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Research article

Anisotropic Robin problems with indefinite potential

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Abstract: We considered a nonlinear elliptic boundary value problem driven by the variable (anisotropic) (p, q)-Laplacian with Robin boundary condition and a superlinear reaction which does not satisfy the Ambrosetti-Rabinowitz condition. Using critical point theory, truncation and comparison techniques and critical groups, we showed the existence of five nontrivial smooth solutions all with sign information and ordered.

This paper is dedicated with esteem to Professor Patrizia Pucci on the occasion of her anniversary.

Keywords: variable exponents; regularity theory; constant sign and nodal solutions; critical point theory; critical groups

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear Robin problem with variable exponents (anisotropic problem):

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) + \xi(z)|u(z)|^{p(z)-2}u(z) = f(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_{p,q}} + \beta(z)|u|^{p(z)-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Let

$$\mathcal{D}_1 = \left\{ r \in C(\overline{\Omega}) : 1 < \min_{\overline{\Omega}} r \right\}.$$

For $r \in \mathcal{D}_1$, by $\Delta_{r(z)}$, we denote the $r(z)$-Laplace differential operator defined by

$$\Delta_{r(z)}u = \operatorname{div}\left(|Du|^{r(z)-2}Du\right) \text{ for all } u \in W_0^{1,r(z)}(\Omega)$$

(anisotropic operator). We mention that in contrast to the isotropic r-Laplacian (that is, $r(z)$ is constant), the anisotropic operator is not homogeneous, and this is a source of difficulties in the study of boundary value problems since we cannot use scaling arguments. Problem (1.1) is driven by the sum of two such operators with different variable exponents $p, q \in C^{0,1}(\overline{\Omega}) \cap \mathcal{D}_1$ that satisfy $1 < q(z) < p(z)$ for all $z \in \overline{\Omega}$. There is also a potential term $\xi(x)|u(z)|^{p-2}u(z)$ that is indefinite, where the coefficient $\xi \in L^\infty(\Omega)$ is, in general, sign changing. This means that the differential operator of (1.1) is not coercive. In the reaction (right hand side) of (1.1), $f(z,x)$ is a Carathéodory function; that is, $z \to f(z,x)$ is measurable and $x \to f(z,x)$ is continuous (thus $f(\cdot,\cdot)$ is jointly measurable). If $p_+ = \max_{\overline{\Omega}} p(\cdot)$, then we assume that $f(z,\cdot)$ is $(p_+ - 1)$-superlinear, but need not satisfy the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). Instead, we employ a weaker condition which incorporates in our setting superlinear reactions with "slower" growth as $x \to +\infty$, which fail to satisfy the AR-condition. In the boundary condition, the conormal derivative $\frac{\partial u}{\partial n_{p,q}}$ is interpreted using the nonlinear Green's identity, which holds in the present variable setting using the results of Fan [1] on boundary trace embedding theorems for variable Sobolev spaces. In particular, if $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_{p,q}} = \left[|Du|^{p(z)-2} + |Du|^{q(z)-2}\right]\frac{\partial u}{\partial n}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$ (see also [2] and [[3], pp. 34–35]).

We prove a multiplicity theorem for problem (1.1), producing five nontrivial smooth solutions, all with sign information (two positive, two negative, and one nodal (sign changing)). Such a multiplicity result was proved by [4] for Dirichlet problems. We mention also the works of [2, 5–10], which deal with parametric problems, consider positive solutions, and prove existence and multiplicity theorems that are global in the parameter $\lambda > 0$.

2. Mathematical background and hypotheses

In the study of problem (1.1), we use variable Lebesgue and Sobolev spaces. These are a particular case of generalized Orlicz spaces, and their theory can be found in the book of [11].

Given $r \in C(\overline{\Omega})$, we set

$$r_- = \min_{x\in\overline{\Omega}} r \quad \text{and} \quad r_+ = \max_{x\in\overline{\Omega}} r.$$

Let $L^0(\Omega)$ denote the space of all measurable functions $u : \Omega \to \mathbb{R}$. As always, we identify two such functions, which differ on a Lebesgue-null set only. Suppose $r \in \mathcal{D}_1$. Then, the variable Lebesgue space

$L^{r(z)}(\Omega)$ is defined by

$$L^{r(z)}(\Omega)=\left\{u\in L^0(\Omega):\int_\Omega |u|^{r(z)}\mathrm{d}z<\infty\right\}.$$

We equip this space with the so-called "Luxemburg norm", $\|\cdot\|_{r(z)}$, defined by

$$\|u\|_{r(z)}=\inf\left\{\lambda>0:\int_\Omega\left[\frac{|u(z)|}{\lambda}\right]^{r(z)}\mathrm{d}z\le 1\right\}.$$

With this norm, $L^{r(z)}(\Omega)$ becomes a Banach space that is separable and reflexive (in fact, uniformly convex). The conjugate exponent $r'\in\mathcal{D}_1$ of $r(\cdot)$ is defined by

$$r'(z)=\frac{r(z)}{r(z)-1}\quad\text{for all } z\in\overline{\Omega},$$

and we know that

$$L^{r(z)}(\Omega)^*=L^{r'(z)}(\Omega).$$

Moreover, the following Hölder inequality is true

$$\int_\Omega |uy|\mathrm{d}z\le\left[\frac{1}{r_-}+\frac{1}{r'_-}\right]\|u\|_{r(z)}\|y\|_{r'(z)}$$

for all $u\in L^{r(z)}(\Omega)$, $y\in L^{r'(z)}(\Omega)$.

Using the variable Lebesgue spaces, we can define the corresponding variable Sobolev spaces. So, for $r\in\mathcal{D}_1$, the variable Sobolev space $W^{1,r(z)}(\Omega)$ is defined by

$$W^{1,r(z)}(\Omega)=\left\{u\in L^{r(z)}:|Du|\in L^{r(z)}(\Omega)\right\},$$

where Du denotes the weak gradient of u. We equip $W^{1,p(z)}(\Omega)$ with the norm $\|\cdot\|$ defined by

$$\|u\|=\|u\|_{r(z)}+\|Du\|_{r(z)}\quad\text{for all } u\in W^{1,r(z)}(\Omega),$$

with $\|Du\|_{r(z)}=\||Du|\|_{r(z)}$. Evidently, $W^{1,r(z)}(\Omega)$ is a closed subspace of $L^{p(z)}(\Omega)\times L^{p(z)}(\Omega,\mathbb{R}^N)$ and so $W^{1,p(z)}(\Omega)$ is a Banach space that is separable and reflexive (in fact uniformly convex). If $r\in\mathcal{D}_1$, the Sobolev critical exponent $r^*(\cdot)$ corresponding to $r(\cdot)$ is defined by

$$r^*(z)=\begin{cases}\frac{Nr(z)}{N-r(z)} & \text{if } r(z)<N\\ +\infty & \text{if } N\le r(z)\end{cases}$$

for all $z\in\overline{\Omega}$.

The next proposition is the "variable" version of the Sobolev embedding theorem. In what follows the symbol $\hookrightarrow$ denotes continuous and dense embedding.

Proposition 2.1. *If $r\in\mathcal{D}_1$, then*

(*a*) *$W^{1,r(z)}(\Omega)\hookrightarrow L^{q(z)}(\Omega)$ for all $q\in C(\overline{\Omega})$ with $q(z)\le r^*(z)$ for all $z\in\overline{\Omega}$, and the embedding is compact if $q(z)<r^*(z)$ for all $z\in\Omega$ (that is $q_+<r^*_-$);*

(b) $W^{1,\tau(z)}(\Omega) \hookrightarrow W^{1,r(z)}(\Omega)$ *for all* $\tau \in C(\overline{\Omega})$ *with* $r(z) \leq \tau(z)$ *for all* $z \in \overline{\Omega}$.

Let

$$\rho_r(u) = \int_\Omega |u|^{r(z)} dz + \int_\Omega |Du|^{r(z)} dz$$

for all $u \in W^{1,r(z)}(\Omega)$. This is known as the modular function for the variable exponent $r(\cdot)$. There is a close relation between the norm $\|\cdot\|$ and the modular function $\rho_r(\cdot)$.

