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J. Differential Equations 195 (2003) 520–536
Journal of Differential Equations
http://www.elsevier.com/locate/jde

Sublinear singular elliptic problems with two parameters

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Received November 8, 2002

This paper is dedicated with esteem to Professor Philippe G. Ciarlet on his 65th birthday

Abstract

We establish several existence and nonexistence results for the boundary value problem $-\Delta u + K(x)g(u) = \lambda f(x,u) + \mu h(x)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbf{R}^N$, λ and μ are positive parameters, h is a positive function, while f has a sublinear growth. The main feature of this paper is that the nonlinearity g is assumed to be unbounded around the origin. Our analysis shows the importance of the role played by the decay rate of g combined with the signs of the extremal values of the potential $K(x)$ on $\bar{\Omega}$. The proofs are based on various techniques related to the maximum principle for elliptic equations.

MSC: 35B50; 35J65; 58J55

Keywords: Singular elliptic equation; Sublinear boundary value problem; Maximum principle; Positive solution; Extremal solutions

1. Introduction and the main results

Let Ω be a smooth bounded domain in $\mathbf{R}^N$ $(N \geqslant 2)$. In this paper, we study the existence or the nonexistence of solutions to the following boundary value problem:

$$\begin{cases} -\Delta u + K(x)g(u) = \lambda f(x,u) + \mu h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \qquad (P_{\lambda,\mu}) \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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0022-0396/03/$ - see front matter
doi:10.1016/S0022-0396(03)00105-0

Here $K, h \in C^{0,\gamma}(\bar{\Omega})$, with $h > 0$ on Ω and λ, μ are positive real numbers. We suppose that $f : \bar{\Omega} \times [0, \infty) \to [0, \infty)$ is a Hölder continuous function which is positive on $\bar{\Omega} \times (0, \infty)$. We also assume that f is nondecreasing with respect to the second variable and is sublinear, that is,

(f1) the mapping $(0, \infty) \ni s \mapsto \frac{f(x,s)}{s}$ is nonincreasing for all $x \in \bar{\Omega}$;

(f2) $\lim_{s \downarrow 0} \frac{f(x,s)}{s} = +\infty$ and $\lim_{s \to \infty} \frac{f(x,s)}{s} = 0$, uniformly for $x \in \bar{\Omega}$.

We assume that $g \in C^{0,\gamma}(0, \infty)$ is a nonnegative and nonincreasing function satisfying

(g1) $\lim_{s \downarrow 0} g(s) = +\infty$;

(g2) there exists $C, \delta_0 > 0$ and $\alpha \in (0, 1)$ such that $g(s) \leqslant Cs^{-\alpha}$ for all $s \in (0, \delta_0)$.

Obviously, hypothesis (g2) implies the following Keller–Osserman-type condition around the origin

(g3) $\int_0^1 (\int_0^t g(s)\, ds)^{-1/2}\, dt < \infty$.

As proved by Bénilan et al. [2], condition (g3) is equivalent to the *property of compact support*, that is, for every $h \in L^1(\mathbf{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbf{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbf{R}^N)$ and

$$-\Delta u + g(u) = h \quad \text{a.e. in } \mathbf{R}^N.$$

Example. The function $f(s) = s^\alpha (0 < \alpha < 1)$ fulfills (f1)–(f2), while $g(s) = (s^\alpha + s^\beta)^{-1} (0 < \alpha < 1,\ \beta > 0)$ satisfies the assumptions (g1)–(g2).

Our framework includes the Emden–Fowler equation that corresponds to $g(s) = s^{-\gamma},\ \gamma > 0$ (see [19]).

A major factor that makes $(P_{\lambda,\mu})$ difficult to treat is the lack of the usual maximal principle between super- and subsolutions, due to the singular character of the equation.

Denote $\mathscr{E} = \{u \in C^2(\Omega) \cap C(\bar{\Omega}); g(u) \in L^1(\Omega)\}$.

We show in this paper that $(P_{\lambda,\mu})$ has at least one solution in $\mathscr{E}$ for λ, μ belonging to a certain range. We also prove that in some cases, $(P_{\lambda,\mu})$ has no solutions in $\mathscr{E}$, provided that λ and μ are sufficiently small.

Remark 1.1. (i) If $u \in \mathscr{E},\ v \in C^2(\Omega) \cap C(\bar{\Omega})$ and $0 < u < v$ in Ω, then $v \in \mathscr{E}$.

(ii) Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of $(P_{\lambda,\mu})$. Then $u \in \mathscr{E}$ if and only if $\Delta u \in L^1(\Omega)$.

Singular semilinear elliptic equations have been intensively studied in the last decades. Such problems arise in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogenous catalysts, in the theory of heat conduction in electrically conducting materials. For instance, problems of this type characterize some reaction–diffusion processes where $u \geqslant 0$ is viewed as the density of a reactant and the region where $u = 0$ is called the *dead core*, where no reaction takes place (see [1] for the study of a single, irreversible steady-state reaction). Nonlinear singular elliptic equations are also encountered in glacial advance, in transport of coal slurries down conveyor belts and in several other geophysical and industrial contents (see [4] for the case of the incompressible flow of a uniform stream past a semi-infinite flat plate at zero incidence). For more details, we also refer to [3,8,14,16] and the references therein.

Many authors considered the problem

$$\begin{cases} -\Delta u + K(x)u^{-\alpha} = \lambda u^p & \text{in } \Omega, \\ u>0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

for $\lambda \geqslant 0$ and $\alpha, p \in (0,1)$. When $K<0$ and $\lambda=0$, problem (1) was studied in [6,9,11,13,17]. For $K \equiv -1$, it was proved in [7] that (1) has at least one solution for all $\lambda \geqslant 0$ and $0<p<1$. Moreover, if $p \geqslant 1$, there exists $\tilde{\lambda}$ such that (1) has a solution for $\lambda \in [0, \tilde{\lambda})$ and no solution for $\lambda > \tilde{\lambda}$.

If $K \equiv 1$, it was established in [20] that there exists $\bar{\lambda}>0$ such that (1) has at least one solution in $\mathscr{E}$ provided that $\lambda > \bar{\lambda}$ and no solution exists if $\lambda < \bar{\lambda}$.

