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Some useful tools in the study of nonlinear elliptic problems

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This paper is dedicated to the memory of Professor Haim Brezis.

Abstract

This paper gives an overview of some basic aspects concerning the qualitative analysis of nonlinear, nonhomogeneous elliptic problems. We are concerned with two classes of elliptic equations with Dirichlet boundary condition. The first problem is driven by a general nonhomogeneous differential operator, which includes several usual operators (such as the (p, q)-Laplace operator introduced by P. Marcellini). Next, we focus on differential operators with unbalanced growth in the nonautonomous case. Our analysis will point out some relevant differences between balanced and unbalanced growth problems. The presentation is done in the context of Dirichlet problems but a similar analysis can be developed for other boundary conditions, such as Neumann or Robin.

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1. Introduction

The aim of this paper is to present an overview of the theory of nonlinear, nonhomogeneous elliptic problems. We will focus on two classes of equations. The first one deals with the following boundary value problem

$$\left\{\begin{aligned}&-\operatorname{div} a(z, Du(z)) = f(z, u(z)) \text{ in } \Omega,\\ &u|_{\partial\Omega} = 0.\end{aligned}\right\} \tag{1}$$

In this problem, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$ and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous map which satisfies certain other regularity and growth conditions, listed in the hypotheses in Section 2. The prototype differential operator of this kind is the (p, q)-Laplacian, that is,

$$\Delta_p u + \mu\Delta_q u \text{ with } 1 < q < p < +\infty,\ \mu \geq 0.$$

For this operator, $a(y) = |y|^{p-2}y + \mu|y|^{q-2}y$ for all $y \in \mathbb{R}^N$. The reaction $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is in general a Carathéodory functions (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$ the function $x \mapsto f(z, x)$ is continuous). Such a function is jointly measurable and so, if $u : \Omega \to \mathbb{R}$ is measurable, then the mapping $z \mapsto f(z, u(z))$ is measurable. In general, $f(z, \cdot)$ satisfies certain polynomial growth conditions and some hypotheses concerning its behavior as $x \to \pm\infty$ and as $x \to 0$. For problem (1) the functional framework within which we work, is provided by the standard Lebesgue and Sobolev spaces. In the next section we discuss the main aspects of the theory of these spaces which are relevant to the analysis of problem (1).

The second class of problems that we will focus on, are those of unbalanced growth. The most characteristic family of such problems is that of (p, q)-equations (according with the terminology of P. Marcellini, see [22–27]), namely problems of the form

$$\left\{\begin{aligned}&-\Delta_p^a u(z) - \Delta_q u(z) = f(z, u(z)) \text{ in } \Omega,\\ &u|_{\partial\Omega} = 0,\ 1 < q < p < \infty.\end{aligned}\right\} \tag{2}$$

Now, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$. For $a \in L^\infty(\Omega) \setminus \{0\}$ with $a(z) \geq 0$, we denote by Δ_p^a the weighted p-Laplace differential operator with weight $a(\cdot)$ defined by

$$\Delta_p^a u = \operatorname{div}(a(z)|Du|^{p-2}Du).$$

The differential operator of (2) is related to the so-called double phase integral functional defined by

$$u \mapsto \int_\Omega (a(z)|Du|^p + |Du|^q)dz.$$

The density function of this integral functional is

$$\eta(z, t) = a(z)t^p + t^q \text{ for all } z \in \Omega, \text{ all } t \geq 0.$$

We do not assume that the weight function $a(\cdot)$ is bounded away from zero (that is, we do not require that $0 < \operatorname{essinf}_\Omega a$) and so $\eta(z, \cdot)$ exhibits unbalanced growth, namely we have

$$t^q \leq \eta(z, t) \leq c_0(t^p + t^q) \text{ for a.a. } z \in \Omega, \text{ all } t \geq 0, \text{ some } c_0 > 0.$$

This type of growth has important consequences, the most profound of which is that the functional framework provided by the standard Lebesgue and Sobolev spaces, is not appropriate and we have to consider generalized Orlicz spaces. In the next section we discuss some main aspects of the theory of these spaces. We mention that double phase operators are suitable to describe diffusion-type processes in a space, where certain subdomains are distinguished from others. For example, we can describe a composite material having on $\{z \in \Omega : a(z) = 0\}$ an energy density with q-growth, while on $\{z \in \Omega : a(z) > 0\}$ has growth of order p. In purely mathematical terms, the operator has varying ellipticity depending on the point $z \in \Omega$.

In the next section we discuss the relevant functional settings for the analysis of problems (1) and (2). In Section 3, we present some basic results from the nonlinear regularity theory. We will see that there is a remarkable difference between balanced and unbalanced problems. Then in Section 4, we discuss nonlinear versions of the Hopf maximum principle and state related comparison principles. Finally, in Section 5 we have gathered some results which are useful in the study of nonlinear elliptic problems starting with the spectral properties of the p-Laplacian and of the weighted p-Laplacian.

We point out that our presentation is done in the context of Dirichlet problems. A similar analysis can be done if we have other boundary conditions, such as Neumann or Robin.

1.1. A remark

In the literature, we usually assume $1 < p < q$. However, since the first author in his works views the p-operator as the dominant one and in order to be consistent with the joint works mentioned in the references, we assume throughout this paper that $1 < q < p$.

2. Functional setting

For the balanced growth problems (see (1)), we work with the standard Lebesgue and Sobolev spaces.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. We denote by $L^0(\Omega)$ the linear space of all measurable functions $u : \Omega \to \mathbb{R}$. We identify two such functions which differ only on a Lebesgue-null set. Then for $1 \leq p < \infty$, we define the Lebesgue space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ u \in L^0(\Omega);\ \|u\|_p = \left(\int_\Omega |u(z)|^p dz \right)^{1/p} < \infty \right\}.$$

For $p = \infty$, the Lebesgue space $L^\infty(\Omega)$ is defined by

$$L^\infty(\Omega) = \left\{ u \in L^0(\Omega);\ \|u\|_\infty = \operatorname{esssup}_\Omega |u| < \infty \right\},$$

where

$$\operatorname{esssup}_\Omega |u| = \inf\{M > 0;\ |u(z)| \leq M \text{ for a.a. } z \in \Omega\}.$$

We know that $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are norms and the spaces $L^p(\Omega)$ and $L^\infty(\Omega)$ are Banach spaces. Note that there is a difference in the way these norms are defined. The norm $\|\cdot\|_p$ is defined via an average process (an integral), while $\|\cdot\|_\infty$ is defined in a pointwise fashion. This distinguishes the structures of the spaces $L^p(\Omega)$ and $L^\infty(\Omega)$. So, $L^p(\Omega)$ ($1 \le p < \infty$) is separable, while $L^\infty(\Omega)$ is not. On the other hand, if we set

$$L^p(\Omega)_+ = \{u \in L^p(\Omega) : 0 \le u(z) \text{ for a.a. } z \in \Omega\}$$

and

$$L^\infty(\Omega)_+ = \{u \in L^\infty(\Omega) : 0 \le u(z) \text{ for a.a. } z \in \Omega\},$$

then we have $\operatorname{int} L^p(\Omega)_+ = \emptyset$ but $\emptyset \ne \operatorname{int} L^\infty(\Omega)_+ = \{u \in L^\infty(\Omega)_+;\ 0 < \operatorname{essinf}_\Omega u\}$, where $\operatorname{essinf}_\Omega u := \sup\{c \in \mathbb{R} : c \le u(z) \text{ for a.a. } z \in \Omega\}$.

If $1 < p < \infty$, then $L^p(\Omega)$ is separable, reflexive (in fact, uniformly convex). Recall that a uniformly convex Banach space exhibits the so-called "Kadec–Klee property", namely, if X is the uniformly convex Banach space with norm $\|\cdot\|_X$, then

$$u_n \xrightarrow{w} u \text{ in } X,\ \|u_n\|_X \to \|u\|_X \Rightarrow u_n \to u \text{ in } X.$$

Using the Lebesgue spaces, we can define the corresponding Sobolev spaces. So for $1 \le p \le \infty$, we define

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : Du \in L^p(\Omega, \mathbb{R}^N)\},$$

with Du being the weak gradient of u, that is, there exist functionals $\frac{\partial u}{\partial z_k} \in L^p(\Omega)$ such that

$$Du = \left(\frac{\partial u}{\partial z_k}\right)_{k=1}^N \text{ and } \int_\Omega \frac{\partial u}{\partial z_k} h dz = -\int_\Omega u \frac{\partial h}{\partial z_k} dz \text{ for all } k = 1, \dots, N, \text{ all } h \in C_c^1(\Omega).$$

We equip $W^{1,p}(\Omega)$ with the norm

$$\|u\|_{1,p} = \|u\|_p + \|Du\|_p.$$

We can also use the equivalent norms

$$|u|_{1,p} = \left(\|u\|_p^p + \|Du\|_p^p\right)^{1/p} \text{ if } 1 \le p < \infty,$$

$$|u|_{1,\infty} = \max\{\|u\|_\infty, \|Du\|_\infty\} \text{ if } p = \infty.$$

Note that

$$W^{1,\infty}(\Omega) = C^{0,1}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is Lipschitz continuous}\}.$$

Inductively we can define higher order Sobolev spaces

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : Du \in W^{k-1,p}(\Omega, \mathbb{R}^N)\}\ k \ge 1,\ 1 \le p \le \infty.$$

Also, we set

$$W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,p}}, \quad 1 \le p \le \infty. \tag{3}$$

Remark 1. Since uniform convergence preserves continuity, functions in $W_0^{1,\infty}(\Omega)$ are necessarily $C^1(\Omega)$. This means that piecewise affine functions do not belong to $W_0^{1,\infty}(\Omega)$. This is the reason why many authors when $p = \infty$, instead of taking the $\|\cdot\|_{1,\infty}$-closure in (3), take the closure with respect to the w^*-topology on $W^{1,\infty}(\Omega)$.