Proposition 2.2. *If* $r \in \mathcal{D}_1$, *then*

(a) $\|u\| = \eta \Leftrightarrow \rho_r\left(\frac{u}{\eta}\right) = 1$.
(b) $\|u\| < 1$ $(resp. = 1, > 1) \Leftrightarrow \rho_r(u) < 1$ $(resp. = 1, > 1)$.
(c) $\|u\| < 1 \Rightarrow \|u\|^{r_+} \leq \rho_r(u) \leq \|u\|^{r_-}$.
(d) $\|u\| > 1 \Rightarrow \|u\|^{r_-} \leq \rho_r(u) \leq \|u\|^{r_+}$.
(e) $\|u\| \to 0$ $(resp. \to +\infty) \Leftrightarrow \rho_r(u) \to 0$ $(resp. \to +\infty)$.

Remark 2.1. *The same result is also true for the variable Lebesgue space* $L^{r(z)}(\Omega)$ *with the Luxemburg norm* $\|\cdot\|_{r(z)}$ *and the modular function.*

$$\rho_r(u) = \int_\Omega |u|^{r(z)} dz \quad \textit{for all } u \in L^{r(z)}(\Omega)$$

Consider the C^1-functional $\ell_r : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\ell_r(u) = \int_\Omega \frac{1}{r(z)} |Du|^{r(z)} dz.$$

Then we have

$$\ell_r'(u) = A_r(u) \quad \text{for all } u \in W^{1,r(z)}(\Omega),$$

with $A_r : W^{1,r(z)}(\Omega) \longrightarrow W^{1,r(z)}(\Omega)^*$ being defined by

$$\langle A_r(u), h \rangle = \int_\Omega |Du|^{r(z)-2} (Du, Dh)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W^{1,r(z)}(\Omega).$$

This operator has the following properties (see [[12], pp. 683]).

Proposition 2.3. *If* $r \in \mathcal{D}_1$, *then* $A_r : W^{1,r(z)}(\Omega) \to W^{1,r(z)}(\Omega)^*$ *is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (thus maximal monotone) and of type* $(S)_+$, *that is,*

$$\textit{if } u_n \xrightarrow{w} u \textit{ in } W^{1,r(z)}(\Omega) \textit{ and } \limsup_{n\to\infty} \langle A_r(u_n), u_n - u \rangle \leq 0$$
$$\textit{then } u_n \to u \textit{ in } W^{1,r(z)}(\Omega).$$

Our assumptions on the two variable exponents p, q, the potential function $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$, are the following:

H_0 : $p, q \in C^{0,1}(\overline{\Omega}) \cap \mathcal{D}_1$, $q(z) < p(z)$ for all $z \in \overline{\Omega}$, $p_+ < \frac{Np_-}{N-p_-}$, $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,1}(\partial\Omega)$, $\beta \geq 0$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

Let

$$V = A_p + A_q : W^{1,p(z)}(\Omega) \to W^{1,p(z)}(\Omega)^*.$$

Then from Proposition 2.3, we infer the following result.

Proposition 2.4. *The operator $V(\cdot)$ is bounded, continuous, monotone (thus maximal monotone) and of type $(S)_+$.*

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : 0 \leq u(z) \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\text{int}C_+ = \{u \in C_+ : 0 < u(z) \text{ for all } z \in \overline{\Omega}\}.$$

If $u \in L^0(\Omega)$, then we set $u^\pm = \max\{\pm u, 0\}$. We have

$$u = u^+ - u^-, \ \ |u| = u^+ + u^-$$

and if $u \in W^{1,r(z)}(\Omega)$, then $u^\pm \in W^{1,r(z)}(\Omega)$.

Given $u, v \in L^0(\Omega)$ with $u(z) \leq v(z)$ for a.a $z \in \Omega$, then we define

$$[u, v] = \{h \in W^{1,r(z)}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a } z \in \Omega\},$$

$$\text{int}_{C^1(\overline{\Omega})}[u, v] = \text{interior in } C^1(\overline{\Omega}) \text{ of } [u, v] \cap C^1(\overline{\Omega}).$$

Suppose X is a Banach space and $\varphi \in C^1(X)$, $c \in \mathbb{R}$. We define

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \ \text{(the critical set of } \varphi),$$

$$\varphi^c = \{u \in X : \varphi(u) \leq c\}.$$

We say that $\varphi(\cdot)$ satisfies the "C-condition" if it has the following property:

> "Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$, such that
> $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$,
> admits a strongly convergent subsequence."

This is a compactness-type condition on $\varphi(\cdot)$ to compensate for the fact that the ambient space X need not be locally compact (since X is, in general, infinite dimensional).

Let $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $H_k(Y_1, Y_2)$, we denote the k^{th}-relative singular homology group with integer coefficients. If $\varphi \in K_\varphi$ is isolated and $c = \varphi(u)$, then the critical groups of $\varphi(\cdot)$ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \text{ for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology, implies that this definition is independent of the isolating neighborhood U (see [3]).

The hypotheses on the reaction $f(z, x)$ are the following:

(H_1): $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a $z \in \Omega$ and

(i) $|f(z,x)| \leq a(z)[1 + |x|^{r(z)-1}]$ for a.a $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$ and $r \in C(\overline{\Omega})$, such that $p_+(z) < r(z) < p_-^* = \frac{Np_-}{N-p_-}$ for all $z \in \overline{\Omega}$;

(ii) if $F(z,x) = \int_0^x f(z,s)\mathrm{d}s$, then

$$\lim_{x\to\pm\infty} \frac{F(z,x)}{|x|^{p_+}} = +\infty \text{ uniformly for a.a } z \in \Omega$$

and there exists $\gamma \in C(\overline{\Omega})$, such that

$$\gamma(z) \in \left((r_- - p_-)\max\left\{\frac{N}{p_-}, 1\right\}, p_+^*\right) \text{ for all } z \in \overline{\Omega},$$

$$0 < \beta_0 \leq \liminf_{x\to\pm\infty} \frac{f(z,x)x - p_+F(z,x)}{|x|^{\gamma(z)}} \text{ uniformly for a.a } z \in \Omega;$$

(iii) there exist $\eta_- < 0 < \eta_+$, $\tau \in \mathcal{D}_1$ with $\tau_+ < q_-$ and $\delta, c > 0$, such that

$$f(z,\eta_+) - \xi(z)\eta_+^{p(z)-1} \leq -\theta < 0 < \theta \leq f(z,\eta_-) + \xi(z)|\eta_-|^{p(z)-1} \text{ for a.a } z \in \Omega;$$

$$c|x|^{\tau(z)} \leq f(z,x)x \leq \tau(z)F(z,x) \text{ for a.a } z \in \Omega \text{ all } |x| \leq \delta;$$

(iv) if $\rho = \max\{\eta_+, |\eta_-|\}$, then there exists $\widehat{\xi}_\rho > 0$ such that, for a.a $z \in \Omega$, the function

$$x \mapsto f(z,x) + \widehat{\xi}_\rho |x|^{p(z)-2}x$$

is nondecreasing on $[-\rho, \rho]$.

Remark 2.2. *Hypotheses $H_1(ii)$ implies that $f(z,\cdot)$ is $(p_+ - 1)$-superlinear, that is,*

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p_+-2}x} = +\infty \quad \textit{uniformly for a.a } z \in \Omega.$$

We do not assume the AR-condition, which is common in the literature when dealing with superlinear problems. We recall that the AR-condition says that there exist $\widehat{\beta} > p_+$ and $M > 0$, such that

$$0 < \widehat{\beta}F(z,x) \leq f(z,x)x \text{ for a.a } z \in \Omega, \text{ all } |x| \geq M, \tag{2.1}$$

$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, \pm M). \tag{2.2}$$

Integrating (2.1) and using (2.2) we obtain

$$\begin{aligned} &\widehat{c}|x|^{\widehat{\beta}} \leq F(z,x) \text{ for a.a } z \in \Omega, \text{all } |x| \geq M, \text{ some } \widehat{c} > 0, \\ \Rightarrow\ & f(z,x) \text{ has at least } (\widehat{\beta} - 1)\text{-growth as } x \to \pm\infty. \end{aligned}$$

Consider the following function

$$f(z,x) = \begin{cases} |x|^{\tau(z)-2}x - \widehat{\eta}|x|^{s(z)-2}s & \text{if } |x| \leq 1 \\ |x|^{p_+-2}x\ln|x| - \mu|x|^{p(z)-2}x & \text{if } 1 < |x|. \end{cases}, \mu = \widehat{\eta} - 1.$$

If $s \in C(\overline{\Omega})$ satisfies $\tau(z) < s(z)$ for all $z \in \overline{\Omega}$ and $\widehat{\eta} > \|\xi\|_\infty$, then this function satisfies hypotheses H_1, but fails to satisfy the AR-condition.