In [18], it is shown that for λ sufficiently large, problem (1) has at least one solution $u_\lambda \in \mathscr{E} \cap C^{1,1-\gamma}(\bar{\Omega})$ and

$$c_1 \operatorname{dist}(x, \partial\Omega) \leqslant u_\lambda(x) \leqslant c_2 \operatorname{dist}(x, \partial\Omega)$$

for any $x \in \Omega$ and for some constants $c_1, c_2 > 0$ independent of x.

A fundamental role will be played in our analysis by the numbers

$$K^* = \max_{x \in \bar{\Omega}} K(x), \quad K_* = \min_{x \in \bar{\Omega}} K(x).$$

Our main results are the following.

Theorem 1.1. *Assume that $K_* > 0$ and f satisfies* (f1)–(f2). *If $\int_0^1 g(s)\,ds = +\infty$, then $(P_{\lambda,\mu})$ has no solution in $\mathscr{E}$ for any $\lambda, \mu > 0$.*

Theorem 1.2. *Assume that $K_* > 0$, f satisfies* (f1)–(f2) *and g satisfies* (g1)–(g2). *Then there exists $\lambda_*, \mu_* > 0$ such that*

$(P_{\lambda,\mu})$ has at least one solution in $\mathscr{E}$ if $\lambda > \lambda_*$ or $\mu > \mu_*$.

$(P_{\lambda,\mu})$ has no solution in $\mathscr{E}$ if $\lambda < \lambda_*$ and $\mu < \mu_*$.

Moreover, if $\lambda > \lambda_*$ or $\mu > \mu_*$, then $(P_{\lambda,\mu})$ has a maximal solution in $\mathscr{E}$ which is increasing with respect to λ and μ (Fig. 1).

Theorem 1.3. *Assume that $K^* \leqslant 0$, f satisfies* (f1)–(f2) *and g satisfies* (g1)–(g2). *Then $(P_{\lambda,\mu})$ has a unique solution $u_{\lambda,\mu} \in \mathscr{E}$ for any $\lambda, \mu > 0$. Moreover, $u_{\lambda,\mu}$ is increasing with respect to λ and μ.*

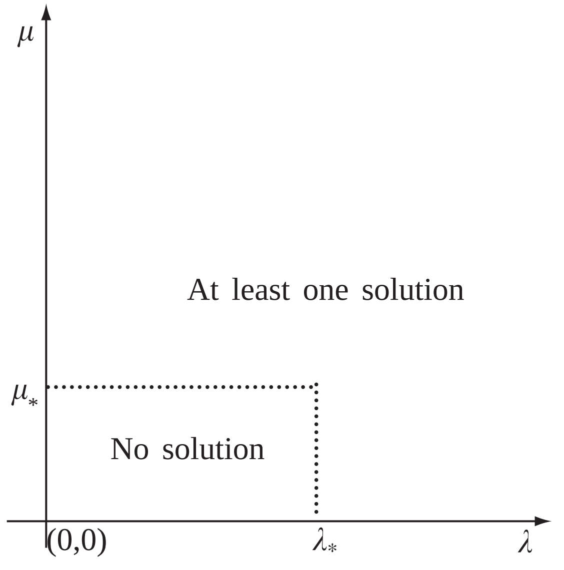

Fig. 1. The dependence on λ and μ in Theorem 1.2.

Theorems 1.2 and 1.3 also show the role played by the sublinear term f and the sign of $K(x)$. Indeed, if f becomes linear then the situation changes radically. First, by the results in [9], the problem

$$\begin{cases} -\Delta u - u^{-\alpha} = -u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution, for any $\alpha > 0$. Next, as shown in [5], the problem

$$\begin{cases} -\Delta u + u^{-\alpha} = u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has no solution, provided $0 < \alpha < 1$ and $\lambda_1 \geqslant 1$ (that is, if Ω is "small"), where λ_1 denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$.

Theorem 1.4. *Assume that* $K^* > 0 > K_*$, f *satisfies* (f1)–(f2) *and* g *verifies* (g1)–(g2). *Then there exists* $\lambda_*, \mu_* > 0$ *such that* $(P_{\lambda,\mu})$ *has at least one solution* $u_{\lambda,\mu} \in \mathscr{E}$ *if* $\lambda > \lambda_*$ *or* $\mu > \mu_*$. *Moreover, for* $\lambda > \lambda_*$ *or* $\mu > \mu_*$, $u_{\lambda,\mu}$ *is increasing with respect to* λ *and* μ.

As it was pointed out in [6], problems related to multiplicity or to uniqueness become difficult even in simple cases. In this sense we also refer to [15], where the existence of radial symmetric solutions of the problem

$$\begin{cases} \Delta u + \lambda(u^p - u^{-\alpha}) = 0 & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{2}$$

is studied, where $0 < \alpha,\ p < 1,\ \lambda > 0$ and B_1 is the unit ball in $\mathbf{R}^N$. Using a bifurcation theorem of Crandall and Rabinowitz, it has been shown in [15] that there exists $\lambda_1 > \lambda_0 > 0$ such that (2) has no solutions for $\lambda < \lambda_0$, one solution for $\lambda = \lambda_0$ or $\lambda > \lambda_1$, two solutions for $\lambda_1 \geqslant \lambda > \lambda_0$.

2. Auxiliary results

Let φ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Lemma 2.1 (see Lazer and McKenna [13]). $\int_\Omega \varphi_1^{-s}\, dx < +\infty$ *if and only if* $s < 1$.

