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Quasilinear parabolic problem with variable exponent: Qualitative analysis and stabilization

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We discuss the existence and uniqueness of the weak solution of the following nonlinear parabolic problem:

$$(P_T) \qquad \begin{cases} u_t - \nabla \cdot \mathbf{a}(x, \nabla u) = f(x,u) & \text{in } Q_T \stackrel{\text{def}}{=} (0,T) \times \Omega, \\ u = 0 & \text{on } \Sigma_T \stackrel{\text{def}}{=} (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

which involves a quasilinear elliptic operator of Leray–Lions type with variable exponents. Next, we discuss the global behavior of solutions and in particular the convergence to a stationary solution as $t \to \infty$.

Keywords: Leray–Lions operator with variable exponents; parabolic equation; local and global in time existence; stabilization.

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1. Introduction and Main Results

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a bounded domain with smooth boundary (at least C^2). Our main goal in this paper is to study the existence, uniqueness and global behavior of the weak solutions to the following problem involving a quasilinear elliptic operator of Leray–Lions type with variable exponents:

$$(P_T) \quad \begin{cases} u_t - \nabla \cdot \mathbf{a}(x, \nabla u) = f(x, u) & \text{in } Q_T := (0,T) \times \Omega, \\ u = 0 & \text{on } \Sigma_T := (0,T) \times \partial\Omega, \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

We assume that $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ satisfies

(f_1) $0 \not\equiv f \colon (x,s) \to f(x,s)$ is a Carathéodory function and $t \mapsto f(x,t)$ is locally Lipschitz uniformly in $x \in \Omega$;

and $\mathbf{a}(x,\xi) = (a_j(x,\xi))_j$ with $a_j(x,\xi) = \phi(x,|\xi|)\xi_j$, $j = 1, \ldots, d$ be defined for all $\xi \in \mathbb{R}^d$, such that ϕ is differentiable on $\Omega \times (0,\infty)$ and $\phi(x,s) > 0$ for $(x,s) \in \Omega \times (0,\infty)$. We assume that $\mathbf{a}$ satisfies the following structural conditions:

(A1) $a_j(x,\mathbf{0}) = 0$, for all a.e. $x \in \Omega$,
(A2) $a_j \in C^1(\Omega \times (\mathbb{R}^d \backslash \{\mathbf{0}\})) \cap C^0(\Omega \times \mathbb{R}^d)$,
(A3) $\sum_{i,j=1}^d \frac{\partial a_j(x,\xi)}{\partial \xi_i} \eta_i \eta_j \geq \gamma |\xi|^{p(x)-2} \cdot |\eta|^2$, $\forall\, x \in \Omega$, , $\forall\, \xi \in \mathbb{R}^d \backslash \{\mathbf{0}\}$, $\forall\, \eta \in \mathbb{R}^d$,
(A4) $\sum_{i,j=1}^d |\frac{\partial a_j(x,\xi)}{\partial \xi_i}| \leq \Gamma |\xi|^{p(x)-2}$, $\forall\, x \in \Omega$, $\xi \in \mathbb{R}^d \backslash \{\mathbf{0}\}$,

where $p : \Omega \mapsto]1, +\infty[$ is a Lebesgue measurable function satisfying $1 < p^- := \inf_{\bar\Omega} p(x) \leq p(x) \leq p^+ := \sup_{\bar\Omega} p(x) < \infty$. More precisely, throughout the paper, we assume that

$$p \in C(\overline{\Omega}) \cap \mathcal{P}^{\log}(\Omega) \quad \text{such that } 1 < p^- \leq p^+ < d,$$

where

$$\mathcal{P}^{\log}(\Omega) \overset{\text{def}}{=} \left\{ q \in \mathcal{P}(\Omega) \, : \, \frac{1}{q} \text{ globally log-Hölder continuous} \right\}.$$

In particular, for any $p \in \mathcal{P}^{\log}(\Omega)$, there exists a function w such that for all $(x,y) \in \Omega^2$

$$|p(x) - p(y)| \leq w(|x-y|) \quad \text{and} \quad \limsup_{t \to 0^+} (-w(t) \ln t) < +\infty.$$

The natural function spaces to study problem (P_T) are the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and the corresponding Sobolev space $W^{1,p(x)}(\Omega)$ defined by

$$L^{p(x)}(\Omega) = \left\{ u \,|\, u \text{ is measurable on } \Omega \text{ and } \int_\Omega |u|^{p(x)} dx < \infty \right\}$$

and

$$W^{1,p(x)}(\Omega) \stackrel{\text{def}}{=} \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}.$$

We recall that $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are normed linear spaces equipped respectively with the (Luxemburg) norms

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\}$$

and

$$\|u\|_{W^{1,p(x)}} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Under the above assumptions on p, we define $\mathbb{W} \stackrel{\text{def}}{=} W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Since Ω is a bounded domain, the Poincaré inequality holds and a natural norm of $\mathbb{W}$ is $\|u\|_{\mathbb{W}} = \|\nabla u\|_{L^{p(x)}(\Omega}$.

Now, we define the even function $\Phi : \Omega \times \mathbb{R} \to \mathbb{R}$ by $\Phi(x,t) = \int_0^t \phi(x,|s|)s\,\mathrm{d}s$, which is increasing on $\mathbb{R}^+$, and for a.e. $x \in \Omega$, $\xi \to A(x,\xi) \stackrel{\text{def}}{=} \Phi(x,|\xi|)$.

Furthermore, from (**A1**), (**A3**) and (**A4**), A is strictly convex and satisfies for any fixed $x \in \Omega$

$$\frac{\gamma}{p^+ - 1}\,|\xi|^{p(x)} \le A(x,\xi) \le \frac{\Gamma}{p^- - 1}\,|\xi|^{p(x)} \quad \text{for all } \xi \in \mathbb{R}^d. \tag{1.1}$$

These inequalities are a direct consequence of Taylor's formula combined with (**A3**) and (**A4**), which yield

$$\frac{\gamma}{p^+ - 1}|\xi|^{p(x)} \le A(x,\xi) - A(x,\mathbf{0}) - \langle \partial_\xi A(x,\mathbf{0}), \xi\rangle \le \frac{\Gamma}{p^- - 1}|\xi|^{p(x)}$$

for all $(x,\xi) \in \Omega \times \mathbb{R}^d$. Similarly, using (**A4**), we deduce that there exists a positive constant c_1 such that

$$|\mathbf{a}(x,\xi)| \le c_1|\xi|^{p(x)-1},\ \phi(x,|\xi|) \le c_1|\xi|^{p(x)-2} \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^d. \tag{1.2}$$

We assume that

(**A5**) $(0,\infty) \ni t \to \Phi(x,\sqrt{t})$ is convex for a.e. $x \in \Omega^+ \stackrel{\text{def}}{=} \{z \in \Omega;\, p(z) \ge 2\}$.

From the above assumption together with $\Phi(x,0) = 0$ and $\Phi(x,\cdot)$ increasing, we obtain a Clarkson-type inequality for the function Φ (see [26, Lemma 2.1] for the proof). More precisely, we have

$$\begin{aligned}&\frac{1}{2}\left[\int_{\Omega^+} \Phi(x,|\nabla u|)\,\mathrm{d}x + \int_{\Omega^+} \Phi(x,|\nabla v|)\,\mathrm{d}x\right]\\ &\qquad \ge \int_{\Omega^+} \Phi\left(x, \frac{|\nabla(u+v)|}{2}\right)\mathrm{d}x + \int_{\Omega^+} \Phi\left(x, \frac{|\nabla(u-v)|}{2}\right)\mathrm{d}x\end{aligned} \tag{1.3}$$

for all u, $v \in \mathbb{W}$.

From the pointwise version of (1.3) (see [26, Lemma 2.1, p. 459]) and from (1.1), we deduce that for $x \in \Omega^+$, $A(x, \cdot)$ is $p(x)$-uniformly convex (see [12, 17, Definition 2.2]), that is,

$$\text{for any } \xi, \eta \in \mathbb{R}^d, \quad A\left(x, \frac{\xi+\eta}{2}\right) \le \frac{1}{2}(A(x,\xi) + A(x,\eta)) - c_0|\xi - \eta|^{p(x)} \tag{1.4}$$

with $c_0 = \frac{\gamma}{2^{p^+}(p^+-1)}$.

Remark 1.1. From the Clarkson inequality (see [9, p. 96]), relation (1.4) is satisfied by $A(x,\xi) = |\xi|^{p(x)}$ with p satisfying $p(x) \ge 2$. In the case $1 < p(x) < 2$, a similar inequality (namely, the Morawetz inequality) can be derived (see [22, Lemma 4.1]).

Remark 1.2. (1) Prototype examples satisfying conditions (**A1**)–(**A5**) below are: (a) $\phi(x,t) = \phi_1(x,t) \overset{\text{def}}{=} t^{p(x)-2}$, (b) $\phi(x,t) = \phi_2(x,t) \overset{\text{def}}{=} (1+t^2)^{\frac{p(x)-2}{2}}$.

(2) Assumption (**A4**) can be replaced by (**A4′**) for the validity of the results in Sec. 1: there exists $c > 0$ such that $|\phi(x,s)s| \le c(1 + |s|^{p(x)-1})$ for all $x \in \Omega$ and $s \in \mathbb{R}$.

There is an abundant literature devoted to questions on existence and uniqueness of solutions to (P_T) for $\phi(x,\xi) = |\xi|^{p(x)-2}$ and $p(x) \equiv p$ (see for instance [6] and references therein). More recently, parabolic and elliptic problems with variable exponents have been studied quite extensively, see for example [1, 4, 2, 5, 11, 21, 27, 31]. The importance of investigating these problems lies in their occurrence in modeling various physical problems involving strong anisotropic phenomena related to electrorheological fluids (an important class of non-Newtonian fluids) [1, 30, 31], image processing [11], elasticity [35], the processes of filtration in complex media [3], stratigraphy problems [22] and also mathematical biology [19].

Regarding the existing literature on quasilinear parabolic equations with variable exponent, we consider in the present paper a more general class of operators of Leray–Lions type than the $p(x)$-Laplace operator and prove new local existence and regularity results for (P_T) (see Theorems 1.2 and B.2). By constructing new barrier functions, we also provide global existence results (see Theorems 1.3 and 1.4) for the general class of quasilinear parabolic problems considered in the paper that are not known in the literature.

A natural issue is to consider the asymptotic behavior of the obtained global solutions. For this purpose, introducing a homogeneity condition, say (**A7**), we prove new uniqueness results for the stationary equation associated to (P_T) (see Theorem 1.5). The proof of the uniqueness results uses the extension of convexity properties of the associated energy functional, in the spirit of works [10] and [13], proved in [23]. Combining Theorems 1.5 and B.2, we are able to prove under suitable assumptions the asymptotic convergence to the stationary solution for global solutions to (P_T) (see Theorem 1.6), extending significantly former results proved in [21]. More precisely, regarding the main results established in [21], we first extend by using monotone methods the local existence results for the general class of operators satisfying (**A1**)–(**A5**) (see Theorems 1.2 and 1.1 below). Next, in respect to [21], using new barrier functions and new uniqueness results for the stationary equation (see Theorem 1.5 below), we improve essentially global existence results and asymptotic convergence to a stationary solution (see below Theorems 1.3, 1.4 and 1.6, respectively). We point out that even if restricting to the $p(x)$-operator, these results are new.

We now state the main results that we will prove in the next sections.

We first consider the following problem:

$$(L_T)\qquad \begin{cases} u_t - \nabla\cdot \mathbf{a}(x,\nabla u) = h(x,t) & \text{in } Q_T,\\ u = 0 & \text{on } \Sigma_T,\\ u(x,0) = u_0(x) & \text{in } \Omega,\end{cases}$$

where $T > 0$, $h \in L^2(Q_T) \cap L^q(Q_T)$, $q > \dfrac{d}{p^-}$.

Considering the initial data $u_0 \in \mathbb{W} \cap L^\infty(\Omega)$, we study the weak solutions of problem (L_T) defined as follows.

Definition 1.1. A weak solution to (L_T) is any function $u \in L^\infty(0,T;\mathbb{W})$ such that $u_t \in L^2(Q_T)$ and satisfying for any $\varphi \in C_0^\infty(Q_T)$

$$\int_0^T\!\!\int_\Omega u_t\varphi \,\mathrm{d}x\mathrm{d}t + \int_0^T\!\!\int_\Omega \mathbf{a}(x,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t = \int_0^T\!\!\int_\Omega h(x,t)\varphi\,\mathrm{d}x\mathrm{d}t$$

and $u(0,.) = u_0$ a.e. in Ω.

We define in a similar way the notion of weak solutions to the problem (P_T) as follows.

Definition 1.2. A solution to (P_T) is a function $u \in L^\infty(0,T;\mathbb{W})$ such that $u_t \in L^2(Q_T)$, $f(\cdot,u) \in L^\infty(0,T,L^2(\Omega))$ and for any $\varphi \in C_0^\infty(Q_T)$

$$\int_0^T\!\!\int_\Omega u_t\varphi \,\mathrm{d}x\mathrm{d}t + \int_0^T\!\!\int_\Omega \mathbf{a}(x,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t = \int_0^T\!\!\int_\Omega f(x,u)\varphi\,\mathrm{d}x\mathrm{d}t$$

and $u(0,\cdot) = u_0$ a.e. in Ω.

According to the above definitions, we establish the following local existence result proved in Sec. 2 and which extends [21, Theorem 2.3].

Theorem 1.1. *Assume that conditions* (**A1**)–(**A5**) *are satisfied and* $p^- > \frac{2d}{d+2}$. *Let* $T > 0$, $u_0 \in \mathbb{W} \cap L^\infty(\Omega)$ *and* $h \in L^2(Q_T) \cap L^q(Q_T)$, $q > \frac{d}{p^-}$. *Then problem* (L_T) *admits a unique solution* u *in the sense of Definition* 1.1. *Moreover*, $u \in C([0,T],\mathbb{W})$.

Next, we deal with the local existence of solutions of problem (P_T) proved in Sec. 3 (Sec. 3.1) which improves [21, Theorem 2.4].

Theorem 1.2. *Assume that conditions* (**A1**)–(**A5**) *are fulfilled. Let* $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ *satisfying* (f_1) *and*:

(f_2) *there exists* $s_0 \in \mathbb{R}$ *such that* $x \mapsto f(x, s_0) \in L^2(\Omega) \cap L^q(\Omega),\ q > \frac{d}{p^-}$.

Assume in addition that one of the following hypotheses holds:

(H1) *there exists a nondecreasing locally Lipschitz function* L_0 *such that*

$$|f(x, v)| \leq L_0(v), \quad a.e.\ (x, v) \in \Omega \times \mathbb{R};$$

(H2) *there exist two nondecreasing locally Lipschitz functions* L_1 *and* L_2 *such that*

$$L_1(v) \leq f(x, v) \leq L_2(v), \quad a.e.\ (x, v) \in \Omega \times \mathbb{R}.$$

Then, for any $u_0 \in \mathbb{W} \cap L^\infty(\Omega)$, *there exists* $\tilde{T} \in (0, +\infty]$ *such that for any* $T \in [0, \tilde{T})$, *problem* (P_T) *admits a unique solution* u *in the sense of Definition* 1.2. *Moreover, for any* $r > 1$, *we have* $u \in C([0, T]; L^r(\Omega)) \cap C([0, T]; \mathbb{W})$.

In Appendix B, using the theory of m-accretive operators, we also provide additional regularity results for weak solutions to (P_T) (see in particular Theorem B.2).

Under additional hypothesis about the growth of f and regularity of the initial data, we are able to prove the existence of global solutions. Precisely, we have the following results showed in Sec. 3 (Sec. 3.2) which gives sharper conditions on f than those in [21, Theorem 2.5].

Theorem 1.3. *Assume that conditions* (**A1**)–(**A5**) *are fulfilled. Let* $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ *satisfying* (f_1) *and*:

(f_3) *there exists* $s_1 \in \mathbb{R}$ *such that* $x \mapsto f(x, s_1) \in L^\infty(\Omega)$;

(f_4) *uniformly with respect to* $x \in \Omega$, *we have*

$$\limsup_{|s| \to \infty} \frac{|f(x, s)|}{|s|^{p^- - 1}} < \gamma \Lambda^{p^-} (p_c)^-$$

where $\Lambda := (\sup_{\|u\|_{\mathbb{W}}=1} \|u\|_{L^{p^-}(\Omega)})^{-1}$ *and* $(p_c)^- = \frac{p^-}{p^+ - 1}$;

(f_5) *uniformly with respect to* $x \in \Omega$, *we have*

$$\liminf_{s \to 0} \frac{|f(x, s)|}{|s|^{p^- - 1}} > \Gamma \Lambda^{p^-} (p^-)_c$$

where $(p^-)_c = \frac{p^-}{p^- - 1}$.

