BLOW-UP SOLUTIONS FOR PARTIAL LAPLACE EQUATIONS WITH KELLER–OSSERMAN CONDITION

By

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Abstract. We investigate the existence, asymptotic boundary behavior and uniqueness of viscosity solutions $u \in C^0(\Omega)$ of equations $\mathcal{M}_{\mathbf{a}}(D^2u) = f(u) + h(x)$ in $\Omega \subset \mathbb{R}^n$ such that $u(x) \to \infty$ as $x \to \partial\Omega$. Such solutions are referred to as large or boundary blow-up solutions. Here, Ω is a smooth bounded domain, $\mathcal{M}_{\mathbf{a}}$ is a weighted partial trace operator, f is a non-decreasing function that satisfies the Keller–Osserman condition, and h is a continuous function in Ω. The main difficulty in the investigation rests on the possibility that $\mathcal{M}_{\mathbf{a}}$ is very degenerate elliptic, and h is unbounded as well as sign-changing in Ω. To the best of our knowledge, large solutions to equations involving partial trace operators have not been investigated before.

1 Introduction

This paper is concerned with the infinite boundary-value problem

$$(P_H) \qquad \begin{cases} \mathcal{H}u = h(x) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set, $\mathcal{H}$ is a second-order fully nonlinear (degenerate) elliptic operator and h is a real-valued function defined in Ω. Moreover, $u = \infty$ means that $u(x) \to \infty$ as $x \to \partial\Omega$. Solutions of (P_H) are referred to as large solutions, or boundary blow-up solutions.

If $u \in C^2(\Omega)$, let $\lambda_1(D^2u) \leq \cdots \leq \lambda_n(D^2u)$ be the eigenvalues of the Hessian matrix D^2u in non-decreasing order. As in [29], we define the weighted partial trace operator

$$(1.1) \qquad \mathcal{M}_{\mathbf{a}}u \equiv M_{\mathbf{a}}(D^2u) := \sum_{i=1}^{n} a_i\lambda_i(D^2u)$$

with $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$. It is a (degenerate) elliptic operator if $\mathbf{a} \geq \mathbf{0}$, namely $a_i \geq 0$ for all $i = 1, \ldots, n$, which is non-uniformly elliptic if $\mathbf{a} \geq \mathbf{0}$ and $a_i = 0$ for some $i = 1, \ldots, n$, even if $a_i > 0$ for the remaining i's. See [29].

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If $\mathbf{a} = \mathbf{1} := (1, \dots, 1)$, we obtain the more familiar Laplace operator $\mathcal{M}_{\mathbf{1}} \equiv \Delta$, uniformly elliptic. Let instead $\mathbf{a} \geq \mathbf{0}$ such that either $a_i = 0$ or $a_i = 1$, but $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \neq \mathbf{1}$. Then $\mathcal{M}_{\mathbf{a}}u$ is a partial trace of the Hessian matrix D^2u, that is to say a partial sum of eigenvalues, and so $\mathcal{M}_{\mathbf{a}}$ is a degenerate, non-uniformly elliptic operator. Taking a positive integer $k < n$, we obtain the sum of the smallest or largest k eigenvalues of D^2u, respectively $\mathcal{M}_k^-u = \lambda_1(D^2u) + \cdots + \lambda_k(D^2u)$ and $\mathcal{M}_k^+u = \lambda_{n-k+1}(D^2u) + \cdots + \lambda_n(D^2u)$, considered by Caffarelli–Li–Nirenberg [12] and Harvey–Lawson [38]. Such operators arise, for instance, in geometric problems involving mean partial curvature, and in differential games as two-player zero-sum games. For further properties and known results on partial trace operators we refer to [29, 45, 70, 71].

We are interested in the case $\mathcal{H}u := \mathcal{M}_{\mathbf{a}}u - f(u)$, where f is a real-valued function defined in $\mathbb{R}$, namely in the problem:

$$(P_{\mathbf{a}}) \qquad \begin{cases} \mathcal{M}_{\mathbf{a}}u = f(u) + h(x) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous, Ω is a bounded open set in $\mathbb{R}^n$, and

$$\text{(f-1)} \qquad f \text{ non-decreasing}, \quad f(0) = 0, \quad f(t) > 0 \text{ for } t > 0.$$

The study of large solutions of elliptic equations has a long history, but one can safely state that a systematic and wide investigation of such solutions began in 1957 with the independent works of J. B. Keller [44] and R. Osserman [62]. In the papers [44, 62], it was shown that, for the equation $\Delta u = f(u)$ to admit a large solution in Ω, it is sufficient that

$$\text{(f-2)} \qquad \int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)\,ds, \ ;t \geq 0.$$

This, a nowadays classical condition, is known as the Keller–Osserman condition. We point out that large solutions to semilinear equations like $\Delta u = f(u)$ arise in various problems of geometry, physics and other application areas. In fact, the first existence result for problem $(P_{\mathbf{a}})$, relying on the Keller–Osserman condition, concerns the Laplacian, namely $\mathcal{M}_{\mathbf{a}}u = \Delta u$. It is stated in Theorem III of [44] in connection with a problem of electrohydrodynamics [43]: the equilibrium of a charged gas in a perfectly conducting container. In that paper J. B. Keller used the existence of large solutions to demonstrate that the density and pressure cannot be made arbitrarily large in the interior region by putting more gas into the container, and that most of the gas accumulates in a thin layer near the surface of the container as the total mass increases. Large solutions have also found applications in geometry and mathematical physics. For instance, in [49], Loewner and Nirenberg used large solutions to $\Delta u = cu^{(n+2)/(n-2)}$ for a positive constant c in a given bounded smooth open set $\Omega \subset \mathbb{R}^n$, $n \geq 3$, to construct a complete Riemannian metric on Ω that is invariant under Möbius transformations.

Since the pioneering works of Keller [44] and Osserman [62], and due to their wide applications the study of large solutions has attracted a lot of attention. An exhaustive list of papers on large solutions is impossible. We only list some of the papers we are largely familiar with: [1, 2, 5, 18, 19, 22, 23, 24, 33, 34, 35, 47, 48, 50, 51, 52, 55, 57, 58, 59, 64, 65, 67, 72, 75], and refer the interested reader to the references therein. Among the aforementioned references, we detail some works that are in line with the problem we wish to investigate. The existence and uniqueness of large solutions to a uniformly elliptic equation

$$\mathrm{Tr}(A(x)D^2u) = |u|^{p-1}u + h(x) \quad \text{with } p > 1$$

was investigated by L. Véron [67]. In the paper [22], G. Díaz and R. Letelier studied large solutions of

$$\mathrm{div}(|Du|^{p-2}Du) = f(u) + h(x), \quad p > 1,$$

using on f a Keller–Osserman type condition adapted to the p-Laplace operator, with a non-negative $h \in C(\Omega)$. S. Alarcón and A. Quaas studied in [2] the existence, asymptotic boundary behavior and uniqueness of solutions of Problem (P_H) when $\mathcal{H}u = H(D^2u)$, with H uniformly elliptic, f satisfying the usual Keller–Osserman condition and $h \in C(\Omega)$ non-positive. In a related work [72], one of the authors, M. E. Amendola and G. Galise show that

$$\mathcal{H}u = \pm|u|^{p-1}u + |u|^{q-1} + h(x),$$

where $\mathcal{H}u = H(Du, D^2u)$, with H uniformly elliptic and "homogeneous" of degree $k \in [p, q]$, $0 < p < q$, has at most one positive large solution on a bounded domain $\Omega \subset \mathbb{R}^n$ with the "local graph property" introduced by M. Marcus and L. Véron [50]. In all the aforementioned works one common feature is that the function h is restricted to have the same sign throughout Ω.

To the best of our knowledge, the first investigation when h is sign-changing has been carried out in a recent paper of J. García-Melián [34] for the equation

$$\Delta u = |u|^{p-1}u + h(x),$$

which was shown to possess a large solution also for unbounded $h \in C(\Omega)$, while uniqueness holds when h is bounded from above in Ω. These results have been extended by one of the authors and G. Porru [57] to Problem (P_H), with $\mathcal{H}u = Lu$, a uniformly elliptic linear operator in non-divergence form with lower-order terms, and f satisfying the Keller–Osserman condition, showing existence and also uniqueness for unbounded $h \in C(\Omega)$. This work has been further extended by the authors of the present paper in [58] to the case where $\mathcal{H}$ is a fully nonlinear uniformly elliptic operator, proving existence, asymptotic boundary behavior and uniqueness of large solutions.

It is worth recalling that Keller–Osserman condition (**f-2**) and its variants have been used in studying various qualitative properties of solutions to elliptic equations. For instance, in the well known paper [66] by Vázquez, it was used to obtain a strong maximum principle. We refer to the excellent monograph of Pucci and Serrin [63] for further results and a comprehensive treatment. See also [26] and [60] for more recent results. The Keller–Osserman condition also appears in the investigation of Harnack inequality to non-negative solutions of equations of the form $Lu = f(u, Du)$. We refer to the papers [21, 42, 54, 56] and their references.

A related issue is the existence and non-existence of entire solutions, that is solutions defined in the whole space $\mathbb{R}^n$. As also stated by Keller in [44, Theorem II] and Osserman in [62, Theorem 1], the Keller–Osserman condition implies the non-existence of entire subsolutions of the equation $\Delta u = f(u)$ when $f > 0$. Conversely, the negation of the Keller–Osserman condition implies the existence of entire subsolutions for the equation $\mathcal{H}u = f(u)$, where $\mathcal{H}$ is a fully nonlinear uniformly elliptic operator. See for instance [27] and [15, 16, 17]. Particular attention has been aroused by the case $f(u) = |u|^{s-1}u$ with $s > 1$, starting from a well known paper of H. Brezis [10], where the existence of entire solutions of the semilinear equation $\Delta u - |u|^{s-1}u = h(x)$ is proved without growth condition on h at infinity. A generalization of this result to the fully nonlinear uniformly elliptic case can be found in [25, 31, 30]. We refer to [4, 14] for related results.

Our investigation needs to overcome several challenges. The standard method used in the literature to establish existence of large solutions exploits some level of regularity of solutions to suitable Dirichlet problems. Unfortunately for the problem $(P_{\mathbf{a}})$, due to the high degeneracy and non-uniform ellipticity of $\mathcal{M}_{\mathbf{a}}$ with $\mathbf{a} \geq \mathbf{0}$, as soon as one of the a_i is zero, no meaningful regularity of solutions to Dirichlet problems is known, in general (see Remark 4.5 in Section 4 below). The case $a_n = 0$ presents a special challenge that leads to some open problems that we will discuss at the end of Section 3. Another obstacle comes from the presence of the possibly sign-changing and unbounded term h. Allowing sign-changing and unbounded h into Problem $(P_{\mathbf{a}})$ demands careful considerations.

The paper is organized as follows. In Section 2 we collect the main assumptions that are at the core of all the discussion in our paper. All the main results of the paper will be presented in Section 3. In Section 4 we will recall some results from the literature that will aid in our work. Section 5 will be devoted to the proof of existence of viscosity solutions to Problem $(P_{\mathbf{a}})$. In Section 6, we will focus on the asymptotic boundary behavior of solutions to $(P_{\mathbf{a}})$. The question of uniqueness of viscosity solutions to $(P_{\mathbf{a}})$ will be dealt with in Section 7. We will conclude the paper with an appendix where we prove a lemma used in Section 5, and subsequent sections.

2 Basic assumptions

For ease of reference to the reader, in this section we will state all the basic assumptions that will be used to state and prove the main results of our paper.

Assumptions on the open set Ω: In this paper, Ω will stand for an open and bounded open subset of $\mathbb{R}^n$.

For our proof of existence of a solution to $(P_{\mathbf{a}})$ we will rely on a geometric condition introduced by Blanc and Rossi in a recent paper [9]: whatever $y \in \partial\Omega$ we take,

$$(\mathcal{G}) \quad \forall r > 0 \;\; \exists \delta > 0 : \forall x \in B_\delta(y) \cap \Omega \;\text{ and }\; \forall \ell \in \mathcal{L}_x \;\; \exists z \in \ell : z \in B_r(y) \cap \partial\Omega,$$

where $\mathcal{L}_x$ is the family of lines passing through the point x.

However, for investigating asymptotic boundary behavior and uniqueness we will need better regularity.

Suppose that Ω has a C^2 boundary $\partial\Omega$. With $d(x)$ denoting the distance between $x \in \Omega$ and the boundary $\partial\Omega$, it is well-known that there is a constant $\mu > 0$ such that $d \in C^2(\overline{\Omega}_\mu)$. Here $\Omega_\mu := \{x \in \Omega : d(x) < \mu\}$. Let $\kappa_1(x), \ldots, \kappa_{n-1}(x)$ be the principal curvatures of $\partial\Omega$ at $x \in \partial\Omega$. We should recall, see [36], that $\kappa_j \leq \mu^{-1}$ on $\partial\Omega$ for all $j = 1, \ldots, n-1$. Given $x \in \Omega_\mu$, let $y(x)$ be the unique point on $\partial\Omega$ such that $d(x) = |x - y(x)|$. In terms of the principal coordinate system at $y(x)$, we note that the gradient of d is $Dd(x) = (0, \ldots, 1)$ and its Hessian

$$D^2 d(x) = \operatorname{diag}\Big(\frac{-\kappa_1(y(x))}{1 - d(x)\kappa_1(y(x))}, \ldots, \frac{-\kappa_{n-1}(y(x))}{1 - d(x)\kappa_{n-1}(y(x))}, 0\Big).$$

We refer to [36, Section 14.6] for details. We recall that a C^2 bounded open set Ω is called uniformly convex if the principal curvatures of $\partial\Omega$ are all bounded away from zero. Thus, a C^2 open bounded set Ω is uniformly convex if and only if there is a constant $\nu > 0$ such that $\kappa_j \geq \nu$ on $\partial\Omega$ for all $j = 1, \ldots, n-1$.

Assumptions on the term f: The conditions (f-1) and (f-2) play an important role in establishing existence of solutions to Problem ($P_{\mathbf{a}}$). In order to study asymptotic boundary behavior and uniqueness, we will need additional conditions on f. But first we will need to point out some useful consequences of the assumptions (f-1) and (f-2) on $f \in C^0(\mathbb{R})$.

We begin with the following two limits:

$$\lim_{t\to\infty} \frac{\sqrt{F(t)}}{f(t)} = 0; \quad \lim_{t\to\infty} \frac{t}{f(t)} = 0. \tag{2.1}$$

See [35, 37] for a proof.

Next, using the positivity of f in $\mathbb{R}_+ := (0, \infty)$ due to (f-1), and the Keller–Osserman condition (f-2), we introduce the function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\int_{\Phi(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = t, \quad t > 0. \tag{2.2}$$

This is a non-increasing function such that

$$\lim_{t\to 0^+} \Phi(t) = \infty, \quad \Phi'(t) = -\sqrt{2F(\Phi(t))}, \quad \Phi''(t) = f(\Phi(t)). \tag{2.3}$$

The following conditions on f will be used to show existence of a viscosity solution to ($P_{\mathbf{a}}$) for an unbounded class of inhomogeneous terms $h \in C^0(\Omega)$, in the investigations of asymptotic boundary behavior, and uniqueness of solutions to Problem ($P_{\mathbf{a}}$).