Lemma 2.2 (see Shi and Yao [18]). *Let* $F : \bar{\Omega} \times (0, \infty) \to \mathbf{R}$ *be a Hölder continuous function with exponent* $\gamma \in (0, 1)$ *on each compact subset of* $\bar{\Omega} \times (0, \infty)$ *which satisfies*

(F1) $\limsup_{s \to +\infty} (s^{-1} \max_{x \in \bar{\Omega}} F(x, s)) < \lambda_1$;
(F2) *for each* $t > 0$, *there exists a constant* $D(t) > 0$ *such that*

$$F(x, r) - F(x, s) \geqslant -D(t)(r - s), \quad \textit{for } x \in \bar{\Omega} \textit{ and } r \geqslant s \geqslant t;$$

(F3) *there exists* $\eta_0 > 0$ *and an open subset* $\Omega_0 \subset \Omega$ *such that*

$$\min_{x \in \bar{\Omega}} F(x, s) \geqslant 0 \quad \textit{for } x \in (0, \eta_0),$$

and

$$\lim_{s \downarrow 0} \frac{F(x, s)}{s} = +\infty \quad \textit{uniformly for } x \in \Omega_0.$$

Then for any nonnegative function $\varphi_0 \in C^{2,\gamma}(\partial\Omega)$, *the problem*

$$\begin{cases} -\Delta u = F(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \varphi_0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

has at least one positive solution $u \in C^{2,\gamma}(G) \cap C(\bar{\Omega})$, *for any compact set* $G \subset \Omega \cup \{x \in \partial\Omega;\ \varphi_0(x) > 0\}$.

Lemma 2.3 (see Shi and Yao [18]). *Let* $F : \bar{\Omega} \times (0, \infty) \to \mathbf{R}$ *be a continuous function such that the mapping* $(0, \infty) \ni s \mapsto \frac{F(x,s)}{s}$ *is strictly decreasing at each* $x \in \Omega$. *Assume that there exists* $v,\ w \in C^2(\Omega) \cap C(\bar{\Omega})$ *such that*

(a) $\Delta w + F(x, w) \leqslant 0 \leqslant \Delta v + F(x, v)$ *in* Ω;
(b) $v, w > 0$ *in* Ω *and* $v \leqslant w$ *on* $\partial\Omega$;
(c) $\Delta v \in L^1(\Omega)$.

Then $v \leqslant w$ in Ω.

We observe that the hypotheses of Lemmas 2.2 and 2.3 are fulfilled for

$$\Phi_{\lambda,\mu}(x, s) = \lambda f(x, s) + \mu h(x), \tag{5}$$

$$\Psi_{\lambda,\mu}(x, s) = \lambda f(x, s) - K(x)g(s) + \mu h(x), \quad \text{provided } K^* \leqslant 0. \tag{6}$$

Lemma 2.4. *Let* f *satisfying* (f1)–(f2) *and* g *satisfying* (g1)–(g2). *Then there exists* $\bar{\lambda} > 0$ *such that the problem*

$$\begin{cases} -\Delta v + g(v) = \lambda f(x, v) + \mu h(x) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{7}$$

has at least one solution $v_{\lambda,\mu} \in \mathscr{E}$ *for all* $\lambda > \bar{\lambda}$ *and for any* $\mu > 0$.

Proof. Let $\lambda, \mu > 0$. According to Lemmas 2.2 and 2.3, the boundary value problem

$$\begin{cases} -\Delta U = \lambda f(x, U) + \mu h(x) & \text{in } \Omega, \\ U > 0 & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega \end{cases} \tag{8}$$

has a unique solution $U_{\lambda,\mu} \in C^{2,\gamma}(\Omega) \cap C(\bar{\Omega})$. Then $\overline{v}_{\lambda,\mu} = U_{\lambda,\mu}$ is a supersolution of (7). The main point is to find a subsolution of (7). For this purpose, let $H : [0, \infty) \to [0, \infty)$ be such that

$$\begin{cases} H''(t) = g(H(t)), & \text{for all } t > 0, \\ H'(0) = H(0) = 0. \end{cases} \tag{9}$$

Obviously, $H \in C^2(0, \infty) \cap C^1[0, \infty)$ exists by our assumption (g2). From (9), it follows that H'' is nonincreasing, while H and H' are nondecreasing on $(0, \infty)$. Using this fact and applying the mean value theorem, we deduce that for all $t > 0$

there exists $\xi_t^1,\ \xi_t^2 \in (0,t)$ such that

$$\frac{H(t)}{t} = \frac{H(t)-H(0)}{t-0} = H'(\xi_t^1) \leqslant H'(t);$$

$$\frac{H'(t)}{t} = \frac{H'(t)-H'(0)}{t-0} = H''(\xi_t^2) \geqslant H''(t).$$

The above inequalities imply

$$H(t) \leqslant tH'(t) \leqslant 2H(t), \quad \text{for all } t>0.$$

Hence,

$$1 \leqslant \frac{tH'(t)}{H(t)} \leqslant 2, \quad \text{for all } t>0. \tag{10}$$

On the other hand, by (g2) and (9), there exists $\eta>0$ such that

$$\begin{cases} H(t) \leqslant \delta_0, & \text{for all } t \in (0,\eta), \\ H''(t) \leqslant CH^{-\alpha}(t), & \text{for all } t \in (0,\eta), \end{cases} \tag{11}$$

which yields

$$H(t) \leqslant ct^{2/(\alpha+1)}, \quad \text{for all } t \in (0,\eta), \tag{12}$$

where $c>0$ is a constant.

Now we look for a subsolution of the form $\underline{v}_{\lambda,\mu} = MH(\varphi_1)$, for some constant $M>0$. We have

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) = \lambda_1 MH'(\varphi_1)\varphi_1 + g(MH(\varphi_1)) - Mg(H(\varphi_1))|\nabla\varphi_1|^2 \quad \text{in } \Omega. \tag{13}$$

Take $M \geqslant 1$. The monotonicity of g leads to

$$g(MH(\varphi_1)) \leqslant g(H(\varphi_1)) \quad \text{in } \Omega,$$

and, by (13),

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leqslant \lambda_1 MH'(\varphi_1)\varphi_1 + g(H(\varphi_1))(1 - M|\nabla\varphi_1|^2) \quad \text{in } \Omega. \tag{14}$$

We claim that

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leqslant 2\lambda_1 MH'(\varphi_1)\varphi_1 \quad \text{in } \Omega. \tag{15}$$

Indeed, by Hopf's maximum principle, there exists $\delta>0$ and $\omega \Subset \Omega$ such that

$$|\nabla\varphi_1| \geqslant \delta \quad \text{in } \Omega\backslash\omega,$$

$$\varphi_1 \geqslant \delta \quad \text{in } \omega.$$

On $\Omega\backslash\omega$, we choose $M \geqslant M_1 = \max\{1, \delta^{-2}\}$. Then, by (14) we obtain

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leqslant \lambda_1 M H'(\varphi_1)\varphi_1 \quad \text{in } \Omega\backslash\omega. \tag{16}$$

Fix $M \geqslant \max\{M_1, \frac{g(H(\delta))}{\lambda_1 H'(\delta)\delta}\}$. Then

$$g(H(\varphi_1)) \leqslant g(H(\delta)) \leqslant \lambda_1 M H'(\delta)\delta \leqslant \lambda_1 M H'(\varphi_1)\varphi_1 \quad \text{in } \omega.$$

From (14) we deduce

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leqslant 2\lambda_1 M H'(\varphi_1)\varphi_1 \quad \text{in } \omega. \tag{17}$$

Hence our claim (15) follows from (16) and (17).

