X \to 2^{X^*}$ is a multi-valued operator. In order that there exists a lower semi-continuous proper convex function Φ on X such that $T = \partial\Phi$, it is necessary and sufficient that T be a maximal cyclically monotone operator. Moreover, in this case T determines Φ uniquely up to an additive constant. We refer to Sect. 2.4.5 for a related result in relationship with the recovery of the Generalized Standard Material.

We conclude with the following condition for the n-cyclically monotonicity of a 2×2 matrix.

Example 8 Let $\mathbf{A}$ be a 2×2 matrix with a positive definite symmetric part S and a skew-symmetric part

$$W = \begin{bmatrix} 0 & -r \\ r & 0 \end{bmatrix} = r \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = rJ,$$

where $r \in \mathbb{R} \setminus \{0\}$. Define $\alpha \in (0, \pi/2)$ such that

$$|r| = \sqrt{\det S}\, \tan\alpha.$$

Then the operator T defined by $Tx = \{\mathbf{A}x\}$ is strictly n-cyclically monotone if and only if $n\alpha < \pi$ [39, 40].

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2.4 Fitzpatrick Sequences

2.4.1 Definition of the Fitzpatrick Function

Let T be a maximal monotone multifunction. Then its associated Fitzpatrick function is defined in [22] by

$$F_{T,2}(x, y) = \langle x, y\rangle - \inf_{y_1 \in T x_1} \langle x - x_1, y - y_1\rangle.$$

According to Sect. 2.2.3 we have

$$\begin{aligned} F_{T,2}(x, y) &= \langle x, y\rangle \quad \text{if } (x, y) \in G(T) \\ F_{T,2}(x, y) &> \langle x, y\rangle \quad \text{if } (x, y) \notin G(T). \end{aligned}$$

We also observe that $F_{T,2}$ is globally lower semi-continuous and convex. Indeed,

$$F_{T,2}(x, y) = \sup_{(x_1,y_1)\in G(T)} \big[\langle x, y_1\rangle + \langle x_1, y\rangle - \langle x_1, y_1\rangle\big]$$

is the supremum of a family of continuous affine real-valued functions, hence $F_{T,2}$ is convex and lower semi-continuous on $X \times X^*$.

2.4.2 Definition of the Fitzpatrick Sequence

For $n \geq 2$ and $(x, y) \in X \times X^*$, let (x_i, y_i) be $n-1$ pairs of the graph $G(T)$ indexed from $i = 1$ to $i = n-1$, insert $(x_n, y_n) = (x, y)$, and close the loop by $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. Then the Fitzpatrick sequence is defined (see Bartz, Bauschke, Borwein, Reich & Wang [7]) by

$$F_{T,n}(x, y) = \langle x, y\rangle + \sup_{y_i \in T x_i} \sum_{i=1}^{n} \langle x_{i+1} - x_i, y_i\rangle.$$

For $n = 2$ we recover the case section 2.4.1 originally proposed by Fitzpatrick [22] to study monotone operators.

2.4.3 Basic Properties

The main properties of Fitzpatrick's sequence are the following:

(i) $F_{T,n}$ is globally convex and lower semi-continuous. Indeed, from the definition, as observed for $F_{T,2}$, the function $F_{T,n}$ is the upper hull of a family of continuous and affine real-valued functions. Thus, $F_{T,n}$ is convex and lower semi-continuous on $X \times X^*$.

(ii) Every function of Fitzpatrick's sequence is bounded from below by the duality product, namely

$$F_{T,n}(x, y) \geq \langle x, y\rangle,$$

with equality if and only if $(x, y) \in G(T)$. This result was essentially unnoticed for several years, until it was rediscovered by Martinez-Legal & Théra [29] and, independently, by Burachik & Svaiter [17].

(iii) Fitzpatrick's sequence is increasing [7]: for all $x \in X$, $y \in X^*$, and $n \geq 2$,

$$F_{T,n}(x, y) \leq F_{T,n+1}(x, y).$$

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(iv) Recursion formula [9]: if $T : X \to 2^{X^*}$ is $(n+1)$-cyclically monotone ($n \geq 2$), then

$$F_{T,n+1}(x,y) = \sup_{\eta \in T\xi} \big[F_{T,n}(\xi, y) + \langle x - \xi, \eta \rangle \big].$$

(v) If $T : X \to 2^{X^*}$ is n-cyclically maximal monotone, then [7]

$$G(T) = \big\{ (x,y) \in X \times X^*;\; F_{T,n}(x,y) = \langle x, y \rangle \big\}.$$
$$F_{T,n}(x,y) > \langle x, y \rangle \quad \text{for all } (x,y) \notin G(T).$$

(vi) Each function of the Fitzpatrick's sequence is a bipotential that represents the constitutive law associated with the multi-valued operator T.