In what follows, we restrict ourselves on Sobolev spaces $W_0^{1,p}(\Omega)$ with $1 \leq p < \infty$. We mention that

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : \gamma_0(u) = 0\},$$

with $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ being the trace operator (on $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure). We know that for all $u \in C^1(\overline{\Omega})$, $\gamma_0(u) = u|_{\partial\Omega}$. So, the trace operator gives meaning to the notion of boundary values for arbitrary Sobolev functions (not necessarily smooth). We know that $\gamma_0 \in \mathcal{L}(W^{1,p}(\Omega), L^p(\partial\Omega))$ and it is compact. Also using the trace operator, we interpret the boundary condition in problems (1) and (2).

Evidently, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ $(1 \leq p < \infty)$ are separable Banach spaces, which are reflexive (in fact, uniformly convex), if $1 < p < \infty$.

For $1 \leq p < \infty$, we define

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p = N. \end{cases}$$

The next theorem, known as the Rellich–Kondrachov embedding theorem, establishes useful embeddings between these spaces. In what follows, $X = W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$.

Theorem 1. *(a) If $1 \leq p < N$, then $X \hookrightarrow L^r(\Omega)$ continuously if $1 \leq r \leq p^*$, compactly if $1 \leq r < p^*$.*

(b) If $p = N$, then $X \hookrightarrow L^r(\Omega)$ compactly if $1 \leq r < \infty$.

(c) If $N < p$, then $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$, $W_0^{1,p}(\Omega) \hookrightarrow C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$, compactly.

Remark 2. The embedding $X \hookrightarrow L^{p^*}(\Omega)$ in (a) is never compact. Also, if Ω is not bounded, then $X \hookrightarrow L^p(\Omega)$ is not compact.

Using the above fundamental embedding theorem, we can deduce some useful equivalent norms on $W^{1,p}(\Omega)$.

Proposition 1. *We set $|u| = \|u\|_r + \|Du\|_p$ for all $u \in W^{1,p}(\Omega)$. Then this is an equivalent norm on $W^{1,p}(\Omega)$ in the following cases:*

(a) If $1 \leq r \leq p^*$ if $1 \leq p < N$.

(b) If $1 \leq r < \infty$ if $p = N$.

(c) If $1 \leq r \leq \infty$ if $N < p$.

For $W_0^{1,p}(\Omega)$, we can do much better, thanks to the so-called "Poincaré inequality".

Theorem 2. *There exists $c = c(p, N, \Omega) > 0$ $(1 \leq p < \infty)$ such that*

$$\|u\|_p \leq c\,\|Du\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

Therefore, $|u| = \|Du\|_p$ is an equivalent norm on $W_0^{1,p}(\Omega)$.

Remark 3. For Neumann problems, where the kernel of the differential operator is non-trivial (the subspace of constant functions), it is useful the so-called "Poincaré–Wirtinger inequality", which says that we can find $c > 0$ such that

$$\left\| u - \frac{1}{|\Omega|_N} \int_\Omega u dz \right\|_p \leq c \, \|Du\|_p \text{ for all } u \in W^{1,p}(\Omega), \ 1 \leq p < \infty,$$

with $|\cdot|_N$ being the Lebesgue measure on $\mathbb{R}^N$. Also, if $\beta \in L^\infty(\partial\Omega) \setminus \{0\}$ and $\beta(z) \geq 0$ for σ-a.a. $z \in \partial\Omega$, then

$$u \mapsto |u| = \left[\|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \right]^{1/p}$$

is an equivalent norm on $W^{1,p}(\Omega)$. This is useful in Robin problems.

As we already mentioned in the Introduction, since we deal with problems with unbalanced growth, we need to use generalized Orlicz spaces.

We assume that $a \in L^\infty(\Omega) \setminus \{0\}$, $a(z) \geq 0$ for a.a. $z \in \Omega$, $\frac{p}{q} < 1 + \frac{1}{N}$ (thus, $p < q^*$) and define

$$\eta(z, t) = a(z)t^p + t^q \text{ for all } z \in \Omega, \ \text{all } t \geq 0.$$

The generalized Lebesgue–Orlicz space $L^\eta(\Omega)$ is defined by

$$L^\eta(\Omega) = \left\{ u \in L^0(\Omega) : \rho_\eta(u) = \int_\Omega \eta(z, |u|) dz < \infty \right\}.$$

The function $\rho_\eta(\cdot)$ is known as the "modular function". We equip $L^\eta(\Omega)$ with the so-called "Luxemburg norm" defined by

$$\|u\|_\eta = \inf \left\{ \lambda > 0 : \rho_\eta\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

Normed this way, $L^\eta(\Omega)$ becomes a Banach space which is separable. In fact, it is uniformly convex (hence reflexive) because for all $z \in \Omega$, $\eta(z, \cdot)$ is a uniformly convex function.

Using $L^\eta(\Omega)$, we can define the corresponding generalized Sobolev–Orlicz space $W^{1,\eta}(\Omega)$ by

$$W^{1,\eta}(\Omega) = \{u \in L^\eta(\Omega) : |Du| \in L^\eta(\Omega)\},$$

with Du being the weak gradient of u. We equip $W^{1,\eta}(\Omega)$ with the norm

$$\|u\|_{1,\eta}(\Omega) = \|u\|_\eta + \|Du\|_\eta \text{ for all } u \in W^{1,\eta}(\Omega),$$

where $\|Du\|_\eta = \| \, |Du| \, \|_\eta$. Also we define

$$W_0^{1,\eta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\eta}}.$$

The spaces $W^{1,\eta}(\Omega)$ and $W_0^{1,\eta}(\Omega)$ are Banach spaces which are separable and reflexive (in fact, uniformly convex). Moreover, on $W_0^{1,\eta}(\Omega)$ the Poincaré inequality holds, namely there exists $\hat{c} > 0$ such that

$$\|u\|_\eta \leq \hat{c}\|Du\|_\eta \text{ for all } u \in W_0^{1,\eta}(\Omega).$$

This means that on $W_0^{1,\eta}(\Omega)$ we can consider the equivalent norm

$$\|u\|_\eta = \|Du\|_\eta \text{ for all } u \in W_0^{1,\eta}(\Omega).$$

We are primarily interested on $W_0^{1,\eta}(\Omega)$ because we are considering Dirichlet problems (see (2)). The next proposition shows that there is a close relationship between the modular function $\rho_\eta(\cdot)$ and the norm $\|\cdot\|$ given above.

Proposition 2. *(a) If $u \in W^{1,\eta}(\Omega) \setminus \{0\}$, then $\|u\| = \theta \Leftrightarrow \rho_\eta(Du) \le 1$.*
(b) $\|u\| < 1$ (resp. $= 1, > 1$) $\Leftrightarrow \rho_\eta(Du) < 1$ (resp., $= 1, > 1$).
(c) $\|u\| < 1 \Rightarrow \|u\|^p \le \rho_\eta(Du) \le \|u\|^q$.
(d) $\|u\| > 1 \Rightarrow \|u\|^q \le \rho_\eta(Du) \le \|u\|^p$.
(e) $\|u\| \to 0$ (resp., $\to +\infty$) $\Leftrightarrow \rho_\eta(Du) \to 0$ (resp., $\to +\infty$).

There are useful embeddings between the generalized Orlicz spaces analogous to the Rellich–Kondrachov theorem (see Theorem 1).

Proposition 3. *We have the following properties.*
(a) $L^\eta(\Omega) \hookrightarrow L^r(\Omega)$ and $W_0^{1,\eta}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ continuously for all $1 \le r \le q$.
(b) $W_0^{1,\eta}(\Omega) \hookrightarrow L^r(\Omega)$ continuously if $1 \le r \le q^*$ and compactly if $1 \le r < q^*$.
Also, if we consider the linear space

$$L_a^p(\Omega) = \left\{ u \in L^0(\Omega) : \int_\Omega a(z)|u|^p dz < \infty \right\}$$

equipped with the seminorm

$$|u| = \left[\int_\Omega a(z)|u|^p dz \right]^{1/p},$$

then $L^\eta(\Omega) \hookrightarrow L_a^p(\Omega)$ and $L^p(\Omega) \hookrightarrow L^\eta(\Omega)$ continuously.

By a "strict weight", we mean a function $\hat{a} \in L^0(\Omega)$ such that

$$0 < \hat{a}(z) < \infty \text{ for a.a. } z \in \Omega.$$

A strict weight $\hat{a}(\cdot)$ belongs to the "p-Muckenhoupt class" denoted by $\hat{a} \in \mathcal{A}_p$, if

$$[\hat{a}]_{\mathcal{A}_p} = \sup_Q \left(\frac{1}{|Q|_N} \int_Q \hat{a}(z) dz \right) \left(\frac{1}{|Q|_N} \int_Q \hat{a}(z)^{1-p'} dz \right)^{p-1} < \infty,$$

where the supremum is taken over the cubes with sides parallel to the coordinate axes (recall that $p' = p/(p-1)$ and $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$). If $\hat{a} \in \mathcal{A}_p$, then the averaging operator

$$\hat{A}_Q u(z) = \frac{1}{|Q|_N} \int_Q u(z') dz' \chi_Q(z)$$

is uniformly bounded on $L^{\hat{\eta}_0}(\Omega)$, with $\hat{\eta}_0(z,t) = \hat{a}(z)t^p$, $z \in \Omega$, $t \ge 0$.