(Asy-1) There exists a constant $A_+ > 0$ such that

$$\liminf_{t\to\infty} \frac{f(A_+t)}{A_+f(t)} > a_n.$$

(Asy-2) Assuming that $a_n > 0$, there is a constant $A_- > 0$ such that

$$\limsup_{t\to\infty} \frac{f(A_-t)}{A_-f(t)} < a_n.$$

For the statement of our other conditions, given $\theta > 1$, let

$$\ell_\theta := \liminf_{t\to\infty} \frac{f(\theta t)}{\theta f(t)}.$$

Using the above notation, we consider the following condition:

(D-C) $\ell_\theta > 1$ for some $\theta > 1$,

This was first introduced by Dindoš in the context of establishing Harnack inequality for non-negative solutions of $\Delta u = f(u)$, see [21]. We refer to (D-C) as Dindoš' condition.

We will also find the need for using a strengthened form of Dindoš condition (D-C) as follows:
(D-C)′ $\ell_\theta > 1$ for all $\theta > 1$.

Let us make a few remarks about the above two conditions; (D-C) and (D-C)′.

Remark 2.1.

(a) If f is regularly varying with index $p > 1$ (the definition will be given below), then f satisfies (D-C)′.

(b) If f satisfies (D-C), then we have

$$\liminf_{t\to\infty} \frac{f(\theta^j t)}{\theta^j f(t)} \geq \ell_\theta^j, \quad \text{and} \quad \limsup_{t\to\infty} \frac{f(\theta^{-j} t)}{\theta^{-j} f(t)} \leq \ell_\theta^{-j} \quad \forall j = 1, 2, 3, \ldots. \tag{2.4}$$

As a result of (b) in Remark 2.1 above we observe that if $\ell_\theta > 1$, then the limit infimum in (2.4) can be made arbitrarily large, and the limit supremum in (2.4) can be made as small as we wish by appropriately taking j large enough. In other words, when f satisfies (D-C), then constants A_+ and A_- that satisfy conditions (Asy-1) and (Asy-2) can be found.

Knowing the relative growth rate of any two solutions of Problem ($P_{\mathbf{a}}$) near the boundary is precursor for the establishment of uniqueness. In this regard the following additional conditions on f will be used:

$$\ell_i := \liminf_{t\to\infty} \frac{F(t)}{tf(t)} > 0. \tag{f-3}$$

For our uniqueness result, we will apply a stronger form of monotonicity of f:

$$t \mapsto \frac{f(t)}{t} \quad \text{is non-decreasing on } (0, \infty). \tag{f-4}$$

Condition (f-4) can be also viewed as a sort of weaker version of condition (D-C)′ at all finite points $t > 0$, in the sense noted in the next remark.

Remark 2.2. Condition (f-4) is equivalent to assuming that $\frac{f(\theta t)}{\theta f(t)} \geq 1$ for all $t > 0$ and all $\theta > 1$. In fact, $s > t > 0$ if and only if $s = \theta t$ for some $\theta > 1$ and so

$$\frac{f(\theta t)}{\theta f(t)} \geq 1 \Leftrightarrow \frac{f(s)}{s} \geq \frac{f(t)}{t}.$$

Finally we introduce a class of functions that have some of the aforementioned properties on the function f.

A function $f : [\alpha, \infty) \to \mathbb{R}$, where $\alpha > 0$, is said to be regularly varying of index $p \in \mathbb{R}$ if and only if

$$\lim_{t\to\infty} \frac{f(\xi t)}{f(t)} = \xi^p, \quad \forall \xi > 0. \tag{2.5}$$

The class of regularly varying functions of index p is denoted by RV_p.

If $f \in \mathrm{RV}_p$ is non-decreasing and of index $p > 1$, then f satisfies the Keller–Osserman condition (f-2), see [5, pp. 13–14], or [53, Remark 2.1]

Assumptions on the term h: Given a non-negative function $\eta \in C^0(\Omega)$ and a function $g : \mathbb{R} \to \mathbb{R}$, we set

$$\Theta_g(\eta) := \limsup_{d(x)\to 0} \frac{\eta(x)}{g(\Phi(d(x)))}.$$

For the identity function $g(t) = t$, we will simply write $\Theta(\eta)$ instead of $\Theta_g(\eta)$. Note that, if η is bounded in Ω, then $\Theta(\eta) = 0$.

Remark 2.3. We will assume in the sequel that $\Theta(h^-) < \infty$. In this case, for $f : \mathbb{R} \to \mathbb{R}$ that satisfies (f-1) and (f-2) we note that $\Theta_f(h^-) = 0$.

This follows from the limits (2.1), (2.3), and the observation that

$$\frac{h^-(x)}{f(\Phi(d(x)))} = \frac{h^-(x)}{\Phi(d(x))} \cdot \frac{\Phi(d(x))}{f(\Phi(d(x)))}.$$

Next we introduce a condition that will allow us to consider unbounded $h \in C^0(\Omega)$ in Problem $(P_{\mathbf{a}})$:

(D-h) The equation

$$\lambda_n(D^2\psi) = -h$$

has a supersolution $\psi \in C^2(\Omega)$ with $\psi \geq 0$ in Ω.

A remark is in order concerning condition (D-h).

Remark 2.4. Let Ω be a bounded open set, and suppose $\sup_\Omega h < \infty$. Then it is easily seen that there is $\psi \in C^2(\overline{\Omega})$ such that (D-h) holds. In fact, let $\alpha := \sup_\Omega h$, and choose β large enough such that $\psi \geq 0$ in Ω, where

$$\psi(x) := \beta - \frac{\alpha}{2}|x|^2.$$

Then $\lambda_n(D^2\psi) = -\alpha \leq -h$ in Ω, and $\psi \in C^2(\overline{\Omega})$.

Suppose further that Ω is uniformly convex. Then (D-h) can be shown to hold for a large class of unbounded functions $h \in C^0(\Omega)$, namely

$$h(x) \leq C d(x)^{\alpha-1}, \quad x \in \Omega, \tag{2.6}$$

for constants $0 < \alpha \leq 1$ and $C > 0$, as it is shown in the Appendix.

We specify that Ω is called uniformly convex when it is of class C^2, and the principal curvatures of $\partial\Omega$ are bounded away from zero (see [36, page 283]), so that there are constants $\tau > 0$ and $\mu > 0$ such that $\tau \leq \kappa_j \leq \mu^{-1}$ on $\partial\Omega$ for all $j = 1, \ldots, n-1$.

3 Main results

Now we are ready to state our main results. Our first result deals with existence of viscosity solutions to Problem $(P_{\mathbf{a}})$.

Theorem 3.1. *Let Ω be a bounded domain of $\mathbb{R}^n$ endowed with the geometric property* $(\mathcal{G})$. *Let $\mathbf{a} \neq 0$ such that $\mathbf{a} \geq \mathbf{0}$. Suppose that $f \in C^0(\mathbb{R})$ satisfies the assumptions* (f-1), (f-2)*, and $h \in C^0(\overline{\Omega})$. Then the boundary blow-up problem $(P_{\mathbf{a}})$ has a viscosity solution.*

The viscosity solution u obtained in Theorem 3.1 when $\mathbf{0} \neq \mathbf{a} \geq \mathbf{0}$ is in general lower semicontinuous, by construction. See Remark 5.2 after the proof. It is known that u is continuous in one of the following cases: $a_1 > a_2 + \cdots + a_n$, $a_n > a_1 + \cdots + a_{n-1}$ and $a_1 a_n > 0$. However, we cannot say that u is continuous in the general case $\mathbf{0} \neq \mathbf{a} \geq \mathbf{0}$.

For a possibly unbounded $h \in C^0(\Omega)$ we also have:

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied, but $h \in C^0(\Omega)$. Suppose that there exists a subsolution $w \in USC(\Omega)$ of the equation*

$$\mathcal{M}_{\mathbf{a}} u = f(u) + h(x)$$

in Ω such that $w = \infty$ on $\partial\Omega$. Then the boundary blow-up problem $(P_{\mathbf{a}})$ has a maximal viscosity solution.

As an application of Theorem 3.2 we state the following existence theorem for unbounded $h \in C^0(\Omega)$, by means of assumptions on $\mathbf{a}, f$ and h which provide the existence of a subsolution as required in Theorem 3.2.

Theorem 3.3. *Let the assumptions of Theorem* 3.2 *hold with $a_n > 0$, and*

(1) *condition* (D-h) *holds with $\psi \in C^2(\Omega) \cap C^0(\overline{\Omega})$, or*

(2) *f satisfies* (Asy-2), $\lim_{t \to -\infty} f(t) = -\infty$*, and $\Theta_f(h) = 0$.*

Then Problem $(P_{\mathbf{a}})$ has a maximal viscosity solution.

Our next set of results deals with the asymptotic boundary behavior of viscosity solutions as well as uniqueness of non-negative solutions to Problem $(P_{\mathbf{a}})$.

Here is now our first result on the asymptotic boundary behavior of subsolutions and supersolutions of $(P_{\mathbf{a}})$.

Theorem 3.4. *Let* $\mathbf{a} \geq 0$ *such that* $a_n > 0$. *Suppose* f *satisfies* (f-1) *and* (f-2), *and* $\Omega \subset \mathbb{R}^n$ *is a* C^2 *bounded open set.*

(1) *If* $h \in C^0(\Omega)$ *satisfies* $\Theta(h^-) < \infty$ *and there is a constant* $A_+ > 0$ *such that* (Asy-1) *holds, then for any subsolution* u *of the equation* $\mathcal{M}_{\mathbf{a}} u = f(u) + h$ *we have*

$$\limsup_{d(x)\to 0} \frac{u^*(x)}{\Phi(d(x))} \leq A_+. \tag{3.1}$$

(2) *If* $h \in C^0(\Omega)$ *satisfies* (D-h), *and there is a constant* $A_- > 0$ *such that* (Asy-2) *holds, then for any supersolution* u *of the equation* $\mathcal{M}_{\mathbf{a}} u = f(u) + h$ *such that* $u = \infty$ *on* $\partial\Omega$, *we have*

$$A_- \leq \frac{\Theta(\psi)}{|\boldsymbol{a}|} + \liminf_{d(x)\to 0} \frac{u_*(x)}{\Phi(d(x))}, \tag{3.2}$$

where ψ *is as given in* (D-h).

For functions f in the class RV_p with $p > 1$, we can find the following exact boundary behavior of viscosity solutions to the boundary-value problem $(\mathrm{P}_{\mathbf{a}})$.

Theorem 3.5. *Let* $\mathbf{a} \geq 0$ *such that* $\mathbf{a} \neq \mathbf{0}$. *Let* f *satisfy* (f-1), *and suppose* $f \in \mathrm{RV}_p$ *for some* $p > 1$. *Then,* Φ *is regularly varying at zero with exponent* $-2/(p-1)$; *that is*

$$\Phi(r) = \Phi(1) r^{-\frac{2}{p-1}} \exp\left(- \int_r^1 \frac{c(s)}{s}\, ds \right), \quad r > 0,$$

with $c \in C^0(\mathbb{R}^+)$ *such that* $c(r) \to 0$ *as* $r \to 0^+$. *Further, assume that* $h \in C^0(\Omega)$ *such that* $\Theta(h^-) < \infty$ *and for which* (D-h) *holds. If* $\Theta(\psi) = 0$, *then for any solution* u *of* $(P_{\mathbf{a}})$ *there holds*

$$\lim_{d(x)\to 0} \frac{u(x)}{\Phi(d(x))} = a_n^{\frac{1}{p-1}}$$

and so

$$u(x) \sim a_n^{\frac{1}{p-1}} \Phi(1) d(x)^{-\frac{2}{p-1}} \exp\left(- \int_{d(x)}^1 \frac{c(s)}{s}\, ds \right) \quad \text{as } d(x) \to 0.$$

For a proof of the above representation of Φ, we refer to the appendix, where we also show a more explicit asymptotic behavior of Φ for a class of functions f in RV_p.

Remark 3.6. We wish to emphasize here Theorem 3.5, and so Corollary 6.2 below, hold even when $a_n = 0$. In this case, any viscosity solution u of $(P_{\mathbf{a}})$ satisfies

$$u(x) = o(\Phi(d(x))) \quad \text{as } d(x) \to 0.$$

This, for instance, is the case for the lower-partial sum operator

$$M_k^-(X) := \lambda_1(X) + \cdots + \lambda_k(X)$$

with $k < n$.

Differently, Theorem 3.4, part (2), and so Corollary 6.1 below, require $a_n > 0$ for the respective conclusions to hold. This assumption is satisfied for the higher-partial sum operator $M_k^+(X) := \lambda_{n-k+1}(X) + \cdots + \lambda_n(X)$, but fails for the aforementioned lower-partial sum operator $M_k^-(X)$, with $k < n$.

Theorem 3.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a C^2 bounded open set. Suppose that $a_n > 0$, and f satisfies* (f-1)*,* (f-2)*,* (f-3) *and* (D-C)$'$*. Also suppose that $h \in C^0(\Omega)$ satisfies $\Theta(h^-) < \infty$, and* (D-h) *with $\Theta(\psi) = 0$. Let u and v be continuous viscosity solutions of $(P_{\mathbf{a}})$. Then*

$$\lim_{d(x)\to 0} \frac{u(x)}{v(x)} = 1, \tag{3.3}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Our last result states that Problem $(\mathrm{P}_{\mathbf{a}})$ admits a unique non-negative viscosity solution provided that $a_n > 0$. This brings us to our main result on uniqueness.

Theorem 3.8. *Suppose that the assumptions of Theorem 3.7 on* $\mathbf{a}$*, f and Ω are satisfied and that* (f-4) *holds as well. Also suppose $h \in C^0(\Omega)$ to be such that $\Theta(h^-) < \infty$ and* (D-h) *holds with $\psi \in C^2(\Omega)$ and $\Theta(\psi) = 0$. Then Problem $(P_{\mathbf{a}})$ admits at most one non-negative continuous viscosity solution.*

Before we conclude this section, we would like to point out two problems we were not able to settle and wish to propose as open problems.

(a) In the case when $a_n > 0$ in the partial trace operator $\mathcal{M}_{\mathbf{a}}$, Theorem 3.8 establishes uniqueness for non-negative viscosity solutions of Problem $(\mathrm{P}_{\mathbf{a}})$. The main difficulty to extend this to all viscosity solutions, regardless of the sign, was the absence of a strong comparison theorem for partial trace equations. We leave this issue as an open problem.

(b) Another natural question to raise, and which we are not able to address at this time, is: does Theorem 3.8 remain true when $a_n = 0$?

4 Notations and auxiliary results

Let $\Omega \subset \mathbb{R}^n$ be an open set, and $\mathcal{S}^n$ be the set of $n \times n$ real symmetric matrices, endowed with the following partial ordering: $X \le Y$ if and only if $Y - X$ is positive semidefinite. We say that $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ is (degenerate) elliptic if

$$X \le Y \Rightarrow H(x,t,\xi,X) \le H(x,t,\xi,Y) \ \ \forall (x,t,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \ \forall\, X, Y \in \mathcal{S}^n. \tag{4.1}$$

Throughout this paper, H will be a continuous mapping. In the case that H is independent of some variable, we simplify by omitting such a variable. For instance, in the case of the trace operator $\mathrm{Tr}(X)$, we simply set $H(X) = \mathrm{Tr}(X)$.

If $u \in C^2(\Omega)$, letting Du be the gradient and D^2u the Hessian matrix of u, we write

$$\mathcal{H}u := H(x, u, Du, D^2u), \tag{4.2}$$

and we think of this operator as acting on $C^2(\Omega)$ and refer to it as a fully nonlinear (degenerate) elliptic operator. Similarly, given $h \in C^0(\Omega)$, the equation

$$\mathcal{H}u = h(x), \tag{4.3}$$

will be said (degenerate) elliptic.

