

Nonlinear eigenvalue problems for quasilinear operators on unbounded domains

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Abstract. We prove several existence results for eigenvalue problems involving the p -Laplacian and a nonlinear boundary condition on unbounded domains. We treat the non-degenerate subcritical case and the solutions are found in an appropriate weighted Sobolev space.

1 Introduction and preliminary results

The growing attention for the study of the p -Laplacian operator Δ_p in the last few decades is motivated by the fact that it arises in various applications. For instance, in Fluid Mechanics, the shear stress $\vec{\tau}$ and the velocity gradient $\nabla_p u$ of certain fluids obey a relation of the form $\vec{\tau}(x) = a(x)\nabla_p u(x)$, where $\nabla_p u = |\nabla u|^{p-2}\nabla u$. Here $p > 1$ is an arbitrary real number and the case $p = 2$ (respectively $p < 2$, $p > 2$) corresponds to a Newtonian (respectively pseudoplastic, dilatant) fluid. The resulting equations of motion then involve $\operatorname{div}(a\nabla_p u)$, which reduces to $a\Delta_p u = a \operatorname{div}(\nabla_p u)$, provided that a is constant. The p -Laplacian appears in the study of flow through porous media ($p = 3/2$, see Showalter-Walkington [24]) or glacial sliding ($p \in (1, 4/3]$, see Pélissier-Reynaud [20]). We also refer to Aronsson-Janfalk [4] for the mathematical treatment of the Hele-Shaw flow of “power-law fluids”. The concept of Hele-Shaw flow refers to the flow between two closely-spaced parallel plates, close in the sense that the gap between the plates is small compared to the dimension of the plates. Quasilinear problems with a variable coefficient also appear in the mathematical model of the torsional creep (elastic for $p = 2$, plastic as $p \rightarrow \infty$, see Bhattacharya-DiBenedetto-Manfredi [5] and Kawohl [18]). This study is based on the observation that a prismatic material rod subject to a torsional moment, at sufficiently high temperature and for an extended period of time, exhibits a permanent deformation, called *creep*. The corresponding equations are derived under the assumptions that the components of strain and stress are linked by a power law referred to as the *creep-law* see Kachanov [16, Chapters IV, VIII], Kachanov [17], and Findley-Lai-Onaran [13]). A nonlinear field equation in Quantum Mechanics involving the p -Laplacian, for $p = 6$, has been proposed in Benci-Fortunato-Pisani [6]. Eigenvalue problems involving the p -Laplacian have been the subject of much recent interest (we refer only to Allegretto-Huang [1], Anane [3], Drábek [9], Drábek-Pohozaev [11], Drábek-Simader [12], García-Peral [15], García-Montefusco-Peral [14]).

Let $\Omega \subset \mathbf{R}^N$ be an unbounded domain with (possible noncompact) smooth boundary $\partial\Omega$. We assume throughout this paper that p, q and m are real numbers satisfying $1 < p < q < p^* = \frac{Np}{N-p}$, if $p < N$ ($p^* = +\infty$ if $p \geq N$), $q \leq m < \frac{p(N-1)}{N-p}$ if $p < N$ ($q \leq m < +\infty$ when $p \geq N$).

Let $C_0^\infty(\Omega)$ be the space of $C_0^\infty(\mathbf{R}^N)$ -functions restricted on Ω .

We define the weighted Sobolev space E as the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_E = \left(\int_{\Omega} \left(|\nabla u(x)|^p + \frac{1}{(1+|x|)^p} |u(x)|^p \right) dx \right)^{1/p}.$$

Denote by $L^p(\Omega; w_1), L^q(\Omega; w_2)$ and $L^m(\partial\Omega; w_3)$ the weighted Lebesgue spaces with weight functions $w_i(x) = (1+|x|)^{\alpha_i}$ ($i = 1, 2, 3$), and the norms defined by

$$\|u\|_{p,w_1}^p = \int_{\Omega} w_1 |u(x)|^p dx, \quad \|u\|_{q,w_2}^q = \int_{\Omega} w_2 |u(x)|^q dx$$

and

$$\|u\|_{m,w_3}^m = \int_{\partial\Omega} w_3 |u(x)|^m dS,$$

where $-N < \alpha_1 < -p$ if $p < N$ ($\alpha_1 < -p$ when $p \geq N$), $-N < \alpha_2 < q \frac{N-p}{p} - N$ if $p < N$ ($-N < \alpha_2 < 0$ when $p \geq N$), and $-N < \alpha_3 < m \frac{N-p}{p} - N + 1$ if $p < N$ ($-N < \alpha_3 < 0$ when $p \geq N$).

We shall use in our paper the following embedding result.

Theorem A. *Under the above assumptions on p, q and m , the space E is compactly embedded in $L^q(\Omega; w_2)$ and also in $L^m(\partial\Omega; w_3)$.*

This theorem is a consequence of Theorem 2 and Corollary 6 of Pflüger [22]. Furthermore, with the same proof as in Pflüger [21, Lemma 2], one can show

Lemma 1 *The quantity*

$$\|u\|_b^p = \int_{\Omega} a(x) |\nabla u|^{p-2} \nabla u \cdot \nabla u dx + \int_{\partial\Omega} b(x) |u|^{p-2} u dS$$

defines an equivalent norm on E .

