NONTRIVIAL SOLUTIONS FOR A MULTIVALUED PROBLEM WITH STRONG RESONANCE

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The Mountain-Pass Theorem of Ambrosetti and Rabinowitz (see [**1**]) and the Saddle Point Theorem of Rabinowitz (see [**21**]) are very important tools in the critical point theory of C^1-functionals. That is why it is natural to ask us what happens if the functional fails to be differentiable. The first who considered such a case were Aubin and Clarke (see [**6**]) and Chang (see [**12**]), who gave suitable variants of the Mountain-Pass Theorem for locally Lipschitz functionals which are defined on reflexive Banach spaces. For this aim they replaced the usual gradient with a generalized one, which was firstly defined by Clarke (see [**13**], [**14**]). As observed by Brezis (see [**12**, p. 114]), these abstract critical point theorems remain valid in non-reflexive Banach spaces.

We apply some of these results to solve a multivalued problem with strong resonance at infinity. We remark that it is not usual to consider nonlinearities which are strongly resonant at $+\infty$ unless they are also strongly resonant at $-\infty$. The literature is very rich in resonant problems; the first who studied such problems (in the smooth case) were Landesman and Lazer (see [**18**]). They found sufficient conditions for the existence of solutions for some single-valued equations with Dirichlet conditions. These problems, which arise frequently in mechanics, were thereafter intensively studied and many applications to concrete situations were given.

1. Abstract framework. Let X be a real Banach space and let $f: X \to \mathbf{R}$ be a locally Lipschitz function. For each $x, v \in X$, we define the *generalized directional derivative* of f at x in the direction v as

$$f^0(x, v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

The *generalized gradient* (the *Clarke subdifferential*) of f at the point x is the subset $\partial f(x)$ of X^* defined by

$$\partial f(x) = \{x^* \in X^*; f^0(x, v) \geq \langle x^*, v \rangle, \text{ for all } v \in X\}.$$

We also define the lower semi-continuous function

$$\lambda(x) = \min\{\|x^*\|; x^* \in \partial f(x)\}.$$

For further properties of these notions we refer to [**12**], [**13**], [**14**].

We say that a point $x \in X$ is a *critical point* of f provided that $0 \in \partial f(x)$, that is $f^0(x, v) \geq 0$ for every $v \in X$. If c is a real number, we say that f satisfies the *Palais-Smale condition* at the level c (in short, $(PS)_c$) if any sequence $(x_n)_n$ in X with the properties $\lim_{n\to\infty} f(x_n) = c$ and $\lim_{n\to\infty} \lambda(x_n) = 0$ is relatively compact.

We shall use in this paper the following result, which is an immediate consequence of the Mountain-Pass Theorem proved in [**12**].

THEOREM 1. *Let $f:X\to\mathbf{R}$ be a locally Lipschitzian function. Suppose that $f(0)=0$ and there is some $v\in X\setminus\{0\}$ such that $f(v)\leq 0$. Moreover, assume that f satisfies the following geometric hypothesis: there exist $R>0$ and $\alpha>0$ such that $R<\|v\|$ and, for each $u\in X$ with $\|u\|=R$, we have $f(u)\geq\alpha$. Let $\mathcal{P}$ be the family of all continuous paths $p:[0,1]\to X$ that join 0 to v and*

$$c=\inf_{p\in\mathcal{P}}\max_{t\in[0,1]}f(p(t)).$$

Then there exists a sequence (x_n) in X such that:

(i) $\lim_{n\to\infty}f(x_n)=c$;

(ii) $\lim_{n\to\infty}\lambda(x_n)=0$.

Moreover, if f satisfies $(\mathrm{PS})_c$ then c is a critical value of f.

The following saddle point type result generalizes the Rabinowitz's theorem (see [**21**]). Its proof is an easy exercise and is left to the reader.

THEOREM 2. *Let $f:X\to\mathbf{R}$ be a locally Lipschitzian function. Assume that $X=Y\oplus Z$, where Z is a finite dimensional subspace of X and for some $z_0\in Z$ there exists $R>\|z_0\|$ such that*

$$\inf_{y\in Y}f(y+z_0)>\max\{f(z);z\in Z,\|z\|=R\},$$

Let

$$K=\{z\in Z;\|z\|\leq R\}$$

and

$$\mathcal{P}=\{p\in C(K,X);p(x)=x \text{ if } \|x\|=R\}.$$

If c is defined as in Theorem 1 and f satisfies $(\mathrm{PS})_c$, then c is a critical value of f.