Assume in addition that

(C1) $u_0 \in \mathbb{W}$ *such that* $\nabla \cdot \mathbf{a}(x, \nabla u_0) \in L^q(\Omega)$ *where* $q > \frac{d}{p^-}$;

Then, for any $T > 0$, *problem* (P_T) *admits a unique weak solution in the sense of Definition* 1.2. *Moreover,* $u \in C([0, T]; \mathbb{W})$.

Furthermore, under the following new hypothesis:

$$(\mathbf{A6}) \quad \sum_{i=1}^{d} \left| \frac{\partial \phi(x,s)}{\partial x_i} \right| \leq C_1(1+|\ln(s)|)\phi(x,s) \quad \text{for all } x \in \Omega,\ s \in (0,\infty)$$

and constructing a suitable supersolution, we prove the existence of bounded global solutions of (P_T) with a regular initial data and releasing hypothesis on f.

Theorem 1.4. *Assume conditions* $(\mathbf{A1})$–$(\mathbf{A6})$ *and* $p \in C^{\beta}(\overline{\Omega})$, $\beta \in (0,1)$. *Let* $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ *satisfying* (f_1), (f_3) *and* (f_4). *Assume in addition*

(C2) $u_0 \in C_0^1(\overline{\Omega})$.

Then, for any $T > 0$, *problem* (P_T) *admits a unique bounded weak solution in the sense of Definition* 1.2. *Moreover,* $u \in C([0,T]; \mathbb{W})$.

Example. A prototype example for f satisfying all conditions (f_1)–(f_5) is $f(x,t) = h(x)t^{r(x)-1}$ with $0 \not\equiv h \in L^\infty(\Omega)$ nonnegative, $r \in C(\overline{\Omega})$ such that $1 < r(x) < p^-$ for all $x \in \overline{\Omega}$.

Remark 1.3. In the case $\phi(x,t) = |t|^{p(x)-2}$, p belonging to $C^1(\overline{\Omega})$ is a sufficient condition to have $(\mathbf{A6})$.

In Sec. 4, we discuss the question of the asymptotic behavior of solutions to (P_T). For this purpose, under suitable conditions, we prove the existence and the uniqueness of the positive stationary solution. Precisely, assuming in addition

(f_6) the function $s \mapsto \frac{f(x,s)}{s^{p^- -1}}$ is nonincreasing for a.e. $x \in \Omega$,

we have the following result, even new for the $p(x)$-operator (i.e. $A(x,\xi) = |\xi|^{p(x)}$).

Theorem 1.5. *Assume that* $p \in C^{\beta}(\overline{\Omega})$, $\beta \in (0,1)$ *such that* $p(\cdot) \not\equiv p^-$. *Let* $f : \Omega \times [0,+\infty) \mapsto [0,+\infty)$ *be a nonnegative function satisfying* (f_1), (f_4)–(f_6) *and* $f(x,0) = 0$. *Assume conditions* $(\mathbf{A1})$–$(\mathbf{A4})$ *and* $(\mathbf{A6})$. *We suppose in addition that*

$(\mathbf{A7})$ $\xi \mapsto A(x,\xi)$ *is* $p(x)$*-homogeneous, that is,*

$$A(x,\xi) = A\left(x, \frac{\xi}{|\xi|}\right) \cdot |\xi|^{p(x)}, \quad \text{for all } \xi \in \mathbb{R}^N \backslash \{\mathbf{0}\};$$

Then, the stationary problem associated with (P_T) *possesses a unique nonnegative and nontrivial weak solution* $u \in \mathbb{W} \cap L^\infty(\Omega)$. *This solution belongs to the class* $C^{1+\alpha}(\overline{\Omega})$, *for some* $\alpha \in (0,1)$, *and satisfies the Hopf maximum principle, namely*

$$u(x) > 0 \text{ for all } x \in \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu}(x) < 0 \text{ for all } x \in \partial\Omega.$$

The above uniqueness result implies the following result which improves significantly [21, Theorem 2.10].

Theorem 1.6. *Assume hypothesis in Theorem* 1.5 *and* $(\mathbf{A5})$ *be satisfied. Let* $u_0 \in C_0^{1,+}(\overline{\Omega})$ *where* $C_0^{1,+}(\overline{\Omega})$ *denotes the interior of the positive cone of* $C_0^1(\overline{\Omega})$. *Then, for* *any $T > 0$, there exists a unique weak solution, $u \in C([0,T], \mathbb{W}) \cap L^\infty(Q_T)$, to* (P_T) *with initial data u_0 and such that $\frac{\partial u}{\partial t} \in L^2(Q_T)$ and $u > 0$ in Q_T. Furthermore, u verifies*

$$u(t) \to u_\infty \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty,$$

where u_∞ is the unique positive stationary solution to (P_T) *given in Theorem* 1.5.

Remark 1.4. Alternatively to the hypothesis $p(.) \not\equiv p^-$ in Theorems 1.5 and 1.6, we can assume instead of (f_6), (f_6'): the function $s \mapsto \frac{f(x,s)}{s^{p^- -1}}$ is strictly decreasing for a.e. $x \in \Omega$.

Remark 1.5. Theorems 1.5 and 1.6 hold for other kind of nonlinearities. Indeed, let $f : \Omega \times (0,\infty) \to \mathbb{R}$ be defined as $f(x,t) = h(x)t^{r(x)-1} - g(x)t^{s(x)-1}$ with $r, s \in C(\overline{\Omega})$ satisfying $1 < r(x) \le p^- \le s(x)$ with $r^+ \ne s^-$ in Ω. Suppose that $h, g \in L^\infty(\Omega)$, are nonnegative and such that $h \not\equiv 0$ and $\frac{h(x)}{g(x)}$ bounded in Ω. Then, statements of Theorems 1.5 and 1.6 hold and their proofs are similar. We point out that any weak solution u satisfies the uniform bound $u(t,x) \le \left(\|\frac{h}{g}\|_{L^\infty(\Omega)} + 1\right)^{\frac{1}{s^- - r^+}}$ in Ω.

Remark 1.6. Under an additional asymptotic super homogeneous growth assumption on f and for initial data large enough, blow up in finite time of solutions can also occur. For instance, let $f(x,v) = v^q$ with $q > p^+$ and define the energy functional

$$E(u) \stackrel{\text{def}}{=} \int_\Omega A(x, \nabla u)\, \mathrm{d}x - \int_\Omega \frac{u^{q+1}}{q+1}\, \mathrm{d}x.$$

Then, using a well-known energy method and for any initial data u_0 satisfying $E(u_0) < 0$, the weak solution to (P_T) blows up in finite time. For further discussions of global behavior of solutions (blow up, localization of solutions, extinction of solutions) to quasilinear anisotropic parabolic equations involving variable exponents, we refer to [2, 5].

Remark 1.7. Condition (**A7**) implies that $A(x,\xi) = \frac{1}{p(x)}\langle a(x,\xi), \xi\rangle$. Examples satisfying (**A1**)–(**A5**) and (**A7**) are given by functions of the form $\phi(x,t) = h(x)t^{p(x)-2}$ where $h \in L^\infty(\Omega)$ such that $h \ge c > 0$.

2. Existence of Solution to (L_T)

For the proof of Theorem 1.1, we first consider the following quasilinear elliptic problem:

$$(P) \quad \begin{cases} u - \lambda \nabla \cdot \mathbf{a}(x, \nabla u) = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $\lambda > 0$ and g a measurable function. So we have the following lemma.

Lemma 2.1. *Assume conditions* (**A1**)–(**A4**). *Let* $g \in L^q(\Omega)$, $q > \frac{d}{p^-}$. *Then for any* $\lambda > 0$, *problem* (P) *admits a unique weak solution* $u \in \mathbb{W}$ *satisfying*

$$\int_\Omega u\varphi \,\mathrm{d}x + \lambda \int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega g\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in \mathbb{W}.$$

Furthermore, $u \in L^\infty(\Omega)$.

Proof. We define the energy functional J_λ associated to (P) given by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega u^2 \,\mathrm{d}x + \lambda \int_\Omega A(x, \nabla u)\,\mathrm{d}x - \int_\Omega gu \,\mathrm{d}x.$$

From (1.1), J_λ is well-defined and continuously differentiable on $\mathbb{W}$. Indeed, $q > \frac{d}{p^-}$ and $1 < p(.) < d$ a.e. in Ω imply that $L^q \subset (L^{p^*(x)})'$. By (1.1), for $\|u\|_{\mathbb{W}} \geq 1$:

$$J_\lambda(u) = \frac{1}{2}\int_\Omega u^2 \,\mathrm{d}x + \lambda \int_\Omega A(x, \nabla u)\mathrm{d}x - \int_\Omega gu \,\mathrm{d}x \geq \frac{\lambda\gamma}{p^+(p^+ - 1)}\|u\|_{\mathbb{W}}^{p^-} - C\|u\|_{\mathbb{W}}.$$

Thus, J_λ is coercive. Hence J_λ admits a global minimizer $u \in \mathbb{W}$ which is a weak solution to (P). By (**A3**), J_λ is strictly convex on $\mathbb{W}$, which guarantees the uniqueness of the critical point and the uniqueness of the solution to (P).

To conclude, Corollary A.2 in Appendix A implies $u \in L^\infty(\Omega)$. □

Proof of Theorem 1.1. This proof follows the proof of [21, Theorem 2.3]. However, the steps 3 and 4 are different due to the more general operator $\mathbf{a}$. For the reader's convenience, we have included the complete proof.

We perform the proof along four steps.

Step 1. Time-discretization of (L_T).
Let $N \in \mathbb{N}^*$, $T > 0$ and set $\Delta_t = \frac{T}{N}$. For $0 \leq n \leq N$, we define $t_n = n\Delta_t$ and for $n \in \{1, \ldots, N\}$, for $(t, x) \in [t_{n-1}, t_n) \times \Omega$

$$h_{\Delta_t}(t, x) = h^n(x) := \frac{1}{\Delta_t}\int_{t_{n-1}}^{t_n} h(s, x)\,\mathrm{d}s.$$

The Jensen's inequality implies that $\|h_{\Delta_t}\|_{L^q(Q_T)} \leq \|h\|_{L^q(Q_T)}$ and we have $h_{\Delta_t} \to h$ in $L^q(Q_T)$.

Now, we define the iterative scheme

$$u^0 = u_0 \text{ and for } n \geq 1, u^n \text{ is solution of}$$
$$\times \begin{cases} \dfrac{u^n - u^{n-1}}{\Delta_t} - \nabla \cdot \mathbf{a}(x, \nabla u^n) = h^n & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The sequence $(u^n)_{n\in\{1,\ldots,N\}}$ is well-defined because existence and uniqueness of $u^1 \in \mathbb{W} \cap L^\infty(\Omega)$ follow from Lemma 2.1 with $g = \Delta_t h^1 + u^0 \in L^q(\Omega)$ and by induction we obtain in the same way the existence of (u^n), for any $n = 2, \ldots, N$.

For $n = 1, \ldots, N$ and $t \in [t_{n-1}, t_n)$, we define the functions

$$u_{\Delta_t}(t) = u^n \quad \text{and} \quad \tilde{u}_{\Delta_t}(t) = \frac{(t - t_{n-1})}{\Delta_t}(u^n - u^{n-1}) + u^{n-1}, \tag{2.2}$$

which satisfy

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \nabla \cdot \mathbf{a}(x, \nabla u_{\Delta_t}) = h_{\Delta_t} \quad \text{in } Q_T. \tag{2.3}$$

Step 2. *A priori* estimates for u_{Δ_t} and $\tilde{u}_{\Delta_t}$.

Multiplying the equation in (2.1) by $(u^n - u^{n-1})$ and summing from $n = 1$ to $N' \leq N$, we get

$$\sum_{n=1}^{N'} \Delta_t \int_\Omega \left(\frac{u^n - u^{n-1}}{\Delta_t} \right)^2 \mathrm{d}x + \sum_{n=1}^{N'} \int_\Omega \mathbf{a}(x, \nabla u^n) \cdot \nabla(u^n - u^{n-1}) \, \mathrm{d}x$$

$$= \sum_{n=1}^{N'} \int_\Omega h^n (u^n - u^{n-1}) \, \mathrm{d}x, \tag{2.4}$$

hence by Young's inequality, we obtain

$$\frac{1}{2} \sum_{n=1}^{N'} \Delta_t \int_\Omega \left(\frac{u^n - u^{n-1}}{\Delta_t} \right)^2 \mathrm{d}x + \sum_{n=1}^{N'} \int_\Omega \mathbf{a}(x, \nabla u^n) \cdot \nabla(u^n - u^{n-1}) \, \mathrm{d}x$$

$$\leq \frac{1}{2} \|h\|^2_{L^2(Q_T)}.$$

Thus, we obtain

$$\left(\frac{\partial \tilde{u}_{\Delta t}}{\partial t} \right)_{\Delta t} \text{ is bounded in } L^2(Q_T) \text{ uniformly in } \Delta_t. \tag{2.5}$$

Since A is strictly convex and from (1.1), we obtain for any N'

$$\begin{aligned} \frac{1}{2} \|h\|^2_{L^2(Q_T)} &\geq \sum_{n=1}^{N'} \int_\Omega \mathbf{a}(x, \nabla u^n) \cdot \nabla(u^n - u^{n-1}) \, \mathrm{d}x \\ &\geq \sum_{n=1}^{N'} \int_\Omega A(x, \nabla u^n) - A(x, \nabla u^{n-1}) \, \mathrm{d}x \\ &= \int_\Omega A(x, \nabla u^{N'}) \, \mathrm{d}x - \int_\Omega A(x, \nabla u^0) \, \mathrm{d}x \\ &\geq \frac{\gamma}{p^+ - 1} \int_\Omega |\nabla u^{N'}|^{p(x)} \, \mathrm{d}x - \int_\Omega A(x, \nabla u^0) \, \mathrm{d}x. \end{aligned}$$

We conclude that

$$(u_{\Delta_t}) \text{ and } (\tilde{u}_{\Delta_t}) \text{ are bounded in } L^\infty(0, T, \mathbb{W}) \text{ uniformly in } \Delta_t. \tag{2.6}$$

Furthermore, using (2.5), we have

$$\sup_{[0,T]} \|u_{\Delta_t} - \tilde{u}_{\Delta_t}\|_{L^2(\Omega)} \leq \max_{n=1,\ldots,N} \|u^n - u^{n-1}\|_{L^2(\Omega)} \leq C \Delta_t^{1/2}. \tag{2.7}$$

Therefore, for $\Delta_t \to 0$, we deduce that there exist $u, v \in L^\infty(0, T, \mathbb{W})$ such that (up to a subsequence)

$$\tilde{u}_{\Delta_t} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0, T, \mathbb{W}), \quad u_{\Delta_t} \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0, T, \mathbb{W}), \tag{2.8}$$

and

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(Q_T). \tag{2.9}$$

Inequality (2.7) implies that $u \equiv v$. By (2.8), for any $r \geq 1$

$$\tilde{u}_{\Delta_t},\ u_{\Delta_t} \rightharpoonup u \quad \text{in } L^r(0, T, \mathbb{W}). \tag{2.10}$$

Step 3. u satisfies (L_T).