2 The main results

Consider the problem

$$(A) \quad \begin{cases} -\operatorname{div}(a(x) |\nabla u|^{p-2} \nabla u) = \lambda f(x) |u|^{p-2} u + g(x) |u|^{q-2} u & \text{in } \Omega, \\ a(x) |\nabla u|^{p-2} \nabla u \cdot n + b(x) |u|^{p-2} u = h(x, u) & \text{on } \partial\Omega, \end{cases}$$

where n denotes the unit outward normal on $\partial\Omega$, $0 < a_0 \leq a \in L^\infty(\Omega)$, while $b : \partial\Omega \rightarrow \mathbf{R}$ is a continuous function satisfying

$$\frac{c}{(1+|x|)^{p-1}} \leq b(x) \leq \frac{C}{(1+|x|)^{p-1}},$$

for some constants $0 < c \leq C$.

Problems of this type arise in the study of physical phenomena related to equilibrium of anisotropic continuous media which possible are somewhere “perfect” insulators, cf. Dautray-Lions [7].

We assume that f and g are nontrivial measurable functions satisfying

$$0 \leq f(x) \leq C(1 + |x|)^{\alpha_1} \quad \text{and} \quad 0 \leq g(x) \leq C(1 + |x|)^{\alpha_2}, \quad \text{for a.e. } x \in \Omega.$$

The mapping $h : \partial\Omega \times \mathbf{R} \rightarrow \mathbf{R}$ is a Carathéodory function which fulfills the assumption

$$(A1) \quad |h(x, s)| \leq h_0(x) + h_1(x)|s|^{m-1},$$

where $h_i : \partial\Omega \rightarrow \mathbf{R}$ ($i = 0, 1$) are measurable functions satisfying

$$h_0 \in L^{m/(m-1)}(\partial\Omega; w_3^{1/(1-m)}) \quad \text{and} \quad 0 \leq h_i \leq C_h w_3 \quad \text{a.e. on } \partial\Omega.$$

We also assume

$$(A2) \quad \lim_{s \rightarrow 0} \frac{h(x, s)}{b(x)|s|^{p-1}} = 0 \quad \text{uniformly in } x.$$

$$(A3) \quad \text{There exists } \mu \in (p, q] \text{ such that}$$

$$\mu H(x, t) \leq t h(x, t) \quad \text{for a.e. } x \in \partial\Omega \text{ and every } t \in \mathbf{R}.$$

$$(A4) \quad \text{There is a nonempty open set } U \subset \partial\Omega \text{ with } H(x, t) > 0 \text{ for } (x, t) \in U \times (0, \infty), \text{ where } H(x, t) = \int_0^t h(x, s) ds.$$

Our first result asserts that under the above hypotheses, problem (A) has at least a solution.

By weak solution of problem (A) we mean a function $u \in E$ such that, for any $v \in E$,

$$\begin{aligned} & \int_{\Omega} a(x) |\nabla u|^{p-2} \nabla u \nabla v \, dx + \int_{\partial\Omega} b(x) |u|^{p-2} u v \, dS \\ &= \lambda \int_{\Omega} f(x) |u|^{p-2} u v \, dx + \int_{\Omega} g(x) |u|^{q-2} u v \, dx + \int_{\partial\Omega} h(x, u) v \, dS. \end{aligned}$$

Define

$$\tilde{\lambda} := \inf_{u \in E; u \neq 0} \left(\frac{\int_{\Omega} a(x) |\nabla u|^p \, dx + \int_{\partial\Omega} b(x) |u|^p \, dS}{\int_{\Omega} f(x) |u|^p \, dx} \right).$$

Our first result is

Theorem 1 *Assume that the conditions (A1)-(A4) hold. Then, for every $\lambda < \tilde{\lambda}$, problem (A) has a nontrivial weak solution.*

In the special case $h(x, s) \equiv 0$ we are able to show also a multiplicity result for problem (A). The statement is the following

Theorem 2 *Assume $h(x, s) \equiv 0$ and $q \geq 2$. Then, for every $\lambda < \tilde{\lambda}$, problem (A) possesses infinitely many solutions.*

Next we prove the existence of an eigensolution to the following eigenvalue problem

$$(B) \quad \begin{cases} -\operatorname{div} (a(x)|\nabla u|^{p-2}\nabla u) = \lambda \left(f(x)|u|^{p-2}u + g(x)|u|^{q-2}u \right) & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)|u|^{p-2}u = \lambda h(x, u) & \text{on } \partial\Omega. \end{cases}$$

We stress that for the next existence result of the paper we drop the assumptions (A2) and (A4). By weak solution of problem (B) we mean a function $u \in E$ such that, for any $v \in E$,

$$\begin{aligned} & \int_{\Omega} a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} b(x)|u|^{p-2}uv \, dS \\ &= \lambda \left[\int_{\Omega} f(x)|u|^{p-2}uv \, dx + \int_{\Omega} g(x)|u|^{q-2}uv \, dx + \int_{\partial\Omega} h(x, u)v \, dS \right]. \end{aligned}$$

We prove

Theorem 3 *Assume that the hypotheses (A1) and (A3) hold. Let d be an arbitrary real number such that $1/d$ is not an eigenvalue λ in problem (B), and satisfying*

$$d > \frac{1}{\bar{\lambda}}. \quad (2.1)$$

Then there exists $\bar{\rho} > 0$ such that for all $r > \rho \geq \bar{\rho}$, the eigenvalue problem (B) has an eigensolution $(u, \lambda) = (u_d, \lambda_d) \in E \times \mathbf{R}$ for which one has

$$\lambda_d \in \left[\frac{1}{d + r^2 \|u_d\|_b^{m-p}}, \frac{1}{d + \rho^2 \|u_d\|_b^{m-p}} \right].$$

3 Problem (A)

Throughout this section we use the same notations as was previously done in the case of problem (A).