Now let $a \in C^{0,1}(\overline{\Omega}) \cap \mathcal{A}_p$, $a(z) > 0$ for all $z \in \Omega$. Set $\eta_0(z,t) = a(z)t^p$ for all $z \in \Omega$, all $t \ge 0$. We consider the corresponding generalized Orlicz spaces $L^{\eta_0}(\Omega)$ and $W_0^{1,\eta_0}(\Omega)$. Both are Banach spaces which are separable and reflexive (in fact, uniformly convex, since $\eta_0(z,\cdot)$ is uniformly convex, see Harjulehto and Hästo [16, pp.63,66]). We know that

$$W_0^{1,\eta_0}(\Omega) \hookrightarrow L^{\eta_0}(\Omega) \text{ compactly}$$

(see Papageorgiou, Rădulescu and Zhang [36, Lemma 2]). This fact leads to a detailed spectral analysis of the operator $(-\Delta_p^a, W_0^{1,\eta_0}(\Omega))$ (see [32]).

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. For $k \in \mathbb{N}$, we denote by $C^k(\overline{\Omega})$ the space of all functions $u \in C^k(\Omega)$ such that for every multiindex $\alpha = (\alpha_m)_{m=1}^N$ with length $|\alpha| = \sum_{m=1}^N \alpha_m \le k$, $D^\alpha u(\cdot)$ is bounded and uniformly continuous on Ω, hence it admits a unique continuous extension on $\overline{\Omega}$. Also we set $C^\infty(\overline{\Omega}) = \cap_{k\in\mathbb{N}} C^k(\overline{\Omega})$. For $\theta \in [0, 1]$, $u \in C^k(\Omega)$ and $|\alpha| \le k$, we define

$$\hat{h}_{\alpha,\theta}(u) = \sup\left\{ \frac{|D^\alpha u(z) - D^\alpha u(z')|}{|z - z'|^\theta} : z, z' \in \Omega,\ z \neq z' \right\}.$$

We denote by $C^{k,\theta}(\overline{\Omega})$ the subspace of all functions $u \in C^k(\overline{\Omega})$ such that $\hat{h}_{\alpha,\theta}(u) < \infty$ for all multiindices α such that $|\alpha| = k$. If $\alpha \equiv 0$, then we obtain the Hölder continuous functions satisfying

$$|u(z) - u(z')| \le c|z - z'|^\theta \text{ for all } z, z' \in \Omega, \text{ some } c = c(u) > 0. \tag{4}$$

If $\theta = 1$, we have the Lipschitz continuous functions. We denote the space of all Hölder continuous functions by $C^{0,\theta}(\overline{\Omega})$. A function satisfying relation (4) with $\theta > 1$ is a constant. The Arzela–Ascoli theorem says that if $k \in \mathbb{N}_0$ and $0 < \gamma < \theta \le 1$, then

$$C^{k,\theta}(\overline{\Omega}) \hookrightarrow C^{k,\gamma}(\overline{\Omega}) \hookrightarrow C^k(\overline{\Omega}) \text{ compactly.}$$

In general it is not true that $C^{k+1}(\overline{\Omega}) \hookrightarrow C^{k,\theta}(\overline{\Omega})$ continuously. We need extra conditions on the domain Ω (for example, if Ω is star-shaped).

Finally, we mention that we denote by $L^p_{loc}(\Omega)$ and $W^{1,p}_{loc}(\Omega)$ the corresponding local Lebesgue and Sobolev spaces. So, $u \in L^p_{loc}(\Omega)$ (resp., $u \in W^{1,p}_{loc}(\Omega)$), if $u \in L^p(K)$ for all compact $K \subseteq \Omega$ (resp., if $u \in W^{1,p}(\Omega')$ for all $\Omega' \subseteq \Omega$ open with $\overline{\Omega'}$ compact if Ω is unbounded).

For the standard Lebesgue and Sobolev spaces, we refer to the books of Adams and Fournier [1], Brezis [5], Papageorgiou, Rădulescu and Repovš [33], Papageorgiou and Winkert [38]. For the generalized Orlicz spaces, we refer to the book of Harjulehto and Hästo [16], and the survey papers of Papageorgiou [31] and Rădulescu [43]. Finally, for more information on the spaces of continuous and differentiable functions, the interested reader can consult the book of Pick, Kufner, John and Fučik [41].

3. Nonlinear regularity theory

The regularity theory for semilinear problems is presented in detail in the books of Gilbarg and Trudinger [13], Han and Lin [15] and also Beck [3] (for vectorial problems). For the nonlinear theory, to the best of our knowledge, there is no book containing the theory and the reader has to consult the original papers. Here we will outline the main ideas and results of the nonlinear regularity, both local and global (that is, up to the boundary of Ω).

We will first consider problem (1) (balanced growth problem). To facilitate the exposition of the main ideas and results, we will start with the simple case of a p-Laplacian equation with a fixed forcing term (right-hand side). So, we consider the following Dirichlet problem

$$\begin{Bmatrix} -\Delta_p u(z) = h(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ \ 1 < p < \infty. \end{Bmatrix} \tag{5}$$

Recall that $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. We say that $u \in W_0^{1,p}(\Omega)$ is a (weak) solution of problem (5) if

$$\int_\Omega |Du|^{p-2}(Du, D\theta)_{\mathbb{R}^N} dz = \int_\Omega h\theta dz \text{ for all } \theta \in W_0^{1,p}(\Omega).$$

Then for a solution $u \in W_0^{1,p}(\Omega)$ of problem (5), we have the following regularity implications:

(a) if $h \in L^r(\Omega)$ with $r > N/p$, then $u \in L^\infty(\Omega)$;

(b) if $h \in L^r(\Omega)$ with $r > N$, then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ (see Guedda and Véron [14], Lieberman [19]).

Let us now replace h by a function depending on u (for simplicity, we drop the z-dependence). So, the problem under consideration is the following

$$\begin{Bmatrix} -\Delta_p u(z) = f(u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ \ 1 < p < \infty. \end{Bmatrix} \tag{6}$$

Therefore in (5) we replace $h(\cdot)$ by $f(u(\cdot))$, which gives us more information. Then for (6) we can say the following:

(a) If $|f(x)| \le c_1(1 + |x|^{r-1})$ for some $c_1 > 0$, all $x \in \mathbb{R}$, with $p \le r < p^*$, then every (weak) solution $u \in W_0^{1,p}(\Omega)$ of (6) belongs to $L^\infty(\Omega)$.

(b) For every weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have

$$u \in C^{1,\alpha}(\overline{\Omega}) \text{ with } \alpha \in (0, 1).$$

Remark 4. In (a) we see that $f(\cdot)$ needs to have subcritical growth, namely $r < p^*$.

From the above facts we see that the nonlinear regularity theory has two steps. First we show the boundedness of the weak solution and then, this property leads to the Hölder regularity up to the boundary $\partial\Omega$.

So, first we try to show the boundedness of the weak solution. We will do this for a general class of problems, which have gradient dependent reaction (convection). We consider the following general Dirichlet problem

$$\begin{Bmatrix} -\operatorname{div} a(z, u, Du) = b(z, u, Du) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{Bmatrix} \tag{7}$$

We introduce the following regularity and growth conditions on the two functions a, b.

H_1: $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions such that

(i) $|a(z, x, y)| \le c_1(|x|^{p^* \frac{p-1}{p}} + |y|^{p-1}) + \mu(z)$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, all $y \in \mathbb{R}^N$, with $c_1 > 0$, $\mu \in L^{p'}(\Omega)$ with $1 < p < N$;

(ii) $c_2|y|^p \le (a(z, x, y), y)_{\mathbb{R}^N}$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, all $y \in \mathbb{R}^N$, some $c_2 > 0$;

(iii) $|b(z, x, y)| \leq c_3(|x|^{p-1} + |y|^{p-1}) + \hat{\mu}(z)$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, all $y \in \mathbb{R}^N$, with $c_3 > 0$, $\hat{\mu} \in L^r(\Omega)$, $r > \frac{N}{p}$.

Remark 5. In fact, the growth condition on the reaction $b(z, x, y)$ can be more general and we can assume that

$$|b(z, x, y)| \leq c_3(|x|^{p^*-1} + |y|^{p\frac{p^*-1}{p^*}} + 1)$$

for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, all $y \in \mathbb{R}^N$, some $c_3 > 0$ (see Ho and Winkert [17]). However, for the purpose of simplifying the presentation of the Moser iteration technique which we illustrate below, we decided to use the more restrictive growth condition $H_1(iii)$.

By a (weak) solution of problem (7), we mean a function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega (a(z, u, Du), D\theta)_{\mathbb{R}^N} dz = \int_\Omega b(z, u, Du)\theta dz \text{ for all } \theta \in W_0^{1,p}(\Omega).$$

We will show the boundedness of u. To do this, we shall use the Moser iteration technique (see Moser [29]). An alternative approach can be found in Ladyzhenskaya and Uraltseva [18, Theorem 7.1, p.286].