Since $p^- > \frac{2d}{d+2}$, Theorem A.1 gives the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Hence, plugging (2.5), (2.6) the compactness Aubin–Simon's result (see [32]) implies that (up to a subsequence),

$$\tilde{u}_{\Delta_t} \to u \in C([0, T], L^2(\Omega)). \tag{2.11}$$

Equation (2.3) multiplied by $(u_{\Delta_t} - u)$ yields

$$\int_0^T \int_\Omega \frac{\partial \tilde{u}_{\Delta_t}}{\partial t}(u_{\Delta_t} - u)\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \mathbf{a}(x, \nabla u_{\Delta_t}) \cdot \nabla(u_{\Delta_t} - u)\,\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_\Omega h_{\Delta_t}(u_{\Delta_t} - u)\mathrm{d}x\mathrm{d}t.$$

Rearranging the terms in the last equations and using (2.7)–(2.10), we have

$$\int_0^T \int_\Omega \left(\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \frac{\partial u}{\partial t}\right)(\tilde{u}_{\Delta_t} - u)\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega (\mathbf{a}(x, \nabla u_{\Delta_t})$$
$$- a(x, \nabla u)) \cdot \nabla(u_{\Delta_t} - u)\,\mathrm{d}x\mathrm{d}t = o_{\Delta_t}(1),$$

where $o_{\Delta_t}(1) \to 0$ as $\Delta_t \to 0^+$. Thus, we get

$$\frac{1}{2}\int_\Omega |\tilde{u}_{\Delta_t}(T) - u(T)|^2 dx + \int_0^T \int_\Omega (\mathbf{a}(x, \nabla u_{\Delta_t})$$
$$- \mathbf{a}(x, \nabla u)) \cdot \nabla(u_{\Delta_t} - u)\,\mathrm{d}x\mathrm{d}t = o_{\Delta_t}(1).$$

Using (2.11), we obtain

$$\int_0^T \int_\Omega (\mathbf{a}(x, \nabla u_{\Delta_t}) - \mathbf{a}(x, \nabla u)) \cdot \nabla(u_{\Delta_t} - u)\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } \Delta_t \to 0^+. \tag{2.12}$$

We now prove that

$$\int_0^T \int_\Omega |\nabla(u_{\Delta_t} - u)|^{p(x)}\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } \Delta_t \to 0^+. \tag{2.13}$$

To establish (2.13), the general form of $\mathbf{a}$ do not allow to use the algebraic inequalities of [33] as in the proof of [21, Theorem 2.3]. The convexity of Φ and assumption (**A3**) bring the arguments to conclude.

Indeed for this purpose, we distinguish two cases: (i) $p < 2$ and (ii) $p \geq 2$. Let us first consider the case $p < 2$. Setting $q(x) = \frac{p(x)(2-p(x))}{2}$ and $\Omega^- = \{x \in \Omega : p(x) < 2\}$, since u, $u_{\Delta_t} \in L^\infty(0,T,\mathbb{W})$ we have from the Hölder inequality

$$\begin{aligned}
&\int_{\Omega^-} |\nabla(u-u_{\Delta_t})|^{p(x)}\,\mathrm{d}x \\
&\quad \leq C\left\|\frac{|\nabla(u-u_{\Delta_t})|^{p(x)}}{(|\nabla u|+|\nabla u_{\Delta_t}|)^{q(x)}}\right\|_{L^{\frac{2}{p(x)}}(\Omega^-)} \|(|\nabla u|+|\nabla u_{\Delta_t}|)^{q(x)}\|_{L^{\frac{2}{2-p(x)}}(\Omega^-)} \\
&\quad \leq \tilde{C}\left\|\frac{|\nabla(u-u_{\Delta_t})|^{p(x)}}{(|\nabla u|+|\nabla u_{\Delta_t}|)^{q(x)}}\right\|_{L^{\frac{2}{p(x)}}(\Omega^-)} \stackrel{\text{def}}{=} \tilde{C}\mathcal{I}.
\end{aligned}$$

Integrating in time the previous inequality and splitting the integral of the right-hand side, we obtain

$$\begin{aligned}
&\int_0^T\int_{\Omega^-} |\nabla(u-u_{\Delta_t})|^{p(x)}\,\mathrm{d}x\mathrm{d}t \\
&\quad \leq C\int_{\mathcal{I}\leq 1}\left(\int_{\Omega^-}\frac{|\nabla(u-u_{\Delta_t})|^2}{(|\nabla u|+|\nabla u_{\Delta_t}|)^{2-p(x)}}\,\mathrm{d}x\right)^{2^{-1}\sup_{\Omega^-} p(x)}\mathrm{d}t \\
&\qquad + C\int_{\mathcal{I}>1}\left(\int_{\Omega^-}\frac{|\nabla(u-u_{\Delta_t})|^2}{(|\nabla u|+|\nabla u_{\Delta_t}|)^{2-p(x)}}\,\mathrm{d}x\right)^{\frac{p^-}{2}}\mathrm{d}t. \qquad (2.14)
\end{aligned}$$

In the other hand, using assumption (**A3**), we deduce that

$$\gamma\int_{\Omega^-}\frac{|\nabla(u-u_{\Delta_t})|^2}{(|\nabla u|+|\nabla u_{\Delta_t}|)^{2-p(x)}}\,\mathrm{d}x \leq \int_\Omega(\mathbf{a}(x,\nabla u_{\Delta_t})-\mathbf{a}(x,\nabla u))\cdot\nabla(u_{\Delta_t}-u)\,\mathrm{d}x. \qquad (2.15)$$

Hence, plugging (2.14), (2.15) and Hölder's inequality, (2.12) implies

$$\int_0^T\int_{\Omega^-}|\nabla(u-u_{\Delta_t})|^{p(x)}\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } \Delta_t \to 0. \qquad (2.16)$$

We now deal with the case $p(x) \geq 2$ and proceed as in [28] (see also [29] for other related issues). From the convexity of Φ, we obtain

$$\int_{\Omega^+}\Phi(x,|\nabla u|)\,\mathrm{d}x \leq \int_{\Omega^+}\Phi\left(x,\frac{|\nabla(u+u_{\Delta_t})|}{2}\right)\mathrm{d}x + \frac{1}{2}\int_{\Omega^+}\mathbf{a}(x,\nabla u)\cdot\nabla(u-u_{\Delta_t})\,\mathrm{d}x$$

and similarly

$$\begin{aligned}
&\int_{\Omega^+}\Phi(x,|\nabla u_{\Delta_t}|)\,\mathrm{d}x \\
&\quad \leq \int_{\Omega^+}\Phi\left(x,\frac{|\nabla(u+u_{\Delta_t})|}{2}\right)\mathrm{d}x + \frac{1}{2}\int_{\Omega^+}\mathbf{a}(x,\nabla u_{\Delta_t})\cdot\nabla(u_{\Delta_t}-u)\,\mathrm{d}x.
\end{aligned}$$

Adding both above relations, we get

$$\begin{aligned}\frac{1}{2}\int_{\Omega^+}&(\mathbf{a}(x,\nabla u)-\mathbf{a}(x,\nabla u_{\Delta_t}))\cdot(\nabla u-\nabla u_{\Delta_t})\,\mathrm{d}x\\ &\geq \int_{\Omega^+}\Phi(x,|\nabla u|)dx+\int_{\Omega^+}\Phi(x,|\nabla u_{\Delta_t}|)\,\mathrm{d}x\\ &\quad -2\int_0^T\int_{\Omega^+}\Phi\left(x,\frac{|\nabla(u+u_{\Delta_t})|}{2}\right)\mathrm{d}x. \end{aligned}\tag{2.17}$$

Using (1.3), we have

$$\begin{aligned}\int_{\Omega^+}&\Phi(x,|\nabla u|)\,\mathrm{d}x+\int_{\Omega^+}\Phi(x,|\nabla u_{\Delta_t}|)\,\mathrm{d}x\\ &\geq 2\int_{\Omega^+}\Phi\left(x,\frac{|\nabla(u+u_{\Delta_t})|}{2}\right)\mathrm{d}x+2\int_{\Omega^+}\Phi\left(x,\frac{|\nabla(u-u_{\Delta_t})|}{2}\right)\mathrm{d}x.\end{aligned}$$

Therefore, plugging the two last inequalities and (1.1), we deduce that

$$\begin{aligned}\int_{\Omega^+}&(\mathbf{a}(x,\nabla u)-\mathbf{a}(x,\nabla u_{\Delta_t}))\cdot\nabla(u-u_{\Delta_t})\,\mathrm{d}x\\ &\geq 4\int_{\Omega^+}\Phi\left(x,\frac{|\nabla(u-u_{\Delta_t})|}{2}\right)\mathrm{d}x\geq\frac{4\gamma}{2^{p^+}(p^+-1)}\int_{\Omega^+}|\nabla(u-u_{\Delta_t})|^{p(x)}\,\mathrm{d}x.\end{aligned}\tag{2.18}$$

Now from (2.12), (2.18) combining with (2.16), we obtain (2.13). This implies that ∇u_{Δ_t} converges to ∇u in $L^{p(x)}(Q_T)$ and u_{Δ_t} converges to u in $\mathbb{W}$. Furthermore,

$$\mathbf{a}(x,\nabla u_{\Delta_t})\to\mathbf{a}(x,\nabla u)\quad\text{in }(L^{p_c(x)}(Q_T))^d\tag{2.19}$$

with $p_c(x)=\frac{p(x)}{p(x)-1}$ the conjugate exponent of p. Indeed, we observe that from (2.13) we get

$$|\nabla u_{\Delta_t}|^{p(x)}\to|\nabla u|^{p(x)}\quad\text{in }L^1((0,T)\times\Omega)\quad\text{as }\Delta_t\to 0^+.$$

Using [9, Theorem 4.9], we have for a subsequence $\{\Delta_{tn}\}$,

$$\nabla u_{\Delta_{tn}}\to\nabla u\text{ a.e.}\quad\text{in }(0,T)\times\Omega\quad\text{and}\quad|\nabla u_{\Delta_{tn}}|^{p(x)}\leq g\in L^1((0,T)\times\Omega).$$

Using the dominated convergence theorem and observing that from (1.2):

$$|\mathbf{a}(x,\nabla u_{\Delta_{tn}})|\leq c_1|\nabla u_{\Delta_{tn}}|^{p(x)-1}\leq g^{\frac{p(x)-1}{p(x)}}\in L^{p_c(x)},$$

we obtain

$$\mathbf{a}(x,\nabla u_{\Delta_{tn}})\to\mathbf{a}(x,\nabla u)\quad\text{in }(L^{p_c(x)}(Q_T))^d$$

from which together with a classical compactness argument we get (2.19).

Finally, Step 1, (2.9) and (2.19) allow to pass to the limit, in the distribution sense, in Eq. (2.3) and we conclude that u is a weak solution of (L_T).

Assume that there exist u and v weak solutions of (L_T). Thus,

$$\int_0^T\int_\Omega \frac{\partial(u-v)}{\partial t}(u-v)\,\mathrm{d}x\mathrm{d}t - \int_0^T (\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla v))\cdot\nabla(u-v)\,\mathrm{d}t = 0.$$

Since $u(0) = v(0)$, the above equality implies that $u \equiv v$ and we deduce the uniqueness.

Step 4. u belongs to $C([0,T];\mathbb{W})$.

Since $u \in C([0,T];L^2(\Omega)) \cap L^\infty([0,T];\mathbb{W})$ and $p \in \mathcal{P}^{\log}(\Omega)$, $u : t \in [0,T] \mapsto \mathbb{W}$ is weakly continuous.

Define $\Psi : \mathbb{W} \ni u \mapsto \int_\Omega \Phi(x,|\nabla u|)\,\mathrm{d}x = \int_\Omega A(x,\nabla u)\,\mathrm{d}x$. Then Ψ is differentiable and $\Psi'(u) = -\nabla\cdot\mathbf{a}(x,\nabla u) \in \mathbb{W}'$. Note that Ψ is a semimodular on $\mathbb{W}$ and thus is weakly lower semicontinuous (see [14, Theorem 2.2.8]). Hence, fixing $t_0 \in [0,T]$, we have

$$\int_\Omega A(x,\nabla u(t_0))\,\mathrm{d}x \le \liminf_{t\to t_0}\int_\Omega A(x,\nabla u(t))\,\mathrm{d}x.$$

From (2.4) with $\sum_{n=N''}^{N'}$ for $1 \le N'' \le N'$ and the convexity of Ψ, it follows that u satisfies for any $t \in [t_0,T]$:

$$\begin{aligned}\int_{t_0}^t\int_\Omega\left(\frac{\partial u}{\partial t}\right)^2\mathrm{d}x\mathrm{d}s + \int_\Omega A(x,\nabla u(t))\,\mathrm{d}x\\ \le \int_{t_0}^t\int_\Omega h\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}s + \int_\Omega A(x,\nabla u(t_0))\,\mathrm{d}x.\end{aligned}\tag{2.20}$$

Passing to the limit, we obtain

$$\limsup_{t\to t_0^+}\int_\Omega A(x,\nabla u(t))\,\mathrm{d}x \le \int_\Omega A(x,\nabla u(t_0))\,\mathrm{d}x.$$

Thus, we get $\lim_{t\to t_0^+}\int_\Omega A(x,\nabla u(t))\,\mathrm{d}x = \int_\Omega A(x,\nabla u(t_0))\,\mathrm{d}x$. Hence from (1.1) and the dominated convergence theorem, we also obtain $\lim_{t\to t_0^+}\int_\Omega \frac{|\nabla u(t)|^{p(x)}}{p(x)}\,\mathrm{d}x = \int_\Omega \frac{|\nabla u(t_0)|^{p(x)}}{p(x)}\,\mathrm{d}x$.

Now, we prove the left continuity. Let $0 < k \le t - t_0$. Multiplying (L_T) by $\tau_k(u)(s) = \frac{u(s+k)-u(s)}{k}$ and integrating over $(t_0,t)\times\Omega$, the convexity gives

$$\begin{aligned}\int_{t_0}^t\int_\Omega \tau_k(u)\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}s + \int_t^{t+k}\int_\Omega\frac{A(x,\nabla u(t))}{k}\,\mathrm{d}x\mathrm{d}s - \int_{t_0}^{t_0+k}\int_\Omega\frac{A(x,\nabla u(t_0))}{k}\,\mathrm{d}x\mathrm{d}s\\ \ge \int_{t_0}^t\int_\Omega \tau_k(u)h\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{2.21}$$

By the dominated convergence theorem as $k \to 0^+$, we have

$$\int_t^{t+k}\int_\Omega \frac{A(x,\nabla u(s))}{k}\,\mathrm{d}x\mathrm{d}s \to \int_\Omega A(x,\nabla u(t))\,\mathrm{d}x,$$

$$\int_{t_0}^{t_0+k}\int_\Omega \frac{A(x,\nabla u(s))}{k}\,\mathrm{d}x\mathrm{d}s \to \int_\Omega A(x,\nabla u(t_0))\,\mathrm{d}x.$$

Hence, (2.21) yields

$$\int_{t_0}^{t}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\mathrm{d}s + \int_\Omega A(x,\nabla u(t))\,\mathrm{d}x \geq \int_{t_0}^{t}\int_\Omega h\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}s + \int_\Omega A(x,\nabla u(t_0))\,\mathrm{d}x.$$

From the above inequality, we deduce that we have the equality in (2.20). Using the dominated convergence theorem, we obtain that

$$\lim_{t\to t_0}\int_\Omega A(x,\nabla u(t))\,\mathrm{d}x = \int_\Omega A(x,\nabla u(t_0))\,\mathrm{d}x$$

and thus

$$\lim_{t\to t_0}\int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x = \int_\Omega |\nabla u(t_0)|^{p(x)}\,\mathrm{d}x.$$

Then from the uniform convexity of $\mathbb{W}$ we deduce that $u \in C([0,T];\mathbb{W})$. □

3. Existence of Solutions to (P_T)

3.1. *Existence of a local solution*

In the first part, we prove Theorem 1.2. For this purpose, we proceed as in the proof of Theorem 1.1 splitting the proof in several steps. The proof of Theorem 1.2 is almost similar as the proof of [21, Theorem 2.5]. Several differences appear in Step 3 due to the general form of **a**. For sake of clarity, we give the entire proof.

Step 1. Existence of barrier functions.

Consider the equations, for $i \in \{0,1,2\}$

$$\begin{cases} \dfrac{dv_i}{dt} = L_i(v_i), \\ v_i(0) = (-1)^i \nu, \end{cases} \tag{3.1}$$

where $\nu = \|u_0\|_\infty$.

For $i \in \{0,1,2\}$, the Cauchy–Lipschitz theorem gives the existence of $T_i^{\max} \in (0,+\infty]$ and a unique maximal solution v_i to (3.1) on $[0,T_i^{\max})$.

If (H1) holds, we take $T \in (0,T_0^{\max})$ otherwise, if (H2) holds, we take $T \in (0,\min(T_1^{\max};T_2^{\max}))$.