In what follows, for notational economy, we define $\gamma_p : W^{1,p(z)}(\Omega) \to \mathbb{R}$ to be the C^1-functional defined by

$$\gamma_p(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{\xi(z)}{p(z)}|u|^{p(z)}dz + \int_{\partial\Omega} \frac{\beta(z)}{p(z)}|u|^{p(z)}d\sigma,$$

with $\sigma(\cdot)$ being the surface measure on $\partial\Omega$. We know that

$$W^{1,p(z)}(\Omega) \hookrightarrow L^{p(z)}(\partial\Omega) \text{ compactly (via the trace map, see [1]).} \tag{2.3}$$

The energy functional for problem (1.1) $\varphi : W^{1,p(z)}(\Omega) \to \mathbb{R}$ is defined by

$$\varphi(u) = \gamma_p(u) + \ell_q(Du) - \int_\Omega F(z,u)dz \quad \text{for all } u \in W^{1,p(z)}(\Omega),$$

with

$$\ell_q(Du) = \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz \text{ for all } u \in W^{1,p(x)}(\Omega).$$

3. Solutions of constant sign

Let $k > \|\xi\|_\infty$. To produce solutions of constant sign for problem (1.1), we also introduce the C^1-functionals $\widehat{\varphi}_\pm : W^{1,p(z)}(\Omega) \to \mathbb{R}$, defined by

$$\widehat{\varphi}_\pm(u) = \gamma_p(u) + \ell_q(Du) + \int_\Omega \frac{k}{p(z)}(u^\mp)^{p(z)}dz - \int_\Omega F(z, \pm u^\pm)dz$$

for all $u \in W^{1,p(z)}(\Omega)$.

First, we produce two constant sign solutions located in the order intervals $[0, \eta_+]$ and $[\eta_-, 0]$.

Proposition 3.1. *If hypotheses H_0 and H_1 hold, then problem* (1.1) *has at least two constant sign solutions*

$$u_0 \in intC_+ \text{ local minimizer of } \varphi, \widehat{\varphi}_+,$$

$$v_0 \in -intC_+ \text{ local minimizer of } \varphi, \widehat{\varphi}_-,$$

$$\eta_- < v_0(z) < 0 < u_0(z) < \eta_+ \text{ for all } z \in \overline{\Omega}.$$

Proof. For $k > \|\xi\|_\infty$, we introduce the Carathéodory function $\widehat{f}_+(z,x)$ defined by

$$\widehat{f}_+(z,x) = \begin{cases} f(z,x^+) + k(x^+)^{p(z)-1} & \text{if } x \le \eta_+ \\ f(z,\eta_+) + k\eta_+^{p(z)-1} & \text{if } \eta_+ < x. \end{cases} \tag{3.1}$$

We set $\widehat{F}(z,x) = \int_0^x \widehat{f}(z,s)ds$ and consider the C^1-functional $\psi_+ : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \gamma_p(u) + \ell_q(Du) + \int_\Omega \frac{k}{p(z)}|u|^{p(z)}dz - \int_\Omega \widehat{F}_+(z,u)dz \text{ for all } u \in W^{1,p(z)}(\Omega).$$

Since $k > \|\xi\|_\infty$, using Proposition 2.9 of [6] and (3.1), we infer that

$$\psi_+(\cdot) \text{ is coercive.}$$

Also using Proposition 2.1 and (2.3), we see that

$$\psi_+(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

Then by the Weierstrass-Tonelli theorem, we can find $u_0 \in W^{1,p(z)}(\Omega)$, such that

$$\psi_+(u_0) = \inf\left\{\psi_+(u) : u \in W^{1,p(z)}(\Omega)\right\}. \tag{3.2}$$

Let $u \in \mathrm{int}C_+$ and choose $t \in (0,1)$ small, such that

$$0 < tu(z) \le \min\{\delta, \eta_+\} \text{ for all } z \in \overline{\Omega}.$$

Using hypothesis $H_1(iii)$, we have

$$\frac{ct^{\tau_+}}{\tau_+}|u(z)| \le F(z, tu(z)) \text{ for a.a } z \in \Omega.$$

Hence, we have

$$\begin{aligned}
\psi_+(tu) &\le \frac{t^{p_-}}{p_-}\left[\gamma_p(u) + k\rho_p(u)\right] + \frac{t^{q_-}}{q_-}\rho_q(Du) - \frac{ct^{\tau_+}}{\tau_+}\rho_\tau(u) \\
&\le \frac{t^{q_-}}{q_-}\left[\gamma_p(u) + k\rho_p(u) + \rho_q(Du)\right] - \frac{ct^{\tau_+}}{\tau_+}\rho_\tau(u) \text{ (since } t \in (0,1) \text{ and } q_- < p_-) \\
&= c_1 t^{q_-} - c_2 t^{\tau_+} \text{ for some } c_1, c_2 > 0.
\end{aligned}$$

Since $\tau_+ < q_-$ (see hypothesis $H_1(iii)$), for $t \in (0,1))$ small, we have

$$\begin{aligned}
&\psi_+(tu) < 0, \\
\Rightarrow\ &\psi_+(u_0) < 0 = \psi_+(0) \text{ (see (3.2))}, \\
\Rightarrow\ &u_0 \ne 0.
\end{aligned}$$

From (3.2) we have

$$\begin{aligned}
&\langle \psi_+'(u_0), h\rangle = 0 \text{ for all } h \in W^{1,p(z)}(\Omega), \\
\Rightarrow\ &\langle V(u_0), h\rangle + \int_\Omega [\xi(z) + k]\,|u_0|^{p(z)-2}u_0 h\mathrm{d}z + \int_{\partial\Omega} \beta(z)|u_0|^{p(z)-2}u_0 h\mathrm{d}\sigma, \\
&= \int_\Omega \widehat{f_+}(z, u_0)h\mathrm{d}z \quad \text{for all } h \in W^{1,p(z)}(\Omega).
\end{aligned} \tag{3.3}$$

For (3.3), we first use the test function $h = -u_0^- \in W^{1,p(z)}(\Omega)$. Then

$$c_3 \min\{\|u_0^-\|^{p_+}, \|u_0^-\|^{p_-}\} \le 0 \text{ for some } c_3 > 0$$
$$\text{(see (3.1) and Proposition 2.9 of [6]).}$$

Next in (3.3), we choose the test function $h = (u_0 - \eta_+)^+ \in W^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}\langle V(u_0), (u_0 - \eta_+)^+\rangle &+ \int_\Omega [\xi(z) + k]\, u_0^{p(z)-1}(u_0 - \eta_+)^+ dz + \int_{\partial\Omega} \beta(z) u_0^{p(z)-1}(u_0 - \eta_+)^+ d\sigma \\ &= \int_\Omega \left[f(z, \eta_+) + k\eta_+^{p(z)-1}\right](u_0 - \eta_+)^+ dz \quad \text{(see (3.1))} \\ &< \int_\Omega [\xi(z) + k]\, \eta_+^{p(z)-1}(u_0 - \eta_+)^+ dz \quad \text{(see hypothesis } H_1(iii)),\end{aligned}$$

$$\begin{aligned}\Rightarrow \langle V(u_0) - V(\eta_+), (u_0 - \eta_+)^+\rangle &+ \int_\Omega [\xi(z) + k]\,(u_0^{p(z)-1} - \eta_+^{p(z)-1})(u_0 - \eta_+)^+ dz \\ &+ \int_{\partial\Omega} \beta(z) u_0^{p(z)-1}(u_0 - \eta_+)^+ d\sigma \\ &\Rightarrow u_0 \leq \eta_+ \text{ (see Proposition 2.4 and recall that } k > \|\xi\|_\infty, \beta \geq 0).\end{aligned}$$

We have proved that

$$u_0 \in [0, \eta_+],\ u_0 \neq 0. \tag{3.4}$$

From (3.4), (3.1) and (3.3), we see that u_0 is a positive solution of problem (1.1).