Proposition 4. *If hypotheses H_1 hold and $u \in W_0^{1,p}(\Omega)$ is a weak solution of (7), then $\|u\|_\infty \leq M$ with $M(\Omega, N, p, \|u\|, c_2, c_3, \|\hat{\mu}\|_r) > 0$.*

Proof. Given that $u = u^+ - u^-$ and $u^+, u^- \geq 0$, we may assume that $u \geq 0$. Let $k \geq 1$, $c > 1$ and $u_c = \min\{u, c\}$. We use the test function $h = uu_c^{kp} \in W_0^{1,p}(\Omega)$. In what follows, we work on $\Omega_1 = \{z \in \Omega : u(z) > 1\}$ and have

$$\begin{aligned} &\int_{\Omega_1} (a(z, u, Du), Du)u_c^{kp} dz + kp \int_{\Omega_1} (a(z, u, Du), Du_c)uu_c^{kp-1} dz \leq \\ &\int_{\Omega_1} c_3(u^{p-1} + |Du|^{p-1} + \hat{\mu}(z))uu_c^{kp} dz. \end{aligned} \tag{8}$$

Using hypothesis $H_1(ii)$, we obtain

$$c_2 \int_{\Omega_1} |Du|^p u_c^{kp} dz \leq \int_{\Omega_1} (a(z, u, Du), Du)u_c^{kp} dz. \tag{9}$$

Since $Du_c = Du$ if $u \leq c$ and $Du_c = 0$ if $u > c$, we see that

$$c_2 kp \int_{\Omega_1} |Du|^p u_c^{kp} dz \leq kp \int_{\Omega_1 \cap \{u \leq c\}} (a(z, u, Du), Du_c)_{\mathbb{R}^N} uu_c^{kp} dz. \tag{10}$$

Now we will estimate the right-hand side of (8). We have the following estimates

$$\begin{aligned} &\int_{\Omega_1} |Du|^{p-1} uu_c^{kp} dz = \int_{\Omega_1} |Du|^{p-1} u_c^{k(p-1)} uu_c^k dz \leq \\ &\varepsilon \int_{\Omega_1} |Du|^p u_c^{kp} dz + c_\varepsilon \int_{\Omega_1} (uu_c^k)^p dz \text{ (using Young's inequality with } \varepsilon > 0), \end{aligned} \tag{11}$$

$$\int_{\Omega_1} u^p u_c^{kp} dz = \int_{\Omega_1} (uu_c^k)^p dz \tag{12}$$

and

$$\begin{aligned}\int_{\Omega_1} \hat{\mu} u u_c^{kp} dz &\le \int_{\Omega_1} |\hat{\mu}|(uu_c^k)^p dz \text{ (since } u(z) > 1 \text{ on } \Omega_1) \\ &\le \|\hat{\mu}\|_r \left(\int_{\Omega_1} (uu_c^k)^{pr'} dz\right)^{1/r'} \\ &\text{(since } r > \frac{N}{p} \Rightarrow r' < \left(\frac{N}{p}\right)' = p^*, \text{ see Theorem 1).}\end{aligned} \tag{13}$$

Using relations (9)–(13) in (8), we have

$$\begin{aligned}& c_2 \left[\int_{\Omega_1} |Du|^p u_c^{kp} dz + kp \int_{\Omega} |Du_c|^p u_c^{kp} dz\right] \\ \le \; & c_3 \varepsilon \int_{\Omega_1} |Du|^p u_c^{kp} dz + c_3(c_3+1)\|uu_c^k\|_p^p + \|\hat{\mu}\|_r \|uu_c^k\|_{pr'}^p.\end{aligned}$$

Choosing $\varepsilon > 0$ small, we obtain

$$\begin{aligned}& \int_{\Omega_1} |Du|^p u_c^{kp} dz + kp \int_{\Omega} |Du_c|^p u_c^{kp} dz \\ \le \; & c_4 \left[\|uu_p^k\|_p^p + \|\hat{\mu}\|_r \|uu_c^k\|_{pr'}^p\right] \text{ for some } c_4 > 0.\end{aligned} \tag{14}$$

We observe that

$$\begin{aligned}\frac{kp+1}{(k+1)^p}|D(uu_c^k)|^p &\le \frac{kp+1}{(k+1)^p}\left(u_c^{kp}|Du|^p + k^p u^p u_c^{(k-1)p}|Du_c|^p\right) \\ &\le u_c^{kp}|Du|^p + kpu_c^{kp}|Du_c|^p \text{ (since } Du_c = 0 \text{ on } \{u > c\}).\end{aligned} \tag{15}$$

Using (15) in (14), we infer that for some $\hat{c} = \hat{c}(\Omega, N, p) > 0$ we have

$$\begin{aligned}& \hat{c}\frac{kp+1}{(k+1)^p}\|uu_c^k\|_{p^*}^p \\ \le \; & \frac{kp+1}{(k+1)^p}\|D(uu_c^k)\|_p^p \text{ (use Theorems 1 and 2)} \\ \le \; & \int_{\Omega} |Du|^p u_c^{kp} dz + kp \int_{\Omega} |Du_c|^p u_c^{kp} dz, \\ \Rightarrow \; & \frac{kp+1}{(k+1)^p}\|uu_c^k\|_{p^*}^p \le c^* \left[\|uu_c^k\|_p^p + \|uu_c^k\|_{pr'}^p\right]\end{aligned}$$

for some $c^* = c^*(\Omega, N, p, \|\hat{\mu}\|_r, c_2, c_3) > 0$.

It follows that

$$\|uu_c^k\|_{p^*} \leq (c^*)^{1/p} \frac{k+1}{(kp+1)^{1/p}} \left[\|uu_c^k\|_p + \|uu_c^k\|_{pr'}\right]. \tag{16}$$

From hypothesis $H_1(iii)$ we know that $r > \frac{N}{p}$, hence $r' < \left(\frac{N}{p}\right)' = \frac{N}{N-p}$ (we recall that for all $q \in (1,\infty)$, $q' = \frac{q}{q-1}$). Therefore $p < pr' < p^* = \frac{Np}{N-p}$ and so we can find $\tau \in (0,1)$ such that

$$\frac{1}{pr'} = \frac{1-\tau}{p} + \frac{\tau}{p^*}. \tag{17}$$

Using the interpolation inequality (see Papageorgiou and Winkert [38, Proposition 2.3.17, p.116]), we have

$$\|uu_c^k\|_{pr'} \leq \|uu_c^k\|_p^{1-\tau} \|uu_c^k\|_{p^*}^{\tau} \leq \gamma \|uu_c^k\|_{p^*}^{\tau\beta} + \xi_\gamma \|uu_c^k\|_p^{(1-\tau)\beta'}, \tag{18}$$

with $\gamma \in (0,1)$ and $\beta > 1$ (using Young's inequality). We choose $\beta > 1$ such that $\tau\beta = 1$. From (17) we have $\tau = \frac{N}{rp}$ and so $\beta = \frac{rp}{N}$. Also, $\xi_\gamma = \gamma^{N/(N-rp)} > 1$ (since $N < rp$) and $(1-\tau)\beta' = 1$. Using these facts in (18), we obtain

$$\|uu_c^k\|_{pr'} \leq \gamma \|uu_c^k\|_{p^*} + \gamma^{N/(N-rp)} \|uu_c^k\|_p. \tag{19}$$

We use (19) in (16) and have

$$\begin{aligned} \|uu_c^k\|_{p^*} &\leq (c^*)^{1/p} \frac{k+1}{(kp+1)^{1/p}} \left[(1+\gamma^{N/(N-rp)})\|uu_c^k\|_p + \gamma \|uu_c^k\|_{p^*}\right] \\ &\leq (c^*)^{1/p} \frac{k+1}{(kp+1)^{1/p}} \left[2\gamma^{N/(N-rp)} \|uu_c^k\|_p + \gamma \|uu_c^k\|_{p^*}\right]. \end{aligned} \tag{20}$$

Let

$$\gamma = \frac{1}{2(c^*)^{1/p}} \frac{(kp+1)^{1/p}}{k+1} < 1$$

and assume without loss of generality that $c^* > 2^{-p}$. Returning to (20) and using this choice of γ, we obtain

$$\|uu_c^k\|_{p^*} \leq 2^{\frac{N}{rp-N}+2} \left((c^*)^{\frac{1}{p}} \frac{k+1}{(k+1)^{\frac{1}{p}}}\right)^{\frac{rp}{rp-N}} \|uu_c^k\|_p. \tag{21}$$

Now let $c \to \infty$. Using Fatou's lemma and the monotone convergence theorem, we obtain

$$\|u\|_{(k+1)p^*} \leq (c^*)^{\frac{\sigma_k}{p}} \zeta_k^{\delta} \|u\|_{(k+1)p},$$

where

$$\sigma_k = \frac{1}{k+1}, \quad \zeta_k = \left(\frac{k+1}{(kp+1)^{\frac{1}{p}}}\right)^{\frac{1}{k+1}}, \quad \delta = \frac{rp}{rp-N}.$$

We perform a bootstrap argument. So, let $\{k_n\}_{n\in\mathbb{N}}$ be such that

$$(k_1+1)p = p^* \text{ and } (k_{n+1}+1)p = (k_n+1)p^* \text{ for all } n \in \mathbb{N}. \tag{22}$$

Then from (21) with $k = k_1$, we have

$$\|u\|_{(k_1+1)p^*} \le (c^*)^{\frac{\sigma_{k_1}}{p}} \zeta_{k_1}^{\delta} \|u\|_{p^*} < \infty.$$

Inductively we obtain

$$\begin{aligned} & \|u\|_{(k_n+1)p^*} \le (c^*)^{\frac{\sigma_{k_n}}{p}} \zeta_{k_n}^{\delta} \|u\|_{p^*} \text{ for all } n \in \mathbb{N}, \\ \Rightarrow \quad & \|u\|_{(k_n+1)p^*} \le (c^*)^{\hat{\sigma}} \left(\prod_{m=1}^{n} \zeta_{k_m} \right)^{\delta} \|u\|_{p^*} \text{ for all } n \in \mathbb{N}, \end{aligned} \tag{23}$$

with $\hat{\sigma} = \frac{1}{p} \sum_{m=1}^{n} \sigma_{k_m}$. Note that

$$\sum_{m \in \mathbb{N}} \sigma_{k_m} \le \sum_{m \in \mathbb{N}} \frac{1}{k_m} < \infty \text{ and } \prod_{m \in \mathbb{N}} \zeta_{k_m} < \infty.$$

Therefore if we pass to the limit as $k \to \infty$, we infer that

$$\begin{aligned} & 0 \le u(z) \le M \text{ for a.a. } z \in \Omega_1, \\ \Rightarrow \quad & \|u\|_\infty \le M \text{ with } M = M(\Omega, N, p, \|\hat{\mu}\|_r, c_2, c_3). \end{aligned}$$

The proof is now complete. □

A careful reading of the above proof reveals that it can be extended to problems with a singular reaction, namely $b : \Omega \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$ satisfies

$$|b(z, x, y)| \le c_3(x^{p-1} + |y|^{p-1} + x^{-\eta}) + \hat{\mu}(z)$$

for a.a. $z \in \Omega$, all $x \ge 0$, all $y \in \mathbb{R}^N$, with $\hat{\mu} \in L^r(\Omega)$, where $r > N/p$. Then again we have that the weak solution of the singular problem is bounded.