We will also consider equations (4.3) when u is only continuous in Ω, passing to a weak formulation in the viscosity sense. An upper semicontinuous function u in Ω, written $u \in USC(\Omega)$, is a viscosity subsolution of (4.3) if for all $x_0 \in \Omega$ and φ of class C^2 in a neighborhood of x_0 such that $u - \varphi$ has a local maximum equal to zero at x_0, which we sometimes describe by saying that φ touches u at x_0 from above, we have $\mathcal{H}\varphi(x_0) \ge h(x_0)$. Similarly, a lower semicontinuous function u in Ω, written $u \in LSC(\Omega)$, is a viscosity supersolution if for all $x_0 \in \Omega$ and φ of class C^2 in a neighborhood of x_0 such that $u - \varphi$ has a local minimum equal to zero at x_0, or in other words, φ touches u at x_0 from below, we have $\mathcal{H}\varphi(x_0) \le h(x_0)$. We say $u \in C^0(\Omega)$ is a viscosity solution of (4.3) if u is a viscosity subsolution and a viscosity supersolution.

If a viscosity subsolution, resp. supersolution of $\mathcal{H}u = h(x)$ belongs to $C^2(\Omega)$, then $\mathcal{H}u \ge h(x)$, resp. $\mathcal{H}u \le h(x)$, in Ω. We will use the same notations for viscosity subsolution, resp. supersolution, belonging only to $USC(\Omega)$, resp. $LSC(\Omega)$, even though in this case the above differential inequalities are not satisfied by u, at points where it is not sufficiently regular, but by the test functions φ, respectively.

Viscosity solutions can be defined also without requiring upper and lower semicontinuity. Given a function $u : \Omega \to \mathbb{R}$, we denote by

$$u^*(x) := \lim_{\delta \to 0^+} \sup_{B_\delta(x)} u, \quad \text{and} \quad u_*(x) := \lim_{\delta \to 0^+} \inf_{B_\delta(x)} u, \tag{4.4}$$

the upper and lower semicontinuous regularization (or envelopes) of u, respectively. We remark that, in fact, $u^* \in USC(\Omega)$ and $u_* \in LSC(\Omega)$.

We say that u is a subsolution (resp. a supersolution) of $\mathcal{H}u = h(x)$ in Ω, written $\mathcal{H}u \geq h(x)$ (resp. $\mathcal{H}u \leq h(x)$), in the general viscosity sense if u^* is a viscosity subsolution (resp. u_* is a viscosity supersolution), according to the preceding definition. So in the general case u is a viscosity solution of the equation $\mathcal{H}u = h(x)$ if $\mathcal{H}u^* \geq h(x)$ and $\mathcal{H}u_* \leq h(x)$ in the viscosity sense.

We will use the following general existence result. See [46, Theorem 4.3].

Theorem 4.1. *Let Ω be an open set. Suppose that $\mathcal{H}$ is a degenerate elliptic operator. Assume that there exist a viscosity subsolution $\underline{U} \in USC(\Omega)$ and a viscosity supersolution $\overline{U} \in LSC(\Omega)$, both locally bounded, of the equation $\mathcal{H}u = h(x)$ in Ω. Then*

$$\begin{aligned} \underline{u} &:= \sup\{u \in USC(\Omega) : u \text{ subsolution, } \underline{U} \leq u \leq \overline{U}\}, \\ \overline{u} &:= \inf\{u \in LSC(\Omega) : u \text{ supersolution, } \underline{U} \leq u \leq \overline{U}\} \end{aligned} \tag{4.5}$$

are viscosity solutions.

Note that $(\underline{u})^* \in USC(\Omega)$ and $(\overline{u})_* \in LSC(\Omega)$, but they are not necessarily continuous in Ω.

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$. We say that $\mathbf{a} \geq \mathbf{0}$ (resp. $\mathbf{a} > \mathbf{0}$) if $a_i \geq 0$ (resp. $a_i > 0$) for all $i = 1, \ldots, n$. Let $\lambda_1(X) \leq \cdots \leq \lambda_n(X)$ be the eigenvalues of matrix $X \in \mathcal{S}^n$. For $\mathbf{a} \geq \mathbf{0}$, the operators $\mathcal{M}_{\boldsymbol{a}}u$ defined in (1.1) are degenerate elliptic. Moreover, in the case $\mathbf{a} > \mathbf{0}$, they are uniformly elliptic, but this is no longer true as soon as one of the a_i is zero. See [29]. However, except in the trivial case $\mathbf{a} = \mathbf{0}$, we will see below that when $\mathbf{a} \geq \mathbf{0}$ these operators are more than degenerate elliptic.

To be precise, let us say that H is non-totally degenerate elliptic, see [6], if

$$H(x, t, \xi, X+rI) \geq H(x, t, \xi, X)+\alpha r \quad \forall (x, t, \xi, X) \in \Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^n \text{ and } \forall r \geq 0, \tag{4.6}$$

for some $\alpha > 0$, where I is the $n \times n$ identity matrix.

Since $\lambda_i(X + rI) = \lambda_i(X) + r$ for all $r \in \mathbb{R}$, then for all $\mathbf{a} \geq \mathbf{0}$ such that $|\mathbf{a}| > 0$, the weighted partial trace operators $\mathcal{M}_{\mathbf{a}}u$, and hence also the operator $\mathcal{M}_{\mathbf{a}}u - f(u)$, are non-totally degenerate elliptic.

For such operators, as a particular case of [6, Theorem 3.1], we have the following comparison principle.

Theorem 4.2. *Let Ω be a bounded open set, $\mathcal{H}u = \mathcal{M}_{\mathbf{a}}u - f(u)$ with $\mathbf{0} \leq \mathbf{a} \neq \mathbf{0}$ and $f \in C^0(\mathbb{R})$ non-decreasing, $h \in C^0(\overline{\Omega})$. Let also $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be such that $\mathcal{H}u \geq h(x)$ and $\mathcal{H}v \leq h(x)$ in Ω in the viscosity sense. Then*

$$\sup_{\Omega}(u - v) \leq \sup_{\partial\Omega}(u - v).$$

We recall the following existence result, Theorem 1 of [9].

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set endowed with condition* $(\mathcal{G})$. *Then, for all $g \in C(\partial\Omega)$, and all $j = 1, \ldots, n$, the Dirichlet problem*

$$(P_0) \qquad \begin{cases} \lambda_j(D^2u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

has a viscosity solution $u \in C(\overline{\Omega})$.

We will use in the sequel the cases $j = 1$ and $j = n$. In these cases the solutions of (P_0) are locally Lipschitz-continuous. See for instance [8] and [29]. The following Lipschitz estimates are deduced from [29, Lemma 5.5] by rescaling.

Theorem 4.4. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and $h \in C^0(\Omega)$. Suppose that $u \in USC(\Omega)$ (resp. $u \in LSC(\Omega)$) is a locally bounded subsolution (resp. supersolution) of the equation $\lambda_1(D^2u) = h(x)$ (resp. $\lambda_n(D^2u) = h(x)$) in Ω. Let $x_0 \in \Omega$, and* $0 < r < \operatorname{dist}(x_0, \partial\Omega)$. *Let $B_r = \{x \in \Omega : |x - x_0| < r\}$. Then there exists a positive constant C, independent of u, such that*

$$\sup_{x,y \in B_{r/2}} |u(x) - u(y)| \leq C\left(\frac{\|u\|_{L^\infty(B_r)}}{r} + r\|h\|_{L^\infty(B_r)}\right)|x - y|. \tag{4.7}$$

We point out that, in fact, interior $C^{1,\alpha}$ estimates have been proved in [61].

Remark 4.5. Let $\mathbf{0} \neq \mathbf{a} \geq \mathbf{0}$. Lemma 5.5 of [29] also provides Hölder estimates for locally bounded subsolutions, resp. supersolutions, of the equation $\mathcal{M}_{\mathbf{a}}u = h(x)$ in the case $a_1 > a_2 + \cdots + a_n > 0$, resp. in the case $a_n > a_1 + \cdots + a_{n-1} > 0$:

$$\sup_{x,y \in B_{r/2}} |u(x) - u(y)| \leq C\left(\frac{\|u\|_{L^\infty(B_r)}}{r^\alpha} + r^{2-\alpha}\|h\|_{L^\infty(B_r)}\right)|x - y|^\alpha, \tag{4.8}$$

with some $\alpha \in (0, 1)$. Theorem 5.3 of [29] yields the same estimate for locally bounded solutions of the equation $\mathcal{M}_{\mathbf{a}}u = h(x)$ in the case $a_1 a_n > 0$. But, as far as we know, there are no similar estimates for a general $\mathbf{a} \geq \mathbf{0}$ such that $\mathbf{a} \neq \mathbf{0}$.

For more properties of partial trace operators, including maximum and comparison principles, existence, uniqueness, regularity, removable singularities, entire solutions, we also refer to [3, 7, 8, 9, 12, 13, 28, 29, 31, 32, 38, 39, 40, 45, 68, 69, 70, 71].

5 Existence results

We solve the boundary blow-up problem $(P_{\mathbf{a}})$ in a bounded open set Ω by approximation with bounded solutions satisfying continuous boundary conditions on $\partial\Omega$. In order to find the approximating functions the basic result is the following.

Theorem 5.1. *Let* $\mathbf{0} \neq \mathbf{a} \geq \mathbf{0}$, $f \in C^0(\mathbb{R})$ *satisfying* (f-1)*, and* Ω *be a bounded open set endowed with the geometric property* $(\mathcal{G})$*. Suppose* $h \in C^0(\Omega)$ *is bounded and* $\phi \in C^0(\partial\Omega)$*. Then the boundary value problem*

$$(P_{\mathbf{a}})_c \qquad \begin{cases} \mathcal{M}_{\mathbf{a}} u = f(u) + h(x) & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

has a viscosity solution $u \in C^0(\overline{\Omega})$.

Proof. The proof makes use of the existence result of Theorem 4.3 and the comparison principle of Theorem 4.2.

Step 1. There exists a subsolution, continuous up to the boundary, of the equation $\lambda_1(D^2u) = f(u) + h(x)$ in Ω such that $u = \phi$ on $\partial\Omega$.

To see this, let $k := f(M_\phi) + M_h$, where

$$M_\phi := \|\phi\|_{L^\infty(\partial\Omega)} \quad \text{and} \quad M_h := \|h\|_{L^\infty(\Omega)}.$$

By Theorem 4.3 the problem

$$\begin{cases} \lambda_1(D^2 v) = 0 & \text{in } \Omega \\ v = \phi - \frac{1}{2}k|x|^2 & \text{on } \partial\Omega \end{cases}$$

has a solution $v \in C^0(\overline{\Omega})$. Set $\underline{U} := v + \frac{1}{2}k|x|^2$. Since by assumption $f(t) \geq 0$ for $t \geq 0$, then

$$\begin{aligned} \lambda_1(D^2\underline{U}) &= \lambda_1(D^2 v + kI) \\ &= \lambda_1(D^2 v) + k = k \geq 0 \qquad \text{in } \Omega, \end{aligned} \tag{5.1}$$

and

$$\underline{U} = v + \frac{1}{2}k|x|^2 = \phi \leq M_\phi \quad \text{on } \partial\Omega. \tag{5.2}$$

By (5.1), (5.2) and the maximum principle, namely using the comparison principle of Theorem 4.2 to compare $\underline{U}$ with the constant function M_ϕ on Ω, we have $\underline{U} \leq M_\phi$ in Ω. From this, since f is non-decreasing, we get $f(M_\phi) \geq f(\underline{U})$ in Ω.

We deduce from (5.1) that $\lambda_1(D^2\underline{U}) \geq k \geq f(\underline{U}) + h(x)$ in Ω. The boundary condition $\underline{U} = \phi$ on $\partial\Omega$ is provided by (5.2), and so we are done.

Step 2. There exists a supersolution, continuous up to the boundary, of the equation $\lambda_n(D^2u) = f(u) + h(x)$ in Ω such that $u = \phi$ on $\partial\Omega$.

We argue in a similar manner as in Step 1, but solving by Theorem 4.3 the problem

$$\begin{cases} \lambda_n(D^2 v) = 0 & \text{in } \Omega, \\ v = \phi + \frac{1}{2}k|x|^2 & \text{on } \partial\Omega, \end{cases}$$

with $k = -f(-M_\phi) + M_h$. By (f-1) we note that $f(t) \le 0$ for $t \le 0$. Therefore $k \ge 0$. We set $\overline{U} = v - \frac{1}{2}k|x|^2$, so that

$$\lambda_n(D^2\overline{U}) = -k \le 0 \qquad \text{in } \Omega, \tag{5.3}$$

and

$$\overline{U} = \phi \ge -M_\phi \quad \text{on } \partial\Omega, \tag{5.4}$$

where M_ϕ is defined as in Step 1. By (5.3), (5.4) and the maximum principle, we have $\overline{U} \ge -M_\phi$ in Ω. From this, since f is non-decreasing, we get $f(-M_\phi) \le f(\overline{U})$.

We deduce from (5.3) that $\lambda_n(D^2\overline{U}) \le -k \le f(\overline{U}) + h(x)$ in Ω. The boundary condition $\overline{U} = \phi$ on $\partial\Omega$ is provided by (5.4), and so we are done.

Step 3. Finally, we show that Problem $(P_{\mathbf{a}})_c$ has a unique viscosity solution $u \in C^0(\overline{\Omega})$.

Let $\mathbf{0} \ne \mathbf{a} \ge \mathbf{0}$, and set $|\mathbf{a}| = a_1 + \cdots + a_n$, so that $|\mathbf{a}| > 0$. The functions $f_{\mathbf{a}} = f/|\mathbf{a}|$ and $h_{\mathbf{a}} = h/|\mathbf{a}|$ satisfy the assumption of this theorem required of f and h, respectively. Therefore Steps 1 and 2 provide a subsolution $\underline{U}_{\mathbf{a}} \in C^0(\overline{\Omega})$ of the equation $|\mathbf{a}|\lambda_1(D^2u) = f(u) + h(x)$ and a supersolution $\overline{U}_{\mathbf{a}} \in C^0(\overline{\Omega})$ of the equation $|\mathbf{a}|\lambda_n(D^2u) = f(u) + h(x)$ in Ω, respectively, such that $\underline{U}_{\mathbf{a}} = \overline{U}_{\mathbf{a}} = \phi$ on $\partial\Omega$.

Since

$$|\mathbf{a}|\lambda_1(X) \le M_{\mathbf{a}}(X) \equiv \sum_{i=1}^n a_i\lambda_i(X) \le |\mathbf{a}|\lambda_n(X), \tag{5.5}$$

then $\underline{U}_{\mathbf{a}} \in C^0(\overline{\Omega})$ and $\overline{U}_{\mathbf{a}} \in C^0(\overline{\Omega})$ are, respectively, a subsolution and a supersolution of the equation $\mathcal{M}_{\mathbf{a}}u = f(u) + h(x)$ in Ω, and $\underline{U}_{\mathbf{a}} = \overline{U}_{\mathbf{a}} = \phi$ on $\partial\Omega$.