The energy functional corresponding to (A) is defined as $F : E \rightarrow \mathbf{R}$

$$F(u) = \frac{1}{p} \int_{\Omega} a(x)|\nabla u|^p \, dx + \frac{1}{p} \int_{\partial\Omega} b(x)|u|^p \, dS - \frac{\lambda}{p} \int_{\Omega} f(x)|u|^p \, dx - \int_{\partial\Omega} H(x, u) \, dS - \frac{1}{q} \int_{\Omega} g(x)|u|^q \, dx$$

where H denotes the primitive function of h with respect to the second variable.

By Lemma 1 we have $\|\cdot\|_b \simeq \|\cdot\|_E$. We may write

$$F(u) = \frac{1}{p} \|u\|_b^p - \frac{\lambda}{p} \int_{\Omega} f(x)|u|^p \, dx - \int_{\partial\Omega} H(x, u) \, dS - \frac{1}{q} \int_{\Omega} g(x)|u|^q \, dx.$$

Since $p < q < p^*$, $-N < \alpha_1 < -p$ and $-N < \alpha_2 < q \frac{N-p}{p} - N$ we can apply Theorem A and we obtain that the embeddings $E \subset L^p(\Omega; w_1)$ and $E \subset L^q(\Omega; w_2)$ are compact. So the functional F is well defined.

We denote by $N_h = h(x, u(x))$, $N_H = H(x, u(x))$ the corresponding Nemytskii operators.

Lemma 2 *The operators*

$$N_h : L^m(\partial\Omega; w_3) \rightarrow L^{m/(m-1)}(\partial\Omega; w_3^{1/(1-m)}), \quad N_H : L^m(\partial\Omega; w_3) \rightarrow L^1(\partial\Omega)$$

are bounded and continuous.

Proof. The proof follows from Theorem 1.1 in [10]. ■

Our hypothesis $\lambda < \tilde{\lambda}$ implies the existence of some $C_0 > 0$ such that, for every $v \in E$

$$\|v\|_b^p - \lambda \int_{\Omega} f(x)|v|^p dx \geq C_0 \|v\|_b^p.$$

Lemma 3 *Under assumptions (A1)-(A4), the functional F is Fréchet differentiable on E and satisfies the Palais-Smale condition.*

Proof. Denote $I(u) = \frac{1}{p}\|u\|_b^p$, $K_H(u) = \int_{\partial\Omega} H(x, u) dS$, $K_{\Psi}(u) = \int_{\Omega} \Psi(x, u) dx$ and $K_{\Phi}(u) = \int_{\Omega} \Phi(x, u) dx$, where $\Phi(x, u) = \frac{1}{p}f(x)|u|^p$ and $\Psi(x, u) = \frac{1}{q}g(x)|u|^q$.

Then the directional derivative of F in the direction $v \in E$ is

$$\langle F'(u), v \rangle = \langle I'(u), v \rangle - \lambda \langle K'_{\Phi}(u), v \rangle - \langle K'_{\Psi}(u), v \rangle - \langle K'_H(u), v \rangle,$$

where

$$\begin{aligned} \langle I'(u), v \rangle &= \int_{\Omega} a(x)|\nabla u|^{p-2} \nabla u \nabla v dx + \int_{\partial\Omega} b(x)|u|^{p-2} uv dS, \\ \langle K'_H(u), v \rangle &= \int_{\partial\Omega} h(x, u) v dS, \\ \langle K'_{\Psi}(u), v \rangle &= \int_{\Omega} g(x)|u|^{q-2} uv dx, \\ \langle K'_{\Phi}(u), v \rangle &= \int_{\Omega} f(x)|u|^{p-2} uv dx. \end{aligned}$$

Clearly, $I' : E \rightarrow E^*$ is continuous. The operator K'_H is a composition of the operators

$$K'_H : E \rightarrow L^m(\partial\Omega; w_3) \xrightarrow{N_h} L^{m/(m-1)}(\partial\Omega; w_3^{1/(1-m)}) \xrightarrow{l} E^*$$

where $\langle l(u), v \rangle = \int_{\partial\Omega} uv dS$. Since

$$\int_{\partial\Omega} |uv| dS \leq \left(\int_{\partial\Omega} |u|^{m'} w_3^{1/(1-m)} dS \right)^{1/m'} \left(\int_{\partial\Omega} |v|^m w_3 dS \right)^{1/m},$$

then l is continuous, by Theorem A. As a composition of continuous operators, K'_H is continuous, too. Moreover, by our assumptions on w_3 , the trace operator $E \rightarrow L^m(\partial\Omega; w_3)$ is compact and therefore, K'_H is also compact.