Let $N \in \mathbb{N}^*$. Set $\Delta_t = \frac{T}{N}$ and consider the family (v_i^n) defined by $v_i^n = v_i(t_n) = v_i(n\Delta_t)$ for $n \in \{1,\dots,N\}$. Hence for any $i \in \{0,1,2\}$

$$v_i^{n+1} = v_i^n + \int_{t_n}^{t_{n+1}} L_i(v_i(s))\,\mathrm{d}s, \quad \forall n \in \{0,\dots,N-1\}.$$

Replacing L_1 (respectively, L_2) by $\min(L_1,0)$ (respectively, $\max(L_2,0)$) in (H2), we can assume that $L_1 \leq 0$ and $L_2 \geq 0$. We get for $n \in \{0,\dots,N\}$, $v_1(T) \leq v_1^n \leq -\nu$ and for $i = 0$ or $i = 2$, $\nu \leq v_i^n \leq v_i(T)$.

Step 2. Semi-discretization in time of (P_T). Introduce the following iterative scheme (u^n) defined as

$$u^0 = u_0 \quad \text{and} \quad \begin{cases} u^n - \Delta_t \nabla \cdot \mathbf{a}(x, \nabla u^n) = u^{n-1} + \Delta_t f(x, u^{n-1}) & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

We just prove the existence of u^1. The conditions (f_1) and (f_2) insure that $f(., u^0) \in L^q(\Omega)$ with $q > \frac{d}{p^-}$. Thus, Lemma 2.1 applied with $g = u^0 + \Delta_t f(x, u^0) \in L^q(\Omega)$ gives the existence of $u^1 \in \mathbb{W} \cap L^\infty(\Omega)$.

Let u_{Δ_t} and $\tilde{u}_{\Delta_t}$ be defined as in (2.2) and for $t < 0$, $u_{\Delta_t}(t) = u_0$. Thus, (2.3) is satisfied with $h_{\Delta_t}(t) \stackrel{\text{def}}{=} f(x, u_{\Delta_t}(t - \Delta_t))$.

Step 3. (u^n) is bounded in $L^\infty(\Omega)$ uniformly in Δ_t.

We first consider the case where (H1) is valid. We claim that for all n, $|u^n| \leq v_0^n$ in Ω. We just prove it in case of $n = 1$. Since L_0 and v_0 are nondecreasing, we get

$$u^1 - v_0^1 - \Delta_t \nabla \cdot \mathbf{a}(x, \nabla u^1) = \int_0^{\Delta_t} (f(x, u_0) - L_0(v_0(s)))\,\mathrm{d}s + u_0 - v_0^0 \leq 0.$$

Multiplying the previous inequality by $(u^1 - v_0^1)^+ = \max(u^1 - v_0^1, 0)$ and integrating on $\omega = \{x \in \Omega \,|\, u^1(x) > v_0^1\}$, we get

$$\int_\omega (u^1 - v_0^1)^2\,\mathrm{d}x + C\Delta_t \int_\omega |\nabla u^1|^{p(x)}\,\mathrm{d}x$$
$$\leq \int_\omega (u^1 - v_0^1)^2\,\mathrm{d}x + \Delta_t \int_\omega \mathbf{a}(x, \nabla u^1)\nabla u^1\,\mathrm{d}x \leq 0.$$

Hence, $u^1 \leq v_0^1$ and by the same method we have $-v_0^1 \leq u^1$.

For (H2), we claim that for all n, $v_1^n \leq u^n \leq v_2^n$ in Ω. Let $n = 1$. Since L_1, L_2, $-v_1$ and v_2 are nondecreasing:

$$u^1 - v_1^1 - \Delta_t \nabla \cdot \mathbf{a}(x, \nabla u^1) = \int_0^{\Delta_t} (f(x, u_0) - L_1(v_1(s)))\,\mathrm{d}s + u_0 - v_1^0 \geq 0,$$

$$u^1 - v_2^1 - \Delta_t \nabla \cdot \mathbf{a}(x, \nabla u^1) = \int_0^{\Delta_t} (f(x, u_0) - L_2(v_2(s)))\,\mathrm{d}s + u_0 - v_2^0 \leq 0.$$

Multiplying the first inequality by $(v_1^1 - u^1)^+$ and the second inequality by $(u^1 - v_2^1)^+$ and integrating respectively on $\omega_1 = \{x \in \Omega \,|\, v_1^1 > u^1(x)\}$ and $\omega_2 = \{x \in \Omega \,|\, v_2^1 < u^1(x)\}$ and using (1.2), we get

$$-\int_{\omega_1} (u^1 - v_1^1)^2\,\mathrm{d}x - C\Delta_t \int_{\omega_1} |\nabla u^1|^{p(x)}\,\mathrm{d}x$$
$$\geq -\int_{\omega_1} (u^1 - v_1^1)^2\,\mathrm{d}x - \Delta_t \int_{\omega_1} \mathbf{a}(x, \nabla u^1)\nabla u^1\,\mathrm{d}x \geq 0,$$

$$\int_{\omega_2} (u^1 - v_2^1)^2\,\mathrm{d}x + C\Delta_t \int_{\omega_2} |\nabla u^1|^{p(x)}\,\mathrm{d}x$$
$$\leq \int_{\omega_2} (u^1 - v_2^1)^2\,\mathrm{d}x + \Delta_t \int_{\omega_2} \mathbf{a}(x, \nabla u^1)\nabla u^1\,\mathrm{d}x \leq 0.$$

Then $v_1^1 \le u^1 \le v_2^1$. By induction, we deduce that for $n \in \{0, \ldots, N\}$, $v_1^n \le u^n \le v_2^n$ in Ω.

Therefore,

$$(u_{\Delta_t}),\ (\tilde{u}_{\Delta_t}) \text{ are bounded in } L^\infty(Q_T) \text{ uniformly in } \Delta_t \tag{3.3}$$

and

$$(h_{\Delta_t}) \text{ is bounded in } L^2(Q_T) \text{ uniformly in } \Delta_t.$$

Indeed, either (H1) holds which implies

$$|f(x, u^n)| \le L_0(u^n) \le L_0(v_0(T))$$

or (H2) holds, we have

$$|f(x, u^n)| \le \max(-L_1(u^n), L_2(u^n)) \le \max(-L_1(v_1(T)), L_2(v_2(T))).$$

Hence

$$\|h_{\Delta_t}\|^2_{L^2(Q_T)} = \Delta_t \sum_{n=1}^{N} \|f(x, u^{n-1})\|^2_{L^2(\Omega)} \le C.$$

Step 4. End of the proof.

By the same computations of Step 2 of the proof of Theorem 1.1, we obtain estimates and we prove that there exists $u \in L^\infty(0, T, \mathbb{W})$ such that

$$\tilde{u}_{\Delta_t}, u_{\Delta_t} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0, T, \mathbb{W}) \quad \text{and} \quad \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(Q_T).$$

Relation (2.5) implies that $(\tilde{u}_{\Delta_t})$ is equicontinuous in $C([0, T]; L^r(\Omega))$ for $1 \le r \le 2$. By the interpolation inequality and (3.3), we obtain that $(\tilde{u}_{\Delta_t})$ is equicontinuous in $C([0, T]; L^r(\Omega))$ for any $r > 1$.

By (2.6) and Step 3, we deduce applying the Ascoli–Arzela theorem that (up to a subsequence) for any $r > 1$

$$\tilde{u}_{\Delta_t} \to u \quad \text{in } C([0, T]; L^r(\Omega)).$$

Since (u_{Δ_t}) is uniformly bounded in $L^\infty(Q_T)$, (f_1) implies

$$\|h_{\Delta_t}(t) - f(., u(t))\|_{L^2(\Omega)} \le C\|u_{\Delta_t}(t - \Delta_t) - u(t)\|_{L^2(\Omega)}.$$

Hence we deduce that $h_{\Delta_t} \to f(\cdot, u)$ in $L^\infty(0, T, L^2(\Omega))$. Next, we follow Step 4 of Theorem 1.1 and obtain that u is a weak solution to (P_T).

Now, we prove the uniqueness of the solution to (P_T). Let w be another weak solution of (P_T). By (f_1), for $t \in [0, T]$:

$$\begin{aligned}
&\frac{1}{2}\|u(t) - w(t)\|^2_{L^2(\Omega)} - \int_0^t \langle \mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla w), u - w\rangle \, \mathrm{d}s \\
&\quad = \int_0^T \int_\Omega (f(x, u) - f(x, w))(u - w) \, \mathrm{d}x\mathrm{d}s \le C \int_0^t \|u(s) - w(s)\|^2_{L^2(\Omega)} \, \mathrm{d}s.
\end{aligned}$$

Since $u \to -\nabla \cdot \mathbf{a}(x, \nabla u)$ is a monotone operator from $\mathbb{W}$ to $\mathbb{W}'$, the second term in the left-hand side is nonnegative. Then, by Gronwall's lemma, we deduce that $u \equiv w$.

Step 5 of the proof of Theorem 1.1 again goes through and completes the proof. □

3.2. *Existence of global solution of (P_T)*

Now, we prove Theorems 1.3 and 1.4. To ensure the existence of barrier functions, we will need the following lemma.

Lemma 3.1. *Assume conditions* (**A1**)–(**A4**). *Let $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be a function satisfying (f_1), (f_3)–(f_5) and $h \in L^q(\Omega)$ with $q > \frac{d}{p^-}$. Assume that f and h are nonnegative functions. Then the stationary problem*

$$(S) \quad \begin{cases} -\nabla \cdot \mathbf{a}(x, \nabla u) = f(x, u) + h & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega \end{cases}$$

admits a nontrivial weak solution $u \in \mathbb{W}$. Furthermore, $u \in L^\infty(\Omega)$.

We define the notion of weak solution to (S) as follows.

Definition 3.1. Any function $w \in \mathbb{W}$ is called a weak solution to (S) if for all $\varphi \in \mathbb{W}$,

$$\int_\Omega \mathbf{a}(x, \nabla w) \cdot \nabla\varphi \, \mathrm{d}x = \int_\Omega (f(x, w) + h)\varphi \, \mathrm{d}x.$$

Proof. Consider the energy functional E associated to (S) given by

$$E(u) = \int_\Omega A(x, \nabla u) \, \mathrm{d}x - \int_\Omega F(x, u) \, \mathrm{d}x - \int_\Omega hu \, \mathrm{d}x,$$

where $F(x, t) = \int_0^t f(x, s) \, \mathrm{d}s$.

Define

$$\alpha_\infty := \sup_{x\in\Omega} \limsup_{|s|\to\infty} \frac{|f(x, s)|}{|s|^{p^- -1}} \quad \text{and } \beta_0 := \inf_{x\in\Omega} \liminf_{s\to 0} \frac{|f(x, s)|}{|s|^{p^- -1}}.$$

By (f_3) and (f_4), for $\varepsilon > 0$, there exists a constant $M = M(\varepsilon)$ large enough such that for any $(x, t) \in \Omega \times \mathbb{R}$, $|F(x, t)| \leq M|t| + \frac{\alpha_\infty+\varepsilon}{p^-}|t|^{p^-}$ and such that $\alpha_\infty + \varepsilon < \gamma\Lambda^{p^-}(p_c)^-$. Hence, E is well-defined and continuous on $\mathbb{W}$. Moreover, by (1.1), for $\|u\|_{\mathbb{W}} \geq 1$, there exists $C > 0$ such that

$$\begin{aligned} E(u) \geq{}& \frac{\gamma}{p^+ - 1} \int_\Omega |\nabla u|^{p(x)} \, \mathrm{d}x - M \int_\Omega |u| \, \mathrm{d}x - \frac{\alpha_\infty + \varepsilon}{p^-} \\ &\times \int_\Omega |u|^{p^-} \, \mathrm{d}x - C\|h\|_{L^q(\Omega)}\|u\|_{\mathbb{W}} \\ \geq{}& \frac{\gamma}{p^+ - 1}\|u\|_{\mathbb{W}}^{p^-} - \frac{\alpha_\infty + \varepsilon}{p^-}\|u\|_{L^{p^-}(\Omega)}^{p^-} - \tilde{M}\|u\|_{\mathbb{W}} \\ \geq{}& \left(\frac{\gamma}{p^+ - 1} - \frac{\alpha_\infty + \varepsilon}{\Lambda^{p^-}p^-}\right)\|u\|_{\mathbb{W}}^{p^-} - \tilde{M}\|u\|_{\mathbb{W}}. \end{aligned}$$

Condition (f_4) implies that E is coercive. Thus, E admits a global minimizer $u \in \mathbb{W}$ which is a weak solution of (S).

We claim that $u \not\equiv 0$. We need to consider two cases $f(.,0) + h \not\equiv 0$ and $h \equiv 0$.

In the first case, this is obvious. For the second case, we establish that there exists $v \in \mathbb{W}$ such that $E(v) < 0$. Consider $\varepsilon > 0$ and $v_\varepsilon \in C_0^1(\Omega)$ such that

$$\beta_0 > \Gamma(\Lambda + \varepsilon)^{p^-}(p^-)_c, \quad \|v_\varepsilon\|_{\mathbb{W}} = 1 \quad \text{and} \quad \|v_\varepsilon\|_{L^{p^-}(\Omega)} > \frac{1}{\Lambda + \varepsilon}. \tag{3.4}$$

By (f_5), for $\eta > 0$, there exists $s_\eta > 0$ small enough such that for any $x \in \Omega$ and $0 < s \leq s_\eta$

$$|f(x,s)| \geq (\beta_0 - \eta)|s|^{p^- - 1}.$$

Hence we obtain using (1.1) and (3.4), $\theta = \theta(\eta) > 0$ small enough such that

$$\begin{aligned} E(\theta v_\varepsilon) &\leq \theta^{p^-}\left(\frac{\Gamma}{p^- - 1}\int_\Omega |\nabla v_\varepsilon|^{p(x)}\,\mathrm{d}x - \frac{\beta_0 - \eta}{p^-}\int_\Omega |v_\varepsilon|^{p^-}\,\mathrm{d}x\right) \\ &\leq \theta^{p^-}\left(\frac{\Gamma}{p^- - 1} - \frac{\beta_0 - \eta}{p^-}\left(\frac{1}{\Lambda + \varepsilon}\right)^{p^-}\right). \end{aligned}$$

Choosing η small enough and using the first inequality in (3.4), we conclude that $E(\theta v_\varepsilon) < 0$ and we deduce that $u \not\equiv 0$. Finally, by Corollary A.2, $u \in L^\infty(\Omega)$. $\square$

Lemma 3.2. *Let $p \in C^\beta(\overline{\Omega})$, $\beta \in (0,1)$. Assume conditions* (**A1**)–(**A4**) *and* (**A6**). *Let $\lambda \in \mathbb{R}^+$ such that*

$$\lambda \geq \lambda^* := \frac{\gamma(p_c)^-}{2|\Omega|^{1/d}C_0}$$

where C_0 is the best embedding constant of $W_0^{1,1}(\Omega) \subset L^{\frac{d}{d-1}}(\Omega)$. Let $w_\lambda \in \mathbb{W} \cap L^\infty(\Omega)$ be the unique solution of

$$(E_\lambda) \quad \begin{cases} -\nabla \cdot \mathbf{a}(x, \nabla w_\lambda) = \lambda & \text{in } \Omega, \\ w_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, there exist two constants C_1 and C_2 which do not depend to λ such that

$$\|w_\lambda\|_{L^\infty} \leq C_1 \lambda^{\frac{1}{(p^- - 1)}} \quad \text{and} \quad w_\lambda(x) \geq C_2 \lambda^{\frac{1}{p^+ - 1 + \mu}} \rho(x) \tag{3.5}$$

where $\mu \in (0,1)$ and $\rho(x) = d(x, \partial\Omega)$ denotes the distance of $x \in \Omega$ to the boundary of Ω.