From this, using Theorem 4.1, we deduce that the equation $\mathcal{M}_{\mathbf{a}}u = f(u) + h(x)$ has a viscosity solution u in Ω, and $\underline{U}_{\mathbf{a}} \le u \le \overline{U}_{\mathbf{a}}$ in Ω. Hence, by the continuity of $\underline{U}_{\mathbf{a}}$ and $\overline{U}_{\mathbf{a}}$, we get

$$\underline{U}_{\mathbf{a}} \le u_* \le u \le u^* \le \overline{U}_{\mathbf{a}} \quad \text{in } \Omega. \tag{5.6}$$

We extend u_*, u and u^* to $\overline{\Omega}$, setting $u_*(x) = u(x) = u^*(x) = \phi(x)$ for $x \in \partial\Omega$. Observing also that

$$\underline{U}_{\mathbf{a}} = \overline{U}_{\mathbf{a}} = \phi \quad \text{on } \partial\Omega,$$

then from (5.6) we obtain that u_*, u and u^* are continuous at the points of the boundary. Hence $u_* \in LSC(\overline{\Omega})$ and $u^* \in USC(\overline{\Omega})$. Moreover, recalling that, by the general definition of viscosity solutions,

$$\mathcal{M}_{\mathbf{a}} u_* \le f(u_*) + h(x), \quad \mathcal{M}_{\mathbf{a}} u^* \ge f(u^*) + h(x) \quad \text{in } \Omega, \tag{5.7}$$

we also have, by the comparison principle,

$$u^* \le u_* \quad \text{in } \Omega. \tag{5.8}$$

Combining (5.8) with the reverse inequality $u_* \le u^*$ of general validity, already used above in (5.6), we get $u^* = u_* = u$ in Ω, and so $u \in C^0(\Omega)$. Since we already know from the above that u is continuous at the points of $\partial\Omega$, we conclude that $u \in C^0(\overline{\Omega})$.

Again by the comparison principle, the solution u is unique, and this finishes the proof. □

Now, with the aid of Theorem 5.1, we prove the existence of solutions to the boundary blow-up problem $(P_{\mathbf{a}})$.

Proof of Theorem 3.1. Let us consider, by invoking Theorem 5.1, the solutions $u_i \in C^0(\overline{\Omega})$, $i \in \mathbb{N}$, of the approximating problems

$$(P_{\mathbf{a}})_i \qquad \begin{cases} \mathcal{M}_{\mathbf{a}} u = f(u) + h(x) & \text{in } \Omega, \\ u = i & \text{on } \partial\Omega. \end{cases}$$

By the comparison principle, Theorem 4.3, the sequence $\{u_i\}_{i\in\mathbb{N}}$ is non decreasing, and we set

$$u(x) := \lim_{i\to\infty} u_i(x), \qquad x \in \Omega. \tag{5.9}$$

Since for all $i \in \mathbb{N}$ we have $\liminf_{x\to\partial\Omega} u(x) \ge \lim_{x\to\partial\Omega} u_i(x) = i$, the function u satisfies the boundary blow-up condition:

$$\lim_{x\to\partial\Omega} u(x) = \infty. \tag{5.10}$$

Step 1. The function $u(x)$ is locally bounded in Ω.

In fact, supposing that $0 \in \Omega$, as we may up to translations, let

$$v(x) := \Phi(R^2 - r^2) + C(R^2 - r^2),$$

where $r = |x|$, for a ball $B_R \Subset \Omega$ centered at 0 of radius R to be chosen suitably small, and $C \in \mathbb{R}_+$ to be chosen suitably large. Note that by (2.3)

$$\lim_{R\to 0^+} \inf_{B_R} v = \lim_{R\to 0^+} \Phi(R) = \infty. \tag{5.11}$$

By direct computation using (2.3), the eigenvalues of the Hessian matrix $D^2 v$ are

$$\lambda_1 = \cdots = \lambda_{n-1} \le 2\sqrt{2F(v)} - 2C; \quad \lambda_n \le 4f(v)r^2 + 2\sqrt{2F(v)} - 2C. \tag{5.12}$$

Therefore

$$\begin{aligned} |\mathbf{a}|\lambda_n(D^2 v(x)) &\le 4|\mathbf{a}|f(v(x))|x|^2 + 2|\mathbf{a}|\sqrt{2F(v(x))} - 2|\mathbf{a}|C \\ &\le |\mathbf{a}|f(v)\left\{4R^2 + \frac{2\sqrt{2F(v)}}{f(v)}\right\} - 2|\mathbf{a}|C, \end{aligned} \tag{5.13}$$

where we can choose $R > 0$ small enough, by (2.1) and (5.11), such that

$$|\mathbf{a}|\left(4R^2 + \frac{2\sqrt{2F(v)}}{f(v)}\right) \le 1,$$

and $2|\mathbf{a}|C \ge -\min_{\overline{B}_R} h$, to obtain

$$M_{\mathbf{a}}(D^2 v) \le |\mathbf{a}|\lambda_n(D^2 v) \le f(v) + h(x). \tag{5.14}$$

That is to say v is a supersolution of the equation $\mathcal{M}_{\mathbf{a}} v = f(v) + h(x)$ in any ball $B_R \Subset \Omega$ with sufficiently small radius R, and by (2.3)

$$\lim_{x \to \partial B_R} v(x) = \infty, \tag{5.15}$$

Comparing the approximating solutions u_i, $i \in \mathbb{N}$, with v on B_R, we get $u_i \le v$ in B_R, and

$$u(x) = \lim_{i \to \infty} u_i(x) \le v(x), \quad x \in B_R. \tag{5.16}$$

Let then K be a compact subset of Ω, and choose R eventually smaller in order that $R < \operatorname{dist}(K, \partial\Omega)$. Covering K with a finite number of balls $B_{R/2}$ with center $x_j \in K$, we set $v_j(x) = v(x - x_j)$. Reasoning as above in B_R, and taking into account that Φ is non-increasing, we have

$$u(x) \le v_j(x) = \Phi(R^2 - |x - x_j|^2) + C(R^2 - |x - x_j|^2) \le \Phi\left(\frac{3}{4}R^2\right) + CR^2, \quad x \in B_{R/2}(x_j),$$

so that

$$u \le \Phi\left(\frac{3}{4}R^2\right) + CR^2 \quad \text{in } K. \tag{5.17}$$

Thus u is locally bounded above in Ω. On the other hand $u \ge u_1$. So u is bounded below in Ω, and the claim of Step 1 is proved.

Step 2. Proof of the theorem for $\mathcal{M}_{\mathbf{a}}u = \lambda_1(D^2u)$, that is $\mathbf{a} = \mathbf{e}_1 = (1, \dots, 0)$.

We start as at beginning with the solutions u_i solving $(P_{\mathbf{e_1}})_i$. Let us call $U(x)$, in this case, the limit function $u(x)$ in (5.9).

By Step 1, $U(x)$ is locally bounded in Ω. By the continuity of f and h, using the local Lipschitz estimate of Theorem 4.4 with $u_i(x)$ instead of u, and $f(u_i(x)) + h(x)$ instead of $h(x)$, note that the u_i's are equi-bounded and equi-Lipschitz on each compact subset of Ω. Therefore, passing to the limit as $i \to \infty$, we also obtain that $U(x)$ is locally Lipschitz continuous. By Dini's Theorem, since $U(x)$ is the limit of the non-decreasing sequence of continuous functions u_i, then the u_i's are locally uniformly convergent. By the stability results in the theory of viscosity solutions, see for instance [20, Section 6], [11, Proposition 2.9], [46, Section 4.3], then $U(x)$ is a continuous viscosity solution of the equation $\lambda_1(D^2u) = f(u) + h(x)$ in Ω. Arguing as in the case of (5.10), $U = \infty$ on $\partial\Omega$, and this finishes the proof.

Step 3. Proof of the theorem for $\mathcal{M}_{\mathbf{a}}u = \lambda_n(D^2u)$, that is $\mathbf{a} = \mathbf{e}_n = (0, \dots, 1)$.

Reasoning as in Step 2, let u_i be the solutions of $(P_{\mathbf{e_n}})_i$. Let us call $V(x)$, in this case, the limit function $u(x)$ in (5.9). The remaining part of the proof is the same as Step 2 with λ_n and $V(x)$ instead of λ_1 and $U(x)$.

Step 4. Proof of the theorem in the general case $\mathbf{0} \neq \mathbf{a} \geq \mathbf{0}$.

As in the proof of Theorem 5.1, Step 3, we observe that for $c \in \mathbb{R}_+$ the functions $f_c = f/c$ and $h_c = h/c$ satisfy the same assumptions of f and h. Therefore Step 2 and Step 3 yield continuous viscosity solutions of the equations

$$c\lambda_1(D^2u) = f(u) + h(x) \quad \text{and} \quad c\lambda_n(D^2u) = f(u) + h(x) \quad \text{in } \Omega,$$

respectively, that go to infinity as $x \to \partial\Omega$.

This shows that the boundary blow-up problems $(P_{c\mathbf{e}_1})$ and $(P_{c\mathbf{e}_n})$ have a continuous viscosity solution for all $c > 0$.

From the hypothesis on $\mathbf{a}$ we deduce that $|\mathbf{a}| = a_1 + \cdots + a_n > 0$. Then we call $U_{\mathbf{a}}$ and $V_{\mathbf{a}}$ such solutions for $c = |\mathbf{a}|$, that is

$$\begin{aligned} |\mathbf{a}|\lambda_1(D^2U_{\mathbf{a}}) &= f(U_{\mathbf{a}}) + h(x) \text{ in } \Omega, \quad U_{\mathbf{a}} = \infty \text{ on } \partial\Omega; \\ |\mathbf{a}|\lambda_n(D^2V_{\mathbf{a}}) &= f(V_{\mathbf{a}}) + h(x) \text{ in } \Omega, \quad V_{\mathbf{a}} = \infty \text{ on } \partial\Omega. \end{aligned} \tag{5.18}$$

By construction, $U_{\mathbf{a}} = \lim_{i\to\infty} u_i$, where

$$|\mathbf{a}|\lambda_1(D^2u_i) = f(u_i) + h(x) \text{ in } \Omega, \quad u_i = i \text{ on } \partial\Omega. \tag{5.19}$$

Observing that, since $\lambda_1 \leq \lambda_n$, the function u_i is a viscosity subsolution of the equation $|\mathbf{a}|\lambda_n(D^2u) = f(u) + h(x)$ in Ω, and $V_{\mathbf{a}}$ is a viscosity solution of the same equation in Ω such that $V_{\mathbf{a}} > u_i$ on $\partial\Omega$, by the comparison principle of Theorem 4.2

we have $u_i \le V_{\mathbf{a}}$ in Ω. Hence, letting $i \to \infty$, we get

$$U_{\mathbf{a}} \le V_{\mathbf{a}}, \quad \text{in } \Omega. \tag{5.20}$$

Moreover, by (5.5), $U_{\mathbf{a}} \in C^0(\Omega)$ and $V_{\mathbf{a}} \in C^0(\Omega)$ are a subsolution and a supersolution, respectively, of the equation $\mathcal{M}_{\mathbf{a}}u = f(u) + h(x)$ such that $U_{\mathbf{a}} \le V_{\mathbf{a}}$ in Ω.

Hence by Theorem 4.1 there exists a viscosity solution u of the same equation such that $U_{\mathbf{a}} \le u \le V_{\mathbf{a}}$ in Ω and so, by (5.18), $u = \infty$ on $\partial\Omega$, finishing the proof.□

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Remark 5.2. The viscosity solution u is the supremum of functions $u_i \in C(\Omega)$, and is in turn lower semicontinuous in Ω. It is known that u is continuous in one of the following cases: $a_1 > a_2 + \cdots + a_n$, $a_n > a_1 + \cdots + a_{n-1}$ and $a_1 a_n > 0$. See the Lipschitz and the Hölder estimates of Remark 4.5 of Section 3.

Theorem 3.1 will be used now to prove Theorem 3.2 in the case $h \in C^0(\Omega)$. The proof is specular to that one of Theorem 3.1.

Proof of Theorem 3.2. Firstly, we construct by homothety an increasing sequence of open subsets Ω_j satisfying condition $(\mathcal{G})$ such that $\Omega_j \Subset \Omega_{j+1} \Subset \Omega$, $j \in \mathbb{N}$, and $\bigcup_j \Omega_j = \Omega$.

So for all $j \in \mathbb{N}$ Theorem 3.1 provides solutions u_j of the approximating problems

$$(P_{\mathbf{a}})_j \qquad \begin{cases} \mathcal{M}_{\mathbf{a}}u = f(u) + h(x) & \text{in } \Omega_j, \\ u = \infty & \text{on } \partial\Omega_j. \end{cases}$$

Let us fix $j \in \mathbb{N}$. We recall that $u_j \in LSC(\Omega)$, see Remark 5.2. Letting $\mathcal{H}u = \mathcal{M}_{\mathbf{a}}u - f(u)$, by the general definition of viscosity solutions, see Section 4, we have $\mathcal{H}(u_j)_* \le h$ in Ω_j, as well as $\mathcal{H}(u_{j+1})^* \ge h$ in Ω_j. Moreover $(u_j)_* = \infty$ on $\partial\Omega_j$, while $(u_{j+1})^* \in USC(\Omega_{j+1})$ is bounded above on $\partial\Omega_j$. Therefore by the comparison principle, Theorem 4.3, we have $(u_{j+1})^* \le (u_j)_*$ in Ω_j, and hence $u_{j+1} \le u_j$ in Ω_j.

Let x be any point in Ω. There is $j_x \in \mathbb{N}$ such that $x \in \Omega_j$ for all $j \ge j_x$. By the above, the sequence $\{u_j(x)\}_{j \ge j_x}$ is non-increasing, and we set

$$u(x) := \lim_{j\to\infty} u_j(x). \tag{5.21}$$

Comparing the subsolution $w \in USC(\Omega)$ of $(P_{\mathbf{a}})$ in Ω, that exists by assumption, and u_j in Ω_j for all $j \ge j_x$, Theorem 4.2 yields $u_j(x) \ge w(x)$, and so, letting $j \to \infty$,

$$u(x) \ge w(x) \quad \text{in } \Omega. \tag{5.22}$$

Therefore, since $\liminf_{x\to\partial\Omega} u(x) \geq \lim_{x\to\partial\Omega} w(x) = \infty$, the function u satisfies the boundary blow-up condition:

$$\lim_{x\to\partial\Omega} u(x) = \infty. \tag{5.23}$$

Step 1. The function $u(x)$ is locally bounded in Ω.

To see this, we know that $u(x)$ is locally bounded from below by (5.22). On the other hand, let K be a compact subset of Ω. By construction there is $j_K \in \mathbb{N}$ such that $K \subset \Omega_j$ for all $j \geq j_K$. Since $u_j(x) \leq u_{j_K}(x)$ for all $x \in \Omega_{j_K}$, and all $j \geq j_K$, then, letting $j \to \infty$, we have $u \leq u_{j_K}$ in K. Hence $u(x)$ is locally bounded above, too, and we are done.

Step 2. The theorem holds for $\mathcal{M}_{\mathbf{a}}u = \lambda_1(D^2u)$, that is $\mathbf{a} = \mathbf{e}_1 = (1, \dots, 0)$.

Let u_j be the viscosity solution of the boundary blow-up problem $(P_{\mathbf{e_1}})_j$, and let us call $U(x)$, in this case, the limit function $u(x)$ in (5.21).

At this point, we can repeat verbatim the proof of Theorem 3.1, Step 2, referring to the non-increasing sequence u_j instead of the non-decreasing sequence u_i addressed there, to show that $U(x)$ is a continuous viscosity solution of the equation $\lambda_1(D^2u) = f(u) + h(x)$ in Ω. Arguing as for the case leading to (5.23), we get $U = \infty$ on $\partial\Omega$, and this finishes the proof.

Step 3. Proof of the theorem for $\mathcal{M}_{\mathbf{a}}u = \lambda_n(D^2u)$, that is $\mathbf{a} = \mathbf{e}_n = (0, \dots, 1)$.