Set $\varphi(u) = f(x)|u|^{p-2}u$. By the proof of Lemma 2 we deduce that the Nemytskii operator corresponding to any function which satisfies (A1) is bounded and continuous. Hence N_h and N_{φ} are bounded and continuous. We note that

$$K'_{\Phi} : E \subset L^p(\Omega; w_1) \xrightarrow{N_{\varphi}} L^{p/(p-1)}(\Omega; w_1^{1/(1-p)}) \xrightarrow{\eta} E^*$$

where $\langle \eta(u), v \rangle = \int_{\Omega} uv dx$. Since

$$\int_{\Omega} |uv| dx \leq \left(\int_{\Omega} |u|^{p/(p-1)} w_1^{1/(1-p)} dx \right)^{(p-1)/p} \left(\int_{\Omega} |v|^p w_1 dx \right)^{1/p},$$

it follows that η is continuous. But K'_Φ is the composition of three continuous operators and by the assumptions on w_1 , the embedding $E \subset L^p(\Omega; w_1)$ is compact. This implies that K'_Φ is compact. In a similar way we obtain that K'_Ψ is compact and the continuous Fréchet differentiability of F follows.

Now, let $u_n \in E$ be a Palais-Smale sequence, i.e.,

$$|F(u_n)| \leq C \text{ for all } n \quad (3.1)$$

and

$$\|F'(u_n)\|_{E^*} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.2)$$

We first prove that $\{u_n\}$ is bounded in E . Remark that (3.2) implies that

$$|\langle F'(u_n), u_n \rangle| \leq \mu \cdot \|u_n\|_b \text{ for } n \text{ large enough.}$$

This and (3.1) imply

$$C + \|u_n\|_b \geq F(u_n) - \frac{1}{\mu} \langle F'(u_n), u_n \rangle. \quad (3.3)$$

But

$$\langle F'(u_n), u_n \rangle = \int_{\Omega} a(x) |\nabla u_n|^p dx + \int_{\partial\Omega} b(x) |u_n|^p dS - \lambda \int_{\Omega} f(x) |u_n|^p dx - \int_{\Omega} g(x) |u_n|^q dx - \int_{\partial\Omega} h(x, u_n) u_n dS.$$

We have

$$\begin{aligned} F(u_n) - \frac{1}{\mu} \langle F'(u_n), u_n \rangle &= \left(\frac{1}{p} - \frac{1}{\mu} \right) \left(\|u_n\|_b^p - \lambda \int_{\Omega} f(x) |u_n|^p dx \right) \\ &\quad - \left(\int_{\partial\Omega} H(x, u_n) dS - \frac{1}{\mu} \int_{\partial\Omega} h(x, u_n) u_n dS \right) - \left(\frac{1}{q} - \frac{1}{\mu} \right) \int_{\Omega} g(x) |u_n|^q dx. \end{aligned}$$

By (A3) we deduce that

$$\int_{\partial\Omega} H(x, u_n) dS \leq \frac{1}{\mu} \int_{\partial\Omega} h(x, u_n) u_n dS. \quad (3.4)$$

Therefore

$$F(u_n) - \frac{1}{\mu} \langle F'(u_n), u_n \rangle \geq \left(\frac{1}{p} - \frac{1}{\mu} \right) C_0 \|u_n\|_b^p. \quad (3.5)$$

Relations (3.3) and (3.5) yield

$$C + \|u_n\|_b \geq \left(\frac{1}{p} - \frac{1}{\mu} \right) C_0 \|u_n\|_b^p.$$

This shows that $\{u_n\}$ is bounded in E .

To prove that $\{u_n\}$ contains a Cauchy sequence we use the following inequalities for $\xi, \zeta \in \mathbf{R}^N$ (see Diaz [8], Lemma 4.10):

$$|\xi - \zeta|^p \leq C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta), \quad \text{for } p \geq 2 \quad (3.6)$$

$$|\xi - \zeta|^2 \leq C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p}, \quad \text{for } 1 < p < 2. \quad (3.7)$$

Then we obtain in the case $p \geq 2$:

$$\begin{aligned}
\|u_n - u_k\|_b^p &= \int_{\Omega} a(x) |\nabla u_n - \nabla u_k|^p dx + \int_{\partial\Omega} b(x) |u_n - u_k|^p dS \\
&\leq C(\langle I'(u_n), u_n - u_k \rangle - \langle I'(u_k), u_n - u_k \rangle) \\
&= C(\langle F'(u_n), u_n - u_k \rangle - \langle F'(u_k), u_n - u_k \rangle + \lambda \langle K'_{\Phi}(u_n), u_n - u_k \rangle - \lambda \langle K'_{\Phi}(u_k), u_n - u_k \rangle \\
&\quad + \langle K'_H(u_n), u_n - u_k \rangle - \langle K'_H(u_k), u_n - u_k \rangle + \langle K'_{\Psi}(u_n), u_n - u_k \rangle - \langle K'_{\Psi}(u_k), u_n - u_k \rangle) \\
&\leq C(\|F'(u_n)\|_{E^*} + \|F'(u_k)\|_{E^*} + |\lambda| \|K'_{\Phi}(u_n) - K'_{\Phi}(u_k)\|_{E^*} \\
&\quad + \|K'_H(u_n) - K'_H(u_k)\|_{E^*} + \|K'_{\Psi}(u_n) - K'_{\Psi}(u_k)\|_{E^*}) \|u_n - u_k\|_b.
\end{aligned}$$

Since $F'(u_n) \rightarrow 0$ and K'_{Φ} , K'_{Ψ} , K'_H are compact, we can assume, passing eventually to a subsequence, that $\{u_n\}$ converges in E .