2. Main results. Let M be a m-dimensional smooth compact Riemann manifold, possibly with smooth boundary ∂M. Particularly, M can be any open bounded smooth subset of $\mathbf{R}^m$. We shall consider the following multivalued elliptic problem

$$\begin{cases}-\Delta_M u(x)-\lambda_1 u(x)\in[\underline{f}(u(x)),\overline{f}(u(x))] & \text{a.e. } x\in M,\\ u=0 & \text{on } \partial M,\\ u\not\equiv 0, & \end{cases}$$

where:

(i) Δ_M is the Laplace–Beltrami operator on M;

(ii) λ_1 is the first eigenvalue of $-\Delta_M$ in $H_0^1(M)$;

(iii) $f\in L^\infty(\mathbf{R})$;

(iv) $\underline{f}(t)=\lim_{\varepsilon\searrow 0}\operatorname{essinf}\{f(s);|t-s|<\varepsilon\}$, $\overline{f}(t)=\lim_{\varepsilon\searrow 0}\operatorname{esssup}\{f(s);|t-s|<\varepsilon\}$.

As proved in [12], the functions $\underline{f}$ and $\bar{f}$ are measurable on $\mathbf{R}$ and, if

$$F(t)=\int_0^t f(s)\,ds,$$

then the Clarke subdifferential of F is given by

$$\partial F(t)=[\underline{f}(t),\bar{f}(t)] \quad \text{a.e. } t\in\mathbf{R}.$$

Let $(g_{ij}(x))_{i,j}$ define the metric on M. We consider on $H_0^1(M)$ the locally Lipschitz functional $\varphi=\varphi_1-\varphi_2$, where

$$\varphi_1(u)=\frac{1}{2}\int_M\Big(\sum_{i,j} g_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}-\lambda_1 u^2\Big)\,dx \quad \text{and} \quad \varphi_2(u)=\int_M F(u)\,dx.$$

By a *solution of the problem* (P) we shall mean any critical point of the energetic functional φ. Denote

$$f(\pm\infty)=\operatorname{ess}\lim_{t\to\pm\infty} f(t) \quad \text{and} \quad F(\pm\infty)=\lim_{t\to\pm\infty} F(t).$$

Our basic hypothesis on f will be

$$f(+\infty)=F(+\infty)=0, \tag{f1}$$

which makes the problem (P) a Landesman–Lazer type one, with strong resonance at $+\infty$.

The following formulates a sufficient condition for the existence of solutions of our problem.

THEOREM A. *Assume that f satisfies* (f1) *and either*

$$F(-\infty)=-\infty \tag{F1}$$

or $-\infty<F(-\infty)\le 0$ and there exists $\eta>0$ such that

$$F \text{ is non-negative on } (0,\eta) \text{ or } (-\eta,0). \tag{F2}$$

Then the problem (P) *has at least one solution.*

For positive values of $F(-\infty)$ it is necessary to impose additional restrictions on f. Our variant for this case is the following theorem.

THEOREM B. *Assume* (f1) *and $0<F(-\infty)<+\infty$. Then the problem* (P) *has at least one solution provided the following conditions are satisfied:*

$$f(-\infty)=0$$

and

$$F(t)\le\frac{\lambda_2-\lambda_1}{2}t^2 \qquad \textit{for each } t\in\mathbf{R}.$$

For the proof of Theorem A we shall make use of the following non-smooth variants of Lemmas 6 and 7 in [15] (see also [3] for Lemma 1) which can be obtained in the same manner.

LEMMA 1. *Assume $f \in L^\infty(\mathbf{R})$ and there exist $F(\pm\infty) \in \bar{\mathbf{R}}$. Moreover, suppose that*
(i) *$f(+\infty) = 0$ if $F(+\infty)$ is finite;*
and
(ii) *$f(-\infty) = 0$ if $F(-\infty)$ is finite.*
Then

$$\mathbf{R} \setminus \{a \,.\, \mathrm{meas}(M); a = -F(\pm\infty)\} \subset \{c \in \mathbf{R};\ \varphi \text{ satisfies } (\mathrm{PS})_c\}.$$

LEMMA 2. *Assume f satisfies* (f1). *Then φ satisfies $(\mathrm{PS})_c$, whenever $c \neq 0$ and $c < -F(-\infty) \,.\, \mathrm{meas}(M)$.*

Here $\mathrm{meas}(M)$ denotes the Riemannian measure of M.

Proof of Theorem A. We shall develop some of the ideas used in [26]. There are two distinct situations.

Case 1. $F(-\infty)$ is finite, that is $-\infty < F(-\infty) \leq 0$. In this case, φ is bounded from below since

$$\varphi(u) = \frac{1}{2} \int_M \Big(\sum_{i,j} g_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \lambda_1 u^2 \Big)\, dx - \int_M F(u)\, dx$$

and, by our hypothesis on $F(-\infty)$,

$$\sup_{u \in H_0^1(M)} \int_M F(u)\, dx < +\infty.$$

Therefore,

$$-\infty < a := \inf_{u \in H_0^1(M)} \varphi(u) \leq 0 = \varphi(0).$$

Choose c small enough in order to have $F(ce_1) < 0$ (note that c may be taken positive if $F > 0$ in $(0, \eta)$ and negative if $F < 0$ in $(-\eta, 0)$). Here $e_1 > 0$ denotes the first eigenfunction of $-\Delta_M$ in $H_0^1(M)$. Hence $\varphi(ce_1) < 0$, so $a < 0$. It follows now from Lemma 2 that φ satisfies $(\mathrm{PS})_a$. The proof ends in this case by applying Theorem 1.