It is sufficient to repeat verbatim the proof of Theorem 3.1, Step 3, with $(P_{\mathbf{e_n}})_j$ and (5.21) instead of $(P_{\mathbf{e_n}})_i$ and (5.9), respectively.

Step 4. Proof of the theorem in the general case $\mathbf{0} \neq \mathbf{a} \geq \mathbf{0}$.

Following the proofs of Theorem 5.1, Step 3, and of Theorem 3.1, Step 4, we find a subsolution $U_{\mathbf{a}}$ and a supersolution $V_{\mathbf{a}}$ of the equation $\mathcal{M}_{\mathbf{a}}u = f(u) + h(x)$ by solving the boundary blow-up problems for the equations $|\mathbf{a}|\lambda_1(u) = f(u) + h(x)$ and $|\mathbf{a}|\lambda_n(u) = f(u) + h(x)$, respectively, as in the above Step 2 and Step 3.

By construction $V_{\mathbf{a}} = \lim_{j\to\infty} u_j$, where

$$|\mathbf{a}|\lambda_n(D^2u_j) = f(u_j) + h(x) \text{ in } \Omega_j, \quad u_j = \infty \text{ on } \partial\Omega_j. \tag{5.24}$$

Since $\lambda_1 \leq \lambda_n$, the function u_j is a viscosity supersolution of the equation $|\mathbf{a}|\lambda_1(D^2u) = f(u) + h(x)$ in Ω_j, while $U_{\mathbf{a}}$ is a viscosity solution of this equation in Ω, and $U_{\mathbf{a}} < u_j = \infty$ on $\partial\Omega_j$. Then, by the comparison principle of Theorem 4.2 we have $U_{\mathbf{a}} \leq u_j$ in Ω_j. Therefore, for any $x \in \Omega$, we have $U_{\mathbf{a}}(x) \leq u_j(x)$ for all $j \geq j_x$, where $j_x \in \mathbb{N}$ is such that $x \in \Omega_{j_x}$. It follows that, letting $i \to \infty$,

$$U_{\mathbf{a}}(x) \leq V_{\mathbf{a}}(x) \quad \text{for all } x \in \Omega. \tag{5.25}$$

We note by Theorem 4.1 that there exists a viscosity solution u of the equation $\mathcal{M}_{\mathbf{a}}u = f(u) + h(x)$ in Ω such that $U_{\mathbf{a}} \leq u \leq V_{\mathbf{a}}$ in Ω and so $u = \infty$ on $\partial\Omega$, since $U_{\mathbf{a}} = \infty = V_{\mathbf{a}}$ on $\partial\Omega$. This completes the proof. □

6 Asymptotic boundary behavior

First we begin by recalling the following eigenvalue inequalities for $X \in \mathcal{S}^n$ (see, for instance, [74, Theorem 7.10]).

$$\lambda_j(X) + \lambda_1(Y) \le \lambda_j(X+Y) \le \lambda_j(X) + \lambda_n(Y), \quad j = 1, \dots, n. \tag{6.1}$$

Consequently, for $X, Z \in \mathcal{S}^n$ we have

$$\begin{aligned} M_{\mathbf{a}}(X+Z) &= \sum_{j=1}^{n} a_j \lambda_j(X+Z) \le \sum_{j=1}^{n} a_j[\lambda_j(X) + \lambda_n(Z)] \\ &= M_{\mathbf{a}}(X) + |\boldsymbol{a}|\lambda_n(Z). \end{aligned} \tag{6.2}$$

Similarly,

$$\begin{aligned} M_{\mathbf{a}}(X+Z) &= \sum_{j=1}^{n} a_j \lambda_j(X+Z) \ge \sum_{j=1}^{n} a_j[\lambda_j(X) + \lambda_1(Z)] \\ &= M_{\mathbf{a}}(X) + |\boldsymbol{a}|\lambda_1(Z). \end{aligned}$$

Hence, we get the following inequalities, valid for all $X, Y \in \mathcal{S}^n$.

$$|\boldsymbol{a}|\lambda_1(Y-X) \le M_{\mathbf{a}}(Y) - M_{\mathbf{a}}(X) \le |\boldsymbol{a}|\lambda_n(Y-X). \tag{6.3}$$

Thus, if $u, v \in C(\Omega)$ with $u - v \in C^2(\Omega)$, then we find that

$$|\boldsymbol{a}|\lambda_1(D^2(u-v)) \le M_{\mathbf{a}}(D^2u) - M_{\mathbf{a}}(D^2v) \le |\boldsymbol{a}|\lambda_n(D^2(u-v)), \tag{6.4}$$

in the viscosity sense.

Proof of Theorem 3.4. We recall that f is assumed to satisfy (f-1) and (f-2). Since Ω is a C^2 bounded open set, we also recall that there is $\mu > 0$ such that $d \in C^2(\overline{\Omega}_\mu)$, and $|Dd| = 1$ in Ω_μ. Here,

$$\Omega_\mu := \{x \in \Omega : d(x) < \mu\}.$$

For any $0 < \rho < \mu$ we consider the following subsets of Ω:

$$\Omega_\rho^- := \{x \in \Omega : \rho < d(x) < \mu\}, \quad \Omega_\rho^+ := \{x \in \Omega : 0 < d(x) < \mu - \rho\}.$$

With the function Φ defined in (2.2), we introduce

$$w_-(x) := A_+\Phi(d(x)-\rho) \quad \text{for } x \in \Omega_\rho^- \quad \text{and} \quad w_+(x) := A_-\Phi(d(x)+\rho) \quad \text{for } x \in \Omega_\rho^+,$$

where A_+ and A_- are the positive constants given in the hypotheses of the theorem.

Note that

$$D^2 w_- = A_+\Phi''(d(x)-\rho)Dd\otimes Dd + A_+\Phi'(d(x)-\rho)D^2 d, \quad x\in\Omega_\rho^-$$

and

$$D^2 w_+ = A_-\Phi''(d(x)+\rho)Dd\otimes Dd + A_-\Phi'(d(x)+\rho)D^2 d, \quad x\in\Omega_\rho^+.$$

Given $x\in\Omega_\mu$, there is a unique $y(x)\in\partial\Omega$ such that $|x-y(x)|=d(x)$.

Let $d_\pm(x) := d(x)\pm\rho$. The eigenvalues of $D^2 w_-$ are

$$A_+\Phi''(d_-(x)), \quad \text{and} \quad -\frac{A_+\Phi'(d_-(x))\kappa_j(y(x))}{1-\kappa_j(y(x))d(x)}, \quad j=1,\dots,n-1.$$

Similarly, the eigenvalues of $D^2 w_+$ are

$$A_-\Phi''(d_+(x)), \quad \text{and} \quad -\frac{A_-\Phi'(d_+(x))\kappa_j(y(x))}{1-\kappa_j(y(x))d(x)}, \quad j=1,\dots,n-1.$$

Here $\kappa_1\le\cdots\le\kappa_{n-1}$ are the principal curvatures of $\partial\Omega$. See [36, Section 14.6].

On recalling the limit in (2.1), we choose μ sufficiently small that, for all $j=1,\dots,n-1$,

$$-\frac{\Phi'(d_\pm(x))}{\Phi''(d_\pm(x))}\frac{\kappa_j(y(x))}{1-\kappa_j(y(x))d(x)} = \frac{\sqrt{2F(\Phi(d_\pm(x)))}}{f(\Phi(d_\pm(x)))}\frac{\kappa_j(y(x))}{1-\kappa_j(y(x))d(x)}\le 1 \quad (x\in\Omega_\mu). \tag{6.5}$$

Thus,

$$-\frac{\Phi'(d_\pm(x))\kappa_j(y(x))}{1-\kappa_j(y(x))d(x)}\le\Phi''(d_\pm(x)), \quad j=1,\dots,n-1 \ (x\in\Omega_\mu). \tag{6.6}$$

On recalling assumption (Asy-1), we fix $\varepsilon>0$ such that

$$a_n < a_n(1+\varepsilon) < \liminf_{t\to\infty}\frac{f(A_+t)}{A_+f(t)}. \tag{6.7}$$

Then, by (2.3) we find the following for $x\in\Omega_\rho^-$:

$$\begin{aligned}\mathcal{M}_{\mathbf{a}}w_- + h^- =& A_+f(\Phi(d_-(x)))\\ &\times\left[a_n + \frac{\sqrt{2F(\Phi(d_-(x)))}}{f(\Phi(d_-(x)))}\sum_{j=1}^{n-1}\frac{a_j\kappa_j(y(x))}{1-\kappa_j(y(x))d(x)} + \frac{h^-(x)}{A_+f(\Phi(d_-(x)))}\right].\end{aligned}$$

Using (2.1) and Remark 2.3, we see that the limit supremum, as $d_-(x)\to 0$, of the expression in the bracket is a_n. Consequently, with μ taken as small as needed, we have

$$\begin{aligned}\mathcal{M}_{\mathbf{a}}w_- + h^- &\le A_+f(\Phi(d_-(x))a_n(1+\varepsilon)\\ &\le f(A_+\Phi(d_-(x)) = f(w_-), \quad \text{by (6.7)}.\end{aligned}$$

Therefore we deduce that

$$\mathcal{M}_{\mathbf{a}} w_- \leq f(w_-) - h^- \leq f(w_-) + h, \quad x \in \Omega_\rho^-. \tag{6.8}$$

Recall that by assumption

$$\mathcal{M}_{\mathbf{a}} u \geq f(u) + h, \quad x \in \Omega, \tag{6.9}$$

in the general viscosity sense, and set $\theta_+ := \max\{u^*(x) : d(x) \geq \mu\}$.

Then from (6.8) we find that

$$\begin{aligned} \mathcal{M}_{\mathbf{a}}(w_- + |\theta_+|) &= \mathcal{M}_{\mathbf{a}} w_- \\ &\leq f(w_- + |\theta_+|) + h, \quad x \in \Omega_\rho^-. \end{aligned} \tag{6.10}$$

We also observe, by construction, that $u^* \leq w_- + |\theta_+|$ on the boundary of Ω_ρ^-. Then, by (6.9) and (6.10), using the comparison principle, Theorem 4.2, we find that $u^* \leq w_- + |\theta_+|$ in Ω_ρ^-. That is

$$\frac{u^*(x)}{\Phi(d(x) - \rho)} - \frac{|\theta_+|}{\Phi(d(x) - \rho)} \leq A_+, \quad x \in \Omega_\rho^-.$$

Letting $\rho \to 0^+$ we find that

$$\frac{u^*(x)}{\Phi(d(x))} - \frac{|\theta_+|}{\Phi(d(x))} \leq A_+, \quad x \in \Omega_\mu.$$

Then we let $d(x) \to 0$ to conclude the assertion in (1) of the theorem.

Next, we proceed to establish (2). This time we use assumption (Asy-2) to fix $\varepsilon > 0$ such that

$$\limsup_{t \to \infty} \frac{f(A_- t)}{A_- f(t)} < (1 - \varepsilon) a_n < a_n. \tag{6.11}$$

Then, for sufficiently small $\rho > 0$, we have the following for $x \in \Omega_\rho^+$:

$$\begin{aligned} \mathcal{M}_{\mathbf{a}} w_+ &= A_- f(\Phi(d_+(x))) \left[a_n + \frac{\sqrt{2F(\Phi(d_+(x)))}}{f(\Phi(d_+(x)))} \sum_{j=1}^{n-1} \frac{a_j \kappa_j(y(x))}{1 - \kappa_j(y(x)) d(x)} \right] \\ &\geq A_- f(\Phi(d_+(x)))(1 - \varepsilon) a_n \\ &\geq f(w_+), \quad \text{by (6.11)}. \end{aligned}$$

Setting $\theta_- := A_- \Phi(\mu)$, then we have

$$\mathcal{M}_{\mathbf{a}}(w_+ - \theta_-) \geq f(w_+ - \theta_-) \quad \text{in } \Omega_\rho^+. \tag{6.12}$$

On the other hand, we use condition (D-h) to take a non-negative $\psi \in C^2(\Omega)$ such that $\lambda_n(D^2 \psi) \leq -h$ in Ω.

Therefore, since $\mathcal{M}_{\mathbf{a}}u \le f(u)+h$ in the general viscosity sense, we have, by (6.2), the following in the viscosity sense:

$$\begin{aligned} \mathcal{M}_{\mathbf{a}}(u_* + |\boldsymbol{a}|^{-1}\psi) &\le \mathcal{M}_{\mathbf{a}}u_* + \lambda_n(D^2\psi) \\ &\le \mathcal{M}_{\boldsymbol{a}}u_* - h, \qquad \text{by (D-h)} \\ &\le f(u_*) \\ &\le f(u_* + |\boldsymbol{a}|^{-1}\psi). \end{aligned} \tag{6.13}$$

Note that, by construction, since $u = \infty$ on $\partial\Omega$, then $w_+ - \theta_- \le u_* + |\boldsymbol{a}|^{-1}\psi$ on $\partial\Omega_\rho^+$. By (6.12 and (6.13), again using the comparison principle, Theorem 4.2, we conclude that $w_+ - \theta_- \le u_* + |\boldsymbol{a}|^{-1}\psi$ in Ω_ρ^+. Therefore, we have

$$A_- \le \frac{u_*(x)}{\Phi(d(x)+\rho)} + \frac{\theta_-}{\Phi(d(x)+\rho)} + \frac{|\boldsymbol{a}|^{-1}\psi(x)}{\Phi(d(x)+\rho)}, \quad x \in \Omega_\rho^+.$$

We let $\rho \to 0$ to conclude that

$$A_- \le \frac{u_*(x)}{\Phi(d(x))} + \frac{\theta_-}{\Phi(d(x))} + \frac{|\boldsymbol{a}|^{-1}\psi(x)}{\Phi(d(x))}, \quad x \in \Omega_\mu.$$

We assume that $\Theta(\psi) < \infty$, for otherwise the inequality (3.2) holds trivially. We now take the limit infimum as $d(x) \to 0$ to get the conclusion in (2) of the theorem. □

Corollary 6.1. *Let* $\mathbf{a} \ge \mathbf{0}$ *such that* $a_n > 0$, *and suppose* $h \in C^0(\Omega)$ *such that either* $\sup_\Omega h < \infty$ *or* Ω *is uniformly convex and* h *satisfies* (2.6). *Also suppose* $\Theta(h^-) < \infty$. *Assume that conditions* (f-1) *and* (f-2) *hold and there are positive constants* A_- *and* A_+ *such that* (Asy-1) *and* (Asy-2) *hold. Then, for any viscosity solution* u *of Problem* ($P_{\mathbf{a}}$), *we have*

$$A_- \le \liminf_{d(x)\to 0} \frac{u_*(x)}{\Phi(d(x))} \le \limsup_{d(x)\to 0} \frac{u^*(x)}{\Phi(d(x))} \le A_+.$$

Proof. Since $\sup_\Omega h < \infty$, or Ω is uniformly convex and h satisfies (2.6), we recall from Remark 2.4 that condition (D-h) is satisfied with a non-negative function $\psi \in C^2(\overline{\Omega})$, so that $\Theta(\psi) = 0$ in Ω. Since $\Theta(h^-) < \infty$, the desired conclusion follows from Theorem 3.4. □

Proof of Theorem 3.5. Note that, since $f \in \mathrm{RV}_p$ for $p > 1$, the function f satisfies (f-2). First let us suppose that $a_n > 0$, and let us take any $0 < \varepsilon < a_n$. Since $f \in \mathrm{RV}_p$ we see that (2.5) holds with $\xi = (a_n \pm \varepsilon)^{\frac{1}{p-1}}$ so that

$$\lim_{t\to\infty} \frac{f((a_n \pm \varepsilon)^{\frac{1}{p-1}} t)}{f(t)} = (a_n \pm \varepsilon)^{\frac{p}{p-1}}.$$

As a consequence, setting $A_\pm := (a_n \pm \varepsilon)^{\frac{1}{p-1}}$, we can write

$$\lim_{t\to\infty} \frac{f(A_\pm t)}{A_\pm f(t)} = a_n \pm \varepsilon.$$

Therefore, we see that both conditions (Asy-1) and (Asy-2) hold with

$$A_+ := (a_n + \varepsilon)^{\frac{1}{p-1}} \quad \text{and} \quad A_- := (a_n - \varepsilon)^{\frac{1}{p-1}},$$

respectively. We point out that (Asy-1) still holds with $A_+ = \varepsilon^{\frac{1}{p-1}}$ when $a_n = 0$. By invoking Theorem 3.4, and using the assumption that $\Theta(\psi) = 0$, we obtain

$$(a_n - \varepsilon)^{\frac{1}{p-1}} \le \liminf_{d(x)\to 0} \frac{u_*(x)}{\Phi(d(x))} \le \limsup_{d(x)\to 0} \frac{u^*(x)}{\Phi(d(x))} \le (a_n + \varepsilon)^{\frac{1}{p-1}}.$$

Letting $\varepsilon \to 0$, we get the desired limit when $a_n > 0$. If, on the other hand, $a_n = 0$, then we see that

$$0 \le \liminf_{d(x)\to 0} \frac{u_*(x)}{\Phi(d(x))} \le \limsup_{d(x)\to 0} \frac{u^*(x)}{\Phi(d(x))} \le \varepsilon^{\frac{1}{p-1}} \quad \forall\, \varepsilon > 0.$$

Again taking the limit as $\varepsilon \to 0$ leads to the desired result. □

A simple consequence of Theorem 3.5 is the following.