If $1 < p < 2$, then we use the estimate

$$\|u_n - u_k\|_b^2 \leq C' |\langle I'(u_n), u_n - u_k \rangle - \langle I'(u_k), u_n - u_k \rangle| (\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}). \quad (3.8)$$

Since $\|u_n\|_b$ is bounded, the same arguments lead to a convergent subsequence. In order to prove the estimate (3.8) we recall the following result: for all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$(x + y)^s \leq C_s(x^s + y^s) \quad \text{for any } x, y \in (0, \infty). \quad (3.9)$$

Then we obtain

$$\begin{aligned}
\|u_n - u_k\|_b^2 &= \left(\int_{\Omega} a(x) |\nabla u_n - \nabla u_k|^p dx + \int_{\partial\Omega} b(x) |u_n - u_k|^p dS \right)^{\frac{2}{p}} \\
&\leq C_p \left[\left(\int_{\Omega} a(x) |\nabla u_n - \nabla u_k|^p dx \right)^{\frac{2}{p}} + \left(\int_{\partial\Omega} b(x) |u_n - u_k|^p dS \right)^{\frac{2}{p}} \right].
\end{aligned} \quad (3.10)$$

Using (3.7), (3.9) and the Hölder inequality we find

$$\begin{aligned}
&\int_{\Omega} a(x) |\nabla u_n - \nabla u_k|^p dx = \int_{\Omega} a(x) (|\nabla u_n - \nabla u_k|^2)^{\frac{p}{2}} dx \\
&\leq C \int_{\Omega} a(x) \left((|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_k|^{p-2} \nabla u_k) (\nabla u_n - \nabla u_k) \right)^{\frac{p}{2}} (|\nabla u_n| + |\nabla u_k|)^{\frac{p(2-p)}{2}} dx \\
&= C \int_{\Omega} \left(a(x) (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_k|^{p-2} \nabla u_k) (\nabla u_n - \nabla u_k) \right)^{\frac{p}{2}} (a(x) (|\nabla u_n| + |\nabla u_k|)^p)^{\frac{2-p}{2}} dx \\
&\leq C \left(\int_{\Omega} a(x) (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_k|^{p-2} \nabla u_k) (\nabla u_n - \nabla u_k) dx \right)^{\frac{p}{2}} \left(\int_{\Omega} a(x) (|\nabla u_n| + |\nabla u_k|)^p dx \right)^{\frac{2-p}{2}} \\
&\leq \tilde{C}_p \left(\int_{\Omega} a(x) |\nabla u_n|^p dx + \int_{\Omega} a(x) |\nabla u_k|^p dx \right)^{\frac{2-p}{2}} \left(\int_{\Omega} a(x) (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_k|^{p-2} \nabla u_k) (\nabla u_n - \nabla u_k) dx \right)^{\frac{p}{2}} \\
&\leq \bar{C}_p \left[\left(\int_{\Omega} a(x) |\nabla u_n|^p dx \right)^{\frac{2-p}{2}} + \left(\int_{\Omega} a(x) |\nabla u_k|^p dx \right)^{\frac{2-p}{2}} \right]
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_{\Omega} a(x) (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_k|^{p-2} \nabla u_k) (\nabla u_n - \nabla u_k) dx \right)^{\frac{p}{2}} \\
& \leq \overline{C}_p \left[\int_{\Omega} a(x) (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_k|^{p-2} \nabla u_k) (\nabla u_n - \nabla u_k) dx \right]^{\frac{p}{2}} \left(\|u_n\|_b^{\frac{(2-p)p}{2}} + \|u_k\|_b^{\frac{(2-p)p}{2}} \right).
\end{aligned}$$

Using the last inequality and (3.9) we have the estimate

$$\begin{aligned}
& \left(\int_{\Omega} a(x) |\nabla u_n - \nabla u_k|^p dx \right)^{\frac{2}{p}} \\
& \leq C'_p \left(\int_{\Omega} a(x) (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_k|^{p-2} \nabla u_k) (\nabla u_n - \nabla u_k) dx \right) (\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}).
\end{aligned} \tag{3.11}$$

In a similar way we can obtain the estimate

$$\left(\int_{\partial\Omega} b(x) |u_n - u_k|^p dS \right)^{\frac{2}{p}} \leq C'_p \left(\int_{\partial\Omega} b(x) (|u_n|^{p-2} u_n - |u_k|^{p-2} u_k) (u_n - u_k) dx \right) (\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}). \tag{3.12}$$

It is now easy to observe that inequalities (3.10), (3.11) and (3.12) imply the estimate (3.8). The proof of Lemma 3 is complete. \blacksquare

Proof of Theorem 1. We have to verify the geometric assumptions of the Mountain-Pass Theorem. We first show that there exist positive constants R and c_0 such that

$$F(u) \geq c_0, \quad \text{for any } u \in E \text{ with } \|u\| = R. \tag{3.13}$$

By Theorem A we obtain some $A > 0$ such that

$$\|u\|_{q,w_2}^q \leq A \|u\|_b^q \quad \text{for all } u \in E.$$

This fact implies that

$$F(u) = \frac{1}{p} \left(\|u\|_b^p - \lambda \|u\|_{p,w_1}^p \right) - \frac{1}{q} \int_{\Omega} g(x) |u|^q dx - \int_{\partial\Omega} H(x, u) dS \geq \frac{C_0}{p} \|u\|_b^p - \frac{A}{q} \|u\|_b^q - \int_{\partial\Omega} H(x, u) dS.$$