We refer to Bauschke, Borwein & Wang [8] for properties of monotone operators in relationship with Fitzpatrick's sequence.

2.4.4 Examples

Example 9 (Bartz, Bauschke, Borwein, Reich & Wang [7]) Assume that the potential Φ is the indicator function i_K of a convex set K. Then the Fitzpatrick sequence of $T := \partial \Phi$ is given by

$$F_{T,n}(x,y) = i_K(x) + i_K^*(y) \quad \text{for all } n \geq 2.$$

By duality, the same property holds for $\Phi = i_K^*$.

Example 10 Let X be a real Hilbert space and let $S : X \to X$ be a linear symmetric single-valued operator. We have seen in Example 6 that if S is positive definite, then the (single-valued) operator T defined by $Tx = \{Sx\}$ is n-cyclically monotone for all $n \geq 2$. The associated Fitzpatrick sequence is [7, 39, 40]

$$F_{T,n}(x,y) = \langle x, y \rangle + \frac{n-1}{2n} \big\langle y - Sx, S^{-1}(y - Sx) \big\rangle.$$

This sequence admits as pointwise limit

$$F_{T,\infty}(x,y) = \frac{1}{2} \langle x, Sx \rangle + \frac{1}{2} \big\langle y, S^{-1} y \big\rangle.$$

Example 11 Let X be a real Hilbert space and let $A : X \to X$ be a linear but not necessarily symmetric operator. Let T be the operator defined by $Tx = \{Ax\}$. Let S be the symmetric part of A. Then the Fitzpatrick sequence associated with T is given by

$$F_{T,k}(x,y) = \langle x, y \rangle + \frac{1}{4} \big\langle y - Sx, H_k^{-1}(y - Sx) \big\rangle,$$

where the matrix H_k is constructed from the matrix $H_2 = S$ by the recursion formula [39]

$$H_k = S - \frac{1}{4} A^t H_{k-1}^{-1} A \quad \text{for } 3 \leq k \leq n.$$

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Example 12 Returning to Example 8 and assuming that $n\alpha < \pi$, the associated Fitzpatrick sequence is, for all $2 \leq k \leq n$ [39, 40],

$$F_{T,k}(x, y) = \langle x, y\rangle + \frac{1}{2}\frac{\sin(k-1)\alpha}{\sin k\alpha}\cos\alpha\big\langle y - Ax, S^{-1}(y - Ax)\big\rangle.$$

We observe that if α tends to 0, then A tends to S and $\frac{\sin(k-1)\alpha}{\sin k\alpha}\cos\alpha$ tends to $(k-1)/k$. Thus, the Fitzpatrick sequence becomes

$$F_{S,k}(x, y) = \langle y, x\rangle + \frac{1}{2}\frac{k-1}{k}\big\langle(y - A\,x), S^{-1}(y - A\,x)\big\rangle,$$

which, as k tends to infinity, tends to

$$F_{S,\infty}(x, y) = \langle y, x\rangle + \frac{1}{2}\big\langle(y - A\,x), S^{-1}(y - A\,x)\big\rangle.$$

This is the sum of the functionals

$$\Phi(x) = \frac{1}{2}\big\langle x, S(x)\big\rangle$$

and

$$\Phi^*(y) = \frac{1}{2}\,y\cdot S^{-1}y,$$

as in Example 10.

2.4.5 Recovery of the Potential of GSM

As stated in Bartz, Bauschke, Borwein, Reich & Wang [7], every n-cyclically monotonicity is captured by the Fitzpatrick function $F_{T,n}(x, y)$. When the constitutive law of a Generalized Standard Material is described by a convex, lower semi-continuous and proper potential Φ, then (by Remark 7) the multi-valued operator $T = \partial\Phi$ is maximal monotone and cyclically monotone, hence maximal cyclically monotone. The Fitzpatrick sequence $\{F_{T,n}(x, y)\}$ admits a pointwise limit $F_{T,\infty}(x, y) = \sup_{n\geq 2} F_{T,n}(x, y)$, which is nothing else than the sum $\Phi(x) + \Phi^*(y)$ of the potential and its conjugate. In such a way we recover the Generalized Standard Materials specific separated potentials.