A useful consequence of this theorem is the following result on (p, q)-equations. So, consider the following Dirichlet problem

$$\left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) = f(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ 1 < q < p < N. \end{array} \right\} \tag{24}$$

Proposition 5. *If $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem (24) and $f \in L^r(\Omega)$ with $r > \frac{N}{p}$, then $u \in L^\infty(\Omega)$ and $\|u\|_\infty \le c\|f\|_r^{\frac{1}{p-1}}$ with $c = c(\Omega, N, p, r) > 0$.*

Also, a similar Moser iteration establishes the boundedness of a weak solution of the double phase problem. So, consider the following problem

$$\left\{ \begin{array}{l} -\Delta_p^a u(z) - \Delta_q u(z) = b(z, u(z), Du(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ 1 < q < p < N. \end{array} \right\} \tag{25}$$

We assume that

(i) $a \in C^{0,1}(\overline{\Omega}) \setminus \{0\}$, $a(z) \ge 0$ for a.a. $z \in \Omega$ and $\frac{p}{q} < 1 + \frac{1}{N}$.

(ii) $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

$$|b(z, x, y)| \le c(|x|^{r-1} + |y|^{\frac{p}{r'}} + 1)$$

for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, all $y \in \mathbb{R}^N$, with $p < r < q^* = \frac{Nq}{N-q}$.

Proposition 6. *Under the above hypotheses, every weak solution $u \in W_0^{1,\eta}(\Omega)$ of problem (25) belongs to $L^\infty(\Omega)$.*

The result is due to Ho and Winkert [17] (Theorem 4.2). In fact, the result in [17] is stated for double phase equations with variable exponents.

We can also have the counterpart of Proposition 5. So, consider the following double phase problem

$$\begin{Bmatrix} -\Delta_p^a u(z) - \Delta_q u(z) = f(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ \ 1 < q < p < N. \end{Bmatrix} \tag{26}$$

As above, we assume that

$$a \in C^{0,1}(\overline{\Omega}) \setminus \{0\}, \ \ a(z) \geq 0 \text{ for a.a. } z \in \Omega \text{ and } \frac{p}{q} < 1 + \frac{1}{N}.$$

We also assume that $f \in L^r(\Omega)$, $r > \frac{N}{p}$. Then as in Proposition 1.3 of Guedda and Véron [14] (p-Laplacian equations) and as in Proposition 2.4 of Perera and Squassina [40] (double phase equations) we obtain the following property.

Proposition 7. *Every weak solution $u \in W_0^{1,\eta}(\Omega)$ of problem (26) belongs to $L^\infty(\Omega)$ and*

$$\|u\|_\infty \leq c\|f\|_r^{\frac{1}{q-1}}$$

with $c = c(\Omega, N, p, r) > 0$.

Local Lipschitz regularity of the solutions of certain problems with nonstandard growth was proved by Lieberman [20]. However, his proof had a gap. A correct proof was produced by Bousquet and Brasco [4]. There is no global regularity theory for double phase problems. There are only local regularity results, see Cupini, Marcellini and Mascolo [8], Marcellini [26], Mingione and Rădulescu [28], and for systems by Di Marco and Marcellini [9].

From Proposition 4, we see that if the solutions of (7) are bounded in $W_0^{1,p}(\Omega)$, then they are also bounded in $L^\infty(\Omega)$.

Now that we have boundedness of the solutions, we can start thinking about Hölder regularity. We start with a local regularity result for equations driven by the p-Laplacian (see Di Benedetto [10] and Tolksdorf [44]). So, we consider the following problem

$$-\Delta_p u(z) = f(z) \text{ in } \Omega. \tag{27}$$

By a solution of problem (27) we mean a function $u \in W_{loc}^{1,p}(\Omega)$ such that

$$\int_\Omega |Du|^{p-2}(Du, D\theta)_{\mathbb{R}^N} dz = \int_\Omega f(z)\theta(z) \text{ for all } \theta \in W_0^{1,p}(\Omega) \text{ with compact support.}$$

This solution is also known as a "weak solution".

Proposition 8. *If $u \in W_{loc}^{1,p}(\Omega) \cap L_{loc}^\infty(\Omega)$ is a distributional solution of problem (27) and $f \in L_{loc}^r(\Omega)$ with $r > N$, then $u \in C_{loc}^{1,\alpha}(\Omega)$ with $\alpha \in (0, 1)$.*

For global (up to the boundary) regularity of the weak solutions of problem (1), we need to strengthen the conditions on the map $a(z, y)$. So, let $\theta \in C^1(0, \infty)$ such that

$\theta(t) > 0$ for all $t > 0$ and

$$0 < c^* \leq \frac{t\theta'(t)}{\theta(t)} \leq \hat{c} \text{ and } c_1 t^{p-1} \leq \theta(t) \leq c_2(t^{s-1}+t^{p-1}) \text{ with } 1 < s < p,\ c_1,\ c_2 > 0. \tag{28}$$

Then the hypotheses on the map $a(z, y)$ defining the differential operator in problem (1), are the following:

H_2: $a(z, y) = a_0(z, |y|)y$ with $t \mapsto a_0(z, t)t$ strictly increasing, $\lim_{t\to 0^+} ta_0(z, t) = 0$ for all $z \in \overline{\Omega}$ and

(i) $a \in C(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$;

(ii) $|\nabla_y a(z, y)| \leq c_3 \frac{\theta(|y|)}{|y|}$ for all $z \in \overline{\Omega}$, all $y \in \mathbb{R}^N \setminus \{0\}$, some $c_3 > 0$;

(iii) $(\nabla_y a(z, y)\xi, \xi)_{\mathbb{R}^N} \geq \frac{\theta(|y|)}{|y|}|\xi|^2$ for all $z \in \Omega$, all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$;

(iv) $|a(z, y) - a(z', y)| \leq c_4(1 + \theta(|y|))|z - z'|^\mu$ for all $z, z' \in \Omega$, all $y \in \mathbb{R}^N$, some $c_4 > 0$ and $\mu \in (0, 1)$.

The above hypotheses lead to the following properties of the map $a(z, y)$.

(a) For all $z \in \overline{\Omega}$, $a(z, \cdot)$ is continuous, strictly monotone, thus maximal monotone;

(b) $|a(z, y)| \leq c_5(1 + |y|^{p-1})$ for all $z \in \overline{\Omega}$, all $y \in \mathbb{R}^N$, some $c_5 > 0$;

(c) $(a(z, y), y)_{\mathbb{R}^N} \geq \frac{c_6}{p-1}|y|^p$ for all $z \in \overline{\Omega}$, all $y \in \mathbb{R}^N$.

The prototype operator satisfying hypotheses H_2 is the weighted (p, q)-Laplace operator defined by

$$u \mapsto \Delta_p^a u + \beta\Delta_q u \text{ with } 1 < q < p,\ \beta \geq 0,$$

where the weight $a \in C^{0,\mu}(\overline{\Omega})$ and $0 < \hat{c}_0 \leq a(z)$ for all $z \in \overline{\Omega}$. If $\beta = 0$, we have the weighted p-Laplacian.

We consider $G_0(z, t) = \int_0^t a_0(s)sds$ for all $t \geq 0$ and then define $G(z, y) = G_0(z, |y|)$ for all $z \in \overline{\Omega}$, all $y \in \mathbb{R}^N$. We see that

$$\nabla_y G(z, y) = a(z, y) \text{ for all } (z, y) \in \overline{\Omega} \times \mathbb{R}^N,$$

that is, $G(z, y)$ is the primitive of $a(z, y)$.

The conditions on the reaction $f(z, x)$ of problem (1) are the following:

H_3: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f(z, x)| \leq \hat{a}(z)(1 + |x|^{r-1})$$

for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\hat{a} \in L^\infty(\Omega)$, $p \leq r < p^*$.

As before, $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1) if

$$\int_\Omega (a(z, Du), D\theta)_{\mathbb{R}^N} dz = \int_\Omega f(z, u)\theta dz \text{ for all } \theta \in W_0^{1,p}(\Omega).$$

The following global regularity result is due to Lieberman [19].

Theorem 3. *If hypotheses H_2 and H_3 hold and $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a weak solution of problem (1), then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.*

Consider the problem

$$\begin{cases} -\operatorname{div} a(z, Du(z)) = f(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{29}$$

For this problem we have the following global regularity result.

Proposition 9. *If hypotheses H_2 hold, $f \in L^r(\Omega)$ with $r > N$ and $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem (29), then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and*

$$\|Du\|_{C^{0,\alpha}(\overline{\Omega})} \le c\|f\|_r^{1/(p-1)} \text{ with } c > 0.$$

We can have a global regularity theorem also for singular problems. We consider the following singular problem

$$\begin{cases} -\operatorname{div} a(z, Du(z)) = k(z) + f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0, \ 0 < \eta < 1, \ c > 0. \end{cases} \tag{30}$$

In this problem it is assumed to have a singularity at $x = 0$. More precisely, we assume that $f(z, x)$ is a Carathéodory function such that for every $\rho > 0$, there exists $\hat{a}_\rho \in L^\infty(\Omega)$ such that

$$|f(z, x)| \le \hat{a}(z) \text{ for a.a. } z \in \Omega, \text{ all } |x| \le \rho, \tag{31}$$

while the term $k(\cdot)$ which corresponds to the singularity, satisfies

$$0 \le k(z) \le c\hat{d}(z)^{-\eta} \text{ for all } z \in \Omega, \tag{32}$$

with $c > 0$, $\eta \in (0, 1)$ and $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \Omega$.