Proof. We first prove the upper bound of (3.5). For that we follow closely the proof of [16, Lemma 2.1], Let u be the solution to (E_λ) for a fixed λ satisfying assumptions of the lemma. By the maximum principle, $u \geq 0$. Using classical regularity results (see [15, 18]), $u \in C^{1,\alpha}(\overline{\Omega})$ and from [34] $u > 0$ in Ω and satisfies the Hopf maximum principle. Now, for $k \geq 0$, set $A_k = \{x \in \Omega : u(x) > k\}$. Using $(u-k)^+$ as a testing function together with (1.1) and the Young inequality, we obtain for any $\epsilon > 0$:

$$\begin{aligned}\frac{\gamma}{p^+-1}\int_{A_k}|\nabla u|^{p(x)}\,\mathrm{d}x &\leq \int_{A_k}\mathbf{a}(x,\nabla u)\cdot\nabla u\,\mathrm{d}x = \lambda\int_{A_k}(u-k)\,\mathrm{d}x\\ &\leq \lambda|A_k|^{\frac{1}{d}}|(u-k)^+|_{L^{\frac{d}{d-1}}(\Omega)} \leq \lambda|A_k|^{1/d}C_0\int_{A_k}|\nabla u|\,\mathrm{d}x\\ &\leq \frac{\lambda|A_k|^{1/d}C_0}{p^-}\int_{A_k}\epsilon^{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\\ &\quad+\frac{\lambda|A_k|^{1/d}C_0}{(p^+)_c}\int_{A_k}\epsilon^{-p_c(x)}\,\mathrm{d}x. \end{aligned} \tag{3.6}$$

Taking $\epsilon = (\frac{\lambda^*}{\lambda})^{1/p^-}$, we have

$$\frac{\lambda|A_k|^{1/d}C_0}{p^-}\int_{A_k}\epsilon^{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x \leq \frac{\gamma}{2(p^+-1)}\int_{A_k}|\nabla u|^{p(x)}\,\mathrm{d}x$$

which implies together with (3.6)

$$\int_{A_k}|\nabla u|^{p(x)}\,\mathrm{d}x \leq \frac{2\lambda C_0(p^+-1)|A_k|^{1/d}}{\gamma(p^+)_c}\int_{A_k}\epsilon^{-p_c(x)}\,\mathrm{d}x \leq \frac{2\lambda C_0(p^+-1)}{\gamma(p^+)_c\epsilon^{(p^-)_c}}|A_k|^{1+\frac{1}{d}}. \tag{3.7}$$

From (1.2), (3.6) and (3.7), we get

$$\int_{A_k}(u-k)\,\mathrm{d}x = \frac{1}{\lambda}\int_{A_k}\mathbf{a}(x,\nabla u)\cdot\nabla u\,\mathrm{d}x \leq \frac{c_1}{\lambda}\int_{A_k}|\nabla u|^{p(x)}\,\mathrm{d}x \leq \tilde{K}|A_k|^{1+\frac{1}{d}}, \tag{3.8}$$

where $\tilde{K} = \frac{2c_1C_0(p^+-1)}{\gamma(p^+)_c\epsilon^{(p^-)_c}}$. By in [25, Lemma 5, Chap. 2] and (3.8), we deduce that

$$\|u\|_{L^\infty(\Omega)} \leq \tilde{K}(d+1)^{\frac{d+1}{d}}|\Omega|^{1/d}$$

from which we easily obtain that $\|u\|_{L^\infty(\Omega)} \leq C_1\lambda^{\frac{1}{p^--1}}$ where C_1 does not depends on λ. Next, we show the lower bound estimate. Since $\partial\Omega$ is C^2, there exists $\ell \in (0,1)$ small enough such that ρ is C^2 and

$$|\nabla\rho| \equiv 1 \quad \text{in } \{x \in \Omega : \rho(x) < 3\ell\} \tag{3.9}$$

(see [24, Lemma 14.16, p. 355]). As in [34], we introduce the following function: let $\kappa > 0$,

$$v_1(x) = \begin{cases}\kappa\rho(x), & \rho(x) < \ell,\\ \kappa\ell + \kappa\displaystyle\int_\ell^{\rho(x)} m(t)\,\mathrm{d}t, & \ell \leq \rho(x) \leq 2\ell,\\ \kappa\ell + \kappa\displaystyle\int_\ell^{2\ell} m(t)\,\mathrm{d}t, & 2\ell \leq \rho(x),\end{cases}$$

where $m(t) = (\frac{2\ell-t}{\ell})^{\frac{2}{p^--1}}$.

Next, we show that for a suitable value of κ, $v_1 \in C^1(\overline{\Omega})$ is a subsolution to (E_λ). Precisely,

$$\nabla \cdot \mathbf{a}(x, \nabla v_1) = \sum_{i=1}^{d} \frac{\partial a_i}{\partial x_i}(x, \nabla v_1) + \sum_{i,j=1}^{d} \frac{\partial a_i}{\partial \xi_j}(x, \nabla v_1) \frac{\partial^2 v_1}{\partial x_i \partial x_j}(x).$$

Using the definition of v_1, (3.9) and noting that for any $i,\, j \in \{1, \dots, d\}$

$$\frac{\partial a_i}{\partial \xi_j}(x, \xi) = \frac{1}{|\xi|} \frac{\partial \phi}{\partial \xi_j}(x, |\xi|)\xi_i \xi_j + \phi(x, |\xi|)\delta_{ij},$$

where δ_{ij} is the Kronecker symbol, we get

$$\nabla \cdot \mathbf{a}(x, \nabla v_1(x)) = \begin{cases} \kappa \displaystyle\sum_{i=1}^{d} \frac{\partial \phi}{\partial x_i}(x, |\kappa|) \frac{\partial \rho}{\partial x_i}(x) + \kappa \phi(x, |\kappa|)\Delta\rho & \text{if } \rho(x) < \ell, \\ 0 & \text{if } 2\ell \leq \rho(x) \end{cases}$$

and in $\{x \in \Omega \,:\, \ell \leq \rho(x) \leq 2\ell\}$,

$$\begin{aligned} \nabla \cdot \mathbf{a}(x, \nabla v_1) = \kappa m(\rho) \sum_{i=1}^{d} \frac{\partial \phi}{\partial x_i}(x, |\kappa| m(\rho)) \frac{\partial \rho}{\partial x_i}(x) + \kappa \sum_{i,j=1}^{d} \frac{\partial a_i}{\partial \xi_j}(x, \kappa m(\rho) \nabla \rho) \\ \times \left(m'(\rho) \frac{\partial \rho}{\partial x_i}(x) \frac{\partial \rho}{\partial x_j}(x) + m(\rho) \frac{\partial^2 \rho}{\partial x_i \partial x_j}(x) \right). \end{aligned}$$

Then, using hypotheses (**A4**), (**A6**) and relations (1.2) and (3.9), we obtain that $|\nabla \mathbf{a}(x, \nabla v_1(x))| \leq C\kappa^{p(x)-1+\mu}$ a.e. on Ω, for any $\mu \in (0, 1)$, where $C = C(\ell, \mu, p, \Omega)$ and independent of κ. Choosing κ such that $2C\kappa^{p^+-1+\mu} = \lambda$, we have that v_1 is a subsolution to (E_λ) and since $w_\lambda \geq v_1$, the lower bound in (3.5) is proved. $\square$

Remark 3.1. About the $C^{1,\alpha}(\overline{\Omega})$-regularity of the solution of (E_λ), we apply [15, Theorem 1.2]. More precisely, we need the condition on $\mathbf{a}$: for $\delta \in (0, 1)$, there exists $\tilde{c} > 0$ such that for any $x,\ y \in \overline{\Omega}$, $\eta \in \mathbb{R}^d$:

$$|\mathbf{a}(x, \eta) - \mathbf{a}(y, \eta)| \leq \tilde{c}|x - y|^\beta (1 + |\eta|^{p^+-1+\delta}),$$

where β is the Hölder exponent of p. Obviously, condition (**A6**) implies the above inequality.

Now, we give the proof of the existence of global solutions.

Proof of Theorems 1.3 and 1.4. We consider the stationary quasilinear elliptic problem associated to (P_T):

$$(P_\infty) \quad \begin{cases} -\nabla \cdot \mathbf{a}(x, \nabla u) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, we claim that if (C1) or (C2) hold then there exist $\underline{u}, \overline{u} \in \mathbb{W} \cap L^\infty(\Omega)$, a sub- and a supersolution of (P_∞) such that $\underline{u} \leq u_0 \leq \overline{u}$.

First, consider that (C1) holds. For $(x, s) \in \Omega \times \mathbb{R}$, define

$$G(x, s) = |\nabla \cdot \mathbf{a}(x, \nabla u_0(x))| + |f(x, s)|.$$

Consider the following problems:

$$\begin{cases} -\nabla \cdot \mathbf{a}(x, \nabla \underline{u}) = -G(x, \underline{u}) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\nabla \cdot \mathbf{a}(x, \nabla \overline{u}) = G(x, \overline{u}) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 3.1 implies the existence of $\underline{u}$ and $\overline{u} \in \mathbb{W} \cap L^\infty(\Omega)$. Moreover,

$$-\nabla \cdot \mathbf{a}(x, \nabla \underline{u}) = -G(\cdot, \underline{u}) \leq -\nabla \cdot \mathbf{a}(x, \nabla u_0) \quad \text{and}$$
$$-\nabla \cdot \mathbf{a}(x, \nabla \overline{u}) = G(\cdot, \overline{u}) \geq -\nabla \cdot \mathbf{a}(x, \nabla u_0) \quad \text{a.e in } \Omega.$$

Hence the weak comparison principle implies $\underline{u} \leq u_0$ and $\underline{u}$ is a subsolution of (P_∞). Similarly, we have that $\overline{u} \geq u_0$ and $\overline{u}$ is a supersolution of (P_∞).

Now, if (C2) holds, we use Lemma 3.2 above. Precisely, from assumptions (f_3) and (f_4), for $\varepsilon > 0$ there exists $M_0 = M_0(\alpha_\infty, \varepsilon) > 0$ such that for any $M \geq M_0$

$$|f(x, s)| \leq M + (\alpha_\infty + \varepsilon)|s|^{p^- - 1} \quad \text{for any } (x, s) \in \Omega \times \mathbb{R}.$$

From Lemma 3.1, there exists a positive solution, $\tilde{w}_{M,\varepsilon} \in \mathbb{W} \cap L^\infty(\Omega)$ to

$$\begin{cases} -\nabla \cdot \mathbf{a}(x, \nabla w) = M + (\alpha_\infty + \varepsilon)|w|^{p^- - 1} & \text{in } \Omega, \\ \tilde{w} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Moreover, we have $\tilde{w}_{M,\varepsilon} \in C^1(\overline{\Omega})$ (see [15, Theorem 1.2; 18, Theorem 4.4]).

Fix $0 < \lambda < M$, let w_λ be the solution of (E_λ). By Lemma 3.2, we deduce that $\|w_\lambda\|_{L^\infty(\Omega)} \to \infty$ as $\lambda \to \infty$. From the weak comparison principle, $w_\lambda \leq \tilde{w}_{M,\varepsilon}$. Therefore,

$$\|\tilde{w}_{M,\varepsilon}\|_{L^\infty(\Omega)} \to \infty \quad \text{as } M \to \infty.$$

Moreover, since $u_0 \in C_0^1(\overline{\Omega})$, there exists $K > 0$ such that for any $x \in \Omega$, $|u_0(x)| \leq K\text{dist}(x, \partial\Omega)$. Hence choosing λ and M large enough, we have by Lemma 3.2: $\tilde{w}_{M,\varepsilon} \geq w_\lambda \geq |u_0|$ in $\overline{\Omega}$.

Set $\overline{u} = \tilde{w}_M$ and $\underline{u} = -\tilde{w}_{M,\varepsilon}$. We deduce for M large enough, $\overline{u}$ and $\underline{u}$ are respectively a super- and a subsolution of (P_∞) such that $\underline{u} \leq u_0 \leq \overline{u}$.

Now, we proceed as in the proof of Theorem 1.2. We define the sequence (u^n) as follows:

$$\begin{cases} u^n - \Delta_t \nabla \cdot \mathbf{a}(x, \nabla u^n) = u^{n-1} + \Delta_t f(x, u^{n-1}) & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega \end{cases}$$

for $n = 1, 2, \ldots, N$ with $u^0 = u_0$. we prove for $n \geq 1$, $\underline{u} \leq u^n \leq \overline{u}$ in Ω. Indeed for $n = 1$, we have

$$\underline{u} - u^1 - \Delta_t(\nabla \cdot \mathbf{a}(x, \nabla \underline{u}) - \nabla \cdot \mathbf{a}(x, \nabla u^1)) \leq \underline{u} - u^0 + \Delta_t(f(x, u^0) - f(x, \underline{u})).$$

Since $s \mapsto f(x, s)$ is Lipschitz on $[-M_1, M_1]$ uniformly in $x \in \Omega$, where M_1 is the maximum of $\|\underline{u}\|_{L^\infty}$ and $\|\overline{u}\|_{L^\infty}$ thus, for Δ_t small enough, the function $\text{Id} - \Delta_t f$

is nondecreasing. Then we have

$$\underline{u} - u^1 - \Delta_t(\nabla \cdot \mathbf{a}(x, \nabla \underline{u}) - \nabla \cdot \mathbf{a}(x, \nabla u^1)) \leq (\mathrm{Id} - \Delta_t f)(\underline{u} - u^0).$$

Hence the right-hand side of the above inequality is nonpositive and thus by the weak comparison principle we have $\underline{u} \leq u^1$. Similarly, we prove $u^1 \leq \overline{u}$.

By induction, for $n \geq 1$, $\underline{u} \leq u^n \leq \overline{u}$ in Ω. Thus, (u^n) is uniformly bounded in $L^\infty(\Omega)$. The rest of the proof follows from steps 3 and 4 of the proof of Theorem 1.2. □

4. Stabilization

4.1. *Existence and uniqueness of the solution of the stationary problem*

In the purpose of investigating the behavior of the global solution to (P_T) as $t \to \infty$, we consider the stationary problem:

$$(P_+) \quad \begin{cases} -\nabla \cdot \mathbf{a}(x, \nabla u) = f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We define the notion of a weak solution as follows:

Definition 4.1. Any positive function $w \in \mathbb{W} \cap L^\infty(\Omega)$ is called a weak solution to (P_+) if for any $\varphi \in \mathbb{W}$,

$$\int_\Omega \mathbf{a}(x, \nabla w) \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f(x, w) \varphi \, \mathrm{d}x.$$

We first discuss the existence and the uniqueness of the weak solution to (P). In the proof of Theorem 1.5, we will use the following ray-strict convexity result on the energy functional proved in [23]. We start by a definition.

Definition 4.2. Let X be vector space. A functional $\mathcal{W} \colon \overset{\bullet}{V} \overset{\mathrm{def}}{=} \{v \colon \Omega \to (0, \infty) \colon v \in X\} \to \mathbb{R}$ will be called *ray–strictly convex* (*strictly convex*, respectively) if it satisfies for all $v_1, v_2 \in \overset{\bullet}{V}$ and for all $\theta \in (0, 1)$

$$\mathcal{W}((1 - \theta)v_1 + \theta v_2) \leq (1 - \theta) \cdot \mathcal{W}(v_1) + \theta \cdot \mathcal{W}(v_2), \tag{4.1}$$

where the inequality is strict unless v_2/v_1 is a constant (always strict if $v_1 \neq v_2$, respectively).

With the above definition, we have the following theorem.

Theorem 4.1. *Let $r \in [1, \infty)$ and $p \colon \Omega \to (1, \infty)$ satisfy $1 < p^- \leq p^+ < \infty$ and $r \leq p^-$. Assume that $A \colon \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}_+$ is continuous satisfying* (**A7**) *and*

(**A8**) $\xi \mapsto N(x, \xi) \overset{\mathrm{def}}{=} A(x, \xi)^{r/p(x)} \colon \mathbb{R}^d \to \mathbb{R}_+$ *is strictly convex for every $x \in \Omega$.*

Then (the restriction of) the functional $\mathcal{W}_A \colon X \overset{\text{def}}{=} \{v \in L^{p(x)/r}(\Omega) \ : \ |v|^{1/r} \in \mathbb{W}\} \to \mathbb{R}_+$ *defined by*

$$\mathcal{W}_A(v) \overset{\text{def}}{=} \int_\Omega A(x,\, \nabla(|v(x)|^{1/r}))\,\mathrm{d}x$$

to the convex cone $\overset{\bullet}{V}$ *is ray-strictly convex on* $\overset{\bullet}{V}$.

Furthermore, if $p(x) \not\equiv r$ *in* Ω, *i.e. if* $r = p^- \equiv p(x) \equiv p^+$ *does not hold in* Ω, *then* $\mathcal{W}_A$ *is even strictly convex on* $\overset{\bullet}{V}$.