Corollary 6.2. *Let* $\mathbf{0} \neq \mathbf{a} \geq \mathbf{0}$. *Let* $h \in C^0(\Omega)$ *such that either* $\sup_\Omega h < \infty$ *or* Ω *is uniformly convex and h satisfies* (2.6). *Also suppose* $\Theta(h^-) < \infty$. *Assume condition* (f-1) *and* $f \in RV_p$ *for some* $p > 1$. *Then for any viscosity solution u of* $(P_{\mathbf{a}})$ *we have*

$$\lim_{d(x)\to 0} \frac{u(x)}{\Phi(d(x))} = a_n^{1/(p-1)}.$$

Proof. We recall from Remark 2.4 that (D-h) holds, and that $\psi \in C^2(\overline{\Omega})$. Thus $\Theta(\psi) = 0$. Therefore, by Theorem 3.5 we conclude that the above limit holds. □

We conclude this section with

Proof of Theorem 3.3. Let us begin with the assumption (1). Set

$$\lambda := \frac{a_n}{n}, \quad \text{and} \quad \Lambda := |\boldsymbol{a}|,$$

and fix $v \in C(\Omega)$, a large viscosity solution to

$$\mathcal{P}_{\lambda,\Lambda}(D^2 v) = f(v) \quad \text{in } \Omega.$$

See [59, Theorem 3.4] for the existence of such a solution. By assumption (D-h), there exists a non-negative function $\psi \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $\lambda_n(D^2\psi) = -h$ in Ω. We now see that $w := v - \psi/|\boldsymbol{a}|$ satisfies

$$\begin{aligned}
\mathcal{M}_{\mathbf{a}} w &= \mathcal{M}_{\mathbf{a}}\Big(v - \frac{\psi}{|\boldsymbol{a}|}\Big) \\
&\geq \mathcal{M}_{\mathbf{a}} v + \lambda_1(D^2(-\psi)) \\
&= \mathcal{M}_{\mathbf{a}} v - \lambda_n(D^2(\psi)) \\
&\geq \mathcal{P}^-_{\lambda,\Lambda}(D^2 v) + h, \qquad \text{by [29, (3.6)]} \\
&= f(v) + h \\
&\geq f(w) + h.
\end{aligned}$$

Here, $\mathcal{P}^-_{\lambda,\Lambda}$ is the Pucci minimal operator with positive real numbers λ and $\Lambda \geq \lambda$ (ellipticity constants) defined as

$$\mathcal{P}^-_{\lambda,\Lambda}(X) = \lambda \sum_{i=1}^{n} \lambda_i^+(X) - \Lambda \sum_{i=1}^{n} \lambda_i^-(X) = \inf_{A \in \mathcal{S}^n_{\lambda,\Lambda}} \mathrm{Tr}(AX),$$

where $\mathcal{S}^n_{\lambda,\Lambda}$ is the set of $n \times n$ real symmetric matrices with eigenvalues between λ and Λ. Thus w is a subsolution of the equation $\mathcal{M}_{\mathbf{a}}(u) = f(u) + h$ such that $w = \infty$ on $\partial\Omega$. According to Theorem 3.2, Problem $(P_{\mathbf{a}})$ admits a maximal viscosity solution.

Now, we turn to the assumption (2). First, we recall that $d \in C^2(\overline{\Omega_\mu})$ for $\mu > 0$ small enough. Let $0 < \sigma < \rho < \mu$. We extend d as a C^2 function to the entire domain Ω by setting

$$\delta = (1 - \varphi)d + \varphi, \tag{6.14}$$

where $\varphi \in C_0^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$ on $\overline{\Omega}$ and

$$\varphi \equiv 1 \ \text{ on } \overline{\Omega}^\rho = \{x \in \Omega : d(x) \geq \rho\}, \quad \varphi \equiv 0 \ \text{ on } \overline{\Omega}_\sigma = \{x \in \Omega : d(x) \leq \sigma\}.$$

By using assumption (Asy-2) we fix $\varepsilon > 0$ such that

$$\limsup_{d(x)\to 0} \frac{f(A_-\Phi(d(x)))}{A_- f(\Phi(d(x)))} < a_n(1-\varepsilon) < a_n,$$

so that, taking a sufficiently small σ, we have

$$a_n(1-\varepsilon)A_- f(\Phi(d(x))) \geq f(A_-\Phi(d(x))), \quad x \in \Omega_\sigma. \tag{6.15}$$

Now let $v(x) := A_-\Phi(\delta(x))$ for $x \in \Omega$. Since $\delta = d$ in Ω_σ, proceeding as in the proof of Theorem 3.4 we find there

$$\mathcal{M}_{\mathbf{a}} v - h = A_- f(\Phi(d(x))) \left[a_n + \frac{\sqrt{2F(\Phi(d(x)))}}{f(\Phi(d(x)))} \sum_{j=1}^{n-1} \frac{a_j \kappa_j(y(x))}{1 - \kappa_j(y(x))d(x)} - \frac{h(x)}{A_- f(\Phi(d(x)))} \right],$$

and, by assumption $\Theta_f(h) = 0$, the limit infimum of the expression in the bracket above is $a_n > 0$. Therefore, eventually taking a smaller $\sigma > 0$, we find in $\mathbf{\Omega}_\sigma$

$$\mathcal{M}_{\mathbf{a}}v - h \geq A_- f(\Phi(d(x)))(1-\varepsilon)a_n \geq f(v), \quad \text{by (6.15)},$$

that is

$$\mathcal{M}_{\mathbf{a}}v \geq f(v) + h \quad \text{in } \mathbf{\Omega}_\sigma. \tag{6.16}$$

On the other hand, noting that $v \in C^2(\overline{\mathbf{\Omega}}^\sigma)$ and $h \in C^0(\overline{\mathbf{\Omega}}^\sigma)$, we set

$$M_v = \max_{\overline{\mathbf{\Omega}}^\sigma} v \ ; \quad m_F := \min_{\overline{\mathbf{\Omega}}^\sigma} \mathcal{M}_{\mathbf{a}}v \ ; \quad M_h := \max_{\overline{\mathbf{\Omega}}^\sigma} h.$$

By the assumption that $f(t) \to -\infty$ as $t \to -\infty$, we find $t_f \in \mathbb{R}$ such that $f(t) \leq m_F - M_h$ for $t \leq t_f$. Therefore, picking $\theta \in \mathbb{R}$ such that $M_v - \theta \leq t_f$, we get in $\overline{\mathbf{\Omega}}^\sigma$

$$\begin{aligned} \mathcal{M}_{\mathbf{a}}(v-\theta) &\geq m_F = m_F - M_h + M_h \\ &\geq f(t_f) + h \\ &\geq f(M_v - \theta) + h \\ &\geq f(v-\theta) + h. \end{aligned} \tag{6.17}$$

Then $w := v - \theta$ is a subsolution of the equation $\mathcal{M}_{\mathbf{a}}u \geq f(u) + h$ in $\mathbf{\Omega}$. In addition, $w = \infty$ on $\partial\mathbf{\Omega}$. Therefore, by Theorem 3.2, we conclude that Problem $(P_{\mathbf{a}})$ admits a solution. □

7 Uniqueness

In this section, we study uniqueness of non-negative solutions to Problem $(P_{\mathbf{a}})$.

We remark that if we assume (f-1) and (f-2), then $0 \leq \ell_i \leq \frac{1}{2}$. See [59, Remark 2.10].

Conditions (f-1), (f-2) and (f-3) yield the following inequalities: for any $\rho > 1$ there is a positive constant $c_\Phi = c_\Phi(\rho)$ such that

$$\Phi(\rho t) \geq c_\Phi \, \Phi(t) \tag{7.1}$$

for sufficiently large t (see [59, Lemma 2.15]); for any $\kappa > 0$ there is a positive constant $c_f = c_f(\kappa)$ such that

$$f(\kappa t) \geq c_f f(t) \tag{7.2}$$

for sufficiently large t (see [59, Lemma 2.12]);

$$\ell_f := \limsup_{t\to\infty} \frac{t}{f(t)(\int_t^\infty ds/F(s)^{1/2})^2} \le \frac{\ell_s}{\ell_i^2}\Big(\frac{1}{2}-\ell_i\Big)^2, \tag{7.3}$$

where

$$\ell_s := \limsup_{t\to\infty} \frac{F(t)}{tf(t)} > 0$$

(see [59, Lemma 2.11]).

We recall, see [59, Remark 2.7], that if (D-C)$'$ holds, then ℓ_θ is a non-decreasing function of θ, and, for any given $\overline{\theta}$ and $\tau \in (1, \ell_{\overline{\theta}})$ there exists $\overline{t}$ such that

$$f(\theta t) \ge \tau\theta f(t) \quad \forall\, \theta > \overline{\theta} \text{ and } \forall\, t > \overline{t}. \tag{7.4}$$

Proof of Theorem 3.7. Since f satisfies condition (D-C)$'$, we recall that there are constants $0 < A_- \le 1 \le A_+ < \infty$ such that f satisfies conditions (Asy-1) and (Asy-2), see Remark 2.1. Therefore we note that Theorem 3.4 applies. So let $u, v \in C(\Omega)$ be two solutions of $(P_{\mathbf{a}})$. It is sufficient to show, following the proof of [59, Theorem 4.4] and [58, Theorem 6.2], that

$$\theta := \limsup_{d(x)\to 0} \frac{u(x)}{v(x)} \le 1. \tag{7.5}$$

For this purpose, let us assume by contradiction that $\theta > 1$.

By definition, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} &\frac{u(x)}{v(x)} \le \theta + \varepsilon \quad \text{if } d(x) \le \delta;\\ &\frac{u(x_0)}{v(x_0)} > \theta - \varepsilon \quad \text{for some } x_0 \in \Omega : d(x_0) < \frac{2}{3}\delta. \end{aligned} \tag{7.6}$$

Here, we will consider ε small enough, $0 < \varepsilon < \varepsilon_0$, where $\theta - \varepsilon_0 > 1$, so that $\theta - \varepsilon > 1$.

Then we set

$$\mathcal{O}_{\varepsilon,r} := \mathcal{O}_\varepsilon \cap B_r(x_0), \quad \text{where } \mathcal{O}_\varepsilon = \{x \in \Omega \,|\, u(x) > (\theta-\varepsilon)v(x)\} \text{ and } r = \frac{1}{2}d(x_0),$$

so that in particular

$$\mathcal{O}_{\varepsilon,r} \subset B_r(x_0) \Subset \Omega_\delta = \{x \in \Omega : d(x) < \delta\} \quad \text{and} \quad r \le d(x) \le 3r \text{ for } x \in B_r(x_0).$$

Eventually taking a smaller δ in order to have $t = v(x)$ sufficiently large, we can apply (7.4) with $\overline{\theta} = \theta - \varepsilon_0$ and $\tau = \frac{\ell_{\overline{\theta}}+1}{2}$ to get

$$f((\theta-\varepsilon)v) \ge (\theta-\varepsilon)f(v) + c_\theta f(v) \quad \text{in } \Omega_\delta \tag{7.7}$$

with a positive constant c_θ.

Then we compare u and $v_\varepsilon = (\theta - \varepsilon)v$ in $\mathcal{O}_{\varepsilon,r}$. Let us first observe that the following hold:

$$\mathcal{M}_{\mathbf{a}} v_\varepsilon = (\theta - \varepsilon)f(v) + (\theta - \varepsilon)h(x), \quad \text{in } \mathbf{\Omega}, \tag{7.8}$$

and

$$\begin{aligned} \mathcal{M}_{\mathbf{a}} u &= f(u) + h(x) \\ &\geq f((\theta - \varepsilon)v) + h(x) \\ &\geq (\theta - \varepsilon)f(v) + c_\theta f(v) + h(x) \quad \text{in } \mathcal{O}_{\varepsilon,r} \ \text{[by (7.7)]}. \end{aligned} \tag{7.9}$$

Letting $\delta > 0$ be small as needed, from Theorem 3.4 we deduce that

$$\frac{1}{2} A_- \Phi(d(x)) \leq v(x) \leq 2A_+ \Phi(d(x)), \quad x \in \mathcal{O}_{\varepsilon,r}, \tag{7.10}$$

and so, using the properties of f and Φ, we find the lower bound:

$$\begin{aligned} f(v(x)) &\geq f\Big(\frac{1}{2} A_- \Phi(d(x))\Big) && \text{[by (7.10), } f \text{ non-decreasing]} \\ &\geq f\Big(\frac{1}{2} A_- \Phi(3r)\Big) && \text{[by } d(x) \leq 3r, \ \Phi \text{ non-increasing]} \\ &\geq f\Big(\frac{1}{2} A_- c_\Phi\, \Phi(r)\Big) && \text{[by (7.1)]} \\ &\geq c_f f(\Phi(r)) && \text{[by (7.2)]} \\ &\geq c_f f(\Phi(d(x))) && \text{[by } d(x) \geq r]. \end{aligned} \tag{7.11}$$

It follows that, for $x \in \mathbf{\Omega}_\delta$ with a sufficiently small $\delta > 0$,

$$c_\theta f(v(x)) \geq c_\theta\, c_f f(\Phi(d(x))) = c' f(\Phi(d(x))). \tag{7.12}$$

Moreover, choosing $\delta > 0$ small as needed, for $y \in B_r(x_0)$ we have

$$\begin{aligned} f(\Phi(r)) &= \frac{f(\Phi(r))}{r^2} \left(\int_{\Phi(r)}^{\infty} \frac{ds}{\sqrt{F(s)}} \right)^2 && \text{[by (2.2)]} \\ &\geq \frac{1}{\ell_f + 1} \frac{\Phi(r)}{r^2} && \text{[by (7.3)]} \\ &\geq \frac{1}{\ell_f + 1} \frac{\Phi(d(y))}{r^2} && \text{[by } d(y) \geq r] \\ &\geq \frac{1}{2A_+(\ell_f + 1)} \frac{v(y)}{r^2} && \text{[by (7.10)]}. \end{aligned} \tag{7.13}$$

We now see that (7.12), together with (7.13), yields

$$c_\theta f(v(x)) \geq C\, \frac{v(y)}{r^2} \tag{7.14}$$

with some $C > 0$, depending on ℓ_f and A_+ only. The inequality (7.14), combined with (7.9), leads to the inequality

$$\mathcal{M}_{\mathbf{a}} u \geq (\theta - \varepsilon) f(v(x)) + h(x) + C \frac{v(y)}{r^2} \tag{7.15}$$

for all $x \in \mathcal{O}_{\varepsilon,r}$ and $y \in B_r(x_0)$.