Since $\varphi_1 > 0$ in Ω, from (10) we have

$$1 \leqslant \frac{H'(\varphi_1)\varphi_1}{H(\varphi_1)} \leqslant 2 \quad \text{in } \Omega. \tag{18}$$

Thus, (15) and (18) yield

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leqslant 4\lambda_1 M H(\varphi_1) = 4\lambda_1 \underline{v}_{\lambda,\mu} \quad \text{in } \Omega. \tag{19}$$

Take $\bar{\lambda} = 4\lambda_1 c^{-1} |\underline{v}_{\lambda,\mu}|_\infty$, where $c = \inf_{x\in\bar{\Omega}} f(x, |\underline{v}_{\lambda,\mu}|_\infty) > 0$. If $\lambda > \bar{\lambda}$, the assumption (f1) produces

$$\lambda \frac{f(x, \underline{v}_{\lambda,\mu})}{\underline{v}_{\lambda,\mu}} \geqslant \bar{\lambda} \frac{f(x, |\underline{v}_{\lambda,\mu}|_\infty)}{|\underline{v}_{\lambda,\mu}|_\infty} \geqslant 4\lambda_1, \quad \text{for all } x \in \Omega.$$

This combined with (19) gives

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leqslant \lambda f(x, \underline{v}_{\lambda,\mu}) \quad \text{in } \Omega.$$

Hence, $\underline{v}_{\lambda,\mu}$ is a subsolution of (7), for all $\lambda > \bar{\lambda}$ and $\mu > 0$.

We now prove that $\underline{v}_{\lambda,\mu} \in \mathscr{E}$, that is $g(\underline{v}_{\lambda,\mu}) \in L^1(\Omega)$. Denote $\Omega_0 = \{x \in \Omega; \varphi_1(x) < \eta\}$. By (11) and (12) it follows that

$$g(\underline{v}_{\lambda,\mu}) = g(MH(\varphi_1)) \leqslant g(H(\varphi_1)) \leqslant CH^{-\alpha}(\varphi_1) \leqslant C_0\, \varphi_1^{-2\alpha/(1+\alpha)} \quad \text{in } \Omega_0,$$

$$g(\underline{v}_{\lambda,\mu}) \leqslant g(MH(\eta)) \quad \text{in } \Omega\backslash\Omega_0.$$

These estimates combined with Lemma 2.1 yield $g(\underline{v}_{\lambda,\mu}) \in L^1(\Omega)$ and so $\Delta \underline{v}_{\lambda,\mu} \in L^1(\Omega)$. Hence,

$$\Delta \bar{v}_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, \bar{v}_{\lambda,\mu}) \leqslant 0 \leqslant \Delta \underline{v}_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, \underline{v}_{\lambda,\mu}) \quad \text{in } \Omega,$$

$$\underline{v}_{\lambda,\mu}, \bar{v}_{\lambda,\mu} > 0 \quad \text{in } \Omega,$$

$$\underline{v}_{\lambda,\mu} = \bar{v}_{\lambda,\mu} \quad \text{on } \partial\Omega,$$

$$\Delta \underline{v}_{\lambda,\mu} \in L^1(\Omega).$$

By Lemma 2.3, it follows that $\underline{v}_{\lambda,\mu} \leqslant \bar{v}_{\lambda,\mu}$ on $\bar{\Omega}$. Now, standard elliptic arguments guarantee the existence of a solution $v_{\lambda,\mu} \in C^2(\Omega) \cap C(\bar{\Omega})$ for (7) such that $\underline{v}_{\lambda,\mu} \leqslant v_{\lambda,\mu} \leqslant \bar{v}_{\lambda,\mu}$ in $\bar{\Omega}$. Since $\underline{v}_{\lambda,\mu} \in \mathscr{E}$, by Remark 1.1 we deduce that $v_{\lambda,\mu} \in \mathscr{E}$. Hence, for all $\lambda > \bar{\lambda}$ and $\mu > 0$, problem (7) has at least a solution in $\mathscr{E}$. The proof of Lemma 2.4 is now complete. $\square$

We shall often refer in what follows to the following approaching problem of $(P_{\lambda,\mu})$:

$$\begin{cases} -\Delta u + K(x)g(u) = \lambda f(x,u) + \mu h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \dfrac{1}{k} & \text{on } \partial\Omega, \end{cases} \qquad (P^k_{\lambda,\mu})$$

where k is a positive integer. We observe that any solution of $(P_{\lambda,\mu})$ is a subsolution of $(P^k_{\lambda,\mu})$.

3. Proof of Theorem 1.1

Suppose to the contrary that there exists λ and μ such that $(P_{\lambda,\mu})$ has a solution $u_{\lambda,\mu} \in \mathscr{E}$ and let $U_{\lambda,\mu}$ be the solution of (8). Since

$$\Delta U_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, U_{\lambda,\mu}) \leqslant 0 \leqslant \Delta u_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, u_{\lambda,\mu}) \quad \text{in } \Omega,$$

by Lemma 2.3 we get $u_{\lambda,\mu} \leqslant U_{\lambda,\mu}$ in $\bar{\Omega}$.