By a weak solution of problem (30) we mean a function $u \in W_0^{1,p}(\Omega)$ such that

$$k\theta \in L^1(\Omega) \text{ for all } \theta \in W_0^{1,p}(\Omega)$$

and

$$\int_\Omega (a(z, Du), D\theta)_{\mathbb{R}^N} dz = \int_\Omega k(z)\theta(z)dz + \int_\Omega f(z, u)\theta dz \text{ for all } \theta \in W_0^{1,p}(\Omega).$$

The next theorem establishes global Hölder regularity for the weak solutions of problem (30) and is due to Giacomoni, Kumar and Sreenadh, see [12, Theorem 1.7].

Theorem 4. *If hypotheses H_2 hold, relations (31) and (32) are satisfied and $u \in W_0^{1,p}(\Omega)$ is a weak solution of (30), then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le M$$

with $M = M(\Omega, N, p, \|u\|_\infty, \eta) > 0$.

The space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$$

is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega} < 0 \right\},$$

where $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward normal on $\partial\Omega$.

Usually $k(z) = cu(z)^{-\eta}$ and we satisfy condition (32) by using the solution of an auxiliary problem. More precisely, we consider the purely singular problem

$$\left\{ \begin{array}{l} -\operatorname{div} a(z, Du(z)) = cu(z)^{-\eta} \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ u > 0. \end{array} \right\} \tag{33}$$

Using the regularization technique (originally due to Crandall, Rabinowitz and Tartar [7]), we show that problem (33) has a unique solution $\overline{u} \in \operatorname{int} C_+$. Hence there exists $\hat{c} > 0$ such that $\hat{c}\hat{d} \le \overline{u}$. Also, if $f \ge 0$, then we can show that every weak solution u of (30) satisfies $\hat{u} \le u$. Therefore $u \ge c\hat{d}$ for some $c > 0$ and we have satisfied condition (32) (see Papageorgiou, Rădulescu and Repovs [34] and Papageorgiou, Rădulescu and Zhang [35]).

Using a standard argument, which flattens $\partial\Omega$ and uses a partition of unit, we show that

$$\hat{d}^{-\eta} \in L^s(\Omega) \text{ for all } s \in [1, \frac{1}{\eta}).$$

In this direction, helpful is the following "Hardy's inequality", see Brezis [5, p. 313].

Proposition 10. *If $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$ and $1 < p < \infty$, then*

$$\left\| \frac{u}{\hat{d}} \right\|_p \le c\|Du\| \textit{ for all } u \in W_0^{1,p}(\Omega),\ \textit{some } c > 0.$$

Conversely,

$$u \in W^{1,p}(\Omega) \textit{ and } \frac{u}{\hat{d}} \in L^p(\Omega) \Rightarrow u \in W_0^{1,p}(\Omega).$$

In fact, we can generalize the first part of the above proposition as follows.

(A): If $1 < p < N$, $\tau \in (0, 1)$ and $\frac{1}{r} = \frac{1}{p} - \frac{1-\tau}{N}$, then

$$\left\| \frac{u}{\hat{d}} \right\|_r \le c\|Du\|_p \text{ for all } u \in W_0^{1,p}(\Omega),\ \text{some } c > 0.$$

If we want to extract information about the pointwise behavior of $u(\cdot)$ near the boundary, we can use the following version of the Hardy inequality:

(B): If $u \in C^{1,\alpha}(\overline{\Omega}) \cap C_0^1(\overline{\Omega})$ with $0 < \alpha \le 1$, then $\frac{u}{\hat{d}} \in C^{0,\beta}(\overline{\Omega})$ with $\beta = \frac{\alpha}{1+\alpha}$ and

$$\left\| \frac{u}{\hat{d}} \right\|_{C^{0,\beta}(\overline{\Omega})} \le c\|u\|_{C^{1,\alpha}(\overline{\Omega})} \text{ for some } c > 0.$$

For more on the Hardy inequality, we refer to the book of Opic and Kufner [30].

4. Maximum and comparison principles

In this section we present some results which are helpful in extracting qualitative information for the solutions of a boundary value problem.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and consider the following differential inequality

$$\operatorname{div} a(z, Du) + b(z, u, Du) \leq 0 \text{ in } \Omega. \tag{34}$$

We assume that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $b : \Omega \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$ satisfy the following structure conditions:

$$(a(z, y), y)_{\mathbb{R}^N} \geq c_1 |y|^p \text{ for all } z \in \Omega, \text{ all } y \in \mathbb{R}^N,$$
$$\text{some } c_1 > 0 \text{ and with } 1 < p < \infty;$$

$$b(z, x, y) \geq 0 \text{ for all } z \in \Omega, \text{ all } x \in \mathbb{R}_+, \text{ all } y \in \mathbb{R}^N.$$

By a solution of problem (34), we mean a function $u \in W^{1,p}_{loc}(\Omega)$ such that $a(\cdot, u(\cdot), Du(\cdot)) \in L^{p'}_{loc}(\Omega)$ and

$$\int_\Omega (a(z, Du), D\theta)_{\mathbb{R}^N} dz \geq \int_\Omega b(z, u, Du)\theta dz \text{ for all } \theta \in C_c^\infty(\Omega), \ \theta \geq 0.$$

Then u is also called a “solution in the sense of distributions”.

We can think of $u(\cdot)$ as an upper solution of the corresponding equation. We say that $u \geq 0$ on $\partial\Omega$, if for every $\delta > 0$, there exists U_δ a neighborhood of $\partial\Omega$ in $\overline{\Omega}$ such that $u \geq -\delta$ in U_δ. An alternative interpretation of the condition $u \geq 0$ on $\partial\Omega$ is to say that $u^- \in W_0^{1,p}(\Omega)$, but in this case, in the definition of the solution, we need to require that $a(\cdot, u(\cdot), Du(\cdot)) \in L^{p'}(\Omega)$.

The next result is known as the “weak maximum principle” and can be found in Theorem 3.2.2 of Pucci and Serrin [42, p. 56].

Proposition 11. *If $a(z, y)$ and $b(z, x, y)$ are as above, $u \in W^{1,p}_{loc}(\Omega)$ is a solution of (34) and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω.*

We can have a stronger version of the above result, provided we restrict further the structure of the maps a and b. In this way we can have a nonlinear version of the Hopf maximum principle for semilinear equations.

Consider the problem

$$-\operatorname{div} a(Du) = b(z, u) \text{ in } \Omega. \tag{35}$$

We assume that

$$a(y) = a_0(|y|)y \text{ for all } y \in \mathbb{R}^N,$$

with $s \mapsto a_0(s)s$ strictly increasing and $\lim_{t\to 0^+} a_0(t)t = 0$.

We also assume that $b(z, x)$ is a Carathéodory function which satisfies the following condition:

“For every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, $f(z, x) + \hat{\xi}_\rho x^{p-1} \geq 0$ for a.a. $z \in \Omega$ and for all $0 \leq x \leq \rho$”.

Theorem 5. *If the above conditions hold, $u \in C^1(\overline{\Omega})$ is a solution in the sense of distributions of (35) and $u(z) \geq 0$ for all $z \in \overline{\Omega}$, then the following properties hold:*

(a) if $u(\cdot)$ vanishes at a point in Ω, then $u \equiv 0$;

(b) if $u(z) > 0$ for all $z \in \Omega$ and $u(z_0) = 0$ for some $z_0 \in \partial\Omega$, then $\frac{\partial u}{\partial n}(z_0) < 0$, with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

According to this theorem, if we have a Dirichlet problem driven by (35) (so $u|_{\partial\Omega} = 0$) and $\hat{u} \in W_0^{1,p}(\Omega)\backslash\{0\}$ is a nonnegative weak solution of the problem, then by Theorem 3 we have $\hat{u} \in C_+$ and, by Theorem 5, $\hat{u} \in \operatorname{int} C_+$. Similar conclusions can be drawn for the Neumann and Robin problems. In this case we have that $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) come

$$\hat{C}_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in \hat{C}_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

Next, we will present some comparison results. We start with the so-called "tangency principle" due to Serrin (see Pucci and Serrin [42, Theorem 2.5.2, p. 35]). We consider the following two partial differential inequalities

$$\operatorname{div} a(z, u, Du) + b(z, u) \geq 0 \text{ in } \Omega, \tag{36}$$

$$\operatorname{div} a(z, v, Dv) + b(z, v) \leq 0 \text{ in } \Omega. \tag{37}$$

Here, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous map such that $a(z, \cdot, \cdot) \in C^1(\mathbb{R}^N \times \mathbb{R}^N)$ for all $z \in \Omega$. Also, $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for any $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that

$$b(z, x) - b(z, u) \geq -\hat{\xi}_\rho(x - u) \text{ for all } z \in \Omega, \text{ all } u \leq x \text{ with } u, x \in [-\rho, \rho].$$

The tangency principle of Serrin establishes the following property.

Proposition 12. *If $u, v \in C^1(\Omega)$ are solutions in the sense of distributions of (36) and (37) respectively, $u(z) \leq v(z)$ for all $z \in \Omega$ and at least one of the matrices*

$$\nabla_y a(z, u, Du) \text{ or } \nabla_y a(z, v, Dv)$$

is positive definite, then either $u \equiv v$ or $u(z) < v(z)$ for all $z \in \Omega$.

