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Solutions with prescribed mass for the p-Laplacian Schrödinger-Poisson system with critical growth

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Abstract

In this paper, we focus on the existence and multiplicity of solutions for the p-Laplacian Schrödinger-Poisson system

$$\begin{cases} -\Delta_p u + \gamma\phi\,|u|^{p-2}u = \lambda\,|u|^{p-2}u + \mu|u|^{q-2}u + |u|^{p^*-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^p, & \text{in } \mathbb{R}^3, \end{cases}$$

with a prescribed mass given by

$$\int_{\mathbb{R}^3} |u|^p dx = a^p,$$

in the Sobolev critical case, where, $1 < p < 3, a > 0$, and $\gamma > 0$, $\mu > 0$ are parameters, $p^* = \frac{3p}{3-p}$ is the Sobolev critical exponent, and $\lambda \in \mathbb{R}$ is an undetermined parameter, acting as a Lagrange multiplier. We investigate this system under the L^p-subcritical perturbation $\mu|u|^{q-2}u$, with $q \in (p, p + \frac{p^2}{3})$, and

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establish the existence of multiple normalized solutions using the truncation technique, concentration-compactness principle, and genus theory. In the L^p-supercritical regime: $q \in (p+\frac{p^2}{3}, p^*)$, we prove two existence results for normalized solutions under different assumptions for the parameters γ, μ, by employing the Pohozaev manifold analysis, concentration-compactness principle and mountain pass theorem. This study presents new contributions regarding the existence and multiplicity of normalized solutions of the p-Laplacian critical Schrödinger-Poisson problem, perturbed with a subcritical term in the whole space $\mathbb{R}^3$, for the first time.

MSC: primary 35J62; secondary 35B65, 35J50

Keywords: p-Laplacian Schrödinger-Poisson system; Normalized solutions; Sobolev critical exponent; Concentration-compactness principle; Genus theory

1. Introduction and main results

In this paper we investigate the following p-Laplacian Schrödinger-Poisson system

$$\begin{cases} -\Delta_p u + \gamma\phi\,|u|^{p-2}u = \lambda\,|u|^{p-2}u + \mu|u|^{q-2}u + |u|^{p^*-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^p\,, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

subject to the prescribed L^p-norm condition:

$$\int_{\mathbb{R}^3} |u|^p dx = a^p, \tag{1.2}$$

where $\lambda \in \mathbb{R}$ is an undetermined parameter, $1 < p < 3, a > 0$ and $\mu, \gamma > 0$ are parameters. The term $\mu|u|^{q-2}u$ is a subcritical perturbation, where $p < q < p^* = \frac{3p}{3-p}$, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator.

The p-Laplacian operator plays a significant role in nonlinear fluid dynamics, where the value of p is related to the flow speed and the materials involved. The system (1.1) consists of a quasilinear Schrödinger equation coupled with a Poisson equation. The Schrödinger-Poisson system originates from quantum mechanics and semiconductor theory, describing the interaction between a charged particle and an electromagnetic field, we refer to [9,24–26,42,40] for more applied background of the Schrödinger-Poisson systems, and p-Laplacian equations.

When $p = 2$, the system (1.1) reduces to the classical Schrödinger-Poisson system, which has been extensively studied in recent decades, following the pioneering work of Benci and Fortunato in [13], we refer to [2,4,5,21,22,15,29,30] for the system with a subcritical term; and to [5,22,31,46,63] for the system with a critical term. Note that, in [44], Ruiz studied the impact of the nonlinear local term on the existence of nontrivial solutions for the subcritical system:

$$\begin{cases} -\Delta u + u + \lambda\phi u = |u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

In [5], Azzollini and Pomponio established the existence of ground state solutions for the critical system:

$$\begin{cases} -\Delta u + u + \lambda\phi u = |u|^{q-2}u + |u|^4 u, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

with $4 < q < 6$. In a more recent study [24], the authors investigated the subcritical quasilinear system:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda\phi|u|^{p-2}u = |u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = |u|^p, & x \in \mathbb{R}^3, \end{cases} \tag{1.5}$$

using variational methods and derived existence results for $1 < p < 3$ and $p < q < p^*$. For further research on p-Laplacian Schrödinger-Poisson systems, we refer to [25,26,39] and the references therein.

As mentioned earlier, there has been increasing attention in recent years on nonlinear p-Laplacian Schrödinger-Poisson systems (1.1), or (1.3)-(1.5), particularly with regard to the existence and multiplicity of ground state solutions, bound state solutions, and sign-changing solutions, *without prescribed mass*. However, from a physical perspective, it is particularly interesting to study solutions to these problems with prescribed L^p-norms. Solutions of this type are commonly referred to as normalized solutions.

Let us consider the classical Schrödinger equation

$$-\Delta u + \lambda u = f(u), \quad x \in \mathbb{R}^3, \tag{1.6}$$

there are many researchers have investigated the existence and multiplicity of normalized solutions, following the pioneering work of Jeanjean [32], by using the minimization methods and constrained mountain pass arguments. For more recent developments on this topic, we refer the interested readers to [6,7,12,36,35,47,48,57], among others.

We observe that there are only a few papers addressing the normalized solution of the p-Laplacian Schrödinger equation. Wang et al. [56] considered the following system:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \mu u + |u|^{s-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2\,dx = \rho, \end{cases}$$

where $1 < p < N$, $\mu \in \mathbb{R}$, and $s \in \left(\frac{N+2}{N}p, p^*\right)$. They considered the L^2-constraint, and by using the Gagliardo-Nirenberg inequality, the L^2-critical exponent is given by $\frac{N+2}{N}p$. Moreover, it is known that $L^2(\mathbb{R}^N) \not\subseteq W^{1,p}(\mathbb{R}^N)$, so the working space is $W^{1,p}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, which is a Hilbert space and plays a crucial role in [55,56].

The first paper to study the p-Laplacian equation with an L^p-constraint is [62], where the system is given by:

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + \mu|u|^{q-2}u + g(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p\,dx = a^p, \end{cases} \tag{1.7}$$

where $g \in C(\mathbb{R}, \mathbb{R})$ and there exist constants α and β such that $p + \frac{p^2}{N} < \alpha \le \beta < p^*$, with the condition that for all $t \in \mathbb{R}$, there is

$$0 < \alpha G(t)t \le g(t)t \le \beta G(t), \quad G(t) = \int_0^t g(\tau)\, d\tau.$$

A simple example is $g(t) = |t|^{r-2}t$ with $p + \frac{p^2}{N} < r < p^*$. Additionally, Wang and Sun [53] considered both the L^2-constraint and the L^p-constraint for the following problem:

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda |u|^{r-2}u + |u|^{q-2}u & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^r\, dx = c, \end{cases}$$

where $1 < p < N$, $\lambda \in \mathbb{R}$, $r = p$ or 2, $p < q < p^*$, and $V(x)$ is a trapping potential satisfying

$$V(x) \in C(\mathbb{R}^N), \quad \lim_{|x| \to +\infty} V(x) = +\infty \quad \text{and} \quad \inf_{x \in \mathbb{R}^N} V(x) = 0.$$

When the nonlinearity g exhibits critical growth, i.e., $g(u) = |u|^{p^*-2}u$, Deng and Wang [23], as well as Feng and Li [27], recently studied the normalized solutions to (1.7), using the concentration-compactness lemma, Schwarz rearrangement, Ekeland's variational principle, and mini-max theorems.

Inspired by the aforementioned works, and recognizing that the L^p-norm is a conserved quantity in the evolution, this paper focuses on searching for solutions to (1.1) with a prescribed L^p-norm, as given in (1.2). To achieve this, we apply the reduction argument introduced in [44], which transforms system (1.1) into the following single equation:

$$-\Delta_p u + \gamma \phi_u |u|^{p-2}u = \lambda |u|^{p-2}u + \mu |u|^{q-2}u + |u|^{p^*-2}u, \quad x \in \mathbb{R}^3, \tag{1.8}$$

where

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|}\, dy.$$

Next, we aim to find solutions to (1.1)-(1.2) as critical points of the action functional:

$$I_\mu(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{\gamma}{2p}\int_{\mathbb{R}^3} \phi_u |u|^p\, dx - \frac{\mu}{q}\|u\|_q^q - \frac{1}{p^*}\|u\|_{p^*}^{p^*},$$

under the L^p-norm constrained manifold:

$$S(a) := \left\{ u \in E : \int_{\mathbb{R}^3} |u|^p = a^p \right\}.$$

It is straightforward to verify that I_μ is a well-defined and C^1-functional on $S(a)$. This approach is relevant from a physical perspective, particularly because the L^p-norm is a conserved quantity in the evolution, and the variational characterization of such solutions is often instrumental in analyzing their orbital stability. For more details, see, for example, [19,37,45] and the references therein.

We note that there are few papers in the literature addressing the existence of normalized solutions for classical Schrödinger-Poisson systems. Recently, Wang and Qian [54] studied the existence of normalized ground states and infinitely many radial solutions for the following system:

$$\begin{cases} -\Delta u + \lambda u + \gamma \phi u = a f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 \, dx = c^2, \end{cases} \tag{1.9}$$

where f is a Sobolev subcritical term. They constructed a specific bounded Palais-Smale sequence when $\gamma < 0$ and $a > 0$. Meanwhile, they obtained a nonexistence result in the case $\gamma < 0$ and $a < 0$, and an existence result when $\gamma > 0$ and $a < 0$, using variational methods.

In [34], Jeanjean and Trung Le specialized in the existence of normalized solutions for the problem (1.9) with $f(u) = |u|^{p-2}u$, which exhibits L^2-supercritical growth:

$$\begin{cases} -\Delta u + \gamma (|x|^{-1} * |u|^2) u = \lambda u + a|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^N} |u|^2 \, dx = c^2. \end{cases} \tag{1.10}$$

The authors showed that problem (1.10) admits two solutions in the case (i): $\gamma > 0, a > 0$, and $p \in \left(\frac{10}{3}, 6\right]$; one solution which is a global minimizer in the case (ii): $\gamma > 0, a < 0$, and $p \in \left(\frac{10}{3}, 6\right]$; and no positive solution of (1.10) in the case (iii): $\gamma < 0, a > 0$, and $p = 6$.

When $\gamma = 1$, $p \in \left(\frac{10}{3}, 6\right)$, Bellazzini, Jeanjean, and Luo [11] investigated the existence of normalized solutions for (1.10) using a mountain-pass argument when $c > 0$ is sufficiently small, and they proved nonexistence when $c > 0$ is not small. In [33], Jeanjean and Luo considered the existence of minimizers with L^2-norm for (1.10) when $p \in \left[3, \frac{10}{3}\right]$, and they obtained a threshold value of $c > 0$ separating existence and nonexistence of minimizers. We point out that Chen and Tang [16] developed an original method for the analysis of normalized solutions to Schrödinger equations with reaction fulfilling L^2-subcritical, L^2-critical, or L^2-supercritical growth. For further results on normalized solutions of Schrödinger-Poisson systems, we refer to [1,20,18,33,34,43,38,60,61,59] and the references therein. For the nonlocal case, we refer to Chen and Tang [17] who established a multiplicity property of solutions with prescribed mass and a nonexistence result in the framework of Kirchhoff equations with Sobolev critical exponent and mixed nonlinearities.

Returning to the problem (1.1)-(1.2), we shall look for normalized solutions using the truncation technique, concentration-compactness principle, and genus theory under the L^p-subcritical perturbation, i.e., $q \in (p, p + \frac{p^2}{3})$. While in the L^p-supercritical perturbation case: $q \in (p + \frac{p^2}{3}, p^*)$, we prove two existence results for normalized solutions under different assumptions for the parameters γ and μ, employing the Pohozaev manifold analysis, the mountain pass theorem and the concentration-compactness principle. To the best of our knowledge, no progress has been made regarding the study of normalized solutions for the p-Laplacian Schrödinger-Poisson equations with Sobolev critical exponents in the literature.

Before presenting the existence result, we recall the definition of ground states. If $\widetilde{u}$ is a solution to (1.1)-(1.2) that has minimal energy among all solutions in $S(a)$, i.e.,

$$(I_\mu|_{S(a)})'(\widetilde{u}) = 0 \quad \text{and} \quad I_\mu(\widetilde{u}) = \inf\{(I_\mu|_{S(a)})'(u) = 0, \quad u \in S(a)\},$$

we say that $\widetilde{u}$ is a ground state of (1.1)-(1.2).

First, we address the existence of multiple normalized ground state solutions in the L^p-subcritical case, where $q \in (p, \bar{p})$, and $\bar{p} = p + \frac{p^2}{3}$, which can be stated as follows:

Theorem 1.1. *Let $\mu, \lambda, a > 0$, and $q \in \left(p, p + \frac{p^2}{3}\right)$. Then, for a given $k \in \mathbb{N}$, there exists $\beta > 0$ independent of k and $\mu_k^* > 0$ large, such that problem (1.1)-(1.2) possesses at least k couples $(u_j, \alpha_j) \in E \times \mathbb{R}$ of weak solutions for $\mu > \mu_k^*$ and*

$$a \in \left(0, \left(\frac{\beta}{\mu}\right)^{\frac{1}{q(1-\delta_q)}}\right) \tag{1.11}$$

with

$$\|u_j\|_p^p = a^p, \ \lambda_j < 0 \text{ for all } j = 1, \dots, k.$$

The second result of this paper addresses the existence and asymptotic behavior of normalized solutions for the L^p-supercritical perturbation when the parameters $\lambda, \mu > 0$ are appropriately small.

Theorem 1.2. *Let $\frac{1+\sqrt{41}}{4} < p < \sqrt[3]{9}$, $p + \frac{p^2}{3} < q < p^*$, $\mu > 0$, and assume that $0 < a < \tilde{a}$, where*

$$\tilde{a} := \left(\frac{(K_a)^{1-\frac{1}{p}}}{4\gamma\tilde{C}\left[C\left(\frac{6}{5}p\right)\right]^{\frac{5}{3}}}\right)^{\frac{1}{2p-1}},$$

where K_a is defined in (4.3), constants $\tilde{C}$ and $C\left(\frac{6}{5}p\right)$ are from (2.9), (2.10), respectively. Then there exist $\Gamma^* > 0$ such that $0 < \gamma < \Gamma^*$, problem (1.1)-(1.2) has a positive normalized ground state solution $u_\lambda \in E$ for some $\lambda < 0$.