Consider the perturbed problem

$$\begin{cases} -\Delta u + K_* g(u+\varepsilon) = \lambda f(x,u) + \mu h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

Since $K_* > 0$, it follows that $u_{\lambda,\mu}$ and $U_{\lambda,\mu}$ are sub- and supersolution for (20), respectively. So, by elliptic regularity, there exists $u_\varepsilon \in C^{2,\gamma}(\bar{\Omega})$ a solution of (20) such that

$$u_{\lambda,\mu} \leqslant u_\varepsilon \leqslant U_{\lambda,\mu} \quad \text{in } \Omega. \tag{21}$$

Integrating in (20) we deduce

$$-\int_\Omega \Delta u_\varepsilon \, dx + K_* \int_\Omega g(u_\varepsilon + \varepsilon) \, dx = \int_\Omega [\lambda f(x, u_\varepsilon) + \mu h(x)] \, dx.$$

Hence,

$$-\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n}\, ds + K_* \int_{\Omega} g(u_\varepsilon + \varepsilon)\, dx \leqslant M, \tag{22}$$

where $M > 0$ is a constant. Since $\frac{\partial u_\varepsilon}{\partial n} \leqslant 0$ on $\partial\Omega$, relation (22) yields $K_* \int_\Omega g(u_\varepsilon + \varepsilon)\, dx \leqslant M$, and so $K_* \int_\Omega g(U_{\lambda,\mu} + \varepsilon)\, dx \leqslant M$. Thus, for any compact subset $\omega \Subset \Omega$ we have

$$K_* \int_\omega g(U_{\lambda,\mu} + \varepsilon)\, dx \leqslant M.$$

Letting $\varepsilon \to 0$, the above relation leads to $K_* \int_\omega g(U_{\lambda,\mu})\, dx \leqslant M$. Therefore,

$$K_* \int_\Omega g(U_{\lambda,\mu})\, dx \leqslant M. \tag{23}$$

Choose $\delta > 0$ sufficiently small and define $\Omega_\delta := \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) \leqslant \delta\}$. Taking into account the regularity of domain, there exists $k > 0$ such that

$$U_{\lambda,\mu} \leqslant k \operatorname{dist}(x, \partial\Omega) \quad \text{for all } x \in \Omega_\delta.$$

Then

$$\int_\Omega g(U_{\lambda,\mu})\, dx \geqslant \int_{\Omega_\delta} g(U_{\lambda,\mu})\, dx \geqslant \int_{\Omega_\delta} g(k \operatorname{dist}(x, \partial\Omega))\, dx = +\infty,$$

which contradicts (23). It follows that the problem $(P_{\lambda,\mu})$ has no solutions in $\mathscr{E}$ and the proof of Theorem 1.1 is now complete.

Remark 3.1. Using the same method as in [20, Theorem 2], we can prove that $(P_{\lambda,\mu})$ has no solution in $C^2(\Omega) \cap C^1(\bar{\Omega})$ as it was pointed out in [6, Remark 2].

4. Proof of Theorem 1.2

We split the proof into several steps.

Step I. Existence of the solutions of $(P_{\lambda,\mu})$ for λ large: By Lemma 2.4, there exists $\bar{\lambda}$ such that for all $\lambda > \bar{\lambda}$ and $\mu > 0$ the problem

$$\begin{cases} -\Delta v + K^* g(v) = \lambda f(x, v) + \mu h(x) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least one solution $v_{\lambda,\mu} \in \mathscr{E}$. Then $v_k = v_{\lambda,\mu} + \frac{1}{k}$ is a subsolution of $(P^k_{\lambda,\mu})$ for all positive integers $k \geqslant 1$.

From Lemma 2.2, let $w \in C^{2,\gamma}(\bar{\Omega})$ be the solution of

$$\begin{cases} -\Delta w = \lambda f(x,w) + \mu h(x) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 1 & \text{on } \partial\Omega. \end{cases}$$

It follows that w is a supersolution of $(P^k_{\lambda,\mu})$ for all $k \geqslant 1$ and

$$\Delta w + \Phi_{\lambda,\mu}(x,w) \leqslant 0 \leqslant \Delta v_1 + \Phi_{\lambda,\mu}(x,v_1) \quad \text{in } \Omega,$$

$$w, v_1 > 0 \quad \text{in } \Omega,$$

$$w = v_1 \quad \text{on } \partial\Omega,$$

$$\Delta v_1 \in L^1(\Omega).$$

Therefore, by Lemma 2.3, $1 \leqslant v_1 \leqslant w$ in $\bar{\Omega}$. Standard elliptic arguments imply that there exists a solution $u^1_{\lambda,\mu} \in C^{2,\gamma}(\bar{\Omega})$ of $(P^1_{\lambda,\mu})$ such that $v_1 \leqslant u^1_{\lambda,\mu} \leqslant w$ in $\bar{\Omega}$. Now, taking $u^1_{\lambda,\mu}$ and v_2 as a pair of super- and subsolutions for $(P^2_{\lambda,\mu})$, we obtain a solution $u^2_{\lambda,\mu} \in C^{2,\gamma}(\bar{\Omega})$ of $(P^2_{\lambda,\mu})$ such that $v_2 \leqslant u^2_{\lambda,\mu} \leqslant u^1_{\lambda,\mu}$ in $\bar{\Omega}$. In this manner, we find a sequence $\{u^n_{\lambda,\mu}\}$ such that

$$v_n \leqslant u^n_{\lambda,\mu} \leqslant u^{n-1}_{\lambda,\mu} \leqslant w \quad \text{in } \bar{\Omega}. \tag{24}$$

Define $u_{\lambda,\mu}(x) = \lim_{n\to\infty} u^n_{\lambda,\mu}(x)$ for all $x \in \bar{\Omega}$. Standard bootstrap arguments imply that $u_{\lambda,\mu}$ is a solution of $(P_{\lambda,\mu})$. From (24), we have $v_{\lambda,\mu} \leqslant u_{\lambda,\mu} \leqslant w$ in $\bar{\Omega}$. Since $v_{\lambda,\mu} \in \mathscr{E}$, by Remark 1.1 it follows that $u_{\lambda,\mu} \in \mathscr{E}$. Consequently, problem $(P_{\lambda,\mu})$ has at least a solution in $\mathscr{E}$ for all $\lambda > \bar{\lambda}$ and $\mu > 0$.

Step II. Existence of the solutions of $(P_{\lambda,\mu})$ for μ large: Let us first notice that g verifies the hypotheses of Theorem 2 in [10]. We also remark that the assumption (g2) and Lemma 2.1 are essential to find a subsolution in the proof of Theorem 2 in [10].