By (A1) and (A2) we deduce that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\frac{1}{q} |g(x)| |u|^q \leq \varepsilon b(x) |u|^p + C_\varepsilon w_3(x) |u|^m.$$

Consequently

$$F(u) \geq \frac{C_0}{p} \|u\|_b^p - \frac{A}{q} \|u\|_b^q - \int_{\partial\Omega} (\varepsilon b(x) |u|^p + C_\varepsilon w_3(x) |u|^m) dS \geq \frac{C_0}{p} \|u\|_b^p - \frac{A}{q} \|u\|_b^q - \varepsilon c_1 \|u\|_b^p - C_\varepsilon C_2 \|u\|_b^m.$$

For $\varepsilon > 0$ and $R > 0$ small enough, we deduce that for every $u \in E$ with $\|u\|_b = R$, $F(u) \geq c_0 > 0$, which yields (3.13).

We verify in what follows the second geometric assumption of the Mountain-Pass Theorem, namely

$$\exists v \in E \text{ with } \|v\| > R \text{ such that } F(v) < c_0. \tag{3.14}$$

Choose $\psi \in C^\infty_\delta(\Omega)$, $\psi \geq 0$, such that $\emptyset \neq \text{supp}\psi \cap \partial\Omega \subset U$. From $\frac{1}{q}g(x)|u|^q \geq c_3s^\mu - c_4$ on $U \times (0, \infty)$ and (A1) we claim that

$$\begin{aligned} F(t\psi) &= \frac{t^p}{p} \left(\|\psi\|_b^p - \lambda \|\psi\|_{p,w_1}^p \right) - \frac{1}{q} \int_\Omega g(x) |t\psi|^q dx - \int_{\partial\Omega} H(x, t\psi) dS \\ &\leq \frac{t^p}{p} \left(\|\psi\|_b^p - \lambda \|\psi\|_{p,w_1}^p \right) - c_3 t^\mu \int_U \psi^\mu dS + c_4 |U| - \frac{t^q}{q} \int_\Omega w_2 \psi^q dx. \end{aligned}$$

Since $q \geq \mu > p$, we obtain $F(t\psi) \rightarrow -\infty$ as $t \rightarrow \infty$. It follows that if $t > 0$ is large enough, $F(t\psi) < 0$, so $v = t\psi$ satisfies (3.14).

By the Ambrosetti-Rabinowitz Theorem, problem (A) has a nontrivial weak solution. \blacksquare

Next we prove the second existence result about problem (A).

Proof of Theorem 2. In order to show the claim we want to apply a classical tool in critical point theory, precisely we will use the Ljusternik-Schnirelmann theory (see [23]). Consider the even functional

$$J(v) = \frac{1}{p} \int_\Omega a(x) |\nabla v|^p dx + \frac{1}{p} \int_{\partial\Omega} b(x) |v|^p dS - \frac{\lambda}{p} \int_\Omega f(x) |v|^p dx,$$

on the closed symmetric manifold

$$M = \{v \in E : \int_\Omega g(x) |v|^q = 1\}.$$

Note that M is only a C^1 -manifold, since we have assumed $1 < p < q$. By our hypotheses on f , g , b and h (note that (A1)-(A4) are easily satisfied), Lemma 3 and Theorem 5.3 in [25], we have that $J|_M$ possesses at least $\gamma(M)$ pairs of critical points (where $\gamma(M)$ stands for the genus of M).

Now we have to estimate $\gamma(M)$. Since $g \not\equiv 0$ there exists an open set $\omega \subset \Omega$ such that $g(x) \geq \delta > 0$ on ω . By the properties of the genus it follows that $\gamma(\omega) \geq \gamma(B)$, where B is the unit ball of $W_0^{1,p}(\omega) \subset E$, but it is well known that the genus of the unit ball of an infinite dimensional Banach space is infinity, so $\gamma(M) = \infty$. Hence there exists a sequence $\{v_n\} \subset E$, such that any v_n (and also $-v_n$) is a constrained critical point of J on M .

By the Lagrange multipliers rule we obtain that there exists a sequence $\{\lambda_n\} \subset \mathbf{R}$ such that

$$\int_\Omega a(x) |\nabla v_n|^p dx + \int_{\partial\Omega} b(x) |v_n|^p dS - \lambda \int_\Omega f(x) |v_n|^p dx = \lambda_n \int_\Omega g(x) |v_n|^q dx.$$

Since $v_n \in M$, using our assumption $\lambda < \tilde{\lambda}$ we find

$$\lambda_n = \|v_n\|_b^p - \lambda \int_\Omega f(x) |v_n|^p dx > 0,$$

so we can apply the usual scaling. Setting $u_n = \lambda_n^{1/(q-p)} v_n$, we have that u_n satisfies for any n the equation

$$\int_\Omega a(x) |\nabla u_n|^p dx + \int_{\partial\Omega} b(x) |u_n|^p dS = \lambda \int_\Omega f(x) |u_n|^p dx + \int_\Omega g(x) |u_n|^q dx,$$

so the claim is proved. \blacksquare

4 Problem (B)

We start with the following auxiliary result.