The regularity theory of Fan [13] (the anisotropic counterpart of the theory of Lieberman [14]), implies that $u_0 \in C_+\backslash\{0\}$.

Let $\rho = \|u_0\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H_1(iv)$. We can always take $\widehat{\xi}_\rho > \|\xi\|_\infty$. Then

$$\begin{aligned}&-\Delta_{p(z)} u_0 - \Delta_{q(z)} u_0 + \left[\widehat{\xi}_\rho + \xi(z)\right] u_0^{p(z)-1} \geq 0 \text{ in } \Omega, \\ &\Rightarrow u_0 \in \text{int} C_+\end{aligned}$$

using Proposition 4 of [4] (see also [[15], pp. 120], for the corresponding isotropic result).

Moreover, we have

$$\begin{aligned}&-\Delta_{p(z)} u_0 - \Delta_{q(z)} u_0 + \left[\widehat{\xi}_\rho + \xi(z)\right] u_0^{p(z)-1} \\ &= f(z, u_0) + \widehat{\xi}_\rho u_0^{p(z)-1} \\ &\leq f(z, \eta_+) + \widehat{\xi}_\rho \eta_+^{p(z)-1} \quad \text{(see (3.4) and hypothesis } H_1(iv)) \\ &\leq -\Delta_{p(z)}\eta_+ - \Delta_{q(z)}\eta_+ + \left[\widehat{\xi}_\rho + \xi(z)\right] \eta_+^{p(z)-1} \\ &\qquad \text{(see hypothesis } H_1(iii) \text{ and recall } \widehat{\xi}_\rho > \|\xi\|_\infty) \\ &\Rightarrow u_0(z) < \eta_+ \text{ for all } z \in \overline{\Omega} \text{ (see [4], Proposition 5).}\end{aligned}$$

Finally, we can say that

$$u_0 \in \text{int}_{C^1(\overline{\Omega})}[0, \eta_+]. \tag{3.5}$$

It is clear from (3.1) that

$$\psi_+|_{[0,\eta_+]} = \widehat{\varphi}_+|_{[0,\eta_+]}. \tag{3.6}$$

From (3.6), (3.5), and (3.2), we see that

$$u_0 \text{ is a } C^1(\overline{\Omega}) \text{ local minimizer of} \widehat{\varphi}_+(\cdot),$$
$$\Rightarrow u_0 \text{ is a } W^{1,p(z)}(\Omega) \text{ local minimizer of } \widehat{\varphi}_+ \cdot$$
$$\text{(see [16], Proposition 3.3).}$$

Since $\varphi|_{C_+} = \widehat{\varphi}_+|_{C_+}$, we conclude u_0 is also a local minimizer of φ.

For the negative solution, we start with the Carathéodory function $\widehat{f}_-(z,x)$ defined by

$$\widehat{f}_-(z,x) = \begin{cases} f(z,\eta_-) - k|\eta_-|^{p(z)-1} & \text{if } x < \eta_- \\ f(z,x) - k(x^-)^{p(z)-1} & \text{if } \eta_- \leq x \end{cases} \quad (k > \|\xi\|_\infty).$$

We set $\widehat{F}_-(z,x) = \int_0^x \widehat{f}_-(z,s)ds$ and introduce the C^1-functional $\psi_- : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\psi_-(u) = \gamma_p(u) + \ell_q(Du) + \int_\Omega \frac{k}{p(z)}|u|^{p(z)}dz - \int_\Omega \widehat{F}_-(z,u)dz \text{ for all } u \in W^{1,p(z)}(\Omega).$$

Working with $\psi_-(\cdot)$ as above, we generate a negative solution

$$v_0 \in \mathrm{int}_{C^1(\overline{\Omega})}[\eta_-, 0].$$

Moreover, v_0 is a local minimizer of $\varphi, \widehat{\varphi}_-$.

Next, using tools from the critical point theory, we generate two more constant sign smooth solutions for problem (1.1).

Proposition 3.2. *If hypotheses H_0 and H_1 hold, then problem* (1.1) *has two more constant sign solutions*

$$\widehat{u} \in intC_+,\ \ \widehat{v} \in -intC_+,\ \ \widehat{u} \neq u_0,\ \ \widehat{v} \neq v_0.$$

Proof. The regularity theory of Fan [13] and Proposition 4 of [4] imply that

$$K_{\widehat{\varphi}_+} \subseteq \mathrm{int}C_+ \cup \{0\}.$$

Since the critical points of $\widehat{\varphi}_+(\cdot)$ are positive solutions of (1.1), we see that we may assume that $K_{\widehat{\varphi}_+}$ is finite (otherwise, we already have an infinity of positive smooth solutions).

Claim: $\widehat{\varphi}_+(\cdot)$ satisfies the C-condition.

Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega)$, such that

$$|\widehat{\varphi}_+(u_n)| \leq c_4 \ \text{ for all } n \in \mathbb{N},\ \text{some } c_4 > 0, \tag{3.7}$$

$$(1 + \|u_n\|)\widehat{\varphi}'_+(u_n) \to 0 \text{ in } W^{1,p(z)}(\Omega)^* \text{ as } n \to \infty. \tag{3.8}$$

From (3.8) we have

$$\begin{aligned} &\Big|\langle V(u_n), h\rangle + \int_\Omega \xi(z)(u_n^+)^{p(z)-1}hdz - \int_\Omega [\xi(z) + k]\,(u_n^-)^{p(z)-1}hdz \\ &+ \int_{\partial\Omega} \beta(z)(|u_n|)^{p(z)-2}u_n h d\sigma - \int_\Omega f(z,u_n^+)hdz\Big| \leq \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \end{aligned} \tag{3.9}$$

for all $h \in W^{1,p(z)}(\Omega)$, all $n \in \mathbb{N}$, with $\varepsilon_n \to 0^+$.

In (3.9) we choose the test function $h = -u_n^- \in W^{1,p(z)}(\Omega)$ and obtain

$$\left|\rho_p(Du_n^-) + \rho_q(Du_n^-) + \int_\Omega [\xi(z) + k]\,(u_n^-)^{p(z)} \mathrm{d}z + \int_{\partial\Omega} \beta(z)(u_n^-)^{p(z)} \mathrm{d}\sigma\right| \leq \varepsilon_n$$

for all $n \in \mathbb{N}$. Since $k > \|\xi\|_\infty$, from Proposition 2.9 of [6], we have

$$u_n^- \to 0 \text{ in } W^{1,p(z)}(\Omega) \text{ as } n \to \infty. \tag{3.10}$$

Next in (3.9) we choose the test function $h = u_n^+ \in W^{1,p(z)}(\Omega)$ and obtain

$$\begin{aligned} \Big|\rho_p(Du_n^+) + \rho_q(Du_n^+) + \int_\Omega \xi(z)(u_n^+)^{p(z)} \mathrm{d}z + \int_{\partial\Omega} \beta(z)(u_n^+)^{p(z)} \mathrm{d}\sigma \\ - \int_\Omega f(z, u_n^+)u_n^+ \mathrm{d}z\Big| \leq \varepsilon_n \end{aligned} \tag{3.11}$$

for all $n \in \mathbb{N}$. From (3.7) and (3.10) we have

$$\begin{aligned} \frac{1}{p_+}\Big[\rho_p(Du_n^+) + \rho_q(Du_n^+) + \int_\Omega \xi(z)(u_n^+)^{p(z)} \mathrm{d}z + \int_{\partial\Omega} \beta(z)(u_n^+)^{p(z)} \mathrm{d}\sigma \\ - \int_\Omega \rho_+ F(z, u_n^+) \mathrm{d}z\Big] \leq c_4, \end{aligned}$$

for all $n \in \mathbb{N}$. Moreover, we can infer that

$$\begin{aligned} \rho_p(Du_n^+) + \rho_q(Du_n^+) + \int_\Omega \xi(z)(u_n^+)^{p(z)} \mathrm{d}z + \int_{\partial\Omega} \beta(z)(u_n^+)^{p(z)} \mathrm{d}\sigma \\ - \int_\Omega p_+ F(z, u_n^+) \mathrm{d}z \leq p_+ c_4, \end{aligned} \tag{3.12}$$

for all $n \in \mathbb{N}$.