Remark 4.1. We note that the function $\xi \mapsto A(x,\xi) = N(x,\xi)^{p(x)/r} \colon \mathbb{R}^d \to \mathbb{R}_+$ is strictly convex for each fixed $x \in \Omega$, thanks to the power function $t \mapsto t^{p(x)/r} \colon \mathbb{R}_+ \to \mathbb{R}_+$ being strictly monotone increasing and convex. Consequently, $A(x,\xi) > A(x,\mathbf{0}) = 0$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, and $A\colon \overline{\Omega} \times \mathbb{S}^{d-1} \to \mathbb{R}_+$ is bounded below and above on the compact set $\overline{\Omega} \times \mathbb{S}^{d-1} \subset \mathbb{R}^d \times \mathbb{R}^d$ by some positive constants; hence, the "coefficient" $A(x,\, \frac{\xi}{|\xi|})$, if $\xi = \nabla(|v|^{1/r}) \neq \mathbf{0}$, is bounded from below and above by some positive constants. Consequently, we recover that the ratio of the functionals in $\int_\Omega A(x,\nabla u)\,\mathrm{d}x$ and $\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x$ is bounded from below and above by the same positive constants as in (1.1).

Conversely, if we assume that $\xi \mapsto A(x,\xi)$ is strictly convex for each fixed $x \in \Omega$, then for any $1 < r \le p^-$, $\xi \mapsto N(x,\xi)$ is strictly convex if $r \neq 1$. Indeed applying [8, Lemma 2.1] to the function $F(\xi) = A(x,\xi)^{1/p(x)}$, we deduce that F is convex and thus, for any $r > 1$, $\xi \mapsto N(x,\xi) = (F(\xi))^r$ is strictly convex.

For the reader's convenience, we give the proof.

Proof of Theorem 4.1. Recalling Definition 4.2, let us consider any $v_1, v_2 \in \overset{\bullet}{V}$ and $\theta \in (0,1)$. Let us denote $v = (1-\theta)v_1 + \theta v_2$; hence, $v \in \overset{\bullet}{V}$. We obtain easily

$$\nabla(v_i(x)^{1/r}) = \frac{v_i^{1/r}}{r}\,\frac{\nabla v_i}{v_i} \quad \text{for } i = 1,2$$

and

$$\begin{aligned}
\nabla(v(x)^{1/r}) &= \frac{1}{r}\,\frac{(1-\theta)\nabla v_1 + \theta\nabla v_2}{((1-\theta)v_1 + \theta v_2)^{1-(1/r)}} \\
&= \frac{v^{1/r}}{r}\,\frac{(1-\theta)\nabla v_1 + \theta \nabla v_2}{v} \\
&= \frac{v^{1/r}}{r}\left((1-\theta)\,\frac{v_1}{v}\cdot\frac{\nabla v_1}{v_1} + \theta\,\frac{v_2}{v}\cdot\frac{\nabla v_2}{v_2}\right)
\end{aligned} \tag{4.2}$$

with the convex combination of coefficients $(1-\theta)\,\frac{v_1}{v}$ and $\theta\,\frac{v_2}{v}$,

$$(1-\theta)\,\frac{v_1}{v} + \theta\,\frac{v_2}{v} = 1.$$

Now, let $x \in \Omega$ be fixed. Since $\xi \mapsto N(x, \xi)$ is strictly convex, by our hypothesis, we may apply the identities from above to conclude that

$$\begin{aligned} N&\left(x, (1-\theta)\,\frac{v_1}{v} \cdot \frac{\nabla v_1}{v_1} + \theta\,\frac{v_2}{v} \cdot \frac{\nabla v_2}{v_2}\right) \\ &\le (1-\theta)\,\frac{v_1}{v} \cdot N\left(x, \frac{\nabla v_1}{v_1}\right) + \theta\,\frac{v_2}{v} \cdot N\left(x, \frac{\nabla v_2}{v_2}\right). \end{aligned} \tag{4.3}$$

The equality holds if and only if

$$\frac{\nabla v_1(x)}{v_1(x)} = \frac{\nabla v_2(x)}{v_2(x)}, \quad \text{which is equivalent to } \nabla\left(\frac{v_2(x)}{v_1(x)}\right) = 0. \tag{4.4}$$

Note that the homogeneity conditions (**A7**) and (**A8**) yield

$$N(x, t\xi) = |t|^r\, N(x, \xi) \quad \text{for all } t \in \mathbb{R},\ \xi \in \mathbb{R}^d. \tag{4.5}$$

Consequently, plugging (4.2), (4.3) and (4.5), we obtain

$$\begin{aligned} N(x, \nabla(v(x)^{1/r})) &= \frac{v}{r^r} \cdot N\left(x, (1-\theta)\,\frac{v_1}{v} \cdot \frac{\nabla v_1}{v_1} + \theta\,\frac{v_2}{v} \cdot \frac{\nabla v_2}{v_2}\right) \\ &\le (1-\theta)\,\frac{v_1}{r^r} \cdot N\left(x, \frac{\nabla v_1}{v_1}\right) + \theta\,\frac{v_2}{r^r} \cdot N\left(x, \frac{\nabla v_2}{v_2}\right) \\ &= (1-\theta) \cdot N\left(x, \nabla(v_1(x)^{1/r})\right) + \theta \cdot N\left(x, \nabla(v_2(x)^{1/r})\right). \end{aligned} \tag{4.6}$$

Finally, by Remark 4.1, we conclude that inequality (4.6) entails

$$A(x, \nabla(v(x)^{1/r})) \le (1-\theta) \cdot A(x, \nabla(v_1(x)^{1/r})) + \theta \cdot A(x, \nabla(v_2(x)^{1/r})). \tag{4.7}$$

We integrate the last inequality (4.7) over Ω to derive the convexity of the restriction of the functional $\mathcal{W}_A$ to the convex cone $\overset{\bullet}{V} \subset X$.

To derive that $\mathcal{W}_A$ is even ray-strictly convex on $\overset{\bullet}{V}$, let us consider any pair $v_1, v_2 \in \overset{\bullet}{V}$ with $v_1 \not\equiv v_2$ in Ω. We observe that the equality in the convexity inequality (4.1) forces both conditions, (4.4) and $p(x)/r = 1$, to hold simultaneously at almost every point $x \in \Omega$. These conditions are then equivalent with $v_2/v_1 \equiv$ const $(\neq 1)$ in Ω and $p(x) \equiv r$ in Ω. Thus, if $p(x) \not\equiv r$ in Ω, then $\mathcal{W}_A$ is even strictly convex on $\overset{\bullet}{V}$. $\square$

A consequence of Theorem 4.1 is the following extension of Díaz–Saa inequality also proved in [23] in a weaker form.

Theorem 4.2. *Let* $r \in [1, \infty)$ *and* $p\colon \Omega \to (1, \infty)$ *satisfy conditions in Theorem* 4.1. *Assume that* $A\colon \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}_+$ *satisfies* (**A1**)–(**A4**), (**A7**). *Then the* *following inequality*:

$$\int_\Omega \left(-\frac{\operatorname{div} \mathbf{a}(x, \nabla v_1(x))}{v_1(x)^{r-1}} + \frac{\operatorname{div} \mathbf{a}(x, \nabla v_2(x))}{v_2(x)^{r-1}} \right) (v_1^r - v_2^r)\, \mathrm{d}x \geq 0 \tag{4.8}$$

holds (*in the sense of distributions*) *for all pairs* $v_1, v_2 \in \mathbb{W}$, *such that* $v_1 > 0,\ v_2 > 0$ *a.e. in* Ω *and both* $v_1/v_2,\ v_2/v_1 \in L^\infty(\Omega)$. *Moreover, if the equality in* (4.8) *occurs, then we have the following two statements*:

(a) $v_2/v_1 \equiv \mathrm{const} > 0$ *in* Ω.
(b) *If also* $p(x) \not\equiv r$ *in* Ω, *then even* $v_1 \equiv v_2$ *holds in* Ω.

Proof. Recalling Definition 4.2, let us consider any pair $w_1, w_2 \in \mathbb{W}$, such that $w_1 > 0$, $w_2 > 0$ a.e. in Ω and both $w_1/w_2,\ w_2/w_1 \in L^\infty(\Omega)$. Consequently, there is a sufficiently small number $\delta \in (0,1)$ such that

$$v \stackrel{\text{def}}{=} (1-\theta)w_1^r + \theta w_2^r \in \overset{\bullet}{V} \quad \text{and} \quad v^{1/r} \in \mathbb{W} \quad \text{for all } \theta \in (-\delta, 1+\delta).$$

The function

$$\theta \mapsto W(\theta) \stackrel{\text{def}}{=} \mathcal{W}(v) = \mathcal{W}_A((1-\theta)w_1^r + \theta w_2^r) \colon (-\delta, 1+\delta) \to \mathbb{R}_+$$

is convex and differentiable with the derivative

$$W'(\theta) = \int_\Omega \mathbf{a}(x, \nabla(v(x)^{1/r})) \cdot \nabla \left(\frac{w_2^r - w_1^r}{v^{1-\frac{1}{r}}} \right) \mathrm{d}x.$$

The monotonicity of the derivative $\theta \mapsto W'(\theta) \colon (-\delta, 1+\delta) \mapsto \mathbb{R}$ yields $W'(0) \leq W'(1)$, which is equivalent with

$$\int_\Omega \mathbf{a}(x, \nabla w_1(x)) \cdot \nabla \left(w_1 - \frac{w_2^r}{w_1^{r-1}} \right) \mathrm{d}x \geq \int_\Omega \mathbf{a}(x, \nabla w_2(x)) \cdot \nabla \left(\frac{w_1^r}{w_2^{r-1}} - w_2 \right) \mathrm{d}x, \tag{4.9}$$

thanks to $v = w_1^r$ if $\theta = 0$, and $v = w_2^r$ if $\theta = 1$.

It is now easy to see that inequality (4.8) is a distributional interpretation of (4.9) after integration by parts.

Finally, let us assume that the equality in (4.8) is valid. This forces $W'(0) = W'(1)$ above; hence, $W'(\theta) = W'(0)$ for all $\theta \in [0,1]$, by the monotonicity of $W' \colon [0,1] \mapsto \mathbb{R}$. It follows that $W \colon [0,1] \to \mathbb{R}$ must be linear, i.e. $W(\theta) = (1-\theta)W(0) + \theta W(1) \in \mathbb{R}$ for all $\theta \in [0,1]$. Recalling our definition of W, Remark 4.1, assumption (**A3**) (which implies the strict convexity of $\xi \mapsto A(x,\xi)$ for each fixed $x \in \Omega$) and Theorem 4.1, we conclude that $w_2/w_1 \equiv \mathrm{const} > 0$ in Ω. This proves statement (a).

To verify statement (b), suppose that the constant above $w_2/w_1 \equiv \mathrm{const} \neq 1$ in Ω. Then the equality in both inequalities, (4.6) and (4.7), is possible only if $p(x) \equiv r$ in Ω. Statement (b) follows. $\square$

Remark 4.2. An alternative proof of Theorem 4.2 is to show a Picone inequality in a similar fashion as in [8] (see Proposition 2.9 and Remark 2.10). Precisely, we can derive the following Picone inequality:

$$\begin{aligned} A(x,\nabla v_2)^{p(x)-r/p(x)}A(x,\nabla v_1)^{r/p(x)} &\geq \left\langle \frac{1}{p(x)}\partial_\xi A(x,\nabla v_2), \nabla(v_1^r/v_2^{r-1})\right\rangle \\ &= \left\langle \frac{1}{p(x)}\mathbf{a}(x,\nabla v_2), \nabla(v_1^r/v_2^{r-1})\right\rangle \end{aligned} \tag{4.10}$$

for any v_1, v_2 satisfying assumptions in Theorem 4.2 and each fixed $x \in \Omega$. Combining (4.10), Remark 1.7 and the Young inequality, we then easily derive (4.9) and (4.8). Note that inequality (4.10) being pointwise is stronger than the Diaz–Saa inequality (4.8) and is of independent interest.

Proof of Theorem 1.5. Let u be a weak solution to (P_+). From (f_1), (f_3) and (f_6), there exists $M > 0$ large enough such that

$$0 \leq f(x,s) \leq M + |s|^{p^- -1} \quad \forall\, s \geq 0 \quad \text{and } x \in \Omega. \tag{4.11}$$

Hence, from [18, Theorem 4.1, p. 312], it follows that u is bounded and from [15, Theorem 1.2], belongs to $C^{1,\alpha}(\overline{\Omega})$. From the Hopf boundary point lemma in [34] (see Theorems 1.2), u satisfies $\partial u/\partial \nu < 0$ on $\partial\Omega$. Therefore, for any pair u, v of weak solutions to (P_+), u/v and v/u belongs to $L^\infty(\Omega)$. Then let $\mathcal{J}_{p^-}$ defined in $\overset{\bullet}{V}$ by

$$\mathcal{J}_{p^-}(w) \stackrel{\text{def}}{=} \int_\Omega A(x,\nabla(w^{1/p^-}))\,\mathrm{d}x - \int_\Omega F(x,w^{1/p^-})\,\mathrm{d}x,$$

where $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$.

From Theorem 4.1 and (f_6), $\mathcal{J}$ is strictly convex on $\overset{\bullet}{V}$. Let $w_1 = u^{p^-}$ and $w_2 = v^{p^-}$. Hence, the function $J_{p^-}: t \to \mathcal{J}_{p^-}(w_2 + t(w_1 - w_2)^+)$ is well defined, convex and differentiable on $[0,1]$. Since u, v are weak solutions to (P_+), $J'_{p^-}(0) = J'_{p^-}(1) = 0$.

According to Theorem 4.2 with $v_1 = w_2 + t(w_1 - w_2)^+$ and $v_2 = w_2$, assertion (b) implies $(w_1 - w_2)^+ \equiv 0$, that is $u \leq v$. Interchanging the role of u and v, we get $u = v$. This proves the uniqueness of the weak solution to (P_+).

Now, supposing that (f_4), (f_5) and $f(x,0) = 0$ are satisfied. Extending f by $f(x,s) = 0$ for $s \leq 0$, we apply Lemma 3.1, with $h = 0$ to obtain the existence of a solution $u \in \mathbb{W} \cap L^\infty(\Omega)$. From the strong maximum principle given by [34, Theorem 1.1], u is positive in Ω and then a weak solution to (P_+). This completes the proof of Theorem 1.5. □

4.2. *Proof of Theorem* 1.6

Let $T > 0$. Note that from assumptions (f_1), (f_6), $f(x,0) = 0$ and since f is locally Lipschitz in respect to the second variable uniformly in $x \in \Omega$, there exists $M > 0$

large enough such that (4.11) is still valid. The existence of $u \in W^{1,2}(0,T;L^2(\Omega)) \cap C([0,T],\mathbb{W}) \cap L^\infty(Q_T)$, unique weak solution to (P_T) with initial data $u_0 \in \mathbb{W} \cap L^\infty(\Omega)$ and T small enough follows from Theorem 1.2. The L^∞ bound is provided by the barrier function v solution to

$$\begin{cases} \dfrac{dv}{dt} = M + v^{p^- -1}, \quad t \in (0,T), \\ v(0) = \|u_0\|_\infty. \end{cases}$$

Note that the uniqueness of the weak solution and the statement $u \le v$ in Q_T follow from the local Lipschitz property of f and the monotonicity of $\nabla \cdot \mathbf{a}(x, \nabla \cdot)$. To get the existence of global weak positive solutions to (P_T), we need to construct a subsolution $\underline{u}$ and a supersolution $\bar{u}$ independent of t:

$$\begin{cases} -\nabla \cdot \mathbf{a}(x, \nabla \underline{u}) = \lambda \underline{u}^{p^- -1} & \text{in } \Omega, \\ \underline{u} = 0 & \text{on } \partial\Omega \end{cases}$$

and $\bar{u} = \tilde{w}_{M,\varepsilon}$ defined in (3.10) where $\lambda > 0$ is small enough and $M > 0$ large enough. From [18, Theorem 4.1], $\underline{u}$ and $\bar{u}$ are bounded. From [15, Theorem 1.2] and [34, Theorems 1.1 and 1.2], they belong to $C_0^{1,+}(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Hence, it is easy to prove that $\|\underline{u}\|_{C^1(\overline{\Omega})} \to 0$ as $\lambda \to 0^+$. From Lemma 3.2, we have that there exist $C_1, C_2 > 0$ independent of M such that $C_1 M^{\frac{1}{p^+ - 1 + \nu}} \rho(x) \le \bar{u}(x)$ for some $0 < \nu < 1$. Therefore, from (f_4) and (f_5) for λ small enough and M large enough we have that $\underline{u}$ (respectively, $\bar{u}$) is a subsolution (respectively, a supersolution) to (P_+) and $\underline{u} \le u_0 \le \bar{u}$. Thus, u is a global weak solution to (P_T) and using the weak comparison principle on the discrete time-approximated scheme (3.2) we obtain $\underline{u} \le u(t) \le \bar{u}$ for any $0 \le t < \infty$. Using Theorem B.2, we obtain that u_1 (respectively, u_2) the weak solution to (P_T) with initial data $\underline{u}$ (respectively, $\bar{u}$) is a mild solution. Then, u_1 (respectively, u_2) belongs to $C([0,\infty), C_0(\overline{\Omega}))$ and since $\underline{u}$ (respectively, $\bar{u}$) is a subsolution (respectively, a supersolution) $[0,\infty) \ni t \to u_1(t)$ (respectively, $[0,\infty) \ni t \to u_2(t)$) is nondecreasing (respectively, nonincreasing). Hence, u_1 and u_2 converge (pointwise) to a positive steady state as $t \to \infty$. From Theorem 1.5, (P_T) admits a unique positive and continuous stationary (weak) solution u_∞ and hence by Dini's theorem we infer that

$$\|u_1(t) - u_\infty\|_{L^\infty(\Omega)} \to 0, \quad \|u_2(t) - u_\infty\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty$$

which reveals $\|u(t) - u_\infty\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$ since $u_1(t) \le u(t) \le u_2(t)$.