Lemma 4 *Under assumption (A1), if $q \leq m$, there exists a number $\bar{\rho} > 0$ such that for each $\rho \geq \bar{\rho}$ the function*

$$v \mapsto \frac{\rho^2}{m} \|v\|_b^m - \frac{1}{p} \|v\|_{p,w_1}^p - \frac{1}{q} \int_{\Omega} g(x) |v|^q dx - \int_{\partial\Omega} H(x, v) dS, \quad v \in E,$$

is bounded from below on E .

Proof. The growth condition for h implies

$$\begin{aligned} \left| \int_{\partial\Omega} H(x, v) dS \right| &\leq \int_{\partial\Omega} \left(h_0(x) |v| + \frac{1}{m} h_1(x) |v|^m \right) dS \\ &\leq \left(\int_{\partial\Omega} h_0^{\frac{m}{m-1}} w_3^{\frac{1}{1-m}} dS \right)^{\frac{m-1}{m}} \|v\|_{L^m(\partial\Omega; w_3)} + C_h \|v\|_{L^m(\partial\Omega; w_3)}^m \leq C_0 + C \|v\|_b^m, \quad v \in E, \end{aligned}$$

with constants $C_0, C > 0$. One obtains also that

$$\frac{1}{q} \left| \int_{\Omega} g(x) |u|^q dx \right| \leq C_2 \|v\|_b^q \leq \bar{C}_0 + \bar{C} \|v\|_b^m, \quad v \in E,$$

with constants $\bar{C}_0, \bar{C} > 0$. Clearly, we can choose now the positive number $\bar{\rho}$ as desired. \blacksquare

In view of Lemma 4 one can find numbers $b_0 > 0$ and $\alpha > 0$ such that

$$\frac{\bar{\rho}^2}{m} \|v\|_b^m + \frac{2}{m} b_0 - \frac{1}{p} \|v\|_{p,w_1}^p - \frac{1}{q} \int_{\Omega} g(x) |v|^q dx - \int_{\partial\Omega} H(x, v) dS \geq \alpha > 0, \quad v \in E. \quad (4.1)$$

With $b_0 > 0$ and $\bar{\rho} > 0$ as above we consider numbers $r > \rho \geq \bar{\rho}$ and a function $\beta \in C^1(\mathbf{R})$ such that

$$\beta(0) = \beta(r) = 0, \quad \beta(\rho) = b_0, \quad (4.2)$$

$$\beta'(t) < 0 \iff t < 0 \text{ or } \rho < t < r, \quad (4.3)$$

$$\lim_{|t| \rightarrow +\infty} \beta(t) = +\infty. \quad (4.4)$$

Lemma 5 *Assume that conditions (A1) and (A3) are fulfilled. Then, for any $d > 0$ satisfying (3), the functional $J : E \times \mathbf{R} \rightarrow \mathbf{R}$ defined by*

$$J(v, t) = \frac{t^2}{m} \|v\|_b^m + \frac{2}{m} \beta(t) - \frac{1}{p} \int_{\Omega} f(x) |v|^p - \frac{1}{q} \int_{\Omega} g(x) |v|^q dx - \int_{\partial\Omega} H(x, v) dx + \frac{d}{p} \|v\|_b^p \quad (4.5)$$

is of class C^1 and satisfies the Palais-Smale condition.

Proof. The property of J to be continuously differentiable has been already justified in the proof of Theorem 1.

In order to check the Palais-Smale condition let the sequences $\{v_n\} \subset E$ and $\{t_n\} \subset \mathbf{R}$ satisfy

$$|J(v_n, t_n)| \leq M, \quad \forall n \geq 1 \quad (4.6)$$

$$J'_v(v_n, t_n) = t_n^2 \|v_n\|_b^{m-p} I'(v_n) - K'_\Phi(v_n) - K'_H(v_n) - K'_\Psi(v_n) + dI'(v_n) \rightarrow 0, \quad (4.7)$$

$$J'_t(v_n, t_n) = \frac{2}{m} (t_n \|v_n\|_b^m + \beta'(t_n)) \rightarrow 0 \quad (4.8)$$

where I, K_Φ, K_H, K_Ψ have been introduced in the proof of Lemma 3.

From (4.1), (4.2), (4.5) and (4.6) we infer that

$$\begin{aligned} M &\geq \frac{t_n^2}{m} \|v_n\|_b^m + \frac{2}{m} \beta(t_n) - \frac{1}{p} \|v_n\|_{p,w_1}^p - \frac{1}{q} \int_\Omega g(x) |v_n|^q dx - \int_{\partial\Omega} H(x, v_n) dx + \frac{d}{p} \|v_n\|_b^p \\ &\geq \frac{t_n^2 - \rho^2}{m} \|v_n\|_b^m + \frac{2}{m} (\beta(t_n) - \beta(\rho)) + \frac{d}{p} \|v_n\|_b^p. \end{aligned}$$

Condition (4.4) in conjunction with the inequality above yields the boundedness of $\{t_n\}$.