Case 2. $F(-\infty) = -\infty$. Then, by Lemma 1, φ satisfies $(\mathrm{PS})_c$ for each $c \neq 0$. Let V be the orthogonal complement of the space spanned by e_1 with respect to $H_0^1(M)$, that is

$$H_0^1(M) = \mathrm{Sp}\{e_1\} \oplus V.$$

For fixed $t_0 > 0$, denote

$$V_0 = \{t_0 e_1 + v; v \in V\} \quad \text{and} \quad a_0 = \inf_{v \in V_0} \varphi(v).$$

Note that φ is coercive on V. Indeed, if $v \in V$, then

$$\varphi(v) \geq \frac{1}{2}\Big(1 - \frac{\lambda_1}{\lambda_2}\Big) \|v\|_{H_0^1}^2 - \int_M F(v) \to +\infty \quad \text{as} \quad \|v\|_{H_0^1} \to +\infty,$$

because the first term has a quadratic growth at infinity (t_0 being fixed), while $\int_M F(v)$ is uniformly bounded (in v), in view of the behaviour of F near $\pm\infty$. Thus, a_0 is attained, because of the coercivity of φ on V. From the boundedness of φ on $H_0^1(M)$ it follows that $-\infty < a \leq 0 = \varphi(0)$ and $a \leq a_0$.

Again, there are two possibilities.

(i) $a < 0$. In this case, by Lemma 2, φ satisfies $(\mathrm{PS})_a$. Hence $a < 0$ is a critical value of φ.

(ii) $a = 0 \leq a_0$. Then, either $a_0 = 0$ or $a_0 > 0$. In the first case, as we have already remarked, a_0 is attained. Thus, there is some $v \in V$ such that

$$0 = a_0 = \varphi(t_0 e_1 + v).$$

Hence, $u = t_0 e_1 + v \in H_0^1(M)\setminus\{0\}$ is a critical point of φ, that is a solution of (P).

If $a_0 > 0$, notice that φ satisfies $(\mathrm{PS})_b$ for each $b \neq 0$. Since $\lim_{t\to+\infty} \varphi(te_1) = 0$, we may apply Theorem 2 to conclude that φ has a critical value $c \geq a_0 > 0$.

Proof of Theorem B. Assume V has the same definition as above, and let

$$V_+ = \{te_1 + v\,; t > 0, v \in V\}.$$

It will be sufficient to show that the functional φ has a non-zero critical point. To do this, we shall make use of two different arguments. If $u = te_1 + v \in V_+$ then

$$\varphi(u) = \frac{1}{2}\int_M (|\nabla v|^2 - \lambda_1 v^2) - \int_M F(te_1 + v).$$

In view of the boundedness of F it follows that

$$-\infty < a_+ := \inf_{u \in V_+} \varphi(u) \leq 0.$$

We analyse two distinct situations.

Case 1. $a_+ = 0$. To prove that φ has a critical point, we use the same arguments as in the proof of Theorem A (the second case). More precisely, for some fixed $t_0 > 0$ we define in the same way V_0 and a_0. Obviously, $a_0 \geq 0 = a_+$, since $V_0 \subset V_+$. The proof follows from now on as in Case 2 of Theorem A, by reconsidering the two distinct situations $a_0 > 0$ and $a_0 = 0$.

Case 2. $a_+ < 0$. Let $u_n = t_n e_1 + v_n$ be a minimizing sequence of φ in V_+. We observe that the sequences $(u_n)_n$ and $(v_n)_n$ are bounded. Indeed, this is essentially a compactness condition and may be proved in a similar way to Lemma 1. It follows that there exists $w \in \bar{V}_+$, such that, going eventually to a subsequence,

$$u_n \to w \quad \text{weakly in } H_0^1(M),$$
$$u_n \to w \quad \text{strongly in } L^2(M),$$
$$u_n \to w \quad \text{a.e.}$$

Applying the Lebesgue dominated convergence theorem we obtain

$$\lim_{n\leftarrow\infty} \varphi_2(u_n) = \varphi_2(u).$$

On the other hand,

$$\varphi(w) \leq \liminf_{n\to\infty} \varphi_1(u_n) - \lim_{n\to\infty} \varphi_2(u_n) = \liminf_{n\to\infty} \varphi(u_n) = a_+.$$

It follows that, necessarily, $\varphi(w) = a_+ < 0$. Since the boundary of V_+ is V and

$$\inf_{u\in V} \varphi(u) = 0,$$

we conclude that w is a local minimum of φ on V_+ and $w \in V_+$.

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