From (3.11) and (3.12) we obtain

$$\int_\Omega \left[f(z, u_n^+)u_n^+ - p_+ F(z, u_n^+)\right] \mathrm{d}z \leq c_5 \text{ for some } c_5 > 0 \text{ all } n \in \mathbb{N}. \tag{3.13}$$

On account of hypotheses $H_1(i), (ii)$ we can find $\beta_1 \in (0, \beta_0)$ and $c_6 > 0$, such that

$$\beta_1 x^{r(z)} - c_6 \leq f(z, x)x - p_+ F(z, x) \text{ for a.a } z \in \Omega, \text{ all } x \geq 0. \tag{3.14}$$

We use (3.14) in (3.13) and obtain

$$\begin{aligned} &\rho_\gamma(u_n^+) \leq c_7 \text{ for some } c_7 > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow\ &\{u_n^+\}_{n \in \mathbb{N}} \subset L^{\gamma(z)}(\Omega) \text{ is bounded (see Proposition 2.2).} \end{aligned} \tag{3.15}$$

From hypothesis $H_1(iii)$, we see that we can always assume that

$$\begin{aligned} &\gamma(z) < r(z) < p(z)^* \text{ for all } z \in \overline{\Omega}, \\ \Rightarrow\ &\gamma_- < r_- < p_-^*. \end{aligned}$$

Let $t \in (0, 1)$ be such that

$$\frac{1}{r_-} = \frac{1-t}{\gamma_-} + \frac{t}{p_-^*}. \tag{3.16}$$

Using the interpolation inequality (see [[12], pp. 116]), we have

$$\begin{aligned} &\|u_n^+\|_{r_-} \le \|u_n^+\|_{\gamma_-}^{1-t} \|u_n^+\|_{p_-^*}^t, \\ \Rightarrow\ &\|u_n^+\|_{r_-}^{r_-} \le c_6 \|u_n^+\|^{tr_-} \text{ for some } c_8 > 0, \text{ all } n \in \mathbb{N}. \end{aligned} \tag{3.17}$$

Here, we have used (3.15), that $L^{\gamma(z)}(\Omega) \hookrightarrow L^{\gamma_-}(\Omega)$ and that $W^{1,p(z)}(\Omega) \hookrightarrow L^{p_-^*}(\Omega)$.

In (3.9) we choose the test function $h = u_n^+ \in W^{1,p(z)}(\Omega)$ and obtain

$$\begin{aligned} \rho_p(Du_n^+) + \rho_q(Du_n^+) + \int_\Omega \xi(z)(u_n^+)^{p(z)} \mathrm{d}z + \int_{\partial\Omega} \beta(z)(u_n^+)^{p(z)} \mathrm{d}\sigma \\ - \int_\Omega f(z, u_n^+) u_n^+ \mathrm{d}z \le \varepsilon_n, \text{ for all } n \in \mathbb{R}^N. \end{aligned} \tag{3.18}$$

If $r_- \to (p_-^*)^-$, then $\gamma_- > p_+$ (see hypotheses H_0). Thus, we may assume that $p_+ < \gamma_-$, and so from (3.15) and since

$$L^{\gamma(z)}(\Omega) \hookrightarrow L^{\gamma_-}(\Omega) \hookrightarrow L^{p_+}(\Omega) \hookrightarrow L^{p(z)}(\Omega),$$

we infer that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq L^{p(z)}(\Omega)$ is bounded. Then from (3.17) and (3.18), we have

$$\rho_p(Du_n^+) + \rho_q(Du_n^+) \le c_9 \left[1 + \|u_n^+\|^{tr_-}\right] \text{ for some } c_9 > 0, \text{ all } n \in \mathbb{N}. \tag{3.19}$$

First suppose that $p_-^* \neq N$. Then

$$p_-^* = \frac{Np_-}{N - p_-} \text{ if } p_- < N; \quad p_-^* = \infty \text{ if } p_- > N.$$

From (3.16), we have

$$\begin{aligned} &tr_- = \frac{p_-^*(r_- - \gamma_-)}{p_-^* - \gamma_-} \text{ if } p_- < N \quad \text{and} \quad tr_- = r_- - \gamma_- \text{ if } p_- > N, \\ &\Rightarrow tr_- < p_- \quad \text{(see hypothesis } H_1(ii)), \end{aligned}$$

$$\Rightarrow \{u_n^+\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega) \text{ is bounded} \quad \text{(see (3.19) and Proposition 2.1).} \tag{3.20}$$

Then (3.10) and (3.20) imply that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega) \text{ is bounded.}$$

Thus, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p(z)}(\Omega), \quad u_n \to u \text{ in } L^{r(z)}(\Omega), L^{p(z)}(\partial\Omega). \tag{3.21}$$

In (3.9) we choose the test function $h = u_n - u \in W^{1,p(z)}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.21). Then

$$\begin{aligned} &\lim_{n\to\infty} \langle V(u_n), u_n - u \rangle = 0, \\ \Rightarrow\ &u_n \to u \text{ in } W^{1,p(z)}(\Omega) \text{ (see Proposition 2.4).} \end{aligned}$$

Therefore, $\widehat{\varphi}_+(\cdot)$ satisfies the C-condition, and this proves the Claim.

Recall that $K_{\widehat{\varphi}_+}$ is finite and u_0 is a local minimizer of $\widehat{\varphi}_+$ (see the proof of Proposition 3.1). Thus, using Theorem 5.7.6, of [3], we can find $\rho \in (0,1)$ small such that

$$\widehat{\varphi}_+(u_0) < \inf \{\widehat{\varphi}_+(u) : \|u - u_0\| = \rho\} = \widehat{m}_+. \tag{3.22}$$

Additionally, if $u \in \mathrm{int}C_+$, then on account of hypothesis $H_1(ii)$, we have

$$\widehat{\varphi}_+(tu) \to -\infty \text{ as } t \to +\infty. \tag{3.23}$$

Then, from (3.22), (3.23), the "Claim" and the mountain pass theorem imply the existence of $\widehat{u} \in W^{1,p(z)}(\Omega)$, such that

$$\begin{aligned} &\widehat{u} \in K_{\widehat{\varphi}_+} \subseteq \mathrm{int}C_+ \cup \{0\}, \quad \widehat{\varphi}_+(u_0) < \widehat{m}_+ \leq \widehat{\varphi}_+(\widehat{u}),\\ \Rightarrow &\widehat{u} \neq u_0. \end{aligned}$$

From Theorem 6.5.6, of [3], we have

$$C_1(\widehat{\varphi}_+, \widehat{u}) \neq 0. \tag{3.24}$$

On the other hand, on account of hypothesis $H_1(iii)$, we have

$$C_k(\widehat{\varphi}_+, 0) = 0 \text{ for all } k \in \mathbb{N}_0, \tag{3.25}$$

(see [4], proof of Proposition 10). Then from (3.24), (3.25) we infer that

$$\widehat{u} \neq 0.$$

Thus, $\widehat{u} \in \mathrm{int}C_+$ is the second positive solution of (1.1) distinct from u_0.

Similarly, working with $\widehat{\varphi}_-(\cdot)$, we produce a second negative solution $\widehat{v} \in -\mathrm{int}C_+$ distinct from v_0.