According to this result, there exists $\bar{\mu} > 0$ such that the problem

$$\begin{cases} -\Delta v + K^* g(v) = \mu h(x) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least a solution $v_\mu \in \mathscr{E}$ provided that $\mu > \bar{\mu}$. Fix $\lambda > 0$ and denote $v_k = v_\mu + \frac{1}{k}$, $k \geqslant 1$. Hence, v_k is a subsolution of $(P^k_{\lambda,\mu})$, for all $k \geqslant 1$. Similarly to the previous step we obtain a solution $u_{\lambda,\mu} \in \mathscr{E}$ for all $\lambda > 0$ and $\mu > \bar{\mu}$.

Step III. Nonexistence for λ, μ small: Let $\lambda, \mu > 0$. Since $K_* > 0$, the assumption (g1) implies $\lim_{s \downarrow 0} \Psi_{\lambda,\mu}(x,s) = -\infty$, uniformly for $x \in \bar{\Omega}$. So, there exists $c > 0$

such that

$$\Psi_{\lambda,\mu}(x,s)<0 \quad \text{for all } (x,s)\in\bar{\Omega}\times(0,c). \tag{25}$$

Let $s\geqslant c$. From (f1) we deduce

$$\frac{\Psi_{\lambda,\mu}(x,s)}{s}\leqslant\lambda\frac{f(x,s)}{s}+\mu\frac{h(x)}{s}\leqslant\lambda\frac{f(x,c)}{c}+\mu\frac{|h|_\infty}{s}$$

for all $x\in\bar{\Omega}$. Fix $\mu<\frac{c\lambda_1}{2|h|_\infty}$ and let $M=\sup_{x\in\bar{\Omega}}\frac{f(x,c)}{c}>0$. From the above inequality we have

$$\frac{\Psi_{\lambda,\mu}(x,s)}{s}\leqslant\lambda M+\frac{\lambda_1}{2}, \quad \text{for all } (x,s)\in\bar{\Omega}\times[c,+\infty). \tag{26}$$

Thus, (25) and (26) yield

$$\Psi_{\lambda,\mu}(x,s)\leqslant a(\lambda)s+\frac{\lambda_1}{2}s, \quad \text{for all } (x,s)\in\bar{\Omega}\times(0,+\infty). \tag{27}$$

Moreover, $a(\lambda)\to 0$ as $\lambda\to 0$. If $(P_{\lambda,\mu})$ has a solution $u_{\lambda,\mu}$, then

$$\begin{aligned}\lambda_1\int_\Omega u^2_{\lambda,\mu}(x)\,dx&\leqslant\int_\Omega|\nabla u_{\lambda,\mu}|^2\,dx=-\int_\Omega u_{\lambda,\mu}(x)\Delta u_{\lambda,\mu}(x)\,dx\\&\leqslant\int_\Omega u_{\lambda,\mu}(x)\Psi(x,u_{\lambda,\mu}(x))\,dx.\end{aligned}$$

Using (27), we get

$$\lambda_1\int_\Omega u^2_{\lambda,\mu}(x)\,dx\leqslant\left[a(\lambda)+\frac{\lambda_1}{2}\right]\int_\Omega u^2_{\lambda,\mu}(x)\,dx.$$

Since $a(\lambda)\to 0$ as $\lambda\to 0$, the above relation leads to a contradiction for $\lambda,\mu>0$ sufficiently small.

Step IV. Existence of a maximal solution of $(P_{\lambda,\mu})$: We show that if $(P_{\lambda,\mu})$ has a solution $u_{\lambda,\mu}\in\mathscr{E}$, then it has a maximal solution. Let $\lambda,\mu>0$ be such that $(P_{\lambda,\mu})$ has a solution $u_{\lambda,\mu}\in\mathscr{E}$. If $U_{\lambda,\mu}$ is the solution of (8), by Lemma 2.3 we have $u_{\lambda,\mu}\leqslant U_{\lambda,\mu}$ in $\bar{\Omega}$. For any $j\geqslant 1$, denote

$$\Omega_j=\left\{x\in\Omega;\operatorname{dist}(x,\partial\Omega)>\frac{1}{j}\right\}.$$

Set $U_0=U_{\lambda,\mu}$ and, for any $j\geqslant 1$, let U_j be the solution of

$$\begin{cases}-\Delta\zeta+K(x)g(U_{j-1})=\lambda f(x,U_{j-1})+\mu h(x) & \text{in } \Omega_j,\\ \zeta=U_{j-1} & \text{in } \Omega\backslash\Omega_j.\end{cases}$$

Using the fact that $\Psi_{\lambda,\mu}$ is nondecreasing with respect to the second variable, we get

$$u_{\lambda,\mu} \leqslant U_j \leqslant U_{j-1} \leqslant U_0 \quad \text{in } \bar{\Omega}.$$

If $\bar{u}_{\lambda,\mu}(x) = \lim_{j\to\infty} U_j(x)$ for all $x \in \bar{\Omega}$, by standard elliptic arguments (see [12]) it follows that $\bar{u}_{\lambda,\mu}$ is a solution of $(P_{\lambda,\mu})$. Since $u_{\lambda,\mu} \leqslant \bar{u}_{\lambda,\mu}$ in $\bar{\Omega}$, by Remark 1.1 we have $\bar{u}_{\lambda,\mu} \in \mathscr{E}$. Moreover, $\bar{u}_{\lambda,\mu}$ is a maximal solution of $(P_{\lambda,\mu})$.