Next, let us consider for all $y \in B_r(x_0)$ the polynomial

$$p(x) = \sigma v(y)\Big(1 - \frac{|x - x_0|^2}{r^2}\Big)$$

with a constant $\sigma > 0$ to be suitably chosen in the sequel.

Since $p \in C^2(\mathbb{R}^n)$ and $D^2 p = -\frac{2\sigma v(y)}{r^2} I$, we use (6.4) to find that

$$\begin{aligned} \mathcal{M}_{\boldsymbol{a}}(u + p) &= M_{\boldsymbol{a}}(D^2 u + D^2 p) \\ &\geq \mathcal{M}_{\mathbf{a}}(D^2 u) + |\boldsymbol{a}| \lambda_1(D^2 p) \\ &= \mathcal{M}_{\boldsymbol{a}} u - \frac{2\sigma v(y)}{r^2} |\mathbf{a}| \\ &\geq (\theta - \varepsilon) f(v) + h(x) + C \frac{v(y)}{r^2} - \frac{2\sigma v(y)}{r^2} |\mathbf{a}|. \end{aligned} \tag{7.16}$$

On choosing $\sigma = \frac{C}{2|a|}$ and using (7.15), we obtain

$$\begin{aligned} \mathcal{M}_{\mathbf{a}}(u + p) &\geq (\theta - \varepsilon) f(v(x)) + h(x) + C \frac{v(y)}{r^2} - \frac{2\sigma v(y)}{r^2} |\mathbf{a}| \\ &= (\theta - \varepsilon) f(v) + (\theta - \varepsilon) h(x). \end{aligned} \tag{7.17}$$

Let $\psi_\varepsilon := \frac{\theta - \varepsilon - 1}{|\boldsymbol{a}|} \psi$. Using (6.4) and (7.8) we see that the following holds in $\mathbf{\Omega}$:

$$\begin{aligned} \mathcal{M}_{\mathbf{a}}(v_\varepsilon + \psi_\varepsilon) &\leq \mathcal{M}_{\mathbf{a}} v_\varepsilon + (\theta - \varepsilon - 1)\lambda_n(D^2 \psi) \\ &\leq (\theta - \varepsilon) f(v) + (\theta - \varepsilon) h - (\theta - \varepsilon - 1) h, \quad \text{by (D-h)} \\ &= (\theta - \varepsilon) f(v) + h. \end{aligned} \tag{7.18}$$

Comparing (7.17) and (7.18) we see that $u + p$ and $v_\varepsilon - \psi_\varepsilon$ are a subsolution and a supersolution of the same equation, and we estimate the difference on $\mathcal{O}_{\varepsilon,r}$. By the comparison principle such difference has a maximum on the boundary,

$$\begin{aligned} \partial\mathcal{O}_{\varepsilon,r} &= (B_r(x_0) \cap \partial\mathcal{O}_\varepsilon) \cup (\mathcal{O}_\varepsilon \cap \partial B_r(x_0)) \\ &= \{x : |x - x_0| < r, u(x) = (\theta - \varepsilon) v(x)\} \cup \{x : |x - x_0| = r, u(x) > (\theta - \varepsilon) v(x)\}, \end{aligned}$$

so that for some $y_0 \in \partial\mathcal{O}_{\varepsilon,r}$ we have

$$\begin{aligned} u(x_0) + p(x_0) - &(\theta - \varepsilon) v(x_0) - \frac{\theta - \varepsilon - 1}{|\boldsymbol{a}|} \psi(x_0) \\ &\leq u(y_0) + p(y_0) - (\theta - \varepsilon) v(y_0) - \frac{\theta - \varepsilon - 1}{|\boldsymbol{a}|} \psi(y_0). \end{aligned} \tag{7.19}$$

We claim that $y_0 \in \partial B_r(x_0)$. Indeed, let $\tau_0 = u(x_0) - (\theta - \varepsilon)v(x_0) > 0$. Supposing by contradiction that $y_0 \in B_r(x_0)$, then $u(y_0) = (\theta - \varepsilon)v(y_0)$, and from (7.19) we would get

$$\tau_0 + p(x_0) < u(x_0) + p(x_0) - \frac{\theta - \varepsilon - 1}{|\boldsymbol{a}|}\psi(x_0) < p(y_0) - \frac{\theta - \varepsilon - 1}{|\boldsymbol{a}|}\psi(y_0),$$

from which we get

$$\tau_0 + p(x_0) - p(y_0) \le \frac{\theta - \varepsilon - 1}{|\boldsymbol{a}|}(\psi(x_0) - \psi(y_0)).$$

But then, for sufficiently small $r > 0$, we get a contradiction since $y_0 \to x_0$, by continuity, $p(x_0) - p(y_0) \to 0$ and $\psi(x_0) - \psi(y_0) \to 0$ as $r \to 0$.

Turning to (7.19) with $y_0 \in \partial B_r(x_0)$, we have $p(y_0) = 0$, and since

$$u(x_0) - (\theta - \varepsilon)v(x_0) \ge 0$$

we get

$$\sigma v(y) = p(x_0) \le u(y_0) - (\theta - \varepsilon)v(y_0) + \frac{\theta - \varepsilon - 1}{|\boldsymbol{a}|}(\psi(x_0) - \psi(y_0)) \tag{7.20}$$

and so, using (7.6) and (7.20), we obtain

$$\begin{aligned} \sigma v(y) &\le (\theta + \varepsilon)v(y_0) - (\theta - \varepsilon)v(y_0) + \frac{\theta - \varepsilon - 1}{|\boldsymbol{a}|}(\psi(x_0) - \psi(y_0)) \\ &= 2\varepsilon v(y_0) + \frac{\theta - \varepsilon - 1}{|\boldsymbol{a}|}(\psi(x_0) - \psi(y_0)). \end{aligned}$$

Letting $r \to 0$, we have $y_0 \to x_0$, and therefore by continuity we have

$$\sigma v(x_0) \le 2\varepsilon v(x_0).$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, and $v(x_0) > 0$, the above inequality with $y = y_0$ and $\varepsilon < \sigma/2$ yields a contradiction, which proves (7.5), and we are done. □

The following lemma can be obtained as an immediate consequence of Theorem 4.2. Since this form of Theorem 4.2 will be used in the uniqueness proof, we isolate it for convenience.

Lemma 7.1. *Suppose* $\mathcal{M}_{\mathbf{a}}u \ge f(u) + h$ *in* Ω *and* $\mathcal{M}_{\mathbf{a}}v \le f(v) + h$ *in* Ω *for some* $u, v \in C(\Omega)$. *If*

$$\lim_{d(x)\to 0} \frac{u(x)}{v(x)} < 1, \tag{7.21}$$

then $u \le v$ *in* Ω.

Proof. Assume the contrary so that $u > v$ at some point in Ω. We consider the non-empty open set

$$\mathcal{O} := \{x \in \Omega : u(x) > v(x)\}.$$

In view of (7.21) we see that $\mathcal{O} \subset\subset \Omega$. Since $u = v$ on $\partial\mathcal{O}$, we invoke Theorem 4.2 to conclude that $u \leq v$ in $\mathcal{O}$, which is an obvious contradiction. □

We now use Theorem 3.7 to prove the following result on uniqueness.

Proof of Theorem 3.8. Let $u, v \in C(\Omega)$ be two non-negative solutions of $(P_{\mathbf{a}})$. It is enough to show that $u \leq v$ in Ω. Let $w_\varepsilon := (1+\varepsilon)v + \frac{\varepsilon}{|a|}\psi$ for $\varepsilon > 0$. Then, as a consequence of (3.3) and the assumption that $\Theta(\psi) = 0$, we find that

$$\lim_{d(x)\to 0} \frac{u(x)}{w_\varepsilon(x)} = \frac{1}{1+\varepsilon} < 1.$$

Moreover, from (6.3) we see that, since $\psi \in C^2(\Omega)$,

$$\mathcal{M}_{\mathbf{a}} w_\varepsilon \leq \mathcal{M}_{\mathbf{a}}((1+\varepsilon)v) + \varepsilon\lambda_n(D^2\psi). \tag{7.22}$$

Therefore, by (7.22) the following holds in Ω:

$$\begin{aligned}
\mathcal{M}_{\mathbf{a}} w_\varepsilon &\leq (1+\varepsilon)\mathcal{M}_{\mathbf{a}} v + \varepsilon\lambda_n(D^2\psi) \\
&\leq (1+\varepsilon)f(v) + (1+\varepsilon)h - \varepsilon h \quad \text{by (D-h)} \\
&\leq f((1+\varepsilon)v) + h \\
&\leq f(w_\varepsilon) + h.
\end{aligned}$$

By Lemma 7.1, we find that $u \leq w_\varepsilon$ in Ω. Letting $\varepsilon \to 0$ we conclude that $u \leq v$ in Ω. □

8 Appendix

In this appendix, we provide a proof of the assertion made in Remark 2.4, showing that the condition (D-h) can be obtained with $h \in C^0(\Omega)$ unbounded above, provided that Ω is a bounded, uniformly convex open set. In other words, we prove the following.

Lemma 8.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and uniformly convex open subset with C^2 boundary and suppose that $h : \Omega \to \mathbb{R}$ such that*

$$h(x) \leq Cd(x)^{\alpha-1} \quad (x \in \Omega), \tag{8.1}$$

for some $0 < \alpha \leq 1$. Then there is a non-negative $\psi \in C^2(\Omega)$ such that

$$\lambda_n(D^2\psi) \leq -h \quad \text{in } \Omega.$$

Proof. Since Ω is a C^2 bounded open set, we recall that the distance function d belongs to $C^2(\overline{\Omega}_\mu))$ for some $\mu > 0$, where

$$\Omega_\mu = \{x \in \Omega : d(x) < \mu\}$$

(see [36, Lemma 14.16]). Consequently the function d^α also belongs to $C^2(\overline{\Omega}_\mu)$. Fix $0 < \rho < \mu$ and consider the closed set $K := \{x \in \overline{\Omega} : d(x) \le \rho\}$.

By the Whitney extension theorem [73], see also Hestenes [41], we can extend the function d^α on K to a C^2 function δ_α in $\mathbb{R}^n$.

We observe that in $\Omega \cap \{d(x) \ge \rho\}$ the Hessian matrix $D^2\delta_\alpha(x)$ is bounded, therefore, $\lambda_n(D^2\delta_\alpha)$ is bounded on $\Omega \cap \{d(x) \ge \rho\}$ as well.

Suppose, as we may up to a translation, that $0 \in \Omega$, and let B_R be a ball centered at 0 of radius $R > 0$ such that $\Omega \Subset B_R$. Then we set

$$\overline{\delta}_\alpha(x) = \delta_\alpha(x) + c_R\left(1 - \frac{|x|^2}{R^2}\right) \tag{8.2}$$

where $c_R > 0$ is chosen sufficiently large to have $\delta_\alpha \ge 0$ in Ω and

$$\lambda_n(D^2\overline{\delta}_\alpha) \le -c_1 \quad \text{in } \Omega \cap \{d(x) \ge \rho\} \quad \text{for some constant } c_1 > 0. \tag{8.3}$$

Hence

$$\lambda_n(D^2\overline{\delta}_\alpha) \le -c_1\rho^{\alpha-1}d^{\alpha-1} \quad \text{in } \Omega \cap \{d(x) \ge \rho\}. \tag{8.4}$$

In $\Omega \cap \{d(x) < \rho\} \subset \Omega_\mu$, the Hessian matrix of $D^2\delta_\alpha(x) = D^2d^\alpha(x)$ is

$$D^2\phi = \phi''(d(x))\, Dd \otimes Dd + \phi'(d(x))\, D^2d,$$

where $\phi(r) = r^\alpha$, so that $\phi'(r) = \alpha r^{\alpha-1} > 0$ and $\phi''(r) = \alpha(\alpha-1)r^{\alpha-2} < 0$ for $r > 0$.

Consequently the eigenvalues of $D^2\delta_\alpha$ in $\Omega \cap \{d(x) \le \rho\}$ are

$$\alpha(\alpha-1)d^{\alpha-2}(x) < 0, \quad -\alpha d^{\alpha-1}(x)\frac{k_j(y(x))}{1 - k_j(y(x))d(x)} < 0, \quad j = 1, \dots, n-1,$$

where $y(x)$ is the unique point of $\partial\Omega$ such that $|x - y(x)| = d(x)$. See (4.7) with $\rho = 0$ and the subsequent lines. Exploiting the uniform convexity of Ω we can find a constant $c_2 > 0$ such that

$$\lambda_n(D^2\delta_\alpha(x)) \le -c_2\, d^{\alpha-1}(x) \quad \text{in } \Omega \cap \{d(x) < \rho\}, \tag{8.5}$$

and a fortiori

$$\lambda_n(D^2\overline{\delta}_\alpha(x)) \le -c_2\, d^{\alpha-1}(x) \quad \text{in } \Omega \cap \{d(x) < \rho\}. \tag{8.6}$$

From (8.4) and (8.6) it follows that there exists a constant $c > 0$ such that

$$\lambda_n(D^2\overline{\delta}_\alpha(x)) \le -c\, d^{\alpha-1}(x) \quad \text{for all } x \in \Omega. \tag{8.7}$$

From this, taking $\psi = \frac{C}{c}\overline{\delta}_\alpha$, where C is the constant in (8.1), finally we get

$$\lambda_n(D^2\psi) = \lambda_n(\frac{C}{c}D^2\overline{\delta}_\alpha) \le -C\, d^{\alpha-1} \le -h \quad \text{in } \Omega, \tag{8.8}$$

and so condition (D-h) is satisfied. □

We also give a proof of the representation formula of Φ stated in Theorem 3.5.

Lemma 8.2. *Let f satisfy* (f-1)*, and suppose $f \in RV_p$ for some $p > 1$. Then*

$$\Phi(r) = \Phi(1) r^{-\frac{2}{p-1}} \exp\left(-\int_r^1 \frac{c(s)}{s}\, ds\right), \quad r > 0,$$

with $c \in C^0(\mathbb{R}^+)$ such that $c(r) \to 0$ as $r \to 0^+$.