Finally, we present an existence result for normalized solutions under the L^p-supercritical perturbation, when the parameter $\mu > 0$ is large.

Theorem 1.3. *If $q \in (\bar{p}, p^*)$, there exists $\mu^* = \mu^*(a) > 0$ large, such that as $\mu > \mu^*$, problem (1.1)-(1.2) possesses a couple $(u_a, \lambda) \in E \times \mathbb{R}$ of weak solutions with $\|u_a\|_p^p = a^p$, $\lambda < 0$.*

Remark 1.4. (i) We note that, in [24–26] and [56], the authors studied the existence of positive solutions to the quasilinear Schrõdinger-Poisson systems (1.4) and (1.5) without the prescribed mass. In this paper, to the best of our knowledge, we present new contributions regarding the existence and multiplicity of normalized solutions of the p-Laplacian critical Schrödinger-Poisson problem, perturbed with a subcritical term in the whole space $\mathbb{R}^3$.

(ii) We extend and improve the results concerning normalized solutions of the classical Schrõdinger-Poisson systems from the references [11,18,33,34,43,54,59–61] to the p-Laplacian cases with the Sobolev critical nonlinearities.

Finally, let us outline the ideas and methods used in this paper to obtain our main results. For the L^p-subcritical perturbation, where $q \in (p, p + \frac{p^2}{3})$, it is challenging to establish the boundedness of the (PS) sequence using the approach from [32]. To overcome this issue, we apply the truncation technique to restore the loss of compactness in the (PS) sequence caused by critical growth. To apply the concentration-compactness principle and obtain the multiplicity of normalized solutions for (1.1)-(1.2), we utilize genus theory. For the L^p-supercritical perturbation, where $q \in \left(p + \frac{p^2}{3}, p^*\right)$, we employ the Pohozaev manifold and mountain pass theorem to prove the existence of positive ground state solutions for (1.1)-(1.2) when $\mu > 0$ is small. When $\mu > 0$ is large, we use a fiber map and the concentration-compactness principle to show that the (PS) sequence is strongly convergent, thereby obtaining a normalized solution to (1.1)-(1.2).

Structure of the paper, and notation. This paper is organized as follows. In Section 2 we provide some preliminary results that will be frequently referenced in the sequel. Section 3 is concerned with the multiplicity of normalized ground state solutions for system (1.1)-(1.2) when $q \in (p, p + \frac{p^2}{3})$, and the proof of Theorem 1.1 is completed. Section 4 is dedicated to prove the existence of normalized positive ground state solutions for problem (1.1)-(1.2) when $q \in (p + \frac{p^2}{3}, p^*)$, and Theorem 1.2 is established if $\mu, \gamma > 0$ are suitably small. In Section 5, we provide another existence result for problem (1.1)-(1.2) with $q \in (p + \frac{p^2}{3}, p^*)$, when the parameter $\mu > 0$ is large, and complete the proof of Theorem 1.3.

Throughout this paper, we denote $B_r(z)$ the open ball of radius r with center at z in $\mathbb{R}^3$, and $\|u\|_p$ is the usual norm of the space $L^p(\mathbb{R}^3)$ for $p \geq 1$. Moreover, we denote by $C, C_i > 0, i = 1, 2, \cdots$, different positive constants whose values may vary from line to line and are not essential to the problem.

2. Preliminary stuff

In this section, we first give the functional space setting and introduce some notations and useful preliminary results, which are important to prove the main results. Let $E := W^{1,p}\left(\mathbb{R}^3\right)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^3} |\nabla u|^p + |u|^p dx\right)^{\frac{1}{p}}.$$

And the homogeneous Sobolev space $D^{1,p}(\mathbb{R}^3)$ is defined by

$$D^{1,p}(\mathbb{R}^3)=\left\{u\in L^{p^*}(\mathbb{R}^3):\int_{\mathbb{R}^3}|\nabla u|^p dx<+\infty\right\},$$

endowed with the norm

$$\|u\|^p:=\|u\|^p_{D^{1,p}(\mathbb{R}^3)}=\|\nabla u\|_p^p=\int_{\mathbb{R}^3}|\nabla u|^p dx.$$

The work space $E_r:=W_r^{1,p}\left(\mathbb{R}^3\right)$ is defined by

$$E_r:=W_r^{1,p}\left(\mathbb{R}^3\right)=\left\{u\in W^{1,p}\left(\mathbb{R}^3\right):\ u \text{ is radially symmetric and decreasing}\right\}.$$

The standard norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_{p,\Omega}$ and by $\|\cdot\|_p$ if $\Omega=\mathbb{R}^3$.

Let us now comment on the critical problem in the whole space, namely

$$-\Delta_p u=|u|^{p^*-2}u \quad \text{in } \mathbb{R}^3, \quad u\in D^{1,p}(\mathbb{R}^3). \tag{2.1}$$

We know that all the regular radial solutions to (2.1) are given by the following expression:

$$U_\varepsilon(x)=\frac{C_{N,p}\varepsilon^{\frac{3-p}{p(p-1)}}}{\left(\varepsilon^{\frac{p}{p-1}}+|x|^{\frac{p}{p-1}}\right)^{\frac{3-p}{p}}} \tag{2.2}$$

with $\varepsilon>0$, and $C_{N,p}$ a normalized constant. Note that, by [52], it follows that the family of functions given above are minimizers to

$$S=\inf_{u\in D^{1,p}(\mathbb{R}^3)\backslash\{0\}}\frac{\|\nabla u\|_p^p}{\|u\|_{p^*}^p}. \tag{2.3}$$

Let us set

$$u_\varepsilon(x)=\psi(x)U_\varepsilon(x),$$

where $\psi\in C_0^\infty(\mathbb{R}^3)$ satisfies

$$\psi(x)=\begin{cases}1, & \text{for } |x|\le R,\\ 0\le\psi(x)\le 1, & \text{for } R<|x|<2R,\\ 0, & \text{for } |x|\ge 2R,\end{cases}$$

with $R>0$. By the direct calculations as in [14,53], we can obtain the following important estimates:

$$\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^p dx=S_p^{\frac{3}{p}}+O\left(\varepsilon^{\frac{3-p}{p-1}}\right), \tag{2.4}$$

$$\int_{\mathbb{R}^3} |u_\varepsilon|^{p^*}\,dx = S_p^{\frac{3}{p}} + O\left(\varepsilon^{\frac{3}{p-1}}\right), \tag{2.5}$$

$$\int_{\mathbb{R}^3} |u_\varepsilon|^{p}dx = \begin{cases} O\left(\varepsilon^{\frac{3-p}{p-1}}\right), & \sqrt{3} < p < 3, \\ O\left(\varepsilon^{p}\right) + K_1\varepsilon^p|\ln\varepsilon|, & p = \sqrt{3}, \\ O\left(\varepsilon^{\frac{3-p}{p-1}}\right) + K_2\varepsilon^p, & 1 < p < \sqrt{3}, \end{cases} \tag{2.6}$$

$$\int_{\mathbb{R}^3} |u_\varepsilon|^{q}dx = \begin{cases} O\left(\varepsilon^{\frac{q(3-p)}{p(p-1)}}\right), & q < \frac{3(p-1)}{3-p}, \\ O\left(\varepsilon^{\frac{3}{p}}\right) + K_3\varepsilon^{\frac{3}{p}}|\ln\varepsilon|, & q = \frac{3(p-1)}{3-p}, \\ O\left(\varepsilon^{\frac{q(3-p)}{p(p-1)}}\right) + K_4\varepsilon^{3-\frac{q(3-p)}{p}}, & q > \frac{3(p-1)}{3-p}, \end{cases} \tag{2.7}$$

where $\varepsilon > 0$ is small and K_i $(i = 1, 2, 3, 4)$ denote positive constants independent of ε. For $1 < p < 3$, if $\frac{6p}{5} < \frac{3(p-1)}{3-p}$, then $\frac{1+\sqrt{41}}{4} < p < 3$. Combining this with (2.7), we deduce that

$$\int_{\mathbb{R}^3} |u_{\varepsilon,p}|^{\frac{6p}{5}}dx = \begin{cases} O\left(\varepsilon^{\frac{6(3-p)}{5(p-1)}}\right) + K_4\varepsilon^{3-\frac{6(3-p)}{5}}, & \frac{3}{2} < p < \frac{1+\sqrt{41}}{4}, \\ O\left(\varepsilon^{\frac{3}{p}}\right) + K_3\varepsilon^{\frac{3}{p}}|\ln\varepsilon|, & p = \frac{1+\sqrt{41}}{4}, \\ O\left(\varepsilon^{\frac{6(3-p)}{5(p-1)}}\right), & \frac{1+\sqrt{41}}{4} < p < 3. \end{cases} \tag{2.8}$$

In the following, we recall some useful inequalities, which play an important part in the proof of our main results.

Proposition 2.1. *(Hardy-Littlewood-Sobolev inequality [41]) Let $l, r > 1$ and $0 < \mu < N$ be such that $\frac{1}{r} + \frac{1}{l} + \frac{\mu}{N} = 2$, $f \in L^r(\mathbb{R}^N)$ and $h \in L^l(\mathbb{R}^N)$. Then there exists a constant $C(N, \mu, r, l) > 0$ such that*

$$\left| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)h(y)|x-y|^{-\mu}dxdy \right| \le C(N,\mu,r,l)\|f\|_r\|h\|_l.$$

From Proposition 2.1, with $l = r = \frac{6}{5}p$, we have that:

$$\int_{\mathbb{R}^3} \phi_u\,|u|^p\,dx \le \tilde{C}\|u\|_{\frac{6}{5}p}^{2p} \tag{2.9}$$

Next, we introduce the following Gagliardo-Nirenberg-Sobolev inequality.

Lemma 2.2. *([62]). Let $q \in (p, p^*)$. Then there exists a constant $C(q) > 0$ such that*

$$\|u\|_q^q \leq C(q) \|\nabla u\|_p^{q\delta_q} \|u\|_p^{q(1-\delta_q)}, \quad \forall u \in E, \tag{2.10}$$

where $\delta_q = \frac{3(q-p)}{pq}$.

Lemma 2.3. *(Sobolev inequality [23]). Let $1 < p < 3$. Then there exists an optimal constant $S > 0$ such that*

$$\mathcal{S}\|u\|_{p^*}^p \leq \|\nabla u\|_p^p, \quad \forall u \in D^{1,p}(\mathbb{R}^3). \tag{2.11}$$

Lemma 2.4. *([24]). If $u_n \rightharpoonup u$ in E_r, then*

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p \, dx \to \int_{\mathbb{R}^3} \phi_u |u|^p \, dx, \tag{2.12}$$

and

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{p-2} u_n \varphi dx \to \int_{\mathbb{R}^3} \phi_u |u|^{p-2} u \varphi dx, \quad \forall \varphi \in E_r. \tag{2.13}$$

In the sequel, we define a useful fiber map (e.g. [62]) preserving the L^p-norm

$$(t \star u)(x) := e^{\frac{3t}{p}} u(e^t x), \quad x \in \mathbb{R}^3, \; t \in \mathbb{R}. \tag{2.14}$$

By simple calculation, we can infer that

$$\|(t \star u)\|_p^p = \|u\|_p^p, \tag{2.15}$$

$$\|(t \star u)\|_q^q = e^{q\delta_q t} \|u\|_q^q, \tag{2.16}$$

and

$$\|\nabla (t \star u)(x)\|_p^p = e^{pt} \|\nabla u(x)\|_p^p. \tag{2.17}$$

Next, we introduce an auxiliary functional $\Psi_u^\mu(t) := I_\mu(t \star u)$ by

$$\begin{aligned} I(u,t) &:= I_\mu(t \star u) \\ &= \frac{1}{p} e^{pt} \|\nabla u\|_p^p + \frac{\gamma}{2p} e^t \int_{\mathbb{R}^3} \phi_u |u|^p \, dx - \frac{\mu}{q} e^{q\delta_q t} \|u\|_q^q - \frac{1}{p^*} e^{p^* t} \|u\|_{p^*}^{p^*}. \end{aligned} \tag{2.18}$$

Besides, we have the fact that

$$q\delta_q \begin{cases} < p, & \text{as } \; p < q < \bar{p}; \\ = p, & \text{as } \; q = \bar{p}; \\ > p, & \text{as } \; \bar{q} < q < p^*, \end{cases}$$

where $\bar{p} := p + \frac{p^2}{3}$, $p^* := \frac{3p}{3-p}$.

The Pohozaev manifold plays an important role in the proof of our main results, which can be derived form [62] and [63].