Let us check the boundedness of $\{v_n\}$ along a subsequence. Without loss of generality we may admit that $\{v_n\}$ is bounded away from 0. From (22) we deduce that the sequence $\{t_n \|v_n\|_b^m\}$ is bounded. Therefore it is sufficient to argue in the case where $t_n \rightarrow 0$. From (4.6) it turns out that

$$\frac{1}{p} \|v_n\|_{p,w_1}^p + \int_\Omega H(x, v_n) dx + \frac{1}{q} \int_\Omega g(x) |v_n|^q dx - \frac{d}{p} \|v_n\|_b^p$$

is bounded. By (4.7) we deduce that

$$\frac{1}{\|v_n\|_b} (-\langle K'_\Phi(v_n), v_n \rangle - \langle K'_H(v_n), v_n \rangle - \langle K'_\Psi(v_n), v_n \rangle + d\|v_n\|_b^p) \rightarrow 0.$$

Then, for n sufficiently large, assumption (A3) allows us to write

$$\begin{aligned} M + 1 + \|v_n\|_b &\geq d \left(\frac{1}{p} - \frac{1}{\mu} \right) \|v_n\|_b^p + \left(\frac{1}{\mu} - \frac{1}{q} \right) \|v_n\|_{L^q(\Omega, w_2)}^q \\ &\quad + \int_{\partial\Omega} \left(\frac{1}{\mu} h(x, v_n) v_n - H(x, v_n) \right) dS + \left(\frac{1}{\mu} - \frac{1}{p} \right) \|v_n\|_{p,w_1}^p \\ &\geq \left(\frac{1}{p} - \frac{1}{\mu} \right) (d\|v_n\|_b^p - \|v_n\|_{p,w_1}^p) \geq \left(\frac{1}{p} - \frac{1}{\mu} \right) \left(d - \frac{1}{\tilde{\lambda}} \right) \|v_n\|_b^p. \end{aligned}$$

By (3), this establishes the boundedness of $\{v_n\}$ in E .

In view of the compactness of the mappings K'_Φ, K'_H, K'_Ψ (see the proof of Lemma 3), by (4.7) we get that

$$(d + t_n^2 \|v_n\|_b^{m-p}) I'(v_n)$$

converges in E as $n \rightarrow \infty$. The boundedness of $\{t_n\}$ and $\{v_n\}$ ensures that $\{I'(v_n)\}$ is convergent in E^* along a subsequence. Assume that $p \geq 2$. Inequality (3.6) shows that

$$\|u_n - u_k\|_b^p \leq C \left[\int_\Omega a(x) (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_k|^{p-2} \nabla u_k) \cdot (\nabla u_n - \nabla u_k) dx + \right.$$

$$+ \int_{\Gamma} b(x)(|u_n|^{p-2}u_n - |u_k|^{p-2}u_k)(u_n - u_k) \, d\Gamma \Big] =$$

$$C \langle I'(u_n) - I'(u_k), u_n - u_k \rangle \leq C \|I'(u_n) - I'(u_k)\|_b^* \|u_n - u_k\|_b \quad \text{if } p \geq 2.$$

Consequently, if $p \geq 2$, $\{v_n\}$ possesses a convergent subsequence. Proceeding in the same way with inequality (3.7) in place of (3.6) we obtain the result for $1 < p < 2$. \blacksquare

In the proof of Theorem 3 we shall make use of the following variant of the Mountain Pass Theorem (see Motreanu [19])

Lemma 6 *Let E be a Banach space and let $J : E \times \mathbf{R} \rightarrow \mathbf{R}$ be a C^1 functional verifying the hypotheses*
(a) there exist constants $\rho > 0$ and $\alpha > 0$ such that $J(v, \rho) \geq \alpha$, for every $v \in E$;
(b) there is some $r > \rho$ with $J(0, 0) = J(0, r) = 0$.

Then the number

$$c := \inf_{g \in \mathcal{P}} \max_{0 \leq \tau \leq 1} J(h(\tau))$$

is a critical value of J , where

$$\mathcal{P} := \{g \in C([0, 1]; E \times \mathbf{R}); \, g(0) = (0, 0), \, g(1) = (0, r)\}.$$

Proof of Theorem 3. We apply Lemma 6 to the function J defined in (4.5). It is clear that assertion (a) is verified with $\rho > 0$ and $\alpha > 0$ described in Lemma 4 and (4.1). Due to relation (4.2), condition (b) in Lemma 6 holds. Lemma 5 ensures that the functional J satisfies the Palais-Smale condition. Therefore Lemma 6 yields a nonzero element $(u, t) \in E \times \mathbf{R}$ such that

$$J'_v(u, t) = \left(d + t^2 \|u\|_b^{m-p}\right) I'(u) - K'_\Phi(u) - K'_H(u) - K'_\Psi(u) = 0, \quad (4.9)$$

$$J'_t(u, t) = \frac{2}{m} (t \|u\|_b^m + \beta'(t)) = 0. \quad (4.10)$$

From (4.10) it follows that

$$t\beta'(t) \leq 0. \quad (4.11)$$

Combining (4.11) and (4.3) we derive that if $t \neq 0$, then $u \neq 0$ and

$$\rho \leq t \leq r. \quad (4.12)$$

Therefore for each d in (3) such that $1/d$ is not an eigenvalue in (B) and each $r > \rho \geq \bar{\rho}$ we deduce that there exists a critical point $(u, t) = (u_d, t_d) \in E \times \mathbf{R}_+$ of J , where $t = t_d$ verifies (4.12). Consequently, relation (4.9) establishes that $u_d \in E$ is an eigenfunction in problem (B) where the corresponding eigenvalue is

$$\lambda_d = \frac{1}{d + t_d^2 \|u_d\|_b^{m-p}},$$

with $t = t_d$ satisfying (4.12). This completes the proof. \blacksquare

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