Conversely, if a multi-valued operator T is maximal cyclically monotone, Rockafellar [34, 35] and Moreau [31] established independently constructive theorems to prove the existence of a proper, convex, and lower semi-continuous potential Φ such that $T = \partial\Phi$. In fact, the method for retrieving $\Phi(x)$ consists in fixing y in $F_{T,\infty}(x, y)$. By duality, $\Phi^*(y)$ is recovered by fixing x. Actually, the construction of the Fitzpatrick sequence is a clever rewriting of the Moreau-Rockafellar theorem.

3 Extended Dirichlet-Neumann Problem

Let Ω be a bounded, connected, open subset of $\mathbb{R}^N$ whose boundary Γ is Lipschitz continuous, the set Ω being locally on a single side of Γ. Let $\boldsymbol{n}$ denote the outer unit normal to Γ. The boundary Γ is composed of two disjoints subsets Γ_0 and Γ_1, the $d\Gamma$-measure of Γ_0 being positive.

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Assume that there are given three functions: $Q \in L^2(\Omega)$, $q \in L^2(\Gamma_1)$, and $\theta_0 \in L^2(\Gamma_0)$.

Consider the following extended Dirichlet-Neumann problem: find $(\mathbf{y}, \theta) \in H(\mathrm{div}; \Omega) \times H^1(\Omega)$ such that

$$\begin{cases} \text{(i)} & \mathrm{div}\, \mathbf{y} + Q = 0 \quad \text{in } \Omega, \\ \text{(ii)} & \mathrm{grad}\, \theta = \mathbf{x} \quad \text{in } \Omega, \\ \text{(iii)} & \mathbf{y} \in T\mathbf{x}, \end{cases} \tag{24}$$

where T is a maximal strictly n-monotone multifunction relating the vectors $\mathbf{x}(\omega)$ and $\mathbf{y}(\omega)$ for each point $\omega \in \Omega$. Mixed boundary conditions of Dirichlet-Neumann type are imposed as follows:

$$\begin{cases} \text{(iv)} & \langle \mathbf{y}, \mathbf{n} \rangle = q \quad \text{on } \Gamma_1, \\ \text{(v)} & \mathrm{tr}\, \theta = \theta_0 \quad \text{on } \Gamma_0. \end{cases} \tag{25}$$

A basic result of this paper is the following property.

Theorem 3 (Extended Variational Principle) *Among all functions θ and vector fields $\mathbf{y}$ satisfying the above conditions* (i), (ii), (iv), *and* (v), *the minimum of the energy functional*

$$J(\theta, \mathbf{y}) = \int_\Omega F_{T,n}(\mathrm{grad}\, \theta, \mathbf{y})\, d\Omega - \int_{\Gamma_0} \theta_0 \langle \mathbf{y}, \mathbf{n} \rangle\, d\Gamma - \int_{\Gamma_1} q\theta\, d\Gamma - \int_\Omega Q\theta\, d\Omega$$

is attained when the constitutive law (iii) *is additionally satisfied.*

Proof Indeed, the Fitzpatrick function being globally convex, the integral functional J is convex. Next, since

$$F_{T,n}(\mathbf{x}, \mathbf{y}) \geq \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathrm{grad}\, \theta, \mathbf{y} \rangle = \mathrm{div}\, (\theta \mathbf{y}) - \theta \ \mathrm{div}\, \mathbf{y},$$

then the Green formula (2) yields

$$\begin{aligned} J(\theta, \mathbf{y}) &\geq \int_\Omega \mathrm{div}\, (\theta\, \mathbf{y})\, d\Omega - \int_\Omega \theta (\mathrm{div}\, \mathbf{y} + Q)\, d\Omega - \int_{\Gamma_0} \theta_0 \langle \mathbf{y}, \mathbf{n} \rangle\, d\Gamma - \int_{\Gamma_1} q\theta\, d\Gamma \\ &= \int_{\Gamma_0} (\theta - \theta_0) \langle \mathbf{y}, \mathbf{n} \rangle\, d\Gamma + \int_{\Gamma_1} \big(\langle \mathbf{y}, \mathbf{n} \rangle - q\big)\, \theta\, d\Gamma - \int_\Omega \theta\, (\mathrm{div}\, \mathbf{y} + Q)\, d\Omega. \end{aligned}$$