Step V. Dependence on λ and μ: We first show the dependence on λ of the maximal solution $\bar{u}_{\lambda,\mu} \in \mathscr{E}$ of $(P_{\lambda,\mu})$. For this purpose, fix $\mu > 0$ and define

$$A := \{\lambda > 0; (P_{\lambda,\mu}) \text{ has at least a solution } u_{\lambda,\mu} \in \mathscr{E}\}.$$

Let $\lambda_* = \inf A$. From the previous steps, we have $A \neq \emptyset$ and $\lambda_* > 0$. Let $\lambda_1 \in A$ and $\bar{u}_{\lambda_1,\mu}$ be the maximal solution of $(P_{\lambda_1,\mu})$. We prove that $(\lambda_1, +\infty) \subset A$. If $\lambda_2 > \lambda_1$ then $\bar{u}_{\lambda_1,\mu}$ is a subsolution of $(P_{\lambda_2,\mu})$. On the other hand,

$$\Delta U_{\lambda_2,\mu} + \Phi_{\lambda_2,\mu}(x, U_{\lambda_2,\mu}) \leqslant 0 \leqslant \Delta \bar{u}_{\lambda_1,\mu} + \Phi_{\lambda_2,\mu}(x, \bar{u}_{\lambda_1,\mu}) \quad \text{in } \Omega,$$

$$U_{\lambda_2,\mu},\ \bar{u}_{\lambda_1,\mu} > 0 \quad \text{in } \Omega,$$

$$U_{\lambda_2,\mu} \geqslant \bar{u}_{\lambda_1,\mu} \quad \text{on } \partial\Omega,$$

$$\Delta \bar{u}_{\lambda_1,\mu} \in L^1(\Omega).$$

By Lemma 2.3, $\bar{u}_{\lambda_1,\mu} \leqslant U_{\lambda_2,\mu}$ in $\bar{\Omega}$. In the same way as in Step IV we find a solution $u_{\lambda_2,\mu} \in \mathscr{E}$ of $(P_{\lambda_2,\mu})$ such that

$$\bar{u}_{\lambda_1,\mu} \leqslant u_{\lambda_2,\mu} \leqslant U_{\lambda_2,\mu} \quad \text{in } \bar{\Omega}.$$

Hence, $\lambda_2 \in A$ and so $(\lambda_*, +\infty) \subset A$. If $\bar{u}_{\lambda_2,\mu} \in \mathscr{E}$ is the maximal solution of $(P_{\lambda_2,\mu})$, the above relation implies $\bar{u}_{\lambda_1,\mu} \leqslant \bar{u}_{\lambda_2,\mu}$ in $\bar{\Omega}$. By the maximum principle, it follows that $\bar{u}_{\lambda_1,\mu} < \bar{u}_{\lambda_2,\mu}$ in Ω. So, $\bar{u}_{\lambda,\mu}$ is increasing with respect to λ.

To prove the dependence on μ, we fix $\lambda > 0$ and define

$$B := \{\mu > 0; (P_{\lambda,\mu}) \text{ has at least one solution } u_{\lambda,\mu} \in \mathscr{E}\}.$$

Let $\mu_* = \inf B$. The conclusion follows in the same manner as above.

The proof of Theorem 1.2 is now complete. □

5. Proof of Theorem 1.3

Let $\lambda, \mu > 0$. We recall that the function $\Psi_{\lambda,\mu}$ defined in (6) verifies the hypotheses of Lemma 2.2, since $K^* \leqslant 0$. So, there exists $u_{\lambda,\mu} \in C^{2,\gamma}(\Omega) \cap C(\bar{\Omega})$ a solution of $(P_{\lambda,\mu})$.

If $U_{\lambda,\mu}$ is the solution of (8), then

$$\Delta u_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, u_{\lambda,\mu}) \leqslant 0 \leqslant \Delta U_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, U_{\lambda,\mu}) \quad \text{in } \Omega,$$

$$u_{\lambda,\mu},\ U_{\lambda,\mu} > 0 \quad \text{in } \Omega,$$

$$u_{\lambda,\mu} = U_{\lambda,\mu} = 0 \quad \text{on } \partial\Omega.$$

Since $\Delta U_{\lambda,\mu} \in L^1(\Omega)$, by Lemma 2.3 we get $u_{\lambda,\mu} \geqslant U_{\lambda,\mu}$ in $\bar{\Omega}$.

We claim that there exists $c > 0$ such that

$$U_{\lambda,\mu} \geqslant c\varphi_1 \quad \text{in } \Omega. \tag{28}$$

Indeed, if not, there exists $\{x_n\} \subset \Omega$ and $\varepsilon_n \to 0$ such that

$$(U_{\lambda,\mu} - \varepsilon_n \varphi_1)(x_n) < 0. \tag{29}$$

Moreover, we can choose the sequence $\{x_n\}$ with the additional property

$$\nabla(U_{\lambda,\mu} - \varepsilon_n \varphi_1)(x_n) = 0. \tag{30}$$

Passing eventually at a subsequence, we can assume that $x_n \to x_0 \in \bar{\Omega}$. From (29), it follows that $U_{\lambda,\mu}(x_0) \leqslant 0$ which implies $U_{\lambda,\mu}(x_0) = 0$, that is $x_0 \in \partial\Omega$. Furthermore, from (30) we have $\nabla U_{\lambda,\mu}(x_0) = 0$. This is a contradiction since $\frac{\partial U_{\lambda,\mu}}{\partial n}(x_0) < 0$, by Hopf's strong maximum principle. Our claim follows and so

$$u_{\lambda,\mu} \geqslant U_{\lambda,\mu} \geqslant c\varphi_1 \quad \text{in } \Omega. \tag{31}$$

Then, $g(u_{\lambda,\mu}) \leqslant g(U_{\lambda,\mu}) \leqslant g(c\varphi_1)$ in Ω. From assumption (g2) and Lemma 2.1 (using the same method as in the proof of Lemma 2.4) it follows that $g(c\varphi_1) \in L^1(\Omega)$. Hence, $u_{\lambda,\mu} \in \mathscr{E}$.

Let us now assume that $u^1_{\lambda,\mu},\ u^2_{\lambda,\mu} \in \mathscr{E}$ are two solutions of $(P_{\lambda,\mu})$. In order to prove the uniqueness, it is enough to show that $u^1_{\lambda,\mu} \geqslant u^2_{\lambda,\mu}$ in $\bar{\Omega}$. This follows by Lemma 2.3.