Proof. Since $f \in RV_p$ for some $p > 1$, we have

$$\lim_{t\to\infty} \frac{F(t)}{tf(t)} = \int_0^1 \lim_{t\to\infty} \frac{f(st)}{f(t)}\, ds = \int_0^1 s^p\, ds = \frac{1}{p+1}. \tag{8.9}$$

Therefore, we have

$$\begin{aligned}
\lim_{r\to0^+} \frac{\Phi(r)}{r\Phi'(r)} &= -\lim_{r\to0^+} \frac{\Phi(r)(2F(\Phi(r)))^{-1/2}}{r} && \text{by (2.3)} \\
&= -\lim_{t\to\infty} \frac{t(2F(t))^{-1/2}}{\int_t^\infty \frac{ds}{\sqrt{2F(s)}}} && \text{by (2.2)} \\
&= \lim_{t\to\infty} \left(1 - \frac{tf(t)}{2F(t)}\right) && \text{by (2.1) and L'Hôpital rule} \\
&= \frac{1-p}{2}, && \text{from (8.9).}
\end{aligned} \tag{8.10}$$

We now write

$$\frac{\Phi'(r)}{\Phi(r)} = \frac{2}{1-p}\frac{1}{r} + \frac{c(r)}{r}, \quad r > 0, \tag{8.11}$$

where

$$c(r) := \frac{r\Phi'(r)}{\Phi(r)} - \frac{2}{1-p}, \quad \text{or} \quad \frac{c(r)}{r} = -\frac{\sqrt{2F(\Phi(r))}}{\Phi(r)} + \frac{2}{p-1}\frac{1}{r}. \tag{8.12}$$

By the limit in (8.10) we see that $c(r) \to 0$ as $r \to 0^+$.

Integrating (8.11) from $t = 1$ to $t = r > 0$, we find that

$$\log \Phi(r) = \frac{2}{1-p} \log r + \log \Phi(1) + \int_1^r \frac{c(s)}{s}\, ds.$$

In other words, we have

$$\begin{aligned} \Phi(r) &= \Phi(1) r^{\frac{2}{1-p}} \cdot \exp\left(\int_1^r \frac{c(s)}{s}\, ds \right) \\ &= \Phi(1) r^{-\frac{2}{p-1}} \cdot \exp\left(- \int_r^1 \frac{c(s)}{s}\, ds \right) \end{aligned}$$

as was to be shown. □

Finally, let us consider the following function.

$$f(t) = \begin{cases} t^p (\log^2 t + k)^\alpha & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

with $p > 1$ and $\alpha \in \mathbb{R}$, where $k = k(p, \alpha)$ is a positive number large enough to have f non-decreasing. Then $f \in \mathrm{RV}_p$, and a straightforward calculation shows that

$$\Phi(r) \sim c_p r^{-\frac{2}{p-1}} |\log r|^{-\frac{2\alpha}{p-1}} \quad \text{as } r \to 0^+,$$

where c_p is a positive constant depending on p.

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References

[1] N. Abatangelo, *Very large solutions for the fractional Laplacian: towards a fractional Keller–Osserman condition*, Adv. Nonlinear Anal. **6** (2017), 383–405.

[2] S. Alarcón and A. Quaas, *Large viscosity solutions for some fully nonlinear equations*, NoDEA Nonlinear Differential Equations Appl. **20** (2013), 1453–1472.

[3] M. E. Amendola, G. Galise and A. Vitolo, *Riesz capacity, maximum principle and removable sets of fully nonlinear second order operators*, Differ Integral Equations **27** (2013), 845–866.

[4] M. E. Amendola, L. Rossi and A. Vitolo, *Harnack inequalities and ABP estimates for nonlinear second order elliptic equations in unbounded domains*, Abstr. Appl. Anal. (2008), Article no. 178534.

[5] C. Bandle and M. Marcus, *"Large" solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behavior*, J. Anal. Math. **58** (1992) 9–24.

[6] M. Bardi and P. Mannucci, *On the Dirichlet problem for non-totally degenerate fully nonlinear elliptic equations*, Commun. Pure Appl. Anal. **5** (2006), 709–731.

[7] I. Birindelli and G. Galise, *The Dirichlet problem for fully nonlinear degenerate elliptic equations with a singular nonlinearity*, Calc. Var. Partial Differential Equations **58** (2019), Article no. 180.

[8] I. Birindelli, G. Galise and H. Ishii, *A family of degenerate elliptic operators: maximum principle and its consequences*, Ann. Inst. H. Poincaré Anal. Non Linéaire **35** (2018), 417–441.

[9] P. Blanc and J. D. Rossi, *Games for eigenvalues of the Hessian and concave/convex envelopes*, J. Math. Pure Appl. **127** (2019), 192–215.

[10] H. Brezis, *Semilinear equations in $\mathbb{R}^N$ without condition at infinity*, Appl. Math. Optim. **12** (1984), 271–282.

[11] L. A. Caffarelli and X. Cabré, *Fully Nonlinear Elliptic Equations*, American Mathematical Society, Providence, RI, 1995.

[12] L. A. Caffarelli, Y. Y. Li and L. Nirenberg, *Some remarks on singular solutions of nonlinear elliptic equations*, J. Fixed Point Theory Appl. **5** (2009), 353–395.

[13] L. A. Caffarelli, Y. Y. Li and L. Nirenberg, *Some remarks on singular solutions of nonlinear elliptic equations. III: viscosity solutions, including parabolic operators*, Comm. Pure Appl. Math. **66** (2013), 109–143.

[14] I. Capuzzo Dolcetta, F. Leoni and A. Vitolo, *The Alexandrov–Bakelman–Pucci weak maximum principle for fully nonlinear equations in unbounded domains*, Comm. Partial Differential Equations **30** (2005), 1863–1881.

[15] I. Capuzzo Dolcetta, F. Leoni and A. Vitolo, *Entire subsolutions of fully nonlinear degenerate elliptic equations*, Bull. Inst. Math. Acad. Sin. (N.S.) **9** (2014), 147–161.

[16] I. Capuzzo Dolcetta, F. Leoni A. Vitolo, *On the inequality $F(x, D^2u) \geq f(u) + g(u)|Du|^p$*, Math. Ann. **365** (2016), 423–448.

[17] I. Capuzzo Dolcetta, F. Leoni and A. Vitolo, *Generalized Keller–Osserman conditions for fully nonlinear degenerate elliptic equations*, J. Math. Sci. **260** (2022), 469–479.

[18] O. Costin, L. Dupaigne and O. Goubet, *Uniqueness of large solutions*, J. Math. Anal. Appl. **395** (2012), 806–812.

[19] O. Costin and L. Dupaigne, *Boundary blow-up solutions in the unit ball: asymptotics, uniqueness and symmetry*, J. Differential Equations **249** (2010), 931–964

[20] M. G. Crandall, H. Ishii and P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), 1–67.

[21] M. Dindoš, *Large solutions for Yamabe and similar problems on domains in Riemannian manifolds*, Trans. Amer. Math. Soc. **363** (2011), 5131–5178.

[22] G. Díaz and R. Letelier, *Explosive solutions of quasilinear elliptic equations: existence and uniqueness*, Nonlinear Anal. **20** (1993), 97–125.

[23] S. Dumont, L. Dupaigne, O. Goubet and V. Rădulescu, *Back to the Keller–Osserman condition for boundary blow-up solutions*, Adv. Nonlinear Stud. **7** (2007), 271–298.

[24] L. Dupaigne, M. Ghergu and V. Rădulescu, *Lane–Emden–Fowler equations with convection and singular potential*, J. Math. Pures Appl. **87** (2007), 563–581.

[25] M. J. Esteban, P. L. Felmer and A. Quaas, *Superlinear elliptic equations for fully nonlinear operators without growth restrictions for the data*, Proc. Edinb. Math. Soc. **53** (2010), 125–141.

[26] P. Felmer, M. Montenegro and A. Quaas, *A note on the strong maximum principle and the compact support principle*, J. Differential Equations **246** (2009), 39–49.

[27] P. L. Felmer, A. Quaas and B. Sirakov, *Solvability of nonlinear elliptic equations with gradient terms*, J. Differential Equations **254** (2013), 4327–4346.

[28] F. Ferrari and G. Galise, *A regularity result for a class of non-uniformly elliptic operators*, Arch. Math. (Basel) **118** (2022), 539–548.

[29] F. Ferrari and A. Vitolo, *Regularity properties for a class of non-uniformly elliptic Isaacs operators*, Adv. Nonlinear Stud. **20** (2020), 213–241.

[30] G. Galise, S. Koike, O. Ley and A. Vitolo, *Entire solutions of fully nonlinear elliptic equations with a superlinear gradient term*, J. Math. Anal. Appl. **441** (2016), 194–210.

[31] G. Galise and A. Vitolo, *Viscosity solutions of uniformly elliptic equations without boundary and growth conditions at infinity*, Int. J. Differ. Equ. (2011), Articel no. 453727.

[32] G. Galise and A. Vitolo, *Removable singularities for degenerate elliptic Pucci operators*, Adv. Differential Equations **22** (2017), 77–100.

[33] J. García-Melián, *Uniqueness of positive solutions for a boundary blow-up problem*, J. Math. Anal. Appl. **360** (2009), 530–536.

[34] J. García-Melián, *Large solutions for an elliptic equation with a nonhomogeneous term*, J. Math. Anal. Appl. **434** (2016), 872–881.

[35] M. Ghergu and V. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*, The Clarendon Press, Oxford University Press, Oxford, 2008.

[36] G. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York–Berlin–Heidelberg, 1977.

[37] F. Gladiali and G. Porru, *Estimates for explosive solutions to p-Laplace equations*, in *Progress in Partial Differential Equations, Vol. 1* Longman, Harlow, 1998, pp. 117–127.

[38] F. R. Harvey and H. Blaine Lawson Jr., *Dirichlet duality and the non-linear Dirichlet problem*, Comm. Pure Appl. Math. **62** (2009), 396–443.

[39] F. R. Harvey and H. Blaine Lawson Jr., *Existence, uniqueness and removable singularities for nonlinear partial differential equations in geometry*, in *Surveys in Differential Geometry. Geometry and Topology*, International Press, Somerville, MA, 2013, pp. 103–156.

[40] F. R. Harvey and H. Blaine Lawson Jr., *Removable singularities for nonlinear subequations*, Indiana Univ. Math. J. **63** (2014), 1525–1552.

[41] M. R. Hestenes, *Extension of the range of a differentiable function*, Duke Math. J. **8** (1941), 183–192.

[42] V. Julin, *Generalized Harnack inequality for nonhomogeneous elliptic equations*, Arch. Ration. Mech. Anal. **216** (2015), 673{702.

[43] J. B. Keller, *Electrohydrodynamics I. The equilibrium of a charged gas in a container*, J. Rational Mech. Anal. **5** (1956), 715–724.

[44] J. B. Keller, *On solutions of* $\Delta u = f(u)$, Comm. Pure Appl. Math. **10** (1957), 503–510.

[45] B. Kindu, A. Mohammed and B. Tsegaw, *A Vázquez-type strong minimum/maximum principle for partial trace operators*, Discrete Contin. Dyn. Syst. Ser. S **16** (2023), 3146–3162.

[46] S. Koike, *A Beginner's Guide to the Theory of Viscosity Solutions*, Mathematical Society of Japan, Tokyo, 2014.

[47] J. López-Gómez and L. Maire, *Uniqueness of large positive solutions*, Z. Angew. Math. Phys. **68** (2017), Article no. 86.

[48] J. López-Gómez, L. Maire and L. Véron, *General uniqueness results for large solutions*, Z. Angew. Math. Phys. **71** (2020), Article no. 109.

[49] C. Loewner and L. Nirenberg, *Partial differential equations invariant under conformal or projective transformations*, in *Contributions to Analysis (A Collection of Papers Dedicated to Lipman Bers)*, Academic Press, New York, 1974, pp. 245–272.

[50] M. Marcus and L. Véron, *Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations*, Ann. Inst. Henri Poincaré Analyse Non Linéare **14** (1997), 237–274.

[51] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptlic equations: the subcritical case*, Arch. Rat. Mech. Anal. **144** (1998), 201–231.

[52] M. Marcus and L. Véron, *The boundary trace and generalized boundary value problems for semilinear elliptic equations with coercive absorption*, Comm. Pure Appl. Math. **56** (2003), 1–43.

[53] A. Mohammed, *Boundary asymptotic and uniqueness of solutions to the p -Laplacian with infinite boundary values*, J. Math. Anal. Appl. **325** (2007), 480{489.

[54] A. Mohammed, G. Porru and A. Vitolo, *Harnack inequality for nonlinear elliptic equations with strong absorption*, J. Differential Equations **263** (2017), 6821–6843.

[55] A. Mohammed and G. Porru, *Large solutions for non-divergence equations with singular lower order terms*, Nonlinear Anal. Real World Appl. **35** (2017), 470–482.

[56] A. Mohammed and G. Porru, *Harnack inequality for non-divergence structure semi-linear elliptic equations*, Advances in Nonlinear Analysis **7** (2018), 259–269.

[57] A. Mohammed and G. Porru, *Large solutions to non-divergence structure semilinear elliptic equations with inhomogeneous term*, Adv. Nonlinear Anal. **8** (2019), 517–532.

[58] A. Mohammed, V. Rădulescu and A. Vitolo, *Blow-up solutions for fully nonlinear equations: existence, asymptotic estimates and uniqueness*, Adv. Nonlinear Anal. **9** (2020), 39–64.

[59] A. Mohammed and A. Vitolo, *Large solutions of fully nonlinear equations: existence and uniqueness*, NoDEA Nonlinear Differential Equations Appl. **26** (2019), Article no. 42.

[60] A. Mohammed and A. Vitolo, *On the strong maximum principle*, Complex Var. Elliptic Equ. **65** (2020), 1299–1314.

[61] A. M. Oberman and L. Silvestre, *The Dirichlet problem for the convex envelope*, Trans. Amer. Math. Soc. **363** (2011), 5871–5886.

[62] R. Osserman, *On the inequality* $\Delta u \geq f(u)$, Pacific J. Math. **7** (1957), 1641–1647.

[63] P. Pucci and J. Serrin, *The Maximum Principle*, Birkhäuser, Basel, 2007.

[64] V. Rădulescu, *Singular phenomena in nonlinear elliptic problems: from blow-up boundary solutions to equations with singular nonlinearities*, in *Handbook of Differential Equations: Stationary Partial Differential Equations. Vol. IV*, Elsevier/North-Holland, Amsterdam, 2007, pp. 485–593.

[65] D. Repovš, *Asymptotics for singular solutions of quasilinear elliptic equations with an absorption term*, J. Math. Anal. Appl. **395** (2012), 78–85.

[66] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. Equ. **12** (1984), 191–202.

[67] L. Véron, *Semilinear elliptic equations with uniform blowup on the boundary*, J. Anal. Math. **59** (1992), 231–250.

[68] A. Vitolo, *Removable singularities for degenerate elliptic equations without conditions on the growth of the solution*, Trans. Am. Math. Soc. **370** (2018), 2679–2705.

[69] A. Vitolo, *Maximum principles for viscosity solutions of weakly elliptic equations*, in *Bruno Pini Mathematical Analysis Seminar 2019*, Università di Bologna, Alma Mater Studiorum, Bologna, 2019 110–136.

[70] A. Vitolo, *Singular elliptic equations with directional diffusion*, Math. Eng. **3** (2021), Articl no. 027.

[71] A. Vitolo, *Lipschitz estimates for partial trace operators with extremal Hessian eigenvalues*, Adv. Nonlinear Anal. **11** (2022), 1182–1200.

[72] A. Vitolo, M. E. Amendola and G. Galise, *On the uniqueness of blow-up solutions of fully nonlinear elliptic equations*, AIMS Conference Publications **2013** (2013), 771–780

[73] H. Whitney, *Analytic extension of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), 63–89.

[74] F. Zhang, *Matrix Theory*, Springer, 1999.

[75] Z. Zhang, *A boundary blow-up elliptic problem with an inhomogeneous term*, Nonlinear Anal. **68** (2008), 3428–3438.

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