We can show the existence of extremal constant sign solutions, that is, we have a smallest positive solution and a largest negative solution. These extremal constant sign solutions will be used in Section 4 to produce a nodal solution.

To obtain the extremal solutions, we consider the following auxiliary anisotropic Robin problem

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_q u(z) + |\xi(z)||u(z)|^{p(z)-2}u(z) = c|u(z)|^{\tau(z)-2}u(z) \text{ in } \Omega,\\ \frac{\partial u}{\partial n_{pq}} + \beta(z)|u(z)|^{p(z)-2}u(z) = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.26}$$

Here, $c > 0$ is as in hypothesis $H_1(iii)$.

Proposition 3.3. *If hypotheses H_0 hold, $\tau \in \mathcal{D}_1$, with $\tau_+ < q_-$, then problem* (3.26) *has a unique positive solution*

$$\overline{u} \in intC_+$$

and since the problem is odd, then

$$\overline{v} = -\overline{u} \in -intC_+$$

is the unique negative solution of (3.26).

Proof. Let $\widehat{\sigma}_+ : W^{1,p(z)}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\widehat{\sigma}_+(u) = \widehat{\gamma}_p(u) + \ell_q(Du) - c\ell_\tau(u^+) \text{ for all } u \in W^{1,p(z)}(\Omega),$$

with

$$\widehat{\gamma}_p(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\mathrm{d}z + \int_\Omega \frac{|\xi(z)|}{p(z)}|u|^{p(z)}\mathrm{d}z + \int_{\partial\Omega} \frac{\beta(z)}{p(z)}|u|^{p(z)}\mathrm{d}\sigma,$$
$$\ell_q(Du) = \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\mathrm{d}z \quad \text{and} \quad \ell_\tau(u^+) = \int_\Omega \frac{1}{\tau(z)}|u^+|^{\tau(z)}\mathrm{d}z.$$

Since $\tau_+ < q_-$ (see hypothesis $H_1(iii)$), we see that $\widehat{\sigma}_+(\cdot)$ is coercive. Additionally, using Proposition 2.1, we show that

$$\widehat{\sigma}_+(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

By the Weierstrass-Tonelli theorem, we can find $\overline{u} \in W^{1,p(z)}(\Omega)$, such that

$$\begin{aligned}
&\widehat{\sigma}_+(\overline{u}) = \inf\left\{\widehat{\sigma}(u) : u \in W^{1,p(z)}(\Omega)\right\},\\
&\Rightarrow \widehat{\sigma}_+(\overline{u}) < 0 = \widehat{\sigma}_+(0) \text{ (since } \tau_+ < q_-\text{, see the proof of Proposition 3.1)},\\
&\Rightarrow \overline{u} \neq 0.
\end{aligned}$$

We have

$$\begin{aligned}
&\langle V(\overline{u}), h\rangle + \int_\Omega |\xi(z)||\overline{u}|^{p(z)-2}\overline{u}h\mathrm{d}z + \int_{\partial\Omega} \beta(z)|\overline{u}|^{p(z)-2}\overline{u}h\mathrm{d}\sigma\\
&= c\int_\Omega |\overline{u}|^{p(z)-2}\overline{u}h\mathrm{d}z \text{ for all } h \in W^{1,p(z)}(\Omega).
\end{aligned} \tag{3.27}$$

In (3.27) we choose the test function $h = -\overline{u} \in W^{1,p(z)}(\Omega)$ and obtain $\overline{u} \geq 0$, $\overline{u} \neq 0$. Thus, $\overline{u}$ is a positive solution of (3.26) and $\overline{u} \in C_+\backslash\{0\}$ (regularity theory). We have

$$\begin{aligned}
&-\Delta_{p(z)}\overline{u} - \Delta_{q(z)}\overline{u} + |\xi(z)|\overline{u}^{p(z)-1} \geq 0 \text{ in } \Omega,\\
&\Rightarrow \overline{u} \in \text{int } C_+ \text{ (see [4])}.
\end{aligned}$$

Next, we show the uniqueness of this positive solution. Suppose that $\overline{y} \in W^{1,p(z)}(\Omega)$ is another positive solution of (3.26). Again, we have that $\overline{y} \in \text{int}C_+$. Evidently,

$$\frac{\overline{u}}{\overline{y}} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\overline{y}}{\overline{u}} \in L^\infty(\Omega).$$

Thus, we can use the variable Diaz-Saa inequality of [17] (Theorem 2.5) and have

$$\begin{aligned}
0 &\leq \int_\Omega \left[\frac{-\Delta_{p(z)}\overline{u} - \Delta_{q(z)}\overline{u}}{\overline{u}^{q_--1}} + \frac{\Delta_{p(z)}\overline{y} + \Delta_{q(z)}\overline{y}}{\overline{y}^{q_--1}}\right](\overline{u}^{q_-} - \overline{y}^{q_-})\,\mathrm{d}z\\
&= \int_\Omega c\left[\frac{1}{\overline{u}^{q_--\tau(z)}} - \frac{1}{\overline{y}^{q_--\tau(z)}}\right](\overline{u}^{q_-} - \overline{y}^{q_-})\,\mathrm{d}z\\
&\Rightarrow \overline{u} = \overline{y}.
\end{aligned}$$

Here we have used that $\tau_+ < q_-$ and so $x \to x^{\tau(z)-q_-}$ is strictly decreasing on $\mathring{\mathbb{R}}_+ = (0, +\infty)$. Therefore, we have uniqueness of the positive solution for problem (3.26). The problem is odd and so $\overline{v} = -\overline{u} \in -\text{int}C_+$ is the unique negative solution of (3.26).

Now let

$$S_+ = \text{set of positive solutions of (1.1)},$$

$$S_- = \text{set of negative solutions of (1.1)}.$$

We know that

$$\emptyset \neq S_+ \subseteq \mathrm{int}C_+ \quad \text{and} \quad \emptyset \neq S_- \subseteq -\mathrm{int}C_+.$$

Moreover, from [18], we know that S_+ is downward directed (that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \leq u_1$, $u \leq u_2$) and S_- is upward directed (that is, if v_1, $v_2 \in S_-$, then we can find $v \in S_-$ such that $v_1 \leq v$, $v_2 \leq v$).

Now we can show that problem (1.1) admits extremal constant sign solutions.

Proposition 3.4. *If hypotheses H_0 and H_1 hold, then there exist $u_* \in S_+$ and $v_* \in S_-$, such that*

$$u_* \leq u \text{ for all } u \in S_+, \quad v \leq v_* \text{ for all } v \in S_-.$$

Proof. Invoking Theorem 5.109 of [19], we can find $\{u_n\}_{n\in\mathbb{N}} \subset S_+$ decreasing, such that

$$\inf_{n\in\mathbb{N}} S_+ = \inf_{n\in\mathbb{N}} u_n.$$

We have

$$\langle V(u_n), h\rangle + \int_\Omega \xi(z)u_n^{p(z)-1}h\mathrm{d}z + \int_{\partial\Omega} \beta(z)u_n^{p(z)-1}h\mathrm{d}\sigma = \int_\Omega f(z, u_n)h\mathrm{d}z \tag{3.28}$$

for all $h \in W^{1,p(z)}(\Omega)$, all $n \in \mathbb{N}$.

$$0 < u_n \leq u_1 \text{ for all } n \in \mathbb{N}. \tag{3.29}$$

In (3.28) we choose the test function $h = u_n \in W^{1,p(z)}(\Omega)$. Then from (3.29) and hypothesis $H_1(i)$, we infer that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega)$ is bounded. From [12] (Proposition A1), we have that $\{u_n\}_{n\in\mathbb{N}} \subseteq L^\infty(\Omega)$ is bounded. Then the anisotropic regularity theory of Fan [13] implies that there exist $\alpha \in (0, 1)$ and $c_{10} > 0$, such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}), \quad ||u_n||_{C^{1,\alpha}(\overline{\Omega})} \leq c_{10} \text{ for all } n \in \mathbb{N}.$$

The compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ (Arzela-Ascoli theorem), implies that we may assume that

$$u_n \to u_* \quad \text{in } C^1(\overline{\Omega}). \tag{3.30}$$

Suppose that $u_* = 0$. From (3.30), we see that we can find $n_0 \in \mathbb{N}$, such that

$$0 < u_n(z) \leq \delta \text{ for all } z \in \overline{\Omega}, \text{ all } n \geq n_0,$$

(here, $\delta > 0$ is as in hypothesis $H_1(iii)$)

$$\Rightarrow cu_n(z)^{\tau-1} \leq f(z, u_n(z)) \text{ for all } z \in \Omega, \text{ all } n \geq n_0. \tag{3.31}$$

(see hypothesis $H_1(iii)$).