If $a(y) = |y|^{p-2}y + y$ for all $y \in \mathbb{R}^N$, with $2 < p$, then the differential operator governing (36) and (37) is the $(p, 2)$-Laplacian $\Delta_p u + \Delta u$. Since $p > 2$, we see that $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and

$$\nabla a(y) = |y|^{p-2}\left[\mathrm{id} + (p-2)\frac{y \otimes y}{|y|^2}\right] + \mathrm{id} \text{ for all } y \in \mathbb{R}^N.$$

Therefore for all $y, \xi \in \mathbb{R}^N$, we have

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq |\xi|^2.$$

So, for the $(p, 2)$-Laplacian, the hypothesis of Proposition 12 is automatically satisfied.
Now suppose that:

- $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory and monotone in $y \in \mathbb{R}^N$ and

$$|a(z, y)| \leq (1 + |y|^{p-1}) \text{ for a.a. } z \in \Omega, \text{ all } y \in \mathbb{R}^N, 1 < p < \infty.$$

- $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and nonincreasing in $x \in \mathbb{R}$.

We consider the following partial differential inequalities.

$$\operatorname{div} a(z, Du) + b(z, u) \geq 0 \text{ in } \Omega, \tag{38}$$

$$\operatorname{div} a(z, Dv) + b(z, v) \leq 0 \text{ in } \Omega. \tag{39}$$

Proposition 13. *If $u, v \in W^{1,p}_{loc}(\Omega) \cap L^p(\Omega)$ are solutions in the sense of distributions of (38) and (39) respectively and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

The above proposition concludes a weak comparison between the two solutions. To have strong comparison for nonlinear problems is a difficult task and requires additional hypotheses on the data of the problem. This is in contrast to semilinear problems (problems driven by a linear elliptic operator), where strong comparison results follow easily using the Hopf maximum principle (see Zeng and Papageorgiou [45] and Papageorgiou and Zhang [39].

We state here two strong comparison results valid also for singular problems. We first state a strong maximum principle for singular problems (see Theorem 5). So, $a : \mathbb{R}^N \to \mathbb{R}^N$ is as in Theorem 5.

Theorem 6. *If $0 < \eta < 1$, $\hat{\xi} > 0$, $\lambda \geq 0$ and $u \in C_+ \setminus \{0\}$ satisfies*

$$-\operatorname{div} a(Du) + \hat{\xi} u^{p-1} - \lambda u^{-\eta} \geq 0 \text{ in } \Omega,$$

then $u \in \operatorname{int} C_+$.

Note that if $\lambda = 0$, then this theorem is a consequence of Proposition 13.

Given $g, h \in L^\infty(\Omega)$, we write $g \prec h$ if for all $K \subseteq \Omega$ compact we have

$$0 < c_K \leq h(z) - g(z) \text{ for a.a. } z \in K.$$

Also, consider another cone in $C^1(\overline{\Omega})$, namely

$$D_+ = \left\{ u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}\Big|_{\partial\Omega \cap \{u=0\}} < 0 \right\}.$$

Theorem 7. *(a) If $\hat{\xi} \in L^\infty(\Omega)_+ \setminus \{0\}$, $\lambda \geq 0$, $0 < \eta < 1$ and $u \in C_+$, $v \in \operatorname{int} C_+$ satisfy*

$$-\operatorname{div} a(Du) + \hat{\xi}(z) u^{p-1} - \lambda u^{-\eta} = g(z) \text{ in } \Omega,$$

$$-\operatorname{div} a(Dv) + \hat{\xi}(z) v^{p-1} - \lambda v^{-\eta} = h(z) \text{ in } \Omega$$

with $g, h \in L^\infty_{loc}(\Omega)$, $g \prec h$, then $v - u \in \operatorname{int} C_+$.

(b) If $\hat{\xi} \in L^\infty(\Omega)_+ \setminus \{0\}$, $\lambda \geq 0$, $0 < \eta < 1$ and $u, v \in C^1(\overline{\Omega})$, $0 \leq u \leq v$ satisfy

$$-\operatorname{div} a(Du) + \hat{\xi}(z) u^{p-1} - \lambda u^{-\eta} = g(z) \text{ in } \Omega,$$

$$-\text{div}\, a(Dv) + \hat{\xi}(z)v^{p-1} - \lambda v^{-\eta} = h(z) \text{ in } \Omega$$

with $g, h \in L^\infty_{loc}(\Omega)$, $0 < \hat{c} \le h(z) - g(z)$ *for a.a.* $z \in \Omega$, *then* $v - u \in D_+$.

These two comparison results can be found in Papageorgiou, Rădulescu and Repovs [34].

In general, the "strict" inequalities between g and h in (a) and (b) in the above theorem, cannot be weakened to the condition

$$g(z) \le h(z) \text{ for a.a. } z \in \Omega,\ \ g \ne h.$$

However, we can state the following result.

Proposition 14. *If* $g, h \in L^\infty(\Omega)$, $g(z) \le h(z)$ *for a.a.* $z \in \Omega$, $g \ne h$, $u, v \in C^1(\overline{\Omega})$, $u \le v$ *and they satisfy*

$$-\text{div}\, a(Du) = g(z) \text{ in } \Omega,\ \ \frac{\partial u}{\partial n}|_{\partial\Omega} < 0,$$

$$-\text{div}\, a(Dv) = h(z) \text{ in } \Omega,\ \ \frac{\partial v}{\partial n}|_{\partial\Omega} < 0,$$

then $v - u \in D_+$.

For double phase problems, although there is no global regularity theory, we can have a strong maximum principle. The result can be found in Proposition 2.4 of Papageorgiou, Vetro and Vetro [37]. As before, we assume that $a \in L^\infty(\Omega)\backslash\{0\}$, $a(z) \ge 0$ for a.a. $z \in \Omega$, $1 < q < p < N$ and $\frac{p}{q} < 1 + \frac{1}{N}$.

Proposition 15. *If* $\hat{\xi} > 0$ *and* $\hat{u} \in W^{1,\eta}_0(\Omega) \setminus \{0\}$ *satisfies* $\hat{u}(z) \ge 0$ *for a.a.* $z \in \Omega$ *and*

$$-\Delta^a_p \hat{u}(z) - \Delta_q \hat{u}(z) + \hat{\xi}\hat{u}(z)^{p-1} \ge 0 \text{ in } \Omega,$$

then $\hat{u} \in W^{1,\eta}(\Omega) \cap L^\infty(\Omega)$ *and* $0 \prec \hat{u}$ *hence,* $0 < \hat{u}(z)$ *for a.a.* $z \in \Omega$.

5. Eigenvalues and other variational tools

In the study of boundary value problems, we often impose conditions on the reaction which describe how the nonlinearity interacts with the spectrum of the differential operator as $x \to \infty$ or $x \to 0$. For this reason, in the first half of this section, we review the main facts concerning the spectral properties of Δ_p.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and $m \in L^\infty(\Omega) \setminus \{0\}$, $m(z) \ge 0$ for a.a. $z \in \Omega$. We consider the following nonlinear eigenvalue problem

$$\left\{\begin{array}{l} -\Delta_p u(z) = \hat{\lambda} m(z)|u(z)|^{p-2}u \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ \ 1 < p < \infty. \end{array}\right\} \tag{40}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an "eigenvalue" of $(-\Delta_p, W^{1,p}_0(\Omega))$ if problem (40) admits a nontrivial solution $\hat{u} \in W^{1,p}_0(\Omega)$, called an "eigenfunction" corresponding to $\hat{\lambda}$. By

Theorem 3 we know that $\hat{u} \in C_0^1(\overline{\Omega})$. Also, if we multiply (40) with $\hat{u}(z)$, integrate over Ω and perform integration by parts (nonlinear Green's theorem, see Proposition 1.5.8 of Papageorgiou, Rădulescu and Repovs [33, p. 31]), we see that every eigenvalue satisfies $\hat{\lambda} \geq 0$. The next proposition characterizes the first element of the spectrum.

Proposition 16. *Problem (40) has a smallest eigenvalue $\hat{\lambda}_1(p, m) > 0$ with the following properties:*

(a) we have

$$\hat{\lambda}_1(p, m) = \inf\left\{ \frac{\|Du\|_p^p}{\int_\Omega m(z)|u|^p dz} : u \in W_0^{1,p}(\Omega),\ \ u \neq 0 \right\} > 0; \tag{41}$$

(b) $\hat{\lambda}_1(p, m)$ is isolated, that is, if $\hat{\sigma}(m)$ denotes the spectrum of (41), then there exists $\varepsilon > 0$ such that

$$(\hat{\lambda}_1(p, m), \hat{\lambda}_1(p, m) + \varepsilon) \cap \hat{\sigma}(m) = \emptyset;$$

(c) $\hat{\lambda}_1(p, m)$ is simple, that is, if $\hat{u}$, $\tilde{u} \in C_0^1(\overline{\Omega})$ are two eigenfunctions corresponding to $\hat{\lambda}_1(p, m) > 0$, then

$$\hat{u} = \theta \tilde{u} \text{ for some } \theta \in \mathbb{R} \setminus \{0\}.$$

The infimum in the variational characterization (41) of $\hat{\lambda}_1(p, m)$ is attained on the corresponding one dimensional eigenspace (see Proposition 16(c)). It is clear that the elements of this eigenspace do not change sign. We denote by $\hat{u}_1 = \hat{u}_1(p, m)$ the positive, L_m^p-normalized (that is, $\int_\Omega m(z)|u|^p dz = 1$) eigenfunction corresponding to $\hat{\lambda}_1 = \hat{\lambda}_1(p, m) > 0$. From Theorem 5 we know that $\hat{u}_1 \in \operatorname{int} C_+$.