Thus, the functional J is non negative over the convex part defined by the conditions (i), (ii), (iv), and (v). The Fitzpatrick function $F_{T,n}(\mathbf{x}, \mathbf{y})$ reduces to the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ if and only if $\mathbf{x}$ and $\mathbf{y}$ satisfy the constitutive law (iii), the minimum zero is attained for $(\theta, \mathbf{y})$ being solution of the above extended Dirichlet-Neumann problem. □

Remark 3 The ordinary Dirichlet-Neumann problem (7) concerns GSM for which the Fitzpatrick function $F_{T,n}(\mathbf{x}, \mathbf{y})$ can be replaced by $\Phi(\mathrm{grad}\, \theta) + \Phi^*(\mathbf{y})$. The functional $J(\theta, \mathbf{y})$ reduced to the function $F(\theta, \mathbf{y})$ defined in (13) and Theorem 1.

We point out that Theorem 3 generalizes the primal-dual two-field variational principle established by Moreau [30, 31] for Generalized Standard Materials.

3.1 Numerical Implementation

The extended Dirichlet-Neumann problem can be solved in the framework of Finite Element Method by implementing an Uzawa-type algorithm. The consideration of Uzawa algorithms

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on infinite-dimensional Hilbert spaces is that they give the right strategy for discretizing the considered partial differential equation. We refer to Arrow, Hurwicz & Uzawa [5], Bacuta [6], Brezzi & Fortin [13], Ciarlet [18], Fortin & Glowinski [23] for advances in the numerical implementation of Uzawa algorithms in partial differential equations.

3.2 Application to Heat Conduction

Assume that problem (24) modells the heat conduction phenomenon for $N=2$. The function θ is the temperature, the vector field $\mathbf{x}$ is the temperature gradient, the vector field $\mathbf{y}$ is the opposite heat flow, the function Q is the heat source applied to the domain Ω, the function q is the heat flux applied over the part Γ_1 of the boundary, and the function θ_0 is the assigned temperature on the complementary part Γ_0. The heat conduction constitutive law enacted between $\mathbf{x}$ and $\mathbf{y}$ is linear but not symmetric. However, this law is assumed to be strictly n-cyclically monotone. The nth Fitzpatrick function is as in Example 12. The integral functional is as in Theorem 3. The numerical approximation of the temperature distribution will be obtained by developing an Uzawa-type algorithm.

If $\alpha \neq 0$, then the Fitzpatrick sequence $F_{A,k}(\mathbf{x},\mathbf{y})$ stops at $k=n$ and $F_{A,n}(\mathbf{x},\mathbf{y})$ is not the sum of a function of $\mathbf{x}$ and a function of $\mathbf{y}$. There is neither primal variational principle to find only θ nor dual variational principle to find $\mathbf{y}$. However, there are mixed variational principles to find simultaneously grad θ and $\mathbf{y}$. Such a case consists in minimizing the functional

$$\int_\Omega F_{A,k}(\operatorname{grad}\theta,\mathbf{y})\,d\Omega - \int_\Omega Q\theta\,d\Omega - \int_{\Gamma_0}\theta_0\,\langle\mathbf{y},\mathbf{n}\rangle\,d\Gamma - \int_{\Gamma_1} q\theta\,d\Gamma.$$

4 Extended Two Fields Primal-Dual Variational Principles in Elasticity

Let Ω be a bounded, connected, open subset of $\mathbb{R}^2$ (or $\mathbb{R}^3$) whose boundary Γ is Lipschitz continuous, the set Ω being locally on a single side of Γ. The boundary Γ is composed of two disjoints subsets Γ_0 and Γ_1, the $d\Gamma$-measure of Γ_0 being positive. The outer unit normal to Γ is denoted $\mathbf{n}$. The inner product in the space $\mathbb{R}^2$ (or $\mathbb{R}^3$) is denoted by a dot between the two vectors.

Assume there are given the three functions: $f \in (L^2(\Omega))^N$, $F \in (L^2(\Gamma_1))^N$, and $u_0 \in (L^2(\Gamma_0))^N$ $(N \in \{2,3\})$.