Appendix A. Regularity Result

We begin by recalling the following compactness embedding.

Theorem A.1. *Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfies $1 \le p^- \le p^+ < d$. Then, $W^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ for any $\alpha \in L^\infty(\Omega)$ such that for all $x \in \Omega$, $\alpha(x) \le p^*(x) = \frac{d\,p(x)}{d - p(x)}$. Also, the previous embedding is compact for $\alpha(x) < p^*(x) - \varepsilon$ a.e. in Ω for any $\varepsilon > 0$.*

Next, we recall the regularity result due to Fan and Zhao [18]:

Proposition A.1 ([18, Theorem 4.1]). *Assume conditions* (**A1**)–(**A4**). *Let* $p \in C(\overline{\Omega})$ *and* $u \in \mathbb{W}$ *satisfying*

$$\int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f(x,u)\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in \mathbb{W},$$

where f *satisfies for all* $(x,t) \in \Omega \times \mathbb{R}$, $|f(x,t)| \le c_1 + c_2|t|^{r(x)-1}$ *with* $r \in C(\overline{\Omega})$ *and* $\forall\, x \in \overline{\Omega}$, $1 < r(x) < p^*(x)$. *Then* $u \in L^\infty(\Omega)$.

For $f(x,\cdot) = f(x)$, we have the following proposition.

Proposition A.2. *Assume conditions* (**A1**)–(**A4**). *Let* $p \in C(\bar{\Omega})$ *with* $p^- < d$ *and* $u \in \mathbb{W}$ *satisfying*

$$\int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in \mathbb{W}, \tag{A.1}$$

where $f \in L^q(\Omega)$, $q > \frac{d}{p^-}$. *Then* $u \in L^\infty(\Omega)$.

To prove Proposition A.2, we have the following regularity lemma (see Fusco and Sbordone [20] and Giacomoni *et al.* [21])

Lemma A.1. *Let* $u \in W_0^{1,p}(\Omega)$, $1 < p < d$, *satisfying for any* B_R, $R < R_0$, *and for all* $\sigma \in (0,1)$, *and any* $k \ge k_0 > 0$

$$\int_{A_{k,\sigma R}} |\nabla u|^p \,\mathrm{d}x \le C\left[\int_{A_{k,R}} \left|\frac{u-k}{R(1-\sigma)}\right|^{p^*} \mathrm{d}x + k^\alpha |A_{k,R}| + |A_{k,R}|^{\frac{p}{p^*}+\varepsilon}\right.$$
$$\left. + \left(\int_{A_{k,R}} \left|\frac{u-k}{R(1-\sigma)}\right|^{p^*} \mathrm{d}x\right)^{\frac{p}{p^*}} |A_{k,R}|^\delta\right]$$

where $A_{k,R} = \{x \in B_R \cap \Omega \,|\, u(x) > k\}$, $0 < \alpha < p^* = \frac{dp}{d-p}$ *and* $\varepsilon, \delta > 0$. *Then* $u \in L^\infty(\Omega)$.

Proof of Proposition A.2. We follow the idea of the proof of [18, Theorem 4.1].

Let $x_0 \in \overline{\Omega}$, B_R the ball of radius R centered in x_0 and $K_R := \Omega \cap B_R$. We define

$$p^+ := \max_{K_R} p(x) \quad \text{and} \quad p^- := \min_{K_R} p(x)$$

and we choose R small enough such that $p^+ < (p^-)^* := \frac{dp^-}{d-p^-}$. Fix $(s,t) \in (\mathbb{R}_+^*)^2$, $t < s < R$ then $K_t \subset K_s \subset K_R$. Define $\varphi \in C^\infty(\Omega)$, $0 \le \varphi \le 1$ such that

$$\varphi = \begin{cases} 1 & \text{in } B_t, \\ 0 & \text{in } \mathbb{R}^d \backslash B_s \end{cases}$$

satisfying $|\nabla\varphi| \lesssim 1/(s-t)$. Let $k \ge 1$, using the same notations as previously $A_{k,\lambda} = \{y \in K_\lambda \,|\, u(y) > k\}$ and taking $\varphi^{p^+}(u-k)^+ \in \mathbb{W}$ in (A.1) as test function, we obtain

$$\int_{A_{k,s}} \mathbf{a}(x,\nabla u)\cdot\nabla u\varphi^{p^+}\,\mathrm{d}x + p^+\int_{A_{k,s}} \mathbf{a}(x,\nabla u)\cdot\nabla\varphi\,(u-k)^+\varphi^{p^+-1}\,\mathrm{d}x$$
$$= \int_{A_{k,s}} f\varphi^{p^+}(u-k)\,\mathrm{d}x. \tag{A.2}$$

Hence by Young's inequality, for $\epsilon > 0$, we have

$$p^+\int_{A_{k,s}} \mathbf{a}(x,\nabla u)\cdot\nabla\varphi(u-k)\varphi^{p^+-1}\,\mathrm{d}x$$
$$\leq \varepsilon\int_{A_{k,s}} |\mathbf{a}(x,\nabla u)|^{\frac{p(x)}{p(x)-1}}\varphi^{(p^+-1)\frac{p(x)}{p(x)-1}}\,\mathrm{d}x + c\varepsilon^{-1}\int_{A_{k,s}}(u-k)^{p(x)}|\nabla\varphi|^{p(x)}\,\mathrm{d}x.$$

Since $|\nabla\varphi| \leq c/(s-t)$ and for any $x \in K_R$, $p^+ \leq (p^+-1)\frac{p(x)}{p(x)-1}$, we have $\varphi^{(p^+-1)\frac{p(x)}{p(x)-1}} \leq \varphi^{p^+}$.

Hence using (1.2), this implies

$$p^+\int_{A_{k,s}} \mathbf{a}(x,\nabla u)\cdot\nabla\varphi\,(u-k)\varphi^{p^+-1}\,\mathrm{d}x$$
$$\lesssim \varepsilon\int_{A_{k,s}} |\nabla u|^{p(x)}\varphi^{p^+}\,\mathrm{d}x + \varepsilon^{-1}\int_{A_{k,s}}\left(\frac{u-k}{s-t}\right)^{p(x)}\mathrm{d}x. \tag{A.3}$$

Using Hölder inequality, we estimate the right-hand side of (A.2) as follows:

$$\int_{A_{k,s}} f\varphi^{p^+}(u-k)\,\mathrm{d}x \leq \|f\|_{L^q}\left(\int_{A_{k,s}}(u-k)^{\frac{q}{q-1}}\,\mathrm{d}x\right)^{\frac{q-1}{q}}.$$

Since $q > \frac{d}{p^-}$, we have $\frac{(p^-)^*}{p^-}\frac{q-1}{q} > 1$. So, applying once again the Hölder inequality, we obtain

$$\int_{A_{k,s}} f\varphi^{p^+}(u-k)\,\mathrm{d}x \lesssim \left(\int_{A_{k,s}}(u-k)^{\frac{(p^-)^*}{p^-}}\,\mathrm{d}x\right)^{\frac{p^-}{(p^-)^*}}|A_{k,s}|^{\delta}, \tag{A.4}$$

where $\delta = \frac{q-1}{q} - \frac{p^-}{(p^-)^*} > 0$. Set $\{u-k > s-t\} = \{x \in K_R \mid u(x)-k > s-t\}$ and its complement as $\{u-k \leq s-t\}$. Now, we split the integral in the right-hand side of (A.4) as follows:

$$\int_{A_{k,s}\cap\{u-k>s-t\}}\left(\frac{u-k}{s-t}\right)^{\frac{(p^-)^*}{p^-}}(s-t)^{\frac{(p^-)^*}{p^-}}\,\mathrm{d}x$$
$$+\int_{A_{k,s}\cap\{u-k\leq s-t\}}\left(\frac{u-k}{s-t}\right)^{\frac{(p^-)^*}{p^-}}(s-t)^{\frac{(p^-)^*}{p^-}}\,\mathrm{d}x$$
$$\lesssim \int_{A_{k,s}}\left(\frac{u-k}{s-t}\right)^{(p^-)^*}\mathrm{d}x + |A_{k,s}| := \mathcal{I}. \tag{A.5}$$

In the same way, the second term in the right-hand side of (A.3) can be estimated as follows.

$$\int_{A_{k,s}\cap\{u-k>s-t\}}\left(\frac{u-k}{s-t}\right)^{p(x)}\,\mathrm{d}x+\int_{A_{k,s}\cap\{u-k\le s-t\}}\left(\frac{u-k}{s-t}\right)^{p(x)}\,\mathrm{d}x\lesssim\mathcal{I}.\tag{A.6}$$

Moreover, we have

$$\int_{A_{k,s}}\mathbf{a}(x,\nabla u)\cdot\nabla u\varphi^{p^+}\,\mathrm{d}x\gtrsim\int_{A_{k,s}}|\nabla u|^{p(x)}\varphi^{p^+}\,\mathrm{d}x\ge 0.\tag{A.7}$$

Finally, plugging (A.3)–(A.7) and we obtain for ε small enough

$$\int_{A_{k,s}}|\nabla u|^{p(x)}\varphi^{p^+}\,\mathrm{d}x\lesssim\mathcal{I}+|A_{k,s}|^{\delta}\mathcal{I}^{\frac{p^-}{(p^-)^*}}$$

where the constant depends on p, R and ε. Moreover, we have

$$\mathcal{I}^{\frac{p^-}{(p^-)^*}}\lesssim\left(\int_{A_{k,s}}\left(\frac{u-k}{s-t}\right)^{(p^-)^*}\,\mathrm{d}x\right)^{\frac{p^-}{(p^-)^*}}+|A_{k,s}|^{\frac{p^-}{(p^-)^*}}.$$

Hence using the Young's inequality, we obtain the following estimate:

$$\begin{aligned}\int_{A_{k,t}}|\nabla u|^{p^-}\,\mathrm{d}x&\le\int_{A_{k,s}}|\nabla u|^{p(x)}\varphi^{p^+}\,\mathrm{d}x+|A_{k,s}|\\&\lesssim\int_{A_{k,s}}\left(\frac{u-k}{s-t}\right)^{(p^-)^*}\,\mathrm{d}x+|A_{k,s}|+|A_{k,s}|^{\frac{p^-}{(p^-)^*}+\delta}\\&\quad+|A_{k,s}|^{\delta}\left(\int_{A_{k,s}}\left(\frac{u-k}{s-t}\right)^{(p^-)^*}\,\mathrm{d}x\right)^{\frac{p^-}{(p^-)^*}}.\end{aligned}$$

By Lemma A.1, we deduce that u bounded in Ω. ☐

Combining Propositions A.1 and A.2, we have the following corollary.

Corollary A.2. *Let $p\in C(\bar{\Omega})$ such that $p^-<d$ and $u\in W_0^{1,p(x)}(\Omega)$ satisfying*

$$\int_\Omega\mathbf{a}(x,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x=\int_\Omega(f(x,u)+g)\varphi\,\mathrm{d}x,\quad\forall\,\varphi\in\mathbb{W},$$

where f satisfies $|f(x,t)|\le c_1+c_2|t|^{r(x)-1}$ with $r\in C(\overline{\Omega})$ and $\forall\,x\in\overline{\Omega}$, $1<r(x)<p^*(x)$ and $g\in L^q$, $q>\frac{d}{p^-}$. Then $u\in L^\infty(\Omega)$.

Appendix B. Existence of Mild Solutions to (L_T) and (P_T)

We use the theory of maximal accretive operators in Banach spaces (see [7, Chaps. 3 and 4]), which provides the existence of mild solutions. More precisely, observing that the operator $A_0(\cdot)\stackrel{\text{def}}{=}-\nabla\cdot\mathbf{a}(x,\nabla(\cdot))$, with Dirichlet boundary conditions, is

m-accretive in $L^\infty(\Omega)$ with

$$\mathcal{D}(A_0) = \{u \in \mathbb{W} \cap L^\infty(\Omega) \mid A_0 u \in L^\infty(\Omega)\},$$

we get the following properties, which essentially follow from Theorem 1.1 with Theorem 4.2 (p. 130) and [7, Theorem 4.4, p. 141]:

Theorem B.1. *Assume conditions* $(\mathbf{A1})$–$(\mathbf{A5})$. *Let* $T > 0$, $h \in L^\infty(Q_T)$ *and let* u_0 *be in* $\mathbb{W} \cap \overline{\mathcal{D}(A_0)}^{L^\infty}$. *Then,*

(i) *the unique weak solution* u *to* (L_T) *belongs to* $\mathcal{C}([0,T]; \mathcal{C}_0(\overline{\Omega}))$;

(ii) *if* v *is another mild solution to* (L_T) *with the initial datum* $v_0 \in \mathbb{W} \cap \overline{\mathcal{D}(A_0)}^{L^\infty}$ *and the right-hand side* $k \in L^\infty(Q_T)$, *then the following estimate holds*:

$$\begin{aligned} &\|u(t) - v(t)\|_{L^\infty(\Omega)} \\ &\quad \le \|u_0 - v_0\|_{L^\infty(\Omega)} + \int_0^t \|h(s) - k(s)\|_{L^\infty(\Omega)}\, \mathrm{d}s, \quad 0 \le t \le T; \end{aligned} \tag{B.1}$$

(iii) *if* $u_0 \in \mathcal{D}(A_0)$ *and* $h \in W^{1,1}(0,T; L^\infty(\Omega))$ *then* $u \in W^{1,\infty}(0,T; L^\infty(\Omega))$ *and* $\nabla \cdot \mathbf{a}(x, \nabla u) \in L^\infty(Q_T)$, *and the following estimate holds*:

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{L^\infty(\Omega)} \le \|\nabla \cdot \mathbf{a}(x, \nabla u_0) + h(0)\|_{L^\infty(\Omega)} + \int_0^T \left\|\frac{\partial h}{\partial t}(t)\right\|_{L^\infty(\Omega)} \mathrm{d}\tau. \tag{B.2}$$

The m-accretivity of A_0 follows from the following proposition.

Proposition B.1. *Assume conditions* $(\mathbf{A1})$–$(\mathbf{A4})$. *Let* $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ *be a function satisfying* (f_1) *and nonincreasing with respect to the second variable. Assume further that* $x \to f(x,0)$ *belongs to* $L^\infty(\Omega)$. *Then,* A_f *defined by* $A_f(u) \overset{\text{def}}{=} -\nabla \cdot \mathbf{a}(\cdot, \nabla u) - f(\cdot, u)$ *is* m*-accretive in* $L^\infty(\Omega)$.