Proposition 2.5. *Let $u \in E \cap L^\infty(\mathbb{R}^3)$ be a weak solution of (1.1), then u satisfies the equality*

$$\frac{3-p}{p}\|\nabla u\|_p^p + \frac{5\gamma}{2p}\int_{\mathbb{R}^3} \phi_u |u|^p \, dx = \frac{3\lambda}{p}\|u\|_p^p + \frac{3\mu}{q}\|u\|_q^q + \frac{3-p}{p}\|u\|_{p^*}^{p^*}. \tag{2.19}$$

Lemma 2.6. *Let $u \in E$ be a weak solution of (1.1)-(1.2), then we can define the following Pohozaev manifold*

$$\mathcal{P}(a) = \{u \in S(a) : P_\mu(u) = 0\},$$

where

$$P_\mu(u) = \|\nabla u\|_p^p + \frac{\gamma}{2p}\int_{\mathbb{R}^3} \phi_u |u|^p \, dx - \mu\delta_q\|u\|_q^q - \|u\|_{p^*}^{p^*}. \tag{2.20}$$

Proof. Since u is the weak solution of (1.1)-(1.2), by (2.19), we have that

$$\frac{3-p}{p}\|\nabla u\|_p^p + \frac{5\gamma}{2p}\int_{\mathbb{R}^3} \phi_u |u|^p \, dx = \frac{3\lambda}{p}\|u\|_p^p + \frac{3\mu}{q}\|u\|_q^q + \frac{3-p}{p}\|u\|_{p^*}^{p^*}.$$

Moreover, since u is the weak solution of system (1.1)-(1.2), we have

$$\|\nabla u\|_p^p + \gamma\int_{\mathbb{R}^3} \phi_u |u|^p \, dx = \lambda\|u\|_p^p + \mu\|u\|_q^q + \|u\|_{p^*}^{p^*}.$$

Combining with (2.20) and the above equality, we obtain that

$$\|\nabla u\|_p^p + \frac{\gamma}{2p}\int_{\mathbb{R}^3} \phi_u |u|^p \, dx = \mu\delta_q\|u\|_q^q + \|u\|_{p^*}^{p^*}.$$

The proof is completed. □

We consider, for any $u \in S(a)$ and $t \in \mathbb{R}$, the fiber Ψ_u^μ introduced in (2.18), and note that

$$\begin{aligned}\left(\Psi_u^\mu\right)'(t) &= e^{pt}\|\nabla u\|_p^p + \frac{\gamma}{2p}e^t\int_{\mathbb{R}^3} \phi_u |u|^p \, dx - \mu\delta_q e^{q\delta_q t}\|u\|_q^q - e^{p^* t}\|u\|_{p^*}^{p^*} \\ &= \|\nabla(t\star u)\|_p^p + \frac{\gamma}{2p}\int_{\mathbb{R}^3} \phi_{t\star u} |t\star u|^p \, dx - \mu\delta_q\|t\star u\|_q^q - \|t\star u\|_{p^*}^{p^*} \\ &= P_\mu(t\star u).\end{aligned} \tag{2.21}$$

Moreover, by direct calculation, we have

$$\left(\Psi_u^\mu\right)''(t) = pe^{pt}\|\nabla u\|_p^p + \frac{\gamma}{2p}e^t\int_{\mathbb{R}^3}\phi_u |u|^p\,dx - \mu q\delta_q^2 e^{q\delta_q t}\|u\|_q^q - p^* e^{p^* t}\|u\|_{p^*}^{p^*}. \tag{2.22}$$

Therefore, we have the following lemma:

Lemma 2.7. *For any $u \in S(a)$, $t \in \mathbb{R}$ is a critical point of $\Psi_u^\mu(t)$ if and only if $(t \star u) \in \mathcal{P}(a)$. Particularly, $u \in \mathcal{P}(a)$ if and only if 0 is a critical point for $\Psi_u^\mu(t)$.*

Now, we recall the following known results.

Lemma 2.8. *([49]). Let $N \geq 2$. The embedding $W_r^{1,p}\left(\mathbb{R}^3\right) \hookrightarrow L^q\left(\mathbb{R}^3\right)$ is compact for any $p < q < p^*$.*

Lemma 2.9. *([27]). (i) The map $(u,t) \in E \times \mathbb{R} \to (t \star u) \in E$ is continuous.*

(ii) For $u \in S(a)$ and $t \in \mathbb{R}$, the map $\varphi \mapsto t \star \varphi$ from $T_u S(a)$ to $T_{t\star u}S(a)$ is a linear isomorphism with inverse $\psi \mapsto (-t \star \psi)$, where

$$T_u S(a) := \left\{\varphi \in S(a) : \int_{\mathbb{R}^3} |u|^{p-2}u\varphi = 0\right\}.$$

Finally, we need a version of linking theorem, see Section 5 in [28].

Definition 2.10. Let X be a topological space and B be a closed subset of X. We say that a class $\mathcal{F}$ of compact of subsets of X is a homotopy-stable family with extended boundary B if for any set A in $\mathcal{F}$ and any $\eta \in C([0,1] \times X; X)$ satisfying $\eta(t,x) = x$ for all $(t,x) \in (\{0\} \times X) \cup ([0,1] \times B)$ we have that $\eta(\{1\} \times A) \in \mathcal{F}$.

Lemma 2.11. *([28]). Let ϕ be a C^1-functional on a complete connected C^1-Finsler manifold X and consider a homotopy-stable family $\mathcal{F}$ with an extended closed boundary B. Set $m = m(\phi, \mathcal{F})$ and let F be a closed subset of X satisfying*

(1) $(A \cap F) \setminus B \neq \emptyset$ for every $A \in \mathcal{F}$;
(2) $\sup \phi(B) \leq m \leq \inf \phi(F)$.

Then, for any sequence of sets $(A_n)_n$ in $\mathcal{F}$ such that $\lim_{n\to\infty}\sup_{A_n}\phi = m$, there exists a sequence $(x_n)_n$ in X such that

$$\lim_{n\to\infty}\phi(x_n) = m, \quad \lim_{n\to\infty}\|d\phi(x_n)\| = 0, \quad \lim_{n\to\infty} dist(x_n, F) = 0, \quad \lim_{n\to\infty} dist(x_n, A_n) = 0.$$

3. Proof of Theorem 1.1

In this section, we aim to prove the multiplicity of normalized solutions to equations (1.1)-(1.2). To start, we recall the definition of a genus. Let X be a Banach space, and let A

be a subset of X. The set A is called symmetric if, for every $u \in A$, it holds that $-u \in A$. We define the set

$$\Sigma := \{A \subset X \setminus \{0\} : A \text{ is closed and symmetric with respect to the origin}\}.$$

For each $A \in \Sigma$, we define the genus $\mathcal{G}(A)$ as follows:

$$\mathcal{G}(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ \inf\{k \in \mathbb{N} : \exists \text{ an odd } \varphi \in C(A, \mathbb{R}^k \setminus \{0\})\}, \\ +\infty, & \text{if no such odd map exists.} \end{cases}$$

Additionally, we denote $\Sigma_k = \{A \in \Sigma : \mathcal{G}(A) \geq k\}$.

To address the issue of compactness loss in the (PS) sequences, we need to make use of the concentration-compactness principle.

Lemma 3.1. *([50], [51]) Let $\{u_n\}$ be a bounded sequence in $D^{1,p}(\mathbb{R}^3)$ that converges weakly and a.e. to some function $u \in D^{1,p}(\mathbb{R}^3)$. Then, the sequences $|\nabla u_n|^p$ and $|u_n|^{p^*}$ converge in the sense of measures, with $|\nabla u_n|^p \rightharpoonup \omega$ and $|u_n|^{p^*} \rightharpoonup \zeta$. Moreover, there exists a countable set J, a family of points $\{z_j\}_{j\in J} \subset \mathbb{R}^3$, and families of positive numbers $\{\zeta_j\}_{j\in J}$ and $\{\omega_j\}_{j\in J}$ such that the following relations hold:*

$$\omega \geq |\nabla u|^p + \sum_{j\in J} \omega_j \delta_{z_j}, \tag{3.1}$$

$$\zeta = |u|^{p^*} + \sum_{j\in J} \zeta_j \delta_{z_j}, \tag{3.2}$$

and

$$\omega_j \geq S \zeta_j^{\frac{p}{p^*}}, \tag{3.3}$$

where δ_{z_j} denotes the Dirac delta mass at the point $z_j \in \mathbb{R}^3$.

Lemma 3.2. *([50],[51]) Let $\{u_n\} \subset D^{1,p}(\mathbb{R}^3)$ be a sequence as in Lemma 3.1, and define the quantities*

$$\omega_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |\nabla u_n|^p dx, \quad \zeta_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |u_n|^{p^*} dx.$$

Then, the following inequalities hold:

$$\omega_\infty \geq S \zeta_\infty^{\frac{p}{p^*}}, \tag{3.4}$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^p dx = \int_{\mathbb{R}^3} d\omega + \omega_\infty, \tag{3.5}$$

and

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{p^*}dx=\int_{\mathbb{R}^3}d\zeta+\zeta_\infty. \tag{3.6}$$

For $u\in S_r(a)$, based on Lemma 2.2 and the Sobolev inequality, we have that

$$\begin{aligned}I_\mu(u)&=\frac{1}{p}\|\nabla u\|_p^p+\frac{\gamma}{2p}\int_{\mathbb{R}^3}\phi_u|u|^p\,dx-\frac{\mu}{q}\|u\|_q^q-\frac{1}{p^*}\|u\|_{p^*}^{p^*}\\&\ge\frac{1}{p}\|\nabla u\|_p^p-\frac{\mu}{q}C(q)\,a^{q(1-\delta_q)}\|\nabla u\|_p^{q\delta_q}-\frac{1}{p^*}S^{-\frac{p^*}{p}}\|\nabla u\|_p^{p^*}\\&:=g\left(\|\nabla u\|_p\right),\end{aligned} \tag{3.7}$$

where

$$g(r)=\frac{1}{p}r^p-\frac{\mu}{q}C(q)a^{q(1-\delta_q)}r^{q\delta_q}-\frac{1}{p^*}S^{-\frac{p^*}{p}}r^{p^*}.$$

Recall that $p<q<\bar{p}$, and we have $q\delta_q<p$. There exists a constant $\beta>0$, such that if $\mu a^{q(1-\delta_q)}\le\beta$, the function g attains its positive local maximum. More specifically, there exist two constants $0<R_1<R_2<+\infty$, such that

$$g(r)>0,\quad\forall r\in(R_1,R_2);\qquad g(r)<0,\quad\forall r\in(0,R_1)\cup(R_2,+\infty).$$

Let $\tau:\mathbb{R}^+\to[0,1]$ be a nonincreasing, smooth function satisfying

$$\tau(r)=\begin{cases}1, & \text{if } r\in[0,R_1],\\ 0, & \text{if } r\in[R_2,+\infty).\end{cases}$$

In the sequel, we consider the truncated functional

$$I_{\mu,\tau}(u)=\frac{1}{p}\|\nabla u\|_p^p+\frac{\gamma}{2p}\int_{\mathbb{R}^3}\phi_u|u|^p\,dx-\frac{\mu}{q}\|u\|_q^q-\frac{\tau\left(\|u\|_p\right)}{p^*}\|u\|_{p^*}^{p^*}.$$

For $u\in S_r(a)$, again using Lemma 2.2 and the Sobolev inequality, we observe that

$$\begin{aligned}I_{\mu,\tau}(u)&\ge\frac{1}{p}\|\nabla u\|_p^p-\frac{\mu}{q}C(q)a^{q(1-\delta_q)}\|\nabla u\|_p^{q\delta_q}-\frac{\tau\left(\|\nabla u\|_p\right)}{p^*}S^{-\frac{p^*}{p}}\|\nabla u\|_p^{p^*}\\&:=\widetilde{g}\left(\|\nabla u\|_p\right)\end{aligned}$$

where

$$\widetilde{g}(r)=\frac{1}{p}r^p-\frac{\mu}{q}C(q)a^{q(1-\delta_q)}r^{q\delta_q}-\frac{\tau(r)}{p^*}S^{-\frac{p^*}{p}}r^{p^*}.$$

By the definition of $\tau(\cdot)$, when $a \in \left(0, \left(\frac{\beta}{\mu}\right)^{\frac{1}{q(1-\delta_q)}}\right)$, it follows that

$$\widetilde{g}(r) < 0, \quad \forall r \in (0, R_1); \quad \widetilde{g}(r) > 0, \quad \forall r \in (R_1, +\infty).$$

For the remainder of the discussion, we will always assume that

$$a \in \left(0, \left(\frac{\beta}{\mu}\right)^{\frac{1}{q(1-\delta_q)}}\right).$$

Without loss of generality, we assume the following conditions hold:

$$\frac{1}{p} r^p - \frac{1}{p^*} S^{-\frac{p^*}{p}} r^{p^*} \geq 0, \quad \forall r \in [0, R_1] \tag{3.8}$$

and

$$R_1 < S^{\frac{3}{p^2}}. \tag{3.9}$$

Lemma 3.3. *The functional $I_{\mu,\tau}$ possesses the following properties:*

(i) $I_{\mu,\tau} \in C^1(E, \mathbb{R})$;
(ii) $I_{\mu,\tau}$ *is coercive and bounded from below on* $S_r(a)$*. Furthermore, if* $I_{\mu,\tau}(u) \leq 0$*, then* $\|\nabla u\|_p \leq R_1$ *and* $I_{\mu,\tau}(u) = I_\mu(u)$;
(iii) The functional $I_{\mu,\tau}$ *restricted to* $S_r(a)$ *satisfies the* $(PS)_c$ *condition for all* $c < 0$*, provided that* $\mu > \mu_1^* > 0$ *is sufficiently large.*

Proof. The conclusions of items *(i)* and *(ii)* can be derived using standard arguments. To prove item *(iii)*, let $\{u_n\}$ be a $(PS)_c$ sequence of $I_{\mu,\tau}$ restricted to $S_r(a)$ with $c < 0$. From item *(ii)*, we know that $\|\nabla u_n\|_p \leq R_1$ for large n, implying that $\{u_n\}$ is a $(PS)_c$ sequence of $I_\mu\big|_{S_r(a)}$ with $c < 0$. This gives us the following:

$$I_\mu(u_n) \to c < 0 \quad \text{and} \quad \|I'_\mu\big|_{S_r(a)}(u_n)\| \to 0 \quad \text{as } n \to \infty.$$

Thus, the sequence $\{u_n\}$ is bounded in E. Therefore, up to a subsequence, there exists a function $u \in E$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in} \quad E, \\ u_n \to u & \text{in} \quad L^q(\mathbb{R}^3), \quad \forall q \in (p, p^*), \\ u_n \to u & \text{a.e. in} \quad \mathbb{R}^3. \end{cases}$$

From the facts $p < q < \bar{p} < p^*$ and using Lemma 2.4, we obtain the following limits:

$$\lim_{n\to\infty} \|u_n\|_q^q = \|u\|_q^q, \quad \lim_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p \, dx = \int_{\mathbb{R}^3} \phi_u |u|^p \, dx.$$

Moreover, we claim that $u \not\equiv 0$. To show this, assume by contradiction that $u \equiv 0$. In this case, we would have $\lim_{n\to\infty} \|u_n\|_q^q = 0$. From (3.8) and the definition of $I_{\mu,\tau}$, we deduce the following:

$$\begin{aligned} 0 > c &= \lim_{n\to\infty} I_{\mu,\tau}(u_n) = \lim_{n\to\infty} I_\mu(u_n) \\ &= \lim_{n\to\infty} \left[\frac{1}{p}\|\nabla u_n\|_p^p + \frac{\gamma}{2p}\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p \, dx - \frac{\mu}{q}\|u_n\|_q^q - \frac{1}{p^*}\|u_n\|_{p^*}^{p^*} \right] \\ &\geq \lim_{n\to\infty} \left[\frac{1}{p}\|\nabla u_n\|_p^p - \frac{\mu}{q}\|u_n\|_q^q - \frac{1}{p^*} S^{-\frac{p^*}{p}} \|\nabla u_n\|_p^{p^*} \right] \\ &\geq -\frac{\mu}{q} \lim_{n\to\infty} \|u_n\|_q^q = 0, \end{aligned}$$

which leads to a contradiction. Therefore, $u \not\equiv 0$.