Consider the following extended Dirichlet-Neumann problem arising in elasticity: find $(u,\mathbf{y}) \in (H^1(\Omega))^N \times H(\mathrm{div};\Omega)$ such that

$$\begin{cases} \text{(i)} & \operatorname{div}\mathbf{y} + f = 0 & \text{in } \Omega, \\ \text{(ii)} & \frac{1}{2}[\nabla u + (\nabla u)^t] = \mathbf{x} & \text{in } \Omega, \\ \text{(iii)} & \mathbf{y} \in T\mathbf{x}, & \end{cases} \tag{26}$$

where T is a maximal strictly n-monotone multifunction relating the symmetric tensor fields $\mathbf{x}(\omega)$ and $\mathbf{y}(\omega)$ for each point $\omega \in \Omega$. Mixed boundary conditions of Dirichlet-Neumann type are imposed as follows:

$$\begin{cases} \text{(iv)} & \mathbf{y}\cdot\mathbf{n} = F & \text{on } \Gamma_1, \\ \text{(v)} & u = u_0 & \text{on } \Gamma_0. \end{cases} \tag{27}$$

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4.1 Extended Variational Principle

Theorem 4 *Among all functions u and vector fields* **y** *satisfying the conditions* (i), (ii), (iv), *and* (v) *of problem* (26)–(27), *the minimum of the energy functional*

$$J(u,\mathbf{y}) = \int_\Omega F_{T,n}\left(\frac{1}{2}\left[\nabla u + (\nabla u)^t\right], \mathbf{y}\right) d\Omega - \int_{\Gamma_0} \langle \mathbf{y}\, u_0, \mathbf{n}\rangle\, d\Gamma$$
$$- \int_{\Gamma_1} F \cdot u\, d\Gamma - \int_\Omega f \cdot u\, d\Omega$$

is attained when additionally the constitutive law (iii) *is satisfied.*

Proof Let us recall the tensorial formula

$$\mathrm{tr}\,(\mathbf{xy}) = \mathrm{tr}\left(\frac{1}{2}\left[\nabla u + (\nabla u)^t\right]\mathbf{y}\right) = \mathbf{y} : \nabla u.$$

Therefore

$$J(u,\mathbf{y}) \geq \int_\Omega \mathbf{y} : \nabla u d\Omega - \int_{\Gamma_0} \mathbf{yn} \cdot u d\Gamma - \int_{\Gamma_1} F \cdot u d\Gamma - \int_\Omega f \cdot u d\Omega.$$

Thus, combining the Green formula (4) with the fact that the Fitzpatrick function $F_{T,n}(\mathbf{x}, \mathbf{y})$ is bounded from below by the inner product, we obtain

$$J(u,\mathbf{y}) \geq - \int_\Omega (\mathrm{div}\, \mathbf{y} + f) \cdot u\, d\Omega + \langle \mathbf{yn}, u\rangle_\Gamma - \int_{\Gamma_0} \mathbf{y}\, u_0 \cdot \mathbf{n}\, d\Gamma - \int_{\Gamma_1} F \cdot u\, d\Gamma$$

Remark that (see [2])

$$\begin{aligned}\langle \mathbf{yn}, u\rangle_\Gamma &= \langle \mathbf{yn}, u\rangle_{\Gamma_0} + \langle \mathbf{yn}, u\rangle_{\Gamma_1} \\ &= \int_{\Gamma_1} \mathbf{yn} \cdot u d\Gamma + \int_{\Gamma_0} \mathbf{yn} \cdot u d\Gamma.\end{aligned}$$

Hence

$$J(u,\mathbf{y}) \geq \int_{\Gamma_0} \big\langle \mathbf{y}(u - u_0), \mathbf{n}\big\rangle d\Gamma + \int_{\Gamma_1} \langle u, \mathbf{yn} - F\rangle\, d\Gamma - \int_\Omega \langle u, \mathrm{div}\; \mathbf{y} + f\rangle\, d\Omega.$$

Therefore, the functional J is non negative over the convex set defined by the conditions (i), (iv) and (v) of the problem (26)–(27). Since the Fitzpatrick function $F_{T,n}(\mathbf{x}, \mathbf{y})$ reduces to the inner-product $\mathrm{tr}\,(\mathbf{xy})$ if and only if $\mathbf{x}$ and $\mathbf{y}$ satisfy the constitutive law (iii), we conclude that the minimum zero is attained for $(u, \mathbf{y})$ being solution of (26)–(27). □

4.2 Special Case of Generalized Standard Materials

The extended variational principle stated in Theorem 4 generalizes the primal-dual two-field variational principle introduced by Moreau [30] for Generalized Standard Materials. In this case, the Fitzpatrick function $F_{T,n}(\mathbf{x}, \mathbf{y})$ is replaced by the sum $\Phi(\mathbf{x}) + \Phi^*(\mathbf{y})$ of the potential and its conjugate, see Moreau [30, 31].