Fix $n \geq n_0$ and introduce the Carathéodory function $e : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$e(z, x) = \begin{cases} c(x^+)^{\tau(z)-1} & \text{if } x \leq u_n(z) \\ cu_n(z)^{\tau(z)-1} & \text{if } u_n(z) < x. \end{cases} \tag{3.32}$$

We set $E(z,x) = \int_0^x e(z,s)\mathrm{d}s$ and consider the C^1-functional $\widehat{b} : W^{1,p(z)} \to \mathbb{R}$ defined by

$$\widehat{b}(u) = \widehat{\gamma}_p(u) + \ell_q(u) - \int_\Omega E(z,u)\mathrm{d}z \text{ for all } u \in W^{1,p(z)}(\Omega).$$

Evidently, $\widehat{b}(\cdot)$ is coercive (see (3.32)) and sequentially weakly lower semicontinuous (use Proposition 2.1). Thus, we can find $\widetilde{u} \in W^{1,p(z)}(\Omega)$

$$\begin{aligned} &\widehat{b}(\widetilde{u}) = \inf\left\{\widehat{b}(u) : u \in W^{1,p(z)}(\Omega)\right\} < 0 = \widehat{b}(0) \text{ (since } \tau_+ < q_-), \\ \Rightarrow\ &\widetilde{u} \neq 0. \end{aligned} \tag{3.33}$$

From (3.33) we have

$$\begin{aligned} &\langle V(\widetilde{u}), h\rangle + \int_\Omega |\xi(z)||\widetilde{u}|^{p(z)-2}\widetilde{u}h\mathrm{d}z + \int_{\partial\Omega} \beta(z)|\widetilde{u}|^{p(z)-2}\widetilde{u}h\mathrm{d}\sigma \\ &= \int_\Omega e(z,\widetilde{u})h\mathrm{d}z \text{ for all } h \in W^{1,p(z)}(\Omega). \end{aligned} \tag{3.34}$$

Choosing $h = -\widetilde{u}^- \in W^{1,p(z)}(\Omega)$, we show that $\widetilde{u} \geq 0, \widetilde{u} \neq 0$.

Next, in (3.34), we choose the test function $h = (\widetilde{u} - u_n)^+ \in W^{1,p(z)}(\Omega)$. Then

$$\begin{aligned} &\langle V(\widetilde{u}), (\widetilde{u} - u_n)^+\rangle + \int_\Omega \xi(z)\widetilde{u}^{p(z)-1}(\widetilde{u} - u_n)^+\mathrm{d}z + \int_{\partial\Omega} \beta(z)\widetilde{u}^{p(z)-1}(\widetilde{u} - u_n)^+\mathrm{d}\sigma \\ &= \int_\Omega c u_n^{\tau(z)-1}(\widetilde{u} - u)^+\mathrm{d}z \quad \text{(see (3.32))} \\ &\leq \int_\Omega f(z,u_n)(\widetilde{u} - u_n)^+\mathrm{d}z \quad \text{(see (3.31))} \\ &\leq \langle V(u_n), (\widetilde{u} - u_n)^+\rangle + \int_\Omega |\xi(z)||u_n|^{p(z)-1}(\widetilde{u} - u_n)^+\mathrm{d}z \\ &+ \int_{\partial\Omega} \beta(z)|u_n|^{p(z)-1}(\widetilde{u} - u_n)^+\mathrm{d}\sigma \ \text{ (since } u_n \in S_+) \\ \Rightarrow\ &\widetilde{u} \leq u_n \text{ for all } n \geq n_0. \end{aligned}$$

This contradicts the hypothesis that $u_* = 0$ (see (3.30)). Thus, $u_* \neq 0$ and using (3.30), we conclude that

$$u_* \in S_+,\ u_* \leq u \quad \text{for all } u \in S_+.$$

Similarly, we produce $v_* \in S_-$, such that $v \leq v_*$ for all $v \in S_-$. In this case, there is an increasing sequence $\{v_n\} \subseteq S_-$, such that

$$\sup S_- = \sup_{n\in\mathbb{N}} v_n.$$

The proof is now complete.

4. Nodal solutions–multiplicity theorem

We use the extremal solutions from Proposition 3.4 to generate a nodal (sign changing solution). The idea is simple. Using truncations of the reaction $f(z,\cdot)$, we restrict our attention to the order interval $[v_*, u_*]$. Any nontrivial solution of (1.1) in this order interval that is distinct from v_* and u_*, is necessarily nodal.

Thus, we consider the two solutions $u_* \in S_+$ and $v_* \in S_-$ and introduce the Carathéodory function $\widehat{j}(z,x)$ defined by

$$\widehat{j}(z,x) = \begin{cases} f(z, v_*(z)) + k|v_*(z)|^{p(z)-2}v_*(z) & \text{if } x < v_*(z) \\ f(z,x) + k|x|^{p(z)-2}x & \text{if } v_*(z) \le x \le u_*(z) \\ f(z, u_*(z)) + ku_*(z)^{p(z)-1} & \text{if } u_*(z) < x, \end{cases} \tag{4.1}$$

with $k > \|\xi\|_\infty$.

In addition, we introduce the positive and negative truncations of $\widehat{j}(z,\cdot)$, namely the Carathéodory functions $\widehat{j}_\pm(z,x)$ defined by

$$\widehat{j}_\pm(z,x) = \widehat{j}(z, \pm x). \tag{4.2}$$

We set

$$\widehat{J}(z,x) = \int_0^x \widehat{j}(z,s)\mathrm{d}s \text{ and } \widehat{J}_\pm(z,x) = \int_0^x \widehat{j}_\pm(z,s)\mathrm{d}s,$$

consider the C^1-functional $\widehat{w}, \widehat{w}_\pm : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \widehat{w}(u) &= \gamma_p(u) + \ell_q(u) + k\ell_p(u) - \int_\Omega \widehat{J}(z,u)\mathrm{d}z \\ \widehat{w}_\pm(u) &= \gamma_p(u) + \ell_q(Du) + k\ell_p(u) - \int_\Omega \widehat{J}_\pm(z,u)\mathrm{d}z \text{ for all } u \in W^{1,p(z)}(\Omega). \end{aligned}$$

Proposition 4.1. *If hypotheses H_0 and H_1 hold, then*

$$K_w \subseteq [v_*, u_*] \cap C^1(\overline{\Omega}),\ K_{w_+} = \{0, u_*\},\ K_{w_-} = \{0, v_*\}.$$

Proof. Let $u \in K_w$. Then

$$\begin{aligned} &\langle V(u), h\rangle + \int_\Omega [\xi(z) + k]\,|u|^{p(z)-2}uh\mathrm{d}z + \int_{\partial\Omega} \beta(z)|u|^{p(z)-2}uh\mathrm{d}\sigma \\ &= \int_\Omega \widehat{j}(z,u)h\mathrm{d}z \quad \text{for all } h \in W^{1,p(z)}(\Omega). \end{aligned} \tag{4.3}$$