Recall that $\hat{\sigma}(m)$ denotes the spectrum of (40). This set is closed in $(0, \infty)$. Employing the Ljusternik–Schnirelmann minimax scheme, we produce a whole sequence $\{\hat{\lambda}_k\}_{k \in \mathbb{N}}$ of distinct eigenvalues of (40) such that $\hat{\lambda}_k \to \infty$. The first element of this sequence is the eigenvalue of Proposition 16. On account of Proposition 16(b) and since $\hat{\sigma}(m)$ is closed, the second element in $\hat{\sigma}(m)$ is given by

$$\hat{\lambda}_2^* = \inf\{\hat{\lambda} \in \hat{\sigma}(m) : \hat{\lambda}_1 < \hat{\lambda}\}.$$

Then $\hat{\lambda}_2^* = \hat{\lambda}_2$ (that is, the second eigenvalue and the second Ljusternik–Schnirelmann eigenvalue coincide). In general, we do not know if $\{\hat{\lambda}_k\}_{k \in \mathbb{N}}$ exhausts $\hat{\sigma}(m)$. This is the case if $p = 2$ (linear eigenvalue problem). Also note that the sequence $\{\hat{\lambda}_k\}_{k \in \mathbb{N}}$ of LS-eigenvalues depends on the choice of the index used in the execution of the LS-minimax scheme. The first two elements of the sequence are independent of the index. For the rest we do not know if this is true.

Proposition 17. *If $m, \hat{m} \in L^\infty(\Omega)$, $0 \leq m(z) \leq \hat{m}(z)$ for a.a. $z \in \Omega$, $m \neq 0$, $m \neq \hat{m}$, then $\hat{\lambda}_1(p, \hat{m}) < \hat{\lambda}_1(p, m)$.*

We can have a variational characterization of $\hat{\lambda}_2$. So, let

$$M = \{u \in W_0^{1,p}(\Omega) : \int_\Omega m(z)|u|^p dz = 1\}$$

and

$$\Gamma = \{\gamma \in C([-1, 1], M) : \gamma(-1) = -\hat{u}_1,\ \gamma(1) = \hat{u}_1\}.$$

Proposition 18. *We have*

$$\hat{\lambda}_2 = \inf_{\gamma \in \Gamma} \max_{-1 \le t \le 1} \|D\gamma(t)\|_p^p.$$

For this eigenvalue we also have a monotonicity property with respect to the weight $m(\cdot)$, but now under more restrictive conditions on the weights.

Proposition 19. *If $m, \hat{m} \in L^\infty(\Omega)$, $0 \le m(z) \le \hat{m}(z)$, $m \ne 0$, $m(z) < \hat{m}(z)$ for a.a. $z \in \Omega$, then $\hat{\lambda}_2(p, \hat{m}) < \hat{\lambda}_2(p, m)$.*

For the double phase problems, relevant is the following eigenvalue problem.

$$\begin{Bmatrix} -\Delta_p^a u(z) = \hat{\lambda} m(z)|u(z)|^{p-2}u(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ 1 < p < N. \end{Bmatrix} \tag{42}$$

Let $a \in C^{0,1}(\overline{\Omega}) \cap \mathcal{A}_p$ (see Section 2) with $a(z) > 0$ for all $z \in \Omega$. Then we know (see Section 2) that if $\eta_0(z, t) = a(z)t^p$, then

$$W_0^{1,\eta_0}(\Omega) \hookrightarrow L^{\eta_0}(\Omega) \text{ compactly.} \tag{43}$$

Using (43) we can have a complete spectral analysis of problem (42).

Proposition 20. *Problem (42) has a smallest eigenvalue $\hat{\lambda}_1^a(p, m) > 0$ given by*

$$\hat{\lambda}_1^a(p, m) = \inf\left\{ \frac{\int_\Omega a(z)|Du|^p dz}{\int_\Omega m(z)|u|^p dz} : u \in W_0^{1,p}(\Omega),\ u \ne 0 \right\} > 0. \tag{44}$$

This eigenvalue is isolated and simple and if $m, \hat{m} \in L^\infty(\Omega)$, $0 \le m(z) \le \hat{m}(z)$ for a.a. $z \in \Omega$, $m \ne 0$, $m \ne \hat{m}$, then

$$\hat{\lambda}_1^a(p, \hat{m}) < \hat{\lambda}_1^a(p, m).$$

In (44) the infimum is realized on the corresponding one dimensional eigenspace which has elements of fixed sign. We denote by $\hat{u}_1$ the positive eigenfunction for $\hat{\lambda}_1^a = \hat{\lambda}_1^a(p, m) > 0$ such that $\|u\|_{\eta_0} = 1$. Set

$$M_0 = \{u \in W_0^{1,\eta_0}(\Omega) : \|u\|_{\eta_0} = 1\}$$

and

$$\Gamma_0 = \{\gamma \in C([-1, 1], M_0) : \gamma(-1) = -\hat{u}_1,\ \gamma(1) = \hat{u}_1\}.$$

Proposition 21. *We have*

$$\hat{\lambda}_2^a = \inf_{\gamma \in \Gamma_0} \max_{-1 \le t \le 1} \rho_a(D\gamma(t)).$$

For the eigenvalue problem (40), we refer to Anane [2], Lindqvist [21], Gasinski and Papageorgiou [11]. For the eigenvalue problem (42), we refer to the recent paper by Papageorgiou, Pudelko and Rădulescu [32].

Suppose that $a(z, y)$ satisfies hypotheses H_2. Recall that $G_0(z,t) = \int_0^t a_0(z,s)ds$ and $G(z, y) = G_0(z, |y|)$ for all $z \in \Omega$, all $y \in \mathbb{R}^N$.

Also let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$|f(z,x)| \le \hat{a}(z)(1 + |x|^{r-1})$$

for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\hat{a} \in L^\infty(\Omega)$, $p \le r < p^*$. We set $F(z,x) = \int_0^x f(z,s)ds$ and consider the C^1-functional $\phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\phi(u) = \int_\Omega G(z, Du)dz - \int_\Omega F(z,u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

The next result, which is an outgrowth of Theorem 3, is a very useful tool in the analysis of nonlinear, balanced growth equations.

Proposition 22. *If $\hat{u} \in W_0^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of $\phi(\cdot)$, that is, there exists $\delta > 0$ such that*

$$\phi(\hat{u}) \le \phi(\hat{u} + h) \text{ for all } h \in C_0^1(\overline{\Omega}),\ \|h\|_{C_0^1(\overline{\Omega})} \le \delta,$$

then $\hat{u}$ is also a local $W_0^{1,p}(\Omega)$-minimizer of $\phi(\cdot)$, that is, there exists $\varepsilon > 0$ such that

$$\phi(\hat{u}) \le \phi(\hat{u} + h) \text{ for all } h \in W_0^{1,p}(\Omega),\ \|h\|_{1,p} \le \varepsilon.$$

This result was first proved for semilinear problems driven by the Laplacian by Brezis and Nirenberg [6]. Since then the result has been generalized in many different ways (see Papageorgiou, Rădulescu and Zhang [35]). For double phase problems there is no corresponding result due to the absence of a global regularity theory. This removes from consideration a powerful analytical tool and makes double phase equations more difficult to deal with.

If X is a Banach space, X^* is its topological dual and we denote by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X). A map $A : X \to X^*$ is said to be of type $(S)_+$ if it has the following property:

If $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to\infty} \langle A(u_n), u_n - u \rangle \le 0$, then $u_n \to u$ in X.

Consider a function $G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ such that

(i) for all $y \in \mathbb{R}^N$, the map $z \mapsto G(z, y)$ is measurable;

(ii) for a.a. $z \in \Omega$, the function $y \mapsto G(z, y)$ is C^1, strictly convex and $G(z, 0) = 0$;

(iii) $|\nabla_g G(z, y)| \le \hat{a}(z)(1 + |y|^{p-1})$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ with $\hat{a} \in L^\infty(\Omega)$, $1 < p < \infty$;

(iv) $(\nabla_y G(z, y), y)_{\mathbb{R}^N} \le pG(z, y)$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$;

(v) $c_0|y|^p \le pG(z, y)$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$, some $c_0 > 0$.

We let $a(z, y) = \nabla_y G(z, y)$ and consider the nonlinear map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by

$$\langle A(u), h \rangle = \int_\Omega (a(z, Du), Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W^{1,p}(\Omega).$$

Proposition 23. *If the above hypotheses hold, then $A(\cdot)$ is continuous, monotone and of type $(S)_+$.*

This proposition is useful in showing that the energy functional of the boundary value problem satisfies the C-condition. If $G(y) = \frac{1}{p}|y|^p + \frac{\mu}{q}|y|^q$ for all $y \in \mathbb{R}^N$, with $1 < q < p$ and $\mu \geq 0$, then we have the (p, q)-Laplacian.

A similar result is also true for the double phase operator. So, let $a \in L^\infty(\Omega) \setminus \{0\}$, $a(z) \geq 0$ for a.a. $z \in \Omega$, $1 < q < p$ and $\frac{p}{q} < 1 + \frac{1}{N}$. We consider the nonlinear, nonhomogeneous map $V : W_0^{1,\eta}(\Omega) \to W_0^{1,\eta}(\Omega)^*$ defined by

$$\langle V(u), h\rangle = \int_\Omega (a(z)|Du|^{p-2} + |Du|^{q-2})(Du, Dh)_{\mathbb{R}^N} \text{ for all } u, h \in W_0^{1,\eta}(\Omega).$$

Proposition 24. *The map $V(\cdot)$ is continuous, bounded (that is, maps bounded sets to bounded ones), strictly monotone and of type $(S)_+$.*

We point out that a continuous monotone map is maximal monotone.

Data availability

No data was used for the research described in the article.

CRediT authorship contribution statement

Nikolaos S. Papageorgiou: Writing – original draft, Visualization, Validation, Supervision, Project administration, Methodology, Investigation, Conceptualization. **Vicenţiu D. Rădulescu:** Writing – review & editing, Validation, Supervision, Software, Investigation, Conceptualization.

Declaration of competing interest

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