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4.3 Numerical Implementation

The extended elasticity problem can be solved in the framework of Finite Element Method by implementing an Uzawa-type algorithm.

4.4 Application to Elasticity

The vector field u is the displacement vector, problem (26) is the equilibrium equation, the tensor field $\mathbf{x}$ is the strain tensor, the tensor field $\mathbf{y}$ is the stress tensor, the vector field f is the specific force applied to the domain Ω, and the vector field F is the force density applied over the part Γ_1 of the boundary. The vector field u_0 is the assigned displacement on the complementary part Γ_0. The dimension is 3. The elastic constitutive law enacted between $\mathbf{x}$ and $\mathbf{y}$ is linear but not symmetric. However it is assumed to be strictly n-cyclically monotone. The nth Fitzpatrick function is as in Example 12. The integral functional is as in Theorem 4. The numerical approximation of the displacement vector will be obtained by developing an Uzawa-type algorithm.

5 Conclusion and Perspectives

The class of n-monotone materials for which the constitutive law is described by an n-cyclically monotone operator is larger than the class of Generalized Standard Materials. The integer n can be regarded as a characteristic of these materials. The equilibrium equation of such n-monotone material is a partial differential equation that can be described by a primal-dual two-fields variational principle, cf. Visintin [41]. The functional of this principle is an integral of the two fields. The integrand involved the nth Fitzpatrick function. In the present paper we deal with a new kind of calculus of variations allowing to solve the thermal or the mechanical problems by Uzawa-type algorithms as easily as for Generalized Standard Materials. However, in mechanical and civil engineering, the constitutive laws of most materials (ductile metals, metal matrix composites, wet clays, plastic soils, granular materials, etc.) are not monotone.

The class of Implicit Standard Materials is larger than the class of n-monotone materials. Every Fitzpatrick's function is globally convex and lower semi-continuous, but the modeling of Implicit Standard Materials only requires bipotentials that are partially convex and lower semi-continuous. It would be interesting to generalize the concept of Fitzpatrick's sequence to the case of non-monotone operators.

We hope that this kind of extension of Fitzpatrick sequences will reveal very helpful to produce relevant bipotentials for representing the non-associated constitutive laws of the Implicit Standard Materials evoked in de Saxcé & Bousshine [36]: unilateral contact with Coulomb's dry friction [37], generalized Drücker-Prager plasticity [36], modified Cam-Clay model [25, 36], non-associated plasticity of soils [10], nonlinear kinematical hardering rules for cyclic plasticity of metals [30], Lemaître's plastic-ductile damage law [26], and shake-down analysis on non-standard elasto-plastic materials [11].

The use of Fitzpatrick sequences allows us to extend the application of energy methods to the resolution of the Dirichlet-Neumann problem

$$\operatorname{div}(A \operatorname{grad} \theta) + Q = 0$$

in the case where the single valued operator A is not cyclically monotone but only n-cyclically monotone for n finite.

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Finally, we point out that an extended primal-dual variational principle based on Fitzpatrick sequences can be developed to study the Cauchy problem

$$\begin{cases} \mathbf{y} \in \dfrac{\partial \mathbf{x}}{\partial t} + T\mathbf{x} \quad \text{in } H^{-1}(\Omega), \text{ a.e. in } (0,T) \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \tag{28}$$

for any fixed $\mathbf{y} \in L^2(0,T;H^{-1}(\Omega))$ and $\mathbf{x}_0 \in L^2(\Omega)$.

As proved in Visintin [41] (extended Brezis-Ekeland-Nayroles principle, see Brezis & Ekeland [12] and Nayroles [32]), problem (28) is equivalent to the null-minimization problem of the convex and lower semi-continuous functional

$$J : \left\{\mathbf{v} \in L^2\big(0,T;H^1(\Omega)\big) \cap H^1\big(0,T;H^{-1}(\Omega)\big);\ \mathbf{v}(0) = \mathbf{x}_0\right\} \to \mathbb{R} \cup \{+\infty\}$$

defined by

$$J(\mathbf{v}) := \int_0^T \big[F_{T,2}(\mathbf{v}, \mathbf{y} - D_t\mathbf{v}) - \langle \mathbf{y}, \mathbf{v}\rangle\big]\, dt + \frac{1}{2}\left|\mathbf{v}(T)\right|^2_{L^2(\Omega)} - \frac{1}{2}\left|\mathbf{x}_0\right|^2_{L^2(\Omega)}.$$

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