Additionally, applying the Lagrange multiplier rule, there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\|I'_\mu(u_n) - \lambda_n \Phi'(u_n)\| \to 0, \quad \text{as } n \to \infty.$$

Hence, we have that

$$-\Delta_p u_n + \gamma\phi_{u_n} |u_n|^{p-2} u_n - \mu|u_n|^{q-2}u_n - |u_n|^{p^*-2}u_n = \lambda_n |u_n|^{p-2} u_n + o_n(1) \text{ in } E_r^{-1}, \tag{3.10}$$

where E_r^{-1} is the dual space of E_r. Thus, we have for $\varphi \in E_r$, that

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi \, dx + \gamma\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{p-2} u_n\varphi \, dx - \mu\int_{\mathbb{R}^3} |u_n|^{q-2}u_n\varphi \, dx \\ &-\int_{\mathbb{R}^3} |u_n|^{p^*-2} u_n\varphi dx = \lambda_n \int_{\mathbb{R}^3} |u_n|^{p^*-2} u_n\varphi \, dx + o_n(1) \end{aligned} . \tag{3.11}$$

and if we choose $\varphi = u_n$, we get

$$\|\nabla u_n\|_p^p + \gamma\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p \, dx - \mu\|u_n\|_q^q - \|u_n\|_{p^*}^{p^*} = \lambda_n\|u_n\|_p^p + o_n(1). \tag{3.12}$$

From (3.12) and the boundedness of $\{u_n\}$ in $D^{1,p}(\mathbb{R}^3)$, we can infer that the sequence $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Therefore, we can assume, without loss of generality, that $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{R}$, up to a subsequence. Consequently, using (3.11), we deduce that u satisfies the following equation:

$$-\Delta_p u + \gamma\phi_u|u|^{p-2}u - \mu|u|^{q-2}u - |u|^{p^*-2}u = \lambda|u|^{p-2}u. \tag{3.13}$$

Indeed, for any $\varphi \in E_r$, it follows by $u_n \rightharpoonup u$ in E_r and $\lambda_n \to \lambda$, that

$$\int_{\mathbb{R}^3} |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx \to \int_{\mathbb{R}^3} |\nabla u|^{p-2}\nabla u \nabla\varphi dx;$$

and

$$\lambda_n \int_{\mathbb{R}^3} |u_n|^{p-2} u_n\varphi dx \to \lambda \int_{\mathbb{R}^3} |u|^{p-2} u\varphi dx,$$

as $n \to \infty$. Since $\{|u_n|^{p^*-2}u_n\}$ is bounded in $L^{\frac{p^*}{p^*-1}}(\mathbb{R}^3)$, and $\{|u_n|^{q-2}u_n\}$ is bounded in $L^{\frac{p^*}{q-1}}(\mathbb{R}^3)$, and $u_n \to u$ a.e. in $\mathbb{R}^3$ we obtain that

$$|u_n|^{p^*-2}u_n \rightharpoonup |u|^{p^*-2}u \text{ in } L^{\frac{p^*}{p^*-1}}(\mathbb{R}^3),\ \textit{and}\ |u_n|^{q-2}u_n \rightharpoonup |u|^{q-2}u \text{ in } L^{\frac{p^*}{p^*-q+1}}(\mathbb{R}^3),$$

and so,

$$\int_{\mathbb{R}^3} |u_n|^{p^*-2}u_n\varphi dx \to \int_{\mathbb{R}^3} |u|^{p^*-2}u\varphi dx \textit{ and } \int_{\mathbb{R}^3} |u_n|^{q-2}u_n\varphi dx \to \int_{\mathbb{R}^3} |u|^{q-2}u\varphi dx,$$

as $n \to \infty$. Recall from Lemma 2.4 that

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{p-2} u_n\varphi\, dx \to \int_{\mathbb{R}^3} \phi_u |u|^{p-2} u\varphi\, dx, \quad \forall\varphi \in E_r.$$

Thus, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla u|^{p-2}\nabla u\nabla\varphi\, dx + \gamma\int_{\mathbb{R}^3} \phi_u |u|^{p-2} u\varphi\, dx - \mu\int_{\mathbb{R}^3} |u|^{q-2}u\varphi\, dx \\ &- \int_{\mathbb{R}^3} |u|^{p^*-2} u\varphi dx = \lambda\int_{\mathbb{R}^3} |u|^{p^*-2} u\varphi\, dx \end{aligned}. \tag{3.14}$$

Therefore, u solves equation (3.13).

In the sequel, by the concentration-compactness principle, we can prove that

$$\int_{\mathbb{R}^3} |u_n|^{p^*} dx \to \int_{\mathbb{R}^3} |u|^{p^*} dx. \tag{3.15}$$

In fact, since $\|\nabla u_n\|_p \le R_1$ for sufficiently large n, by Lemma 3.1, there exist two positive measures, ζ and ω in $\mathcal{M}(\mathbb{R}^3)$, such that

$$|\nabla u_n|^p \rightharpoonup \omega, \quad |u_n|^{p^*} \rightharpoonup \zeta \text{ in } \mathcal{M}(\mathbb{R}^3) \tag{3.16}$$

as $n \to \infty$. Then, by Lemma 3.1, either $u_n \to u$ in $L^{p^*}_{\mathrm{loc}}(\mathbb{R}^3)$, or there exists a (at most countable) set of distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^3$ and positive numbers $\{\zeta_j\}_{j\in J}$ such that

$$\zeta = |u|^{p^*} + \sum_{j\in J} \zeta_j \delta_{x_j}.$$

Moreover, there exists a (at most countable) set $J \subset \mathbb{N}$, a corresponding set of distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^3$, and two sets of positive numbers $\{\zeta_j\}_{j\in J}$ and $\{\omega_j\}_{j\in J}$ such that items (3.1)-(3.3) hold. Now, assume that $J \neq \emptyset$. We divide the proof into three steps.

Step 1. We now prove that $\omega_j = \zeta_j$, where ω_j and ζ_j are the measures from Lemma 3.1.

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with $\varphi \in [0, 1]$, such that $\varphi \equiv 1$ in $B_{1/2}(0)$, and $\varphi \equiv 0$ in $\mathbb{R}^3 \setminus B_1(0)$. For any $\rho > 0$, define

$$\varphi_\rho(x) := \varphi\left(\frac{x - x_j}{\rho}\right) = \begin{cases} 1, & \text{if } |x - x_j| \le \frac{1}{2}\rho, \\ 0, & \text{if } |x - x_j| \ge \rho. \end{cases}$$

By the boundedness of $\{u_n\}$ in E_r, we know that $\{u_n\varphi_\rho\}$ is also bounded in E_r. Hence, it follows that

$$\begin{aligned} o_n(1) &= \left\langle I'_\mu(u_n), u_n\varphi \right\rangle \\ &= \int_{\mathbb{R}^3} |\nabla u_n|^{p-2} \nabla u_n \nabla \left(u_n\varphi_\rho\right) dx + \gamma \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p \varphi_\rho dx \\ &- \mu \int_{\mathbb{R}^3} |u_n|^q \varphi_\rho dx - \int_{\mathbb{R}^3} |u_n|^{p^*} \varphi_\rho dx. \end{aligned} \tag{3.17}$$

It is easy to check that

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u_n|^{p-2} \nabla u_n \nabla \left(u_n\varphi_\rho\right) dx &= \int_{\mathbb{R}^3} |\nabla u_n|^p \varphi_\rho dx + \int_{\mathbb{R}^3} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla\varphi_\rho dx \\ &:= T_1 + T_2 \end{aligned} \tag{3.18}$$

where

$$T_1 = \int_{\mathbb{R}^3} |\nabla u_n|^p \varphi_\rho dx, \ T_2 = \int_{\mathbb{R}^3} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla\varphi_\rho dx.$$

For T_1, by (3.16), we obtain

$$\lim_{\rho\to 0}\lim_{n\to\infty} T_1 = \lim_{\rho\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^p \varphi_\rho dx = \lim_{\rho\to 0} \int_{\mathbb{R}^3} \varphi_\rho \, d\omega = \omega\left(\{x_j\}\right) = \omega_j. \tag{3.19}$$

Using the Hölder inequality, we obtain the following expression:

$$\begin{aligned}\lim_{\rho\to 0}\lim_{n\to\infty} T_2 &= \lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3} u_n |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi_\rho dx\\
&\le \lim_{\rho\to 0}\lim_{n\to\infty}\left(\int_{\mathbb{R}^3}\left|u_n\nabla\varphi_\rho\right|^p dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^3}|\nabla u_n|^p dx\right)^{\frac{p-1}{p}}\\
&\le C\lim_{\rho\to 0}\lim_{n\to\infty}\left(\int_{\mathbb{R}^3}|u_n|^p\left|\nabla\varphi_\rho\right|^p dx\right)^{\frac{1}{p}}\\
&\le C\lim_{\rho\to 0}\lim_{n\to\infty}\left(\int_{\mathbb{R}^3}|u_n|^{p^*}\right)^{\frac{1}{p^*}}\left(\int_{\mathbb{R}^3}\left|\nabla\varphi_\rho\right|^{\frac{pp^*}{p^*-p}}\right)^{\frac{p^*-p}{pp^*}} = 0.\end{aligned}$$

So we have

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^{p-2}\nabla u_n\nabla\left(u_n\varphi_\rho\right)dx = \omega\left(\{x_j\}\right) = w_j.$$

Again by (3.16), we have

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^{p^*}\varphi_\rho dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3}\varphi_\rho d\zeta = \zeta\left(\{x_j\}\right) = \zeta_j. \tag{3.20}$$

By the definition of φ_ρ, and the absolute continuity of the Lebesgue integral, one has that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^q\varphi_\rho\, dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3}|u|^q\varphi_\rho\, dx = \lim_{\rho\to 0}\int_{|x-x_j|\le\rho}|u|^q\varphi_\rho dx = 0. \tag{3.21}$$

Thus, by Proposition 2.1 and Lemma 2.8, we have

$$\begin{aligned}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\varphi_\rho dx &\le C\left(\int_{\mathbb{R}^3}|u_n|^{\frac{6p}{5}}dx\right)^{\frac{5}{6}}\left(\int_{\mathbb{R}^3}|u_n^p\varphi_\rho|^{\frac{6}{5}}\varphi dx\right)^{\frac{5}{6}}\\
&\le C\|u_n\|^p_{\frac{6}{5}p}\left(\int_{\mathbb{R}^3}|u_n|^{\frac{6p}{5}}\left|\varphi_\rho\right|^{\frac{6}{5}}dx\right)^{\frac{5}{6}}\\
&\le C_1\left(\int_{\mathbb{R}^3}|u_n|^{\frac{6p}{5}}\varphi_\rho dx\right)^{\frac{5}{6}}.\end{aligned} \tag{3.22}$$

Therefore,

$$
\begin{aligned}
\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\varphi_\rho dx &\le \lim_{\rho\to 0}\lim_{n\to\infty} C_1\left(\int_{\mathbb{R}^3}|u_n|^{\frac{6p}{5}}\varphi_\rho dx\right)^{\frac{5}{6}}\\
&= \lim_{\rho\to 0} C_1\left(\int_{\mathbb{R}^3}|u|^{\frac{6p}{5}}\varphi_\rho dx\right)^{\frac{5}{6}} \\
&= \lim_{\rho\to 0} C_1\left(\int_{|x-x_j|\le\rho}|u|^{\frac{6p}{5}}\varphi_\rho dx\right)^{\frac{5}{6}} = 0.
\end{aligned}
\tag{3.23}
$$

In summary, by combining (3.17)-(3.19) and (3.21), taking the limit as $\rho\to\infty$, and subsequently taking the limit as $n\to\infty$, we conclude that

$$\omega_j=\zeta_j.$$

Step 2. We now prove that $\omega_\infty=\zeta_\infty$, where ω_∞ and ζ_∞ are defined in Lemma 3.2. Let $\psi\in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with $\psi\in[0,1]$, $\psi\equiv 0$ in $B_{1/2}(0)$, and $\psi\equiv 1$ in $\mathbb{R}^3\setminus B_1(0)$. For any $R>0$, define

$$\psi_R(x):=\psi\left(\frac{x}{R}\right)=\begin{cases}0, & \text{if } |x|\le\frac{1}{2}R,\\ 1, & \text{if } |x|\ge R.\end{cases}$$

Using the boundedness of $\{u_n\}$ and $\{u_n\psi_R\}$ in E_r, we have

$$
\begin{aligned}
o_n(1)&=\left\langle I'_\mu(u_n),u_n\psi_R\right\rangle\\
&=\int_{\mathbb{R}^3}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n\psi_R)dx+\gamma\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\psi_R dx\\
&-\mu\int_{\mathbb{R}^3}|u_n|^q\psi_R dx-\int_{\mathbb{R}^3}|u_n|^{p^*}\psi_R dx.
\end{aligned}
\tag{3.24}
$$

It is easy to derive that

$$
\begin{aligned}
\int_{\mathbb{R}^3}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n\psi_R)dx&=\int_{\mathbb{R}^3}|\nabla u_n|^p\psi_R dx+\int_{\mathbb{R}^3}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_R dx,\\
&:=T_3+T_4
\end{aligned}
$$

where

$$T_3 = \int_{\mathbb{R}^3} |\nabla u_n|^p \, \psi_R dx, \;\; T_4 = \int_{\mathbb{R}^3} u_n \, |\nabla u_n|^{p-2} \, \nabla u_n \nabla \psi_R dx.$$

For T_3, by (3.16) and Lemma 3.2, we infer to

$$\lim_{R\to\infty}\lim_{n\to\infty} T_3 = \lim_{R\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^p \, \psi_R dx = \omega_\infty.$$