In (4.3), we first choose the test function $h = (u - u_*)^+ \in W^{1,p(z)}(\Omega)$. Thus, we have

$$\begin{aligned} &\langle V(u), (u-u_*)^+\rangle + \int_\Omega [\xi(z) + k]\,u^{p(z)-1}(u-u_*)^+\mathrm{d}z + \int_{\partial\Omega} \beta(z)u^{p(z)-1}(u-u_*)^+\mathrm{d}\sigma \\ &= \int_\Omega \left[f(z,u_*) + ku_*^{p(z)-1}\right](u-u_*)^+\mathrm{d}z \quad \text{(see (4.1))} \\ &= \langle V(u_*), (u-u_*)^+\rangle + \int_\Omega [\xi(z)+k]\,u_*^{p(z)-1}(u-u_*)^+\mathrm{d}z \\ &\quad + \int_{\partial\Omega} \beta(z)u_*^{p(z)-1}(u-u_*)^+\mathrm{d}\sigma \\ &\Rightarrow u \le u_* \quad (\text{since } k > \|\xi\|_\infty). \end{aligned}$$

Similarly, using the test function $h = (v_* - u)^+ \in W^{1,p(z)}(\Omega)$ we show that $v_* \leq u$. Therefore,

$$u \in [v_*, u_*] \cap C^1(\overline{\Omega}) \quad \text{(from Fan [13]).}$$

Next, let $u \in K_{\widehat{w}_+}$. Thus, we have

$$\begin{aligned} &\langle V(u), h\rangle + \int_\Omega [\xi(z) + k]\,|u|^{p(z)-2}uh\mathrm{d}z + \int_{\partial\Omega} \beta(z)|u|^{p(z)-2}uh\mathrm{d}\sigma \\ &= \int_\Omega \widehat{j}_+(z,u)h\mathrm{d}z \quad \text{for all } h \in W^{1,p(z)}(\Omega). \end{aligned} \tag{4.4}$$

In (4.4), we first choose the test function $h = -u^- \in W^{1,p(z)}(\Omega)$. Then

$$\begin{aligned} &\rho_p(Du^-) + \int_\Omega [\xi(z) + k]\,(u^-)^{p(z)}\mathrm{d}z + \int_{\partial\Omega} \beta(z)(u^-)^{p(z)}\mathrm{d}\sigma = 0 \\ &\Rightarrow u \geq 0 \quad (\text{since } k > \|\xi\|_\infty). \end{aligned}$$

Moreover, as before using the test function $h = (u - u_*)^+ \in W^{1,p(z)}(\Omega)$ we show that $u \leq u_*$. Hence

$$\begin{aligned} &u \in [0, u_*], \\ &\Rightarrow u = 0 \quad \text{or } u = u_* \quad \text{(due to the extremality of } u_*), \\ &\Rightarrow K_{\widehat{w}_+} = \{0, u_*\}. \end{aligned}$$

Similarly, we show that

$$K_{\widehat{w}_-} = \{0, v_*\}.$$

This completes the proof.

Next, we determine the nature of the critical points $u^*, v^* \in K_{\widehat{w}}$.

Proposition 4.2. *If hypotheses H_0 and H_1 hold, then*

$$u_* \in intC_+ \textit{ and } v_* \in \textit{-}intC_+ \textit{ are local minimizers of } w(\cdot).$$

Proof. From (4.1) and (4.2), and since $k > \|\xi\|_\infty$, we infer that

$$w_+(\cdot) \text{ is coercive.}$$

Additionally, using Proposition 2.1, we see that

$$w_+(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

Thus, we can find $\widetilde{u}_* \in W^{1,p(z)}(\Omega)$, such that

$$w_+(\widetilde{u}_*) = \inf\left\{w_+(u) : u \in W^{1,p(z)}(\Omega)\right\}. \tag{4.5}$$

Since $\tau_+ < q_-$ as before, we have

$$\begin{aligned} &w_+(\widetilde{u}_*) < 0 = w_+(0), \\ &\Rightarrow \widetilde{u}_* \neq 0. \end{aligned}$$

From (4.5) we have

$$\widetilde{u}_* \in K_{\widehat{w}_+} = \{0, u_*\} \quad \text{(see Proposition 4.1)},$$
$$\Rightarrow \widetilde{u}_* = u_* \in \mathrm{int}C_+.$$

Since $\widehat{w}|_{C_+} = \widehat{w}_+|_{C_+}$ (see (4.1),(4.2)), we see that

$$u_* \text{ is a } C^1(\overline{\Omega})\text{-local minimizer of } \widehat{w}(\cdot),$$
$$\Rightarrow u_* \text{ is a } W^{1,p(z)}(\Omega)\text{-local minimizer of } \widehat{w}(\cdot) \quad \text{(see [16])}.$$

Similarly, working with $w_-(\cdot)$ we show that

$$v_* \text{ is a } W^{1,p(z)}(\Omega)\text{-local minimizer of } \widehat{w}(\cdot).$$

The proof is now complete.

Now we are ready to produce a nodal solution.

Proposition 4.3. *If hypotheses H_0 and H_1 hold, then problem* (1.1) *admits a nodal solution*

$$y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega}).$$

Proof. From Proposition 4.1 and the extremality of u_* and v_*, we see that we may assume that

$$K_{\widehat{w}} \subseteq [v_*, u_*] \cap C^1(\overline{\Omega}) \text{ is finite.} \tag{4.6}$$

Otherwise, we already have an infinity of smooth nodal solutions.

Without any loss of generality, we may assume that

$$\widehat{w}(v_*) \leq \widehat{w}(u_*).$$

The reasoning is similar if the opposite inequality holds.

From Proposition 4.2, we know that u_* is a local minimizer of $\widehat{w}(\cdot)$. The functional $\widehat{w}(\cdot)$ is coercive (see (4.1)). Thus, from Proposition 5.1.15 of [3], we can see that $\widehat{w}(\cdot)$ satisfies the C-condition. Then on account of (4.6) and Theorem 5.7.6 of [3], we can find $\rho \in (0, 1)$ small, such that

$$\widehat{w}(v_*) \leq \widehat{w}(u_*) < \inf\{w(u) : \|u - u_*\| = \rho\} = \widehat{m}, \quad \|v_* - u_*\| > \rho. \tag{4.7}$$

Then, through (4.7), and since $\widehat{w}(\cdot)$ satisfies the C-condition, the use of the mountain pass theorem is permited, so we can find $y_0 \in W^{1,p(z)}(\Omega)$, such that

$$y_0 \in K_w \in [v_*, u_*] \cap C^1(\overline{\Omega}), \quad \widehat{m} \leq \widehat{w}(y_0),$$
$$\Rightarrow y_0 \notin \{u_*, v_*\} \quad \text{(see (4.7))}.$$

Theorem 6.5.8 of [3] implies that

$$C_1(\widehat{w}, y_0) \neq 0. \tag{4.8}$$

Note that $0 \in \mathrm{int}_{C^1(\overline{\Omega})}[v_*, u_*]$. Thus, the anisotropic regularity theory and the homotopy invariance property of critical groups imply

$$\begin{aligned} &C_k(w, 0) = C_k(\varphi, 0) \quad \text{for all } k \in \mathbb{N}_0, \\ &\Rightarrow C_k(w, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \end{aligned} \tag{4.9}$$

From (4.8) and (4.9), it follows that $y_0 \neq 0$, and so $y_0 \in C^1(\overline{\Omega})$ is a nodal solution for problem (1.1).

Thus, we can state the following multiplicity theorem for problem (1.1). Note that we have sign information for all the solutions, and they are ordered.

Theorem 4.1. *If hypotheses H_0 and H_1 hold, then problem* (1.1) *has at least five nontrivial smooth solutions*

$$\widehat{u}, u_* \in intC_+, \quad u_* \leq \widehat{u},$$
$$\widehat{v}, v_* \in -intC_+, \quad \widehat{v} \leq v_*,$$
$$y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega}) \text{ is nodal.}$$

Remark 4.1. *It will be interesting to extend this result to double phase problems, that is, problems driven by*

$$u \to -\Delta^a_{p(z)} u - \Delta_{q(z)} u$$

with $-\Delta^a_{p(z)} u = -div\left(a(z)|Du|^{p(z)-2}Du\right)$, *where* $a \in C^{0,1}(\overline{\Omega})\backslash\{0\}$ *and* $\min_{\overline{\Omega}} a = 0$. *For such problems ,there is no global regularity theory. Thus, many of the tools used in this paper are not available for double phase problems.*

Use of AI tools declaration

The authors declare that no artificial intelligence tools were used in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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