Let us show now the dependence on λ of the solution of $(P_{\lambda,\mu})$. For this purpose, let $0 < \lambda_1 < \lambda_2$ and $u_{\lambda_1,\mu},\ u_{\lambda_2,\mu}$ be the unique solutions of $(P_{\lambda_1,\mu})$ and $(P_{\lambda_2,\mu})$, respectively, with $\mu > 0$ fixed. Since $u_{\lambda_1,\mu},\ u_{\lambda_2,\mu} \in \mathscr{E}$ and

$$\Delta u_{\lambda_2,\mu} + \Phi_{\lambda_2,\mu}(x, u_{\lambda_2,\mu}) \leqslant 0 \leqslant \Delta u_{\lambda_1,\mu} + \Phi_{\lambda_2,\mu}(x, u_{\lambda_1,\mu}) \quad \text{in } \Omega,$$

in virtue of Lemma 2.3 we find $u_{\lambda_1,\mu} \leqslant u_{\lambda_2,\mu}$ in $\bar{\Omega}$. So, by the maximum principle, $u_{\lambda_1,\mu} < u_{\lambda_2,\mu}$ in Ω.

The dependence on μ follows similarly.

The proof of Theorem 1.3 is now complete. $\square$

6. Proof of Theorem 1.4

Step I. Existence: Using the fact that $K^* > 0$, from Theorem 1.2 it follows that there exists $\lambda_*, \mu_* > 0$ such that the problem

$$\begin{cases} -\Delta v + K^* g(v) = \lambda f(x, v) + \mu h(x) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

has a maximal solution $v_{\lambda,\mu} \in \mathscr{E}$, provided $\lambda > \lambda_*$ or $\mu > \mu_*$. Moreover, $v_{\lambda,\mu}$ is increasing with respect to λ and μ. Then $v_k = v_{\lambda,\mu} + \frac{1}{k}$ is a subsolution of $(P^k_{\lambda,\mu})$, for all $k \geqslant 1$. On the other hand, by Lemma 2.2, the boundary value problem

$$\begin{cases} -\Delta w + K_* g(w) = \lambda f(x, w) + \mu h(x) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = \frac{1}{k} & \text{on } \partial\Omega \end{cases}$$

has a solution $w_k \in C^{2,\gamma}(\bar{\Omega})$. Obviously, w_k is a supersolution of $(P^k_{\lambda,\mu})$.

Since $K^* > 0 > K_*$, we have

$$\Delta w_k + \Phi_{\lambda,\mu}(x, w_k) \leqslant 0 \leqslant \Delta v_k + \Phi_{\lambda,\mu}(x, v_k) \quad \text{in } \Omega,$$

and

$$w_k,\ v_k > 0 \quad \text{in } \Omega,$$

$$w_k = v_k \quad \text{on } \partial\Omega,$$

$$\Delta v_k \in L^1(\Omega).$$

From Lemma 2.3 it follows that $v_k \leqslant w_k$ in $\bar{\Omega}$. By standard super- and subsolution argument, there exists a minimal solution $u^1_{\lambda,\mu} \in C^{2,\gamma}(\bar{\Omega})$ of $(P^1_{\lambda,\mu})$ such that $v_1 \leqslant u^1_{\lambda,\mu} \leqslant w_1$ in $\bar{\Omega}$. Now, taking $u^1_{\lambda,\mu}$ and v_2 as a pair of super- and subsolutions for $(P^2_{\lambda,\mu})$, we deduce that there exists a minimal solution $u^2_{\lambda,\mu} \in C^{2,\gamma}(\bar{\Omega})$ of $(P^2_{\lambda,\mu})$ such that $v_2 \leqslant u^2_{\lambda,\mu} \leqslant u^1_{\lambda,\mu}$ in $\bar{\Omega}$. Arguing in the same manner, we obtain a sequence $\{u^k_{\lambda,\mu}\}$ such that

$$v_k \leqslant u^k_{\lambda,\mu} \leqslant u^{k-1}_{\lambda,\mu} \leqslant w_1 \quad \text{in } \bar{\Omega}. \tag{32}$$

Define $u_{\lambda,\mu}(x) = \lim_{k\to\infty} u^k_{\lambda,\mu}(x)$ for all $x \in \bar{\Omega}$. With a similar argument to that used in the proof of Theorem 1.2, we find that $u_{\lambda,\mu} \in \mathscr{E}$ is a solution of $(P_{\lambda,\mu})$. Hence, problem $(P_{\lambda,\mu})$ has at least a solution in $\mathscr{E}$, provided that $\lambda > \lambda_*$ or $\mu > \mu_*$.

Step II. Dependence on λ and μ: As above, it is enough to justify only the dependence on λ. Fix $\lambda_* < \lambda_1 < \lambda_2$, $\mu > 0$ and let $u_{\lambda_1,\mu}$, $u_{\lambda_2,\mu} \in \mathscr{E}$ be the solutions of $(P_{\lambda_1,\mu})$ and $(P_{\lambda_2,\mu})$, respectively, that we have obtained in Step I. It follows that $u^k_{\lambda_2,\mu}$ is a supersolution of $(P^k_{\lambda_1,\mu})$. So, Lemma 2.3 combined with the fact that $v_{\lambda,\mu}$ is increasing with respect to $\lambda > \lambda_*$ yield

$$u^k_{\lambda_2,\mu} \geqslant v_{\lambda_2,\mu} + \frac{1}{k} \geqslant v_{\lambda_1,\mu} + \frac{1}{k} \quad \text{in } \bar{\Omega}.$$

Thus, $u^k_{\lambda_2,\mu} \geqslant u^k_{\lambda_1,\mu}$ in $\bar{\Omega}$ since $u^k_{\lambda_1,\mu}$ is the minimal solution of $(P^k_{\lambda_1,\mu})$ which satisfies $u^k_{\lambda_1,\mu} \geqslant v_{\lambda_1,\mu} + 1/k$ in $\bar{\Omega}$. It follows that $u_{\lambda_2,\mu} \geqslant u_{\lambda_1,\mu}$ in $\bar{\Omega}$. By the maximum principle we deduce that $u_{\lambda_2,\mu} > u_{\lambda_1,\mu}$ in Ω.

This concludes the proof. $\square$

Acknowledgments

This work has been completed while V. Rădulescu was visiting the Université de Savoie with a CNRS research position in September–November 2002. He thanks the members of the Laboratoire de Mathématiques for hospitality and he is especially grateful to Prof. Ioan R. Ionescu for this invitation and for many interesting discussions.

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