By virtue of Hölder's inequality, we get

$$\begin{aligned}\lim_{R\to\infty}\lim_{n\to\infty} T_4 &= \lim_{R\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^3} u_n \, |\nabla u_n|^{p-2} \, \nabla u_n \nabla \psi_R dx \\ &\le \lim_{R\to\infty}\lim_{n\to\infty} \left(\int_{\mathbb{R}^3} |u_n \nabla \psi_R|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^3} |\nabla u_n|^p dx \right)^{\frac{p-1}{p}} \\ &\le C \lim_{R\to\infty}\lim_{n\to\infty} \left(\int_{\mathbb{R}^3} |u_n|^p \, |\nabla \psi_R|^p dx \right)^{\frac{1}{p}} \\ &\le C \lim_{R\to\infty}\lim_{n\to\infty} \left(\int_{\mathbb{R}^3} |u_n|^{p^*} \right)^{\frac{1}{p^*}} \left(\int_{\mathbb{R}^3} |\nabla \psi_R|^{\frac{pp^*}{p^*-p}} \right)^{\frac{p^*-p}{pp^*}} = 0.\end{aligned}$$

Hence,

$$\lim_{R\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^{p-2} \, \nabla u_n \nabla \, (u_n \psi_R) dx = \omega_\infty.$$

By Lemma 3.2, we have

$$\lim_{R\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^{p^*} \, \psi_R dx = \zeta_\infty. \tag{3.25}$$

Analogous before, we infer that

$$\lim_{R\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^q \, \psi_R \, dx = \lim_{R\to\infty} \int_{\mathbb{R}^3} |u|^q \, \psi_R \, dx = \lim_{R\to\infty} \int_{|x|>\frac{1}{2}R} |u_n|^q \, \psi_R dx = 0 \tag{3.26}$$

Moreover, we can obtain

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\psi_R dx \le \lim_{R\to\infty}\lim_{n\to\infty} C_1\left(\int_{\mathbb{R}^3}|u_n|^{\frac{6p}{5}}\psi_R dx\right)^{\frac{5}{6}}$$
$$= \lim_{R\to\infty} C_1\left(\int_{\mathbb{R}^3}|u|^{\frac{6p}{5}}\psi_R dx\right)^{\frac{5}{6}} \tag{3.27}$$
$$= \lim_{R\to\infty} C_1\left(\int_{|x|\ge\frac{R}{2}}|u|^{\frac{6p}{5}}\psi_R dx\right)^{\frac{5}{6}} = 0$$

Summing up, from (3.24)-(3.27), taking the limit as $n\to\infty$, and then the limit as $R\to\infty$, we have

$$\omega_\infty = \zeta_\infty.$$

Step 3. We claim that $\zeta_j = 0$ for all $j\in J$ and $\zeta_\infty = 0$.

Assume, by contradiction, that there exists $j_0\in J$ such that $\zeta_{j_0} > 0$ or $\zeta_\infty > 0$. From Step 1, Step 2, and Lemmas 3.1 and 3.2, it follows that

$$\zeta_{j_0} \le (S^{-1}\omega_{j_0})^{\frac{p^*}{p}} = (S^{-1}\zeta_{j_0})^{\frac{p^*}{p}}, \tag{3.28}$$

and

$$\zeta_\infty \le \left(S^{-1}\omega_\infty\right)^{\frac{p^*}{p}} = \left(S^{-1}\zeta_\infty\right)^{\frac{p^*}{p}}, \tag{3.29}$$

Consequently, we get $\zeta_{j_0} \ge S^{3/p}$ or $\zeta_\infty \ge S^{3/p}$. If the former case occurs, we have

$$R_1^p \ge \lim_{n\to\infty}\|\nabla u_n\|_p^p \ge S\lim_{n\to\infty}\left(\int_{\mathbb{R}^3}|u_n|^{p^*}dx\right)^{\frac{p}{p^*}}$$
$$\ge S\lim_{n\to\infty}\left(\int_{\mathbb{R}^3}|u_n|^{p^*}\varphi_\rho dx\right)^{\frac{p}{p^*}} = S\left(\int_{\mathbb{R}^3}\varphi_\rho d\zeta\right)^{\frac{p}{p^*}} \tag{3.30}$$

Taking the limit $\rho\to 0$ in the last inequality, we get

$$R_1^p \ge S\left(\zeta_{j_0}\right)^{\frac{p}{p^*}} = S\left(S^{\frac{3}{p}}\right)^{\frac{p}{p^*}} = S^{\frac{3}{p}},$$

which contradicts (3.9). If the last case happens, we have

$$
\begin{aligned}
R_1^p \geq \lim_{n\to\infty} \|\nabla u_n\|_p^p &\geq S \lim_{n\to\infty} \left(\int_{\mathbb{R}^3} |u_n|^{p^*} dx \right)^{\frac{p}{p^*}} \\
&\geq S \lim_{n\to\infty} \left(\int_{\mathbb{R}^3} |u_n|^{p^*} \psi_R dx \right)^{\frac{p}{p^*}} \\
&\geq S \lim_{n\to\infty} \left(\int_{|x|\geq R} |u_n|^{p^*} dx \right)^{\frac{p}{p^*}} .
\end{aligned} \tag{3.31}
$$

Taking the limits $n \to \infty$ and $R \to \infty$ in (3.31), we infer that

$$
R_1^p \geq S(\zeta_\infty)^{\frac{p}{p^*}} \geq S(S^{3/p})^{\frac{p}{p^*}} = S^{3/p}.
$$

This leads to a contradiction with (3.9). Therefore, we conclude that $\zeta_j = 0$ for all $j \in J$ and $\zeta_\infty = 0$. As a result, by Lemma 3.1, we obtain that $u_n \to u$ in $L^{p^*}_{\mathrm{loc}}(\mathbb{R}^3)$; and by Lemma 3.2, we know that $u_n \to u$ in $L^{p^*}(\mathbb{R}^3)$.

Next, we prove that there exists a constant $\mu_1^* > 0$, independent of $n \in \mathbb{N}$, such that if $\mu > \mu_1^*$, the Lagrange multiplier $\lambda < 0$ in (3.13). Indeed, observe that $\{u_n\} \subset S_r(a)$ and $\|\nabla u_n\|_p \leq R_1$, as shown in the previous part of this proof, along with (2.9)-(2.10), which imply that there exists a constant $Q_1 > 0$, independent of n, such that

$$
Q_1 \leq \|u_n\|_q^q \leq C(q) \|\nabla u_n\|_p^{q\delta_q} \|u_n\|_p^{q(1-\delta_q)} \leq C(q) R_1^{q\delta_q} a^{q(1-\delta_q)}. \tag{3.32}
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p dx &\leq \tilde{C} \|u_n\|_{\frac{6}{5}p}^{2p} \\
&\leq \tilde{C} \left[C\left(\frac{6p}{5}\right)\right]^{\frac{5}{3}} \|\nabla u_n\|_p \|u_n\|_p^{2p-1} \\
&\leq \tilde{C} \left[C\left(\frac{6p}{5}\right)\right]^{\frac{5}{3}} R_1 a^{2p-1} \\
&:= Q_2,
\end{aligned} \tag{3.33}
$$

where $Q_2 = Q_2(R_1, a) > 0$. We define the constant

$$
\mu_1^* = \frac{\gamma(2p-1) Q_2}{2p\left[1-\delta_q\right] Q_1}. \tag{3.34}
$$

By (3.32)-(3.34), we have

$$\mu_1^* > \lim_{n\to\infty} \frac{\gamma\,(2p-1)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx}{2p\left(1-\delta_q\right)\|u_n\|_q^q} = \frac{\gamma\,(2p-1)\int_{\mathbb{R}^3}\phi_u|u|^p\,dx}{2p\left(1-\delta_q\right)\|u\|_q^q} > 0. \tag{3.35}$$

Recall by (3.13) and its Pohozaev identity $P_\mu(u)=0$, we infer that

$$\lambda\|u\|_p^p = \frac{2p-1}{2p}\gamma\int_{\mathbb{R}^3}\phi_u|u|^p\,dx + \mu\left(\delta_q-1\right)\|u\|_q^q. \tag{3.36}$$

Now, if $\mu > \mu_1^*$, we conclude from (3.35), that

$$\mu > \frac{\gamma\,(2p-1)\int_{\mathbb{R}^3}\phi_u|u|^p\,dx}{2p\left(1-\delta_q\right)\|u\|_q^q}$$

Thus, from (3.36), we infer to $\lim_{n\to\infty}\lambda_n=\lambda<0$. Hence, taking into account (3.12), we derive

$$\begin{aligned}
&\lim_{n\to\infty}\left[\|\nabla u_n\|_p^p + \gamma\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx - \lambda\|u_n\|_q^q\right]\\
&=\lim_{n\to\infty}\left[\mu\|u_n\|_q^q + \|u_n\|_{p^*}^{p^*} + o_n(1)\right]\\
&=\mu\|u\|_q^q + \|u\|_{p^*}^{p^*} = \|\nabla u\|_p^p + \gamma\int_{\mathbb{R}^3}\phi_u|u|^p dx - \lambda\|u\|_p^p.
\end{aligned} \tag{3.37}$$

Since $\lambda<0$ for $\mu>\mu_1^*$ large, we obtain by Fatou's Lemma,

$$\begin{aligned}
&\lim_{n\to\infty}\|\nabla u_n\|_p^p + \gamma\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p dx - \lambda\|u_n\|_p^p\\
&\geq \|\nabla u\|_p^p + \gamma\int_{\mathbb{R}^3}\phi_u|u|^p dx + \liminf_{n\to\infty}\left(-\lambda\|u_n\|_p^p\right).
\end{aligned} \tag{3.38}$$

and from (3.37)-(3.38), one has

$$-\lambda\|u\|_p^p \geq \liminf_{n\to\infty}(-\lambda\|u_n\|_p^p). \tag{3.39}$$

But by Fatou's Lemma, we see that

$$\liminf_{n\to\infty}(-\lambda\|u_n\|_p^p) \geq -\lambda\|u\|_p^p. \tag{3.40}$$

Combining (3.39) with (3.40) we get

$$\lim_{n\to\infty}(-\lambda\|u_n\|_p^p) = -\lambda\|u\|_p^p;$$

that is,

$$\lim_{n\to\infty} \|u_n\|_p^p = \|u\|_p^p.$$

Thus, by (3.37) we have

$$\lim_{n\to\infty} \|\nabla u_n\|_p^p = \|\nabla u\|_p^p.$$

Therefore, $u_n \to u$ in E_r and $\|u\|_p = a$. The proof is complete. □

For $\varepsilon > 0$, we define the set

$$I_{\mu,\tau}^{-\varepsilon} = \{u \in E_r \cap S(a) : I_{\mu,\tau}(u) \le -\varepsilon\} \subset E_r.$$

Since $I_{\mu,\tau}(u)$ is continuous and even on E_r, it follows that the set $I_{\mu,\tau}^{-\varepsilon}$ is both closed and symmetric.

Lemma 3.4. *For any fixed $k \in \mathbb{N}$, there exist constants $\varepsilon_k := \varepsilon(k) > 0$ and $\mu_k := \mu(k) > 0$ such that, for $0 < \varepsilon \le \varepsilon_k$ and $\mu \ge \mu_k$, the following inequality holds:*

$$\mathcal{G}(I_{\mu,\tau}^{-\varepsilon}) \ge k.$$

The proof of Lemma 3.4 is similar to that of Lemma 3.2 in [3], so we omit it here. In the following, we define the set

$$\Sigma_k := \{\Omega \subset E_r \cap S(a) : \Omega \text{ is closed and symmetric}, \mathcal{G}(\Omega) \ge k\},$$

and, by Lemma 3.3-(ii), we know that

$$c_k := \inf_{\Omega\in\Sigma_k} \sup_{u\in\Omega} I_{\mu,\tau}(u) > -\infty$$

for all $k \in \mathbb{N}$. To prove Theorem 1.1, we introduce the critical value and define

$$K_c := \{u \in E_r \cap S(a) : I'_{\mu,\tau}(u) = 0, I_{\mu,\tau}(u) = c\}.$$

From this, we can derive the following conclusion:

Lemma 3.5. *If $c = c_k = c_{k+1} = \cdots = c_{k+\ell}$, then we have $\mathcal{G}(K_c) \ge \ell + 1$. In particular, $I_{\mu,\tau}(u)$ admits at least $\ell + 1$ nontrivial critical points.*

Proof. For $\varepsilon > 0$, it is straightforward to verify that $I_{\mu,\tau}^{-\varepsilon} \in \Sigma$. For any fixed $k \in \mathbb{N}$, by Lemma 3.4, there exist constants $\varepsilon_k := \varepsilon(k) > 0$ and $\mu_k := \mu(k) > 0$ such that, if $0 < \varepsilon \le \varepsilon_k$ and $\mu \ge \mu_k$, we obtain

$$\mathcal{G}(I_{\mu,\tau}^{-\varepsilon_k}) \ge k.$$

Thus, $I_{\mu,\tau}^{-\varepsilon_k} \in \Sigma_k$, and furthermore,

$$c_k \leq \sup_{u \in I_{\mu,\tau}^{-\varepsilon_k}} I_{\mu,\tau}(u) = -\varepsilon_k < 0.$$

Assume that $0 > c = c_k = c_{k+1} = \cdots = c_{k+\ell}$. Then, by Lemma 3.3-(iii), $I_{\mu,\tau}(u)$ satisfies the $(PS)_c$-condition at the level $c < 0$. Consequently, K_c is a compact set. By Theorem 2.1 in [3], or Theorem 2.1 in [35], we know that the restricted functional $I_{\mu,\tau}|_{S(a)}$ possesses at least $\ell + 1$ nontrivial critical points. □

Proof of Theorem 1.1. Let $\mu \geq \mu^* = \max\{\mu_1^*, \mu_k\}$. From Lemma 3.3-(ii), we observe that the critical points of $I_{\mu,\tau}(u)$ found in Lemma 3.5 are the critical points of I_μ, which completes the proof. □

4. Proof of Theorem 1.2

In this section, we focus on the L^p-supercritical regime: $\bar{p} < q < p^*$, and complete the proof of Theorem 1.2. To proceed with our arguments, we first present some useful lemmas and define a function $I : E_r \to \mathbb{R}$ by $I(u,t) := I_\mu(t * u)$. From Lemma 2.6, we observe that any critical point of $I_\mu|_{S(a)}$ belongs to $\mathcal{P}(a)$. Consequently, the properties of the manifold $\mathcal{P}(a)$ are closely related to the mini-max structure of $I_\mu|_{S(a)}$.

Lemma 4.1. *Let $\bar{p} < q < p^*$, $\mu, \gamma > 0$ and $u \in S(a)$, then*

(i) $\|\nabla(t \star u)\|_p \to 0^+$ and $I_\mu(t \star u) \to 0^+$ if $t \to -\infty$;
(ii) $\|\nabla(t \star u)\|_p \to +\infty$ and $I_\mu(t \star u) \to -\infty$ if $t \to +\infty$.