Proof. First, let $h \in L^\infty(\Omega)$ and $\lambda > 0$. Then,

$$\begin{cases} u + \lambda A_f(u) = h & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

admits a unique solution, $u \in \mathbb{W} \cap L^\infty(\Omega)$. Indeed, for $\mu > 0$ large enough μ and $-\mu$ are respectively supersolution and subsolution to the above equation and then from the weak comparison principle, $u \in [-\mu, \mu]$ and u is obtained by a minimization argument and a truncation argument. The uniqueness of the solution follows from the strict convexity of the associated energy functional. Next, we prove the accretivity of A_f. Let h and $g \in L^\infty(\Omega)$ and set u and v the unique solutions to

$$\begin{aligned} u + \lambda A_f u &= h \quad \text{in } \Omega, \\ v + \lambda A_f v &= g \quad \text{in } \Omega. \end{aligned}$$

Subtracting the two above equations and using the test function $w \stackrel{\text{def}}{=} (u-v-\|h-g\|_{L^\infty(\Omega)})^+$, we get $u-v \leq \|h-g\|_{L^\infty(\Omega)}$ and reversing the roles of u and v, we get that $\|u-v\|_{L^\infty(\Omega)} \leq \|h-g\|_{L^\infty(\Omega)}$. This proves the proposition. $\square$

The proof of Theorem B.1, similar to the proof of [21, Theorem 2.8], is given below.

Proof of Theorem B.1. We follow the approach developed in the proof of [7, Theorems 4.2 and 4.4]. Let u_0, v_0 be in $\overline{\mathcal{D}(A_0)}^{L^\infty(\Omega)}$. For $z \in \mathcal{D}(A_0)$ and r, k in $L^\infty(Q_T)$, set

$$\vartheta(t,s) = \|r(t)-k(s)\|_{L^\infty(\Omega)} \quad \forall\,(t,s) \in [0,T]\times[0,T];$$

$$\begin{aligned} b(t,r,k) &= \|u_0 - z\|_{L^\infty(\Omega)} + \|v_0 - z\|_{L^\infty(\Omega)} + |t|\|A_0 z\|_{L^\infty(\Omega)} \\ &\qquad + \int_0^{t^+} \|r(\tau)\|_{L^\infty(\Omega)}\,\mathrm{d}\tau + \int_0^{t^-} \|k(\tau)\|_{L^\infty(\Omega)}\,\mathrm{d}\tau, \quad t \in [-T,T], \end{aligned}$$

and

$$\Upsilon(t,s) = b(t-s,r,k) + \begin{cases} \displaystyle\int_0^s \vartheta(t-s+\tau,\tau)\,\mathrm{d}\tau & \text{if } 0 \leq s \leq t \leq T, \\ \displaystyle\int_0^t \vartheta(\tau,s-t+\tau)\,\mathrm{d}\tau & \text{if } 0 \leq t \leq s \leq T, \end{cases}$$

the solution of

$$\begin{cases} \dfrac{\partial\Upsilon}{\partial t}(t,s) + \dfrac{\partial\Upsilon}{\partial s}(t,s) = \vartheta(t,s) & (t,s) \in [0,T]\times[0,T], \\ \Upsilon(t,0) = b(t,r,k) & t \in [0,T], \\ \Upsilon(0,s) = b(-s,r,k) & s \in [0,T]. \end{cases} \tag{B.3}$$

Moreover, let us denote by (u_ϵ^n) the solution of (2.1) with $\Delta_t = \epsilon$, $h = r$, $r^n = \frac{1}{\epsilon}\int_{(n-1)\epsilon}^{n\epsilon} r(\tau,\cdot)\,\mathrm{d}\tau$ and (u_η^n) the solution of (2.1) with $\Delta_t = \eta$, $h = k$, $k^n = \frac{1}{\eta}\int_{(n-1)\eta}^{n\eta} k(\tau,\cdot)\,\mathrm{d}\tau$, respectively. For $(n,m) \in \mathbb{N}^*$, elementary calculations lead to

$$\begin{aligned} &u_\epsilon^n - u_\eta^m + \frac{\epsilon\eta}{\epsilon+\eta}(A_0 u_\epsilon^n - A_0 u_\eta^m) \\ &\qquad = \frac{\eta}{\epsilon+\eta}(u_\epsilon^{n-1} - u_\eta^m) + \frac{\epsilon}{\epsilon+\eta}(u_\epsilon^n - u_\eta^{m-1}) + \frac{\epsilon\eta}{\epsilon+\eta}(r^n - k^m), \end{aligned}$$

and since A_0 is m-accretive in $L^\infty(\Omega)$ we first verify that $F_{n,m}^{\epsilon,\eta} = \|u_\epsilon^n - u_\eta^m\|_{L^\infty(\Omega)}$ obeys

$$F_{n,m}^{\epsilon,\eta} \leq \frac{\eta}{\epsilon+\eta} F_{n-1,m}^{\epsilon,\eta} + \frac{\epsilon}{\epsilon+\eta} F_{n,m-1}^{\epsilon,\eta} + \frac{\epsilon\eta}{\epsilon+\eta}\|r^n - k^m\|_\infty,$$

$$F_{n,0}^{\epsilon,\eta} \leq b(t_n, r_\epsilon, k_\eta) \quad \text{and} \quad F_{0,m}^{\epsilon,\eta} \leq b(-s_m, r_\epsilon, k_\eta),$$

and thus, with an easy inductive argument, that $\digamma^{\epsilon,\eta}_{n,m} \le \Upsilon^{\epsilon,\eta}_{n,m}$ where $\Upsilon^{\epsilon,\eta}_{n,m}$ satisfies

$$\Upsilon^{\epsilon,\eta}_{n,m} = \frac{\eta}{\epsilon+\eta}\Upsilon^{\epsilon,\eta}_{n-1,m} + \frac{\epsilon}{\epsilon+\eta}\Upsilon^{\epsilon,\eta}_{n,m-1} + \frac{\epsilon\eta}{\epsilon+\eta}\|h^n_\epsilon - h^m_\eta\|_\infty,$$
$$\Upsilon^{\epsilon,\eta}_{n,0} = b(t_n, r_\epsilon, k_\eta) \quad \text{and} \quad \Upsilon^{\epsilon,\eta}_{0,m} = b(-s_m, r_\epsilon, k_\eta).$$

For $(t,s) \in (t_{n-1}, t_n) \times (s_{m-1}, s_m)$, set

$$\vartheta^{\epsilon,\eta}(t,s) = \|r_\epsilon(t) - k_\eta(s)\|_\infty,$$
$$\Upsilon^{\epsilon,\eta}(t,s) = \Upsilon^{\epsilon,\eta}_{n,m},$$
$$b_{\epsilon,\eta}(t,r,k) = b(t_n, r_\epsilon, k_\eta)$$

and

$$b_{\epsilon,\eta}(-s,r,k) = b(-s_m, r_\epsilon, k_\eta).$$

Then $\Upsilon^{\epsilon,\eta}$ satisfies the following discrete version of (B.3):

$$\frac{\Upsilon^{\epsilon,\eta}(t,s) - \Upsilon^{\epsilon,\eta}(t-\epsilon,s)}{\epsilon} + \frac{\Upsilon^{\epsilon,\eta}(t,s) - \Upsilon^{\epsilon,\eta}(t,s-\eta)}{\eta} = \vartheta^{\epsilon,\eta}(t,s),$$
$$\Upsilon^{\epsilon,\eta}(t,0) = b_{\epsilon,\eta}(t,r,k) \quad \text{and} \quad \Upsilon^{\epsilon,\eta}(0,s) = b_{\epsilon,\eta}(s,r,k),$$

and from $b_{\epsilon,\eta}(\cdot,r,k) \to b(\cdot,r,k)$ in $L^\infty([0,T])$. Furthermore,

$$\sum_{n=1}^{N_n} \int_{t_{n-1}}^{t_n} \|r(s) - r_n\|_\infty \, \mathrm{d}s \to 0, \quad \text{as } \epsilon \to 0^+,$$
$$\sum_{m=1}^{N_m} \int_{s_{m-1}}^{s_m} \|k(s) - k_m\|_\infty \, \mathrm{d}\tau \to 0 \quad \text{as } \eta \to 0^+.$$

The above statements follow easily from the fact that $r, k \in L^1(0,T; L^\infty(\Omega))$ and a density argument.

We deduce that $\rho_{\epsilon,\eta} = \|\Upsilon^{\epsilon,\eta} - \Upsilon\|_{L^\infty([0,T]\times[0,T])} \to 0$ as $(\epsilon,\eta) \to 0$ (see for instance [7, Chap. 4, Lemma 4.3, p. 136] and [7, Chap. 4, proof of Theorem 4.1, p. 138]). Then from

$$\|u_\epsilon(t) - u_\eta(s)\|_\infty = \digamma^{\epsilon,\eta}(t,s) \le \Upsilon^{\epsilon,\eta}(t,s) \le \Upsilon(t,s) + \rho_{\epsilon,\eta}, \tag{B.4}$$

we obtain with $t = s$, $r = k = h$, $v_0 = u_0$:

$$\|u_\epsilon(t) - u_\eta(t)\|_{L^\infty(\Omega)} \le 2\|u_0 - z\|_{L^\infty(\Omega)} + \rho_{\epsilon,\eta},$$

and since z can be chosen in $\mathcal{D}(A_0)$ arbitrary close to u_0, we deduce that u_ϵ is a Cauchy sequence in $L^\infty(Q_T)$ and then that $u_\epsilon \to u$ in $L^\infty(Q_T)$. Thus, passing to the limit in (B.4) with $r = k = h$, $v_0 = u_0$ we obtain

$$\|u(t) - u(s)\|_{L^\infty(\Omega)} \le 2\|u_0 - z\|_{L^\infty(\Omega)} + |t-s|\|A_0 z\|_{L^\infty(\Omega)} + \int_0^{|t-s|} \|h(\tau)\|_{L^\infty(\Omega)} \, \mathrm{d}\tau$$
$$+ \int_0^{\max(t,s)} \|h(|t-s|+\tau) - h(\tau)\|_{L^\infty(\Omega)} \, \mathrm{d}\tau,$$

which, together with the density $\mathcal{D}(A_0)$ in $L^\infty(\Omega)$ and $h \in L^1(0,T;L^\infty(\Omega))$, yields $u \in C([0,T];L^\infty(\Omega))$. Analogously, from (B.4) with $\varepsilon = \eta = \Delta_t$, $r = k = h$, $v_0 = u_0$ and $t = s + \Delta_t$, we deduce that

$$\begin{aligned}\|u_{\Delta_t}(t) - \tilde{u}_{\Delta_t}(t)\|_{L^\infty(\Omega)} &\le 2\|u_{\Delta_t}(t) - u_{\Delta_t}(t-\Delta_t)\|_{L^\infty(\Omega)} \\ &\le 4\|u_0 - z\|_{L^\infty(\Omega)} + 2\Delta_t\|A_0 z\|_{L^\infty(\Omega)} \\ &\quad + 2\int_0^{\Delta_t} \|h(\tau)\|_{L^\infty(\Omega)}\,\mathrm{d}\tau \\ &\quad + 2\int_0^t \|h(\Delta_t+\tau) - h(\tau)\|_\infty\,\mathrm{d}\tau,\end{aligned}$$

which gives the limit $\tilde{u}_{\Delta_t} \to u$ in $C([0,T];L^\infty(\Omega))$ as $\Delta_t \to 0^+$. Note that since $\tilde{u}_{\Delta_t} \in C([0,T];C_0(\overline{\Omega}))$, the uniform limit u belongs to $C([0,T];C_0(\overline{\Omega}))$. Moreover, passing to the limit in (B.4) with $t = s$, we obtain

$$\|u(t) - v(t)\|_{L^\infty(\Omega)} \le \|u_0 - z\|_{L^\infty(\Omega)} + \|v_0 - z\|_{L^\infty(\Omega)} + \int_0^t \|r(\tau) - k(\tau)\|_\infty\,\mathrm{d}\tau,$$

and (B.1) follows since we can choose z arbitrary close to v_0. Finally, if $A_0u_0 \in L^\infty(\Omega)$ and $h \in W^{1,1}(0,T;L^\infty(\Omega))$ and if we assume (without loss of generality) that $t > s$ then with $z = v_0 = u(t-s)$ and $(r,k) = (h, h(\cdot + t - s))$ in the last above inequality, we obtain

$$\begin{aligned}\|u(t) - u(s)\|_{L^\infty(\Omega)} &\le \|u_0 - u(t-s)\|_{L^\infty(\Omega)} \\ &\quad + \int_0^s \|h(\tau) - h(\tau+t-s)\|_{L^\infty(\Omega)}\,\mathrm{d}\tau. \end{aligned} \tag{B.5}$$

From (B.1) with $v = u_0$, $k = A_0u_0$:

$$\|u_0 - u(t-s)\|_{L^\infty(\Omega)} \le \int_0^{t-s} \|A_0u_0 - h(\tau)\|_{L^\infty(\Omega)}\,\mathrm{d}\tau. \tag{B.6}$$

Using (B.6) and gathering Fubini's theorem and

$$h(\tau) - h(\tau+t-s) = \int_\tau^{\tau+t-s} \frac{\mathrm{d}h}{\mathrm{d}t}(\sigma)\,\mathrm{d}\sigma,$$

the right-hand side of (B.5) is smaller than

$$\begin{aligned}&(t-s)\|A_0u_0 - h(0)\|_{L^\infty(\Omega)} + \int_0^{t-s} \|h(0) - h(\tau)\|_{L^\infty(\Omega)}\,\mathrm{d}\tau \\ &\quad + \int_0^s \|h(\tau) - h(\tau+t-s)\|_{L^\infty(\Omega)}\,\mathrm{d}\tau.\end{aligned}$$

Thus,

$$\begin{aligned}&\|u(t) - u(s)\|_{L^\infty(\Omega)} \\ &\quad \le (t-s)\|A_0u_0 - h(0)\|_{L^\infty(\Omega)} + (t-s)\int_0^T \left\|\frac{\mathrm{d}h}{\mathrm{d}t}(\tau)\right\|_{L^\infty(\Omega)}\mathrm{d}\tau.\end{aligned} \tag{B.7}$$

Dividing the expression (B.7) by $|t-s|$, we get that u is a Lipschitz function and since $\frac{\partial u}{\partial t} \in L^2(Q_T)$, passing to the limit $|t-s| \to 0$ we obtain that $\frac{u(t)-u(s)}{t-s} \to \frac{\partial u}{\partial t}$ as $s \to t$ weakly in $L^2(Q_T)$ and *-weakly in $L^\infty(Q_T)$. Furthermore,

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^\infty(\Omega)} \leq \liminf_{s \to t} \frac{\|u(t)-u(s)\|_{L^\infty(\Omega)}}{|t-s|}.$$

Therefore, we get $u \in W^{1,\infty}(0,T;L^\infty(\Omega))$ as well as inequality (B.2). □

Concerning problem (P_T), we deduce the following similar result.

Theorem B.2. *Assume that conditions in Theorem* B.1 *and hypothesis on f in Theorem* 1.2 *are satisfied. Let $u_0 \in \mathbb{W} \cap \overline{\mathcal{D}(A_0)}^{L^\infty}$. Then, the unique weak solution to (P_T) belongs to $C([0,T];C_0(\overline{\Omega}))$ and*

(i) *there exists $\omega > 0$ such that if v is another weak solution to (P_T) with the initial datum $v_0 \in \mathbb{W} \cap \overline{\mathcal{D}(A_0)}^{L^\infty}$ then the following estimate holds for $T < \tilde{T}$:*

$$\|u(t)-v(t)\|_{L^\infty(\Omega)} \leq e^{\omega t}\|u_0 - v_0\|_{L^\infty(\Omega)}, \quad 0 \leq t \leq T.$$

(ii) *If $u_0 \in \mathcal{D}(A_0)$ then $u \in W^{1,\infty}(0,T;L^\infty(\Omega))$ and $\nabla \cdot \mathbf{a}(x,\nabla u) \in L^\infty(Q_T)$, and the following estimate holds*:

$$\left\| \frac{\partial u}{\partial t}(t) \right\|_{L^\infty(\Omega)} \leq e^{\omega t}\|\nabla \cdot \mathbf{a}(x,\nabla u_0) + f(x,u_0)\|_{L^\infty(\Omega)}.$$

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