Proof. Using (2.17), we have

$$\|\nabla(t * u)\|_p^p = e^{pt} \|\nabla u\|_p^p.$$

It follows that

$$\|\nabla(t * u)\|_p \to 0^+ \quad \text{as} \quad t \to -\infty,$$

and

$$\|\nabla(t * u)\|_p \to +\infty \quad \text{as} \quad t \to +\infty.$$

Next, we note that

$$I(u,t) = I_\mu(t * u) = \frac{1}{p} e^{pt} \|\nabla u\|_p^p + \frac{\gamma}{2p} e^t \int_{\mathbb{R}^3} \phi_u |u|^p \, dx - \frac{\mu}{q} e^{q\delta_q t} \|u\|_q^q - \frac{1}{p^*} e^{p^* t} \|u\|_{p^*}^{p^*}.$$

Moreover, since

$$q\delta_q > p \quad \text{if} \quad \bar{p} < q < p^*,$$

we can infer that

$$I_\mu(t*u) \to 0^+ \quad \text{as} \quad t \to -\infty,$$

and

$$I_\mu(t*u) \to -\infty \quad \text{as} \quad t \to +\infty.$$

This completes the proof. □

Lemma 4.2. *Let $\bar{p} < q < p^*$, $\mu, \gamma > 0$. There exist $K = K_a > 0$ and $\tilde{a} > 0$, where*

$$\tilde{a} := \left(\frac{(K_a)^{1-\frac{1}{p}}}{4\gamma \tilde{C} \left[C\left(\frac{6}{5}p\right)\right]^{\frac{5}{3}}} \right)^{\frac{1}{2p-1}},$$

such that for all $0 < a < \tilde{a}$,

$$P_\mu(u) > 0,\, I_\mu(u) > 0 \text{ for all } u \in \mathcal{A}_a \text{ and } 0 < \sup_{u\in\mathcal{A}_a} I_\mu(u) < \inf_{u\in\mathcal{B}_a} I_\mu(u), \tag{4.1}$$

where

$$\mathcal{A}_a := \{u \in S_r(a) : \|\nabla u\|_p^p \le K_a\} \quad \text{and} \quad \mathcal{B}_a := \{u \in S_r(a) : \|\nabla u\|_p^p = pK_a\}.$$

Proof. Suppose $u, v \in S_r(a)$ such that $\|\nabla u\|_p^p \le K_a \|\nabla v\|_p^p = pK_a$. By Proposition 2.1, Lemma 2.2, and Lemma 2.3, we conclude that for $u \in S_r(a)$, the following holds:

$$\begin{aligned} P_\mu(u) &= \|\nabla u\|_p^p + \frac{\gamma}{2p}\int_{\mathbb{R}^3} \phi_u |u|^p\,dx - \mu\delta_q \|u\|_q^q - \|u\|_{p^*}^{p^*} \\ &\ge \|\nabla u\|_p^p - C(q)\,\mu\delta_q a^{q(1-\delta_q)} \|\nabla u\|_p^{q\delta_q} - S^{-\frac{p^*}{p}} \|\nabla u\|_p^{p^*}, \end{aligned}$$

and

$$\begin{aligned} I_\mu(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{\gamma}{2p}\int_{\mathbb{R}^3} \phi_u |u|^p\,dx - \frac{\mu}{q}\|u\|_q^q - \frac{1}{p^*}\|u\|_{p^*}^{p^*} \\ &\ge \frac{1}{p}\|\nabla u\|_p^p - C(q)\,\frac{\mu}{q} a^{q(1-\delta_q)} \|\nabla u\|_p^{q\delta_q} - \frac{1}{p^*} S^{-\frac{p^*}{p}} \|\nabla u\|_p^{p^*}. \end{aligned}$$

Moreover, we have

$$
\begin{aligned}
I_\mu(v)-I_\mu(u) \geq & \frac{1}{p}\|\nabla v\|_p^p-\frac{1}{p}\|\nabla u\|_p^p-\frac{\gamma}{2p}\int\limits_{\mathrm{R}^3}\phi_u\,|u|^p\,dx-\frac{\mu}{q}\|v\|_q^q-\frac{1}{p^*}\|v\|_{p^*}^{p^*} \\
\geq & \frac{1}{p}\|\nabla v\|_p^p-\frac{1}{p}\|\nabla u\|_p^p-\frac{\gamma}{2p}\tilde{C}\left[C\left(\frac{6}{5}p\right)\right]^{\frac{5}{3}}a^{2p-1}\left(K_a\right)^{\frac{1}{p}} \\
& -\frac{\mu}{q}C(q)\,a^{q(1-\delta_q)}\left(pK_a\right)^{\frac{q\delta_q}{p}}-\frac{1}{p^*}S^{-\frac{p^*}{p}}\left(pK_a\right)^{\frac{p^*}{p}} \\
\geq & K_a-\frac{1}{p}K_a-\frac{\gamma}{2p}\tilde{C}\left[C\left(\frac{6}{5}p\right)\right]^{\frac{5}{3}}\left(\frac{\left(K_a\right)^{1-\frac{1}{p}}}{4\gamma\tilde{C}\left[C\left(\frac{6}{5}p\right)\right]^{\frac{5}{3}}}\right)\left(K_a\right)^{\frac{1}{p}} \\
& -\frac{C(q)\,\mu}{q}\left(\frac{\left(K_a\right)^{1-\frac{1}{p}}}{4\gamma\tilde{C}\left[C\left(\frac{6}{5}p\right)\right]^{\frac{5}{3}}}\right)^{\frac{q(1-\delta_q)}{2p-1}}\left(pK_a\right)^{\frac{q\delta_q}{p}}-\frac{1}{p^*}S^{-\frac{p^*}{p}}\left(pK_a\right)^{\frac{p^*}{p}} \\
= & \frac{8p-9}{8p}K_a-\left(\frac{p^{\frac{q\delta_q}{p}}C(q)\,\mu}{q\left(4\gamma\tilde{C}\left[C\left(\frac{6}{5}p\right)\right]^{\frac{5}{3}}\right)^{\frac{q(1-\delta_q)}{2p-1}}}\left(K_a\right)^{\frac{q(p+2)-2p(p+1)}{p(2p-1)}}\right)K_a \\
& -\left(\frac{p^{\frac{p^*}{p}}}{p^*}S^{-\frac{p^*}{p}}\left(K_a\right)^{\frac{p^*-p}{p}}\right)K_a\geq\frac{8p-9}{8p}K_a>0,
\end{aligned}
\tag{4.2}
$$

for $0<a<\tilde{a}$, and we denote by

$$
K_a:=\min\left\{\left(\frac{q\left(4\gamma\tilde{C}\left[C\left(\frac{6}{5}p\right)\right]^{\frac{5}{3}}\right)^{\frac{q(1-\delta_q)}{2p-1}}}{8p^{\frac{p+q\delta_q}{p}}\mu C(q)}\right)^{\frac{p(2p-1)}{q(p+2)-2p(p+1)}},\ \left(\frac{p^*}{8p^{\frac{p^*+p}{p}}}S^{\frac{p^*}{p}}\right)^{\frac{p}{p^*-p}}\right\}. \tag{4.3}
$$

Then we deduce by (4.2) that (4.1) holds. □

Next, we examine the characterizations of the mountain pass levels for $I(u,t)$ and $I_\mu(u)$. Denote the closed set $I_\mu^d:=\{u\in S_r(a): I_\mu(u)\leq d\}$.

Proposition 4.3. *Assume that $\bar{p}<q<p^*$, $\mu,\gamma>0$. Define*

$$
\widetilde{\sigma}_\mu(a):=\inf_{\widetilde{\xi}\in\widetilde{\Gamma}_a}\max_{t\in[0,1]}I(\widetilde{\xi}(t)),
$$

where

$$\widetilde{\Gamma}_a = \{\widetilde{\xi} \in C([0,1], S_r(a) \times \mathbb{R}) : \widetilde{\xi}(0) \in (\mathcal{A}_a, 0),\ \widetilde{\xi}(1) \in (I_\mu^0, 0)\},$$

and

$$\sigma_\mu(a) := \inf_{\xi \in \Gamma_a} \max_{t\in[0,1]} I_\mu(\xi(t)),$$

where

$$\Gamma_a = \{\xi \in C([0,1], S_r(a)) :\ \xi(0) \in \mathcal{A}_a,\ \xi(1) \in I_\mu^0\}.$$

Then we get

$$\widetilde{\sigma}_\mu(a) = \sigma_\mu(a) > 0.$$

Proof. Since $\Gamma_a \times \{0\} \subset \widetilde{\Gamma}_a$, it is easy to know that $\widetilde{\sigma}_\mu(a) \le \sigma_\mu(a)$. Then we only need to verify $\widetilde{\sigma}_\mu(a) \ge \sigma_\mu(a)$. For $\widetilde{\xi}(t) = (\widetilde{\xi}_1(t), \widetilde{\xi}_2(t)) \in \widetilde{\Gamma}_a$, one has,

$$\widetilde{\xi}(0) = (\widetilde{\xi}_1(0), \widetilde{\xi}_2(0)) \in (A_{k_1}, 0) \text{ and } \widetilde{\xi}(1) = (\widetilde{\xi}_1(1), \widetilde{\xi}_2(1)) \in (I_\mu^0, 0).$$

So, set $\xi(t) = (\widetilde{\xi}_2(t) \star \widetilde{\xi}_1(t))$, we have $\xi(t) \in \Gamma_a$, and so,

$$\max_{t\in[0,1]} I(\widetilde{\xi}(t)) = \max_{t\in[0,1]} I_\mu(\widetilde{\xi}_2(t) \star \widetilde{\xi}_1(t)) = \max_{t\in[0,1]} I_\mu(\xi(t)),$$

which implies that $\widetilde{\sigma}_\mu(a) \ge \sigma_\mu(a) > 0$, and the proof is completed. □

Next, we will show the existence of the $(PS)_{\widetilde{\sigma}_\mu(a)}$ sequence for $I(u,t)$ on $S_r(a) \times \mathbb{R}$, which is demonstrated by a standard argument by using Ekeland's variational principle and constructing pseudo-gradient flow (Proposition 2.2 [32]).

Proposition 4.4. *Let $\bar{p} < q < p^*$, $\mu, \gamma > 0$. There exists a Palais-Smale sequence $\{w_n\} \subset S_r(a)$ for $I_\mu|_{S(a)}$ at level $\sigma_\mu(a)$ and $P_\mu(w_n) \to 0$ as $n \to \infty$.*

Proof. Let

$$X = S(a) \times \mathbb{R}, \quad \mathcal{F} = \{\tilde{\xi}([0,1]) : \tilde{\xi} \in \tilde{\Gamma}_a\}, \quad B := (\mathcal{A}_a, 0) \cup (I_\mu^0, 0),$$

$$F := \{(u,t) \in S(a) \times \mathbb{R} \mid I(u,t) \ge \sigma_\mu(a)\}, \quad A = \tilde{\xi}([0,1]), \quad A_n = \tilde{\xi}_n([0,1]) = \xi_n([0,1]) \times \{0\}.$$

We need to verify that $\mathcal{F}$ is a homotopy stable family of compact subsets of X with extended closed boundaries B and F, satisfying the assumptions (1) and (2) in Lemma 2.11. Specifically, for each $\tilde{\xi} \in \tilde{\Gamma}_a$, since $\tilde{\xi}(0) \in (\mathcal{A}_a, 0)$ and $\tilde{\xi}(1) \in (I_\mu^0, 0)$, we have $\tilde{\xi}(0), \tilde{\xi}(1) \in B$. For any set $A \in \mathcal{F}$ and any $\eta \in C([0,1] \times X; X)$ such that $\eta(t,x) = x$ for all $(t,x) \in (\{0\} \times X) \cup ([0,1] \times B)$, it follows that $\eta(1, \tilde{\xi}(0)) = \tilde{\xi}(0)$ and $\eta(1, \tilde{\xi}(1)) = \tilde{\xi}(1)$. Therefore, we obtain $\eta(\{1\} \times A) \in \mathcal{F}$. Combining $(A \cap F) \setminus B \neq \emptyset$, we conclude that the assumptions (1) and (2) in Lemma 2.11 are satisfied.

Thus, there exists a sequence $\{(u_n, t_n)\} \subset S(a) \times \mathbb{R}$ such that as $n \to +\infty$, ... ,

$$(|t_n| + \text{dist}(u_n, \xi_n([0,1])) \to 0, \tag{4.4}$$

$$I(u_n, t_n) \to \sigma_\mu(a), \tag{4.5}$$

$$(I|_{S(a)\times\mathbb{R}})'(u_n, t_n) \to 0. \tag{4.6}$$

Moreover, we can obtain that $I(u_n, t_n) = I(t_n \star u_n, 0)$ and

$$\left\langle \left(I|_{S(a)\times\mathbb{R}}\right)'(u_n, t_n), (\varphi, s)\right\rangle = \left\langle \left(I|_{S(a)\times\mathbb{R}}\right)'(t_n \star u_n, 0), (t_n \star \varphi, s)\right\rangle, \tag{4.7}$$

for $(\varphi, s) \in E \times \mathbb{R}$ with $\int_{\mathbb{R}^3} v_n \varphi = 0$. Recording that $w_n = t_n \star u_n \in S(a)$, we know from (4.5) that

$$I_\mu(w_n) = I(t_n \star u_n, 0) = I(w_n, t_n) \to \sigma_\mu(a), \quad \text{as } n \to \infty.$$

Taking $(0, 1)$ as a test function in (4.7), we deduce from (4.6) that

$$P(w_n) = \left(I_\mu|_{S(a)}\right)'(w_n) \to 0, \quad \text{as } n \to \infty. \quad \square$$

Remark 4.5. From Proposition 4.4, we can say that there exist a sequence $\{u_n\}$, which is a (PS) sequence for I_μ with the level $\sigma_\mu(a)$, that is

$$I_\mu(u_n) \to \sigma_\mu(a) \quad \text{as} \quad n \to +\infty, \tag{4.8}$$

and

$$(I_\mu|S_r(a))'(u_n) \to 0 \quad \text{as} \quad n \to +\infty. \tag{4.9}$$

Lemma 4.6. *The (PS) sequence $\{u_n\}$ discussed in Remark 4.5 is bounded in E_r. Furthermore, assume that $\sigma_\mu(a) < \frac{1}{3}S^{\frac{3}{p}}$ and $\gamma < \gamma_1^*$ for some $\gamma_1^* > 0$. Then, we have*

$$\lim_{n\to\infty} \lambda_n = \lambda < 0.$$

Proof. From Remark 4.5, we see that $I_\mu(u_n)$ is bounded. In fact, by $P_\mu(u_n) \to 0$ as $n \to \infty$, we have

$$|2I_\mu(u_n) + P_\mu(u_n)| \le C,$$

which implies that,

$$2\|\nabla u_n\|_p^p + \frac{p+1}{2p}\gamma \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p \, dx - \frac{\mu\left(p + q\delta_q\right)}{q}\|u\|_q^q - \frac{p+p^*}{p^*}\|u\|_{p^*}^{p^*} \ge -C. \tag{4.10}$$

In view of (4.10) and $I_\mu(u_n) \le C$, we have

$$\frac{p+1}{2p}\gamma\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx+\frac{\mu\left(p+q\delta_q\right)}{q}\|u_n\|_q^q+\frac{p+p^*}{p^*}\|u_n\|_{p^*}^{p^*}\le(2p+1)\,C,$$

which implies that

$$\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx,\ \|u_n\|_q^q,\ \|u_n\|_{p^*}^{p^*},\quad\forall\, n\in\mathbb{N},$$

are all bounded. Thus, $\|\nabla u_n\|_p\le R_2$ for some $R_2>0$, independent of $n\in\mathbb{N}$. Since $\{u_n\}\subset S_r(a)$, we conclude that $\{u_n\}$ is bounded in E_r. Therefore, by passing to a subsequence, we may assume that $u_n\rightharpoonup u$ for some $u\in E_r$, and consequently, $u_n\to u$ in $L^q(\mathbb{R}^3)$ for all $q\in(p,p^*)$. As a result, it follows from Proposition 5.12 [58] that there exists a sequence $\{\lambda_n\}\subset\mathbb{R}$ such that

$$I_\mu'(u_n)-\lambda_n\Phi'(u_n)\to 0\text{ in }E_r^{-1}\text{ as }n\to\infty.$$

That is, we have

$$-\Delta_p u_n+\gamma\phi_{u_n}|u_n|^{p-2}u_n-\mu|u_n|^{q-2}u_n+|u_n|^{p^*-2}u_n=\lambda_n|u_n|^{p-2}u_n+o_n(1)\ \text{ in }E_r^{-1}.\tag{4.11}$$

Similar to the proof of Lemma 3.3, we know that u solves the equation

$$-\Delta_p u+\gamma\phi_u|u|^{p-2}u-\mu|u|^{q-2}u+|u|^{p^*-2}u=\lambda|u|^{p-2}u.\tag{4.12}$$

Moreover, $u\not\equiv 0$. In fact, argue by contradiction that $u\equiv 0$. Then $u_n\to 0$ in $L^q(\mathbb{R}^3)$, $\forall q\in(p,p^*)$, and by $P_\mu(u_n)=o_n(1)$, (2.9) we have

$$\begin{aligned}o_n(1)&=\|\nabla u_n\|_p^p+\frac{\gamma}{2p}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx-\mu\delta_q\|u_n\|_q^q-\|u_n\|_{p^*}^{p^*}\\&=\|\nabla u_n\|_p^p-\|u_n\|_{p^*}^{p^*}+o_n(1).\end{aligned}$$

We may assume that $\lim_{n\to\infty}\|\nabla u_n\|_p^p=\lim_{n\to\infty}\|u_n\|_{p^*}^{p^*}dx=l\ge 0$. Thus, we have

$$\begin{aligned}\sigma_\mu(a)+o_n(1)&=I_\mu(u_n)\\&=\frac{1}{p}\|\nabla u_n\|_p^p+\frac{\gamma}{2p}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx-\frac{\mu}{q}\|u_n\|_q^q-\frac{1}{p^*}\|u_n\|_{p^*}^{p^*}\\&=\frac{1}{p}l-\frac{1}{p^*}l+o_n(1)\\&=\frac{1}{3}l+o_n(1)\end{aligned}\tag{4.13}$$

On the other hand, by the Sobolev inequality (2.11), we have $l\ge Sl^{\frac{p}{p^*}}$. This gives rise to two possible cases: (i) $l=0$; (ii) $l\ge S^{\frac{3}{p}}$.

If $l=0$, then by (4.13), we obtain $I_\mu(u_n)\to 0$, which contradicts $I_\mu(u_n)\to\sigma_\mu(a)>0$. Now, if the second case, $l\ge S^{\frac{3}{p}}$, holds, then by (4.17), we get $I_\mu(u_n)\to\frac{1}{3}l\ge\frac{1}{3}S^{\frac{3}{p}}$, which contradicts $I_\mu(u_n)\to\sigma_\mu(a)<\frac{1}{3}S^{\frac{3}{p}}$. Thus, $u\not\equiv 0$. Moreover, from (4.11) and $P_\mu(u_n)=o_n(1)$, we obtain

$$\lambda_n\|u_n\|_p^p=\frac{2p-1}{2p}\gamma\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx-\mu\left(\delta_q-1\right)\|u_n\|_q^q+o_n(1)\,. \tag{4.14}$$

Since $\{u_n\}\subset S_r(a)$ is bounded in E_r, by Lemma 2.8 and (4.14), we conclude that the sequence $\{\lambda_n\}$ is bounded and that $\lim_{n\to\infty}\lambda_n=\lambda\in\mathbb{R}$. Using an argument similar to that in (3.32) and (3.33), for all $n\in\mathbb{N}$, we have

$$W_1\le\|u_n\|_q^q\le C(q)\,\|\nabla u_n\|_p^{q\delta_q}\|u_n\|_p^{q(1-\delta_q)}\le C(q)\,R_2^{q\delta_q}a^{q(1-\delta_q)}, \tag{4.15}$$

and

$$\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx\le\tilde{C}\|u_n\|_{\frac{6}{5}p}^{2p}\le\tilde{C}\left[C\left(\frac{6}{5}p\right)\right]^{\frac{5}{3}}R_2a^{2p-1}:=W_2, \tag{4.16}$$

where $W_1>0$, $W_2=W_2(R_2,a)>0$. We define the positive constant

$$\gamma_1^*:=\frac{2p\mu(1-\delta_q)W_1}{(2p-1)W_2}. \tag{4.17}$$

Therefore, if $\gamma<\gamma_1^*$, we get

$$\gamma(2p-1)W_2<2p\mu(1-\delta_q)W_1.$$

Hence, by (4.15), (4.16), we see that

$$\frac{2p-1}{2p}\gamma\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx<\left(1-\delta_q\right)\mu\|u_n\|_q^q. \tag{4.18}$$

Taking the limit in (4.17) as $n\to\infty$, and applying Lemmas 2.4, 2.8, we obtain

$$\frac{2p-1}{2p}\gamma\int_{\mathbb{R}^3}\phi_u|u|^p\,dx<\left(1-\delta_q\right)\mu\|u\|_q^q. \tag{4.19}$$

Consequently, passing the limit in (4.14) as $n\to\infty$, and using (4.19), we deduce that

$$\lambda a^p=\frac{2p-1}{2p}\gamma\int_{\mathbb{R}^3}\phi_u|u|^p\,dx-\mu\left(\delta_q-1\right)\|u\|_q^q<0. \tag{4.20}$$

Thus, we have that $\lambda<0$, if $\gamma<\gamma_1^*$ is small. $\square$

Lemma 4.7. *Let* $\frac{1+\sqrt{41}}{4} < p < \sqrt[3]{9}$, $\bar{p} < q < p^*$, $\mu > 0$, *and suppose that* $0 < a < \tilde{a}$. *Then there exists* $\gamma_2^* > 0$ *such that* $\sigma_\mu(a) < \frac{1}{3} S^{\frac{3}{p}}$ *for* $\gamma \in (0, \gamma_2^*)$ *small enough.*

Proof. Let us define the function

$$v_\varepsilon = a \frac{u_\varepsilon}{\|u_\varepsilon\|_p} \in S(a) \cap E_r,$$

and introduce the function

$$\begin{aligned} \Psi_{v_\varepsilon}^\mu(t) := I_\mu(t * v_\varepsilon) = \frac{1}{p} e^{pt} \|\nabla v_\varepsilon\|_p^p + \frac{\gamma}{2p} e^t \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^p \, dx \\ - \frac{\mu}{q} e^{q\delta_q t} \|v_\varepsilon\|_q^q - \frac{1}{p^*} e^{p^* t} \|v_\varepsilon\|_{p^*}^{p^*}. \end{aligned} \tag{4.21}$$

It is easy to see that $\Psi_{v_\varepsilon}^\mu(t) \to 0^+$ as $t \to -\infty$, and $\Psi_{v_\varepsilon}^\mu(t) \to -\infty$ as $t \to +\infty$. Therefore, we conclude that $\Psi_{v_\varepsilon}^\mu$ attains its global positive maximum at some $t_\varepsilon > 0$, and the critical point t_ε is unique. By the condition $(\Psi_{v_\varepsilon}^\mu)'(t_\varepsilon) = P_\mu(t_\varepsilon * v_\varepsilon) = 0$, we have

$$\begin{aligned} e^{p^* t_\varepsilon} \|v_\varepsilon\|_{p^*}^{p^*} &= e^{pt_\varepsilon} \|\nabla v_\varepsilon\|_p^p + \frac{\gamma}{2p} e^{t_\varepsilon} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^p \, dx - \mu \delta_q e^{q\delta_q t_\varepsilon} \|v_\varepsilon\|_q^q \\ &\le e^{pt_\varepsilon} \|\nabla v_\varepsilon\|_p^p + \frac{\gamma}{2p} e^{t_\varepsilon} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^p \, dx \\ &= e^{pt_\varepsilon} \left(\|\nabla v_\varepsilon\|_p^p + \frac{\gamma}{2p} e^{(1-p)t_\varepsilon} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^p \, dx \right) \\ &\le 2 e^{pt_\varepsilon} \max \left\{ \|\nabla v_\varepsilon\|_p^p, \frac{\gamma}{2p} e^{(1-p)t_\varepsilon} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^p \, dx \right\}. \end{aligned} \tag{4.22}$$

Now, we distinguish the following possible cases.

Case 1. $\|\nabla v_\varepsilon\|_p^p > \frac{\gamma}{2p} e^{(1-p)t_\varepsilon} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^p \, dx$. In this case, we have

$$e^{p^* t_\varepsilon} \|v_\varepsilon\|_{p^*}^{p^*} \le 2 e^{pt_\varepsilon} \|\nabla v_\varepsilon\|_p^p,$$

that is,

$$e^{(p^* - p)t_\varepsilon} \le 2 \frac{\|\nabla v_\varepsilon\|_p^p}{\|v_\varepsilon\|_{p^*}^{p^*}}. \tag{4.23}$$

By $(\Psi_{v_\varepsilon}^\mu)'(t_\varepsilon) = 0$, we obtain that

$$
\begin{aligned}
&e^{(p^*-p)t_\varepsilon}\\
&=\frac{\|\nabla v_\varepsilon\|_p^p}{\|v_\varepsilon\|_{p^*}^{p^*}}+\frac{\gamma}{2p}\frac{e^{(1-p)t_\varepsilon}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^p\,dx}{\|v_\varepsilon\|_{p^*}^{p^*}}-\mu\delta_q e^{(q\delta_q-p)t_\varepsilon}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{p^*}^{p^*}}\\
&\geq\frac{\|\nabla v_\varepsilon\|_p^p}{\|v_\varepsilon\|_{p^*}^{p^*}}-\mu\delta_q\left(\frac{2\|\nabla v_\varepsilon\|_p^p}{\|v_\varepsilon\|_{p^*}^{p^*}}\right)^{\frac{q\delta_q-p}{p^*-p}}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{p^*}^{p^*}}\\
&=\frac{\|u_\varepsilon\|_p^{p^*-p}\|\nabla u_\varepsilon\|_p^p}{a^{p^*-p}\|u_\varepsilon\|_{p^*}^{p^*}}-\mu\delta_q\left(\frac{2\|u_\varepsilon\|_p^{p^*-p}\|\nabla u_\varepsilon\|_p^p}{a^{p^*-p}\|u_\varepsilon\|_{p^*}^{p^*}}\right)^{\frac{q\delta_q-p}{p^*-p}}\frac{\|u_\varepsilon\|_p^{p^*-q}\|u_\varepsilon\|_q^q}{a^{p^*-q}\|u_\varepsilon\|_{p^*}^{p^*}} \qquad (4.24)\\
&=\frac{\|u_\varepsilon\|_p^{p^*-p}\left(\|\nabla u_\varepsilon\|_p^p\right)^{\frac{q\delta_q-p}{p^*-p}}}{a^{p^*-p}\|u_\varepsilon\|_{p^*}^{p^*}}\left[\left(\|\nabla u_\varepsilon\|_p^p\right)^{\frac{p^*-q\delta_q}{p^*-p}}-\frac{\mu\delta_q 2^{\frac{q\delta_q-p}{p^*-p}}a^{q(1-\delta_q)}\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_p^{q(1-\delta_q)}\left(\|u_\varepsilon\|_{p^*}^{p^*}\right)^{\frac{q\delta_q-p}{p^*-p}}}\right]\\
&=\frac{\|u_\varepsilon\|_p^{p^*-p}\left(\|\nabla u_\varepsilon\|_p^p\right)^{\frac{q\delta_q-p}{p^*-p}}}{a^{p^*-p}\|u_\varepsilon\|_{p^*}^{p^*}}\left[\left(\|\nabla u_\varepsilon\|_p^p\right)^{\frac{p^*-q\delta_q}{p^*-p}}-\frac{\mu\delta_q 2^{\frac{q\delta_q-p}{p^*-p}}a^{q(1-\delta_q)}}{\left(\|u_\varepsilon\|_{p^*}^{p^*}\right)^{\frac{q\delta_q-p}{p^*-p}}}\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_p^{q(1-\delta_q)}}\right].
\end{aligned}
$$

Recall some important estimates in (2.4)-(2.8), and from $\frac{1+\sqrt{41}}{4}<p<\sqrt[3]{9}$, we have that there exist positive constants C_1, C_2 and C_3 depending on s and q, such that

$$C_1\leq\|\nabla u_\varepsilon\|_p^p\leq\frac{1}{C_1}, \qquad (4.25)$$

$$C_2\leq\|u_\varepsilon\|_{p^*}^{p^*}\leq\frac{1}{C_2}, \qquad (4.26)$$

and

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_p^{q-p}}=C_3\varepsilon^{3-\frac{(q-1)(3-p)}{p-1}}. \qquad (4.27)$$

By virtue of (4.24)-(4.27), we get

$$e^{(p^*-p)t_\varepsilon}\geq C\frac{\|u_\varepsilon\|_p^{p^*-p}}{a^{p^*-p}}\left[C_1-\mu\delta_q a^{q(1-\delta_q)}2^{\frac{q\delta_q-p}{p^*-p}}\frac{C_3}{C_2}\varepsilon^{3-\frac{q(3-p)(p-\delta_q)}{p(p-1)}}\right]\geq C\frac{\|u_\varepsilon\|_p^{p^*-p}}{a^{p^*-p}}. \qquad (4.28)$$

Case 2. $\|\nabla v_\varepsilon\|_p^p\leq\frac{\gamma}{2p}e^{(1-p)t_\varepsilon}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^p\,dx$. In this case, we get

$$e^{p^*t_\varepsilon}\|v_\varepsilon\|_{p^*}^{p^*}\leq 2e^{pt_\varepsilon}\frac{\gamma}{2p}e^{(1-p)t_\varepsilon}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^p\,dx,$$

that is

$$e^{(p^*-1)t_\varepsilon} \leq \frac{\gamma}{p} \frac{\int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^p \, dx}{\|v_\varepsilon\|_{p^*}^{p^*}}. \tag{4.29}$$

Again using $(\Psi_{v_\varepsilon}^{\mu})'(t_\varepsilon) = 0$, (4.29), (2.9) and Hölder inequality, we derive as

$$\begin{aligned}
& e^{(p^*-p)t_\varepsilon} \\
&= \frac{\|\nabla v_\varepsilon\|_p^p}{\|v_\varepsilon\|_{p^*}^{p^*}} + \frac{\gamma}{2p} e^{(1-p)t_\varepsilon} \frac{\int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^p \, dx}{\|v_\varepsilon\|_{p^*}^{p^*}} - \mu\delta_q e^{(q\delta_q - p)t_\varepsilon} \frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{p^*}^{p^*}} \\
&\geq \frac{\|\nabla v_\varepsilon\|_p^p}{\|v_\varepsilon\|_{p^*}^{p^*}} - \mu\delta_q \left(\frac{\gamma}{p} \frac{\int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^p \, dx}{\|v_\varepsilon\|_{p^*}^{p^*}} \right)^{\frac{q\delta_q - p}{p^*-1}} \frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{p^*}^{p^*}} \\
&\geq \frac{\|\nabla v_\varepsilon\|_p^p}{\|v_\varepsilon\|_{p^*}^{p^*}} - \mu\delta_q \left(\frac{\gamma}{p} \frac{\widetilde{C} \|v_\varepsilon\|_{\frac{6}{5}p}^{2p}}{\|v_\varepsilon\|_{p^*}^{p^*}} \right)^{\frac{q\delta_q - p}{p^*-1}} \frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{p^*}^{p^*}} \\
&\geq \frac{\|\nabla v_\varepsilon\|_p^p}{\|v_\varepsilon\|_{p^*}^{p^*}} - \mu\delta_q \left(\frac{\gamma}{p} \frac{\widetilde{C} \|v_\varepsilon\|_p^{2p-1} \|v_\varepsilon\|_{p^*}}{\|v_\varepsilon\|_{p^*}^{p^*}} \right)^{\frac{q\delta_q - p}{p^*-1}} \frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{p^*}^{p^*}} \\
&\geq \frac{\|\nabla v_\varepsilon\|_p^p}{\|v_\varepsilon\|_{p^*}^{p^*}} - \mu\delta_q \left(\frac{\gamma \widetilde{C}}{p} \right)^{\frac{q\delta_q - p}{p^*-1}} a^{\frac{(2p-1)(q\delta_q - p)}{p^*-1}} \left(\frac{1}{\|v_\varepsilon\|_{p^*}^{p^*-1}} \right)^{\frac{q\delta_q - p}{p^*-1}} \frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{p^*}^{p^*}} \qquad (4.30) \\
&\geq \frac{\|u_\varepsilon\|_p^{p^*-p} \|\nabla u_\varepsilon\|_p^p}{a^{p^*-p} \|u_\varepsilon\|_{p^*}^{p^*}} - \mu\delta_q \left(\frac{\gamma \widetilde{C}}{p} \right)^{\frac{q\delta_q - p}{p^*-1}} a^{\alpha} \frac{\|u_\varepsilon\|_q^q \|u_\varepsilon\|_p^{p^*-p+q\delta_q-q}}{\|u_\varepsilon\|_{p^*}^{p^*-p+q\delta_q}} \\
&= \frac{\|u_\varepsilon\|_p^{p^*-p} \left(\|\nabla u_\varepsilon\|_p^p \right)^{\frac{q\delta_q - p}{p^*-p}}}{a^{p^*-p} \|u_\varepsilon\|_{p^*}^{p^*}} \Bigg[\left(\|\nabla u_\varepsilon\|_p^p \right)^{\frac{p^*-q\delta_q}{p^*-p}} \\
&\quad - \mu\delta_q \left(\frac{\gamma \widetilde{C}}{p} \right)^{\frac{q\delta_q - p}{p^*-1}} a^{\alpha - p^* + p} \frac{\|u_\varepsilon\|_q^q \|u_\varepsilon\|_p^{q\delta_q - q}}{\|u_\varepsilon\|_{p^*}^{q\delta_q - p} \left(\|\nabla u_\varepsilon\|_p^p \right)^{\frac{q\delta_q - p}{p^*-p}}} \Bigg] \\
&= \frac{\|u_\varepsilon\|_p^{p^*-p} \left(\|\nabla u_\varepsilon\|_p^p \right)^{\frac{q\delta_q - p}{p^*-p}}}{a^{p^*-p} \|u_\varepsilon\|_{p^*}^{p^*}} \Bigg[\left(\|\nabla u_\varepsilon\|_p^p \right)^{\frac{p^*-q\delta_q}{p^*-p}} \\
&\quad - \mu\delta_q \left(\frac{\gamma \widetilde{C}}{p} \right)^{\frac{q\delta_q - p}{p^*-1}} \frac{a^{\alpha - p^* + p}}{\|u_\varepsilon\|_{p^*}^{q\delta_q - p} \left(\|\nabla u_\varepsilon\|_p^p \right)^{\frac{q\delta_q - p}{p^*-p}}} \frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_p^{q - q\delta_q}} \Bigg],
\end{aligned}$$

where $\alpha = \frac{(2p-1)(q\delta_q - 1)}{p^*-1} + q - q\delta_q + p - p^*$. By (2.4)-(2.5), we conclude that there exists constant $C_4 > 0$ such that

$$C_4 \le \left(\|\nabla u_\varepsilon\|_p^p\right)^{\frac{q\delta_q - p}{p^*-p}} \|u_\varepsilon\|_{p^*}^{q\delta_q - p} \le \frac{1}{C_4}. \tag{4.31}$$

So, by (4.25)-(4.27) and (4.31), we have

$$\begin{aligned} e^{(p^*-p)t_\varepsilon} &\ge C\frac{\|u_\varepsilon\|_p^{p^*-p}}{a^{p^*-p}}\left[C_1 - \mu\delta_q\left(\frac{\gamma\widetilde{C}}{p}\right)^{\frac{q\delta_q-p}{p^*-1}} a^{m-p^*+p}\frac{C_3}{C_4}\varepsilon^{3-\frac{q(3-p)(p-\delta_q)}{p(p-1)}}\right] \\ &\ge \frac{C\|u_\varepsilon\|_p^{p^*-p}}{a^{p^*-p}}. \end{aligned} \tag{4.32}$$

In what follows we focus on an upper estimate for $\Psi^\mu_{v_\varepsilon}(t)$. We split the argument into two steps.

Step 1. We estimate $\hat{\Psi}^\mu_{v_\varepsilon}(t_\varepsilon)$ as follows:

$$\hat{\Psi}^\mu_{v_\varepsilon}(t) = \frac{1}{p}e^{pt}\|\nabla v_\varepsilon\|_p^p - \frac{1}{p^*}e^{p^*t}\|v_\varepsilon\|_{p^*}^{p^*}.$$

By direct calculation, we deduce that the function $\hat{\Psi}^\mu_{v_\varepsilon}(t_\varepsilon)$ has a unique critical point $\hat{t}_\varepsilon$, which is a strict maximum point and is given by

$$e^{\hat{t}_\varepsilon} = \left(\frac{\|\nabla v_\varepsilon\|_p^p}{\|v_\varepsilon\|_{p^*}^{p^*}}\right)^{\frac{1}{p^*-p}}. \tag{4.33}$$

Applying the fact

$$\sup_{\theta\ge 0}\left(\frac{\theta^p}{p}a - \frac{\theta^{p^*}}{p^*}b\right) = \frac{1}{3}\left(\frac{a}{b^{\frac{p}{p^*}}}\right)^{\frac{3}{p}},$$

for any fixed $a, b > 0$. We can deduce by (2.4), (2.5), that

$$\begin{aligned} \hat{\Psi}^\mu_{v_\varepsilon}(\hat{t}_\varepsilon) &= \frac{1}{3}\left(\frac{\|\nabla v_\varepsilon\|_p^p}{\left(\|v_\varepsilon\|_{p^*}^{p^*}\right)^{\frac{p}{p^*}}}\right)^{\frac{3}{p}} = \frac{1}{3}\left(\frac{\|\nabla u_\varepsilon\|_p^p}{\left(\|u_\varepsilon\|_{p^*}^{p^*}\right)^{\frac{p}{p^*}}}\right)^{\frac{3}{p}} \\ &= \frac{1}{3}\left(\frac{S^{\frac{3}{p}} + O\left(\varepsilon^{\frac{3-p}{p-1}}\right)}{\left(S^{\frac{3}{p}} + O\left(\varepsilon^{\frac{3}{p-1}}\right)\right)^{\frac{p}{p^*}}}\right)^{\frac{3}{p}} = \frac{1}{3}S^{\frac{3}{p}} + O\left(\varepsilon^{\frac{3-p}{p-1}}\right). \end{aligned} \tag{4.34}$$

Step 2. we next estimate for $\Psi^\mu_{v_\varepsilon}(t)$. By (2.9), (4.22)and Hölder inequality, we can deduce that

$$
\begin{aligned}
e^{(p^*-p)t_\varepsilon} &= \frac{\|\nabla v_\varepsilon\|_p^p}{\|v_\varepsilon\|_{p^*}^{p^*}} + \frac{\gamma}{2p} e^{(1-p)t_\varepsilon} \frac{\int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^p \, dx}{\|v_\varepsilon\|_{p^*}^{p^*}} - \mu\delta_q \frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{p^*}^{p^*}} \\
&\le \frac{1}{\|v_\varepsilon\|_{p^*}^{p^*}} \left(\|\nabla v_\varepsilon\|_p^p + \frac{\gamma}{2p} e^{(1-p)t_\varepsilon} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^p \, dx \right) \\
&\le \frac{1}{\|v_\varepsilon\|_{p^*}^{p^*}} \left(\|\nabla v_\varepsilon\|_p^p + \frac{\gamma}{2p} e^{(1-p)t_\varepsilon} \tilde{C} \|v_\varepsilon\|_{\frac{6}{5}p}^{2p} \right) \\
&\le \frac{1}{\|v_\varepsilon\|_{p^*}^{p^*}} \left(\|\nabla v_\varepsilon\|_p^p + \frac{\gamma}{2p} e^{(1-p)t_\varepsilon} \tilde{C} \|v_\varepsilon\|_p^{2p-1} \|v_\varepsilon\|_{p^*} \right) \\
&= \frac{1}{a^{p^*-p} \|u_\varepsilon\|_{p^*}^{p^*}} \left(\|\nabla u_\varepsilon\|_p^p \|u_\varepsilon\|_p^{p^*-p} + \frac{\gamma}{2p} e^{(1-p)t_\varepsilon} \tilde{C} a^p \|u_\varepsilon\|_p^{p^*-1} \|u_\varepsilon\|_{p^*} \right).
\end{aligned}
\tag{4.35}
$$

In view of (2.4), (2.5), (2.7) and (4.35), we know t_ε can not go to $+\infty$, namely, there exists some $t^* \in \mathbb{R}$ such that

$$
t_\varepsilon \le t^*, \quad \text{for all } \ \varepsilon > 0.
$$

Hence, by (2.6)-(2.7), (2.9)-(2.10), (4.27), (4.31)-(4.33), and above inequality, we get

$$
\begin{aligned}
\sup_{t \in \mathbb{R}} \Psi_{v_\varepsilon}^\mu(t) &= \Psi_{v_\varepsilon}^\mu(t_\varepsilon) \\
&= \hat{\Psi}_{v_\varepsilon}^\mu(t_\varepsilon) + \frac{\gamma}{2p} e^{t_\varepsilon} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^p \, dx - \frac{\mu}{q} e^{q\delta_q t_\varepsilon} \|v_\varepsilon\|_q^q \\
&\le \frac{1}{3} S^{\frac{3}{p}} + O\left(\varepsilon^{\frac{3-p}{p-1}} \right) + C\gamma \|v_\varepsilon\|_{\frac{6}{5}p}^{2p} - \frac{\mu}{q} e^{q\delta_q t_\varepsilon} \|v_\varepsilon\|_q^q \\
&\le \frac{1}{3} S^{\frac{3}{p}} + O\left(\varepsilon^{\frac{3-p}{p-1}} \right) + C\gamma \frac{a^{2p}}{\|u_\varepsilon\|_p^{2p}} \|u_\varepsilon\|_{\frac{6}{5}p}^{2p} - \frac{C\mu a^{q(1-\delta_q)}}{q} \frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_p^{q(1-\delta_q)}} \\
&\le \frac{1}{3} S^{\frac{3}{p}} + C_1 \varepsilon^{\frac{3-p}{p-1}} + C_2 \gamma \frac{\|u_\varepsilon\|_{\frac{6}{5}p}^{2p}}{\|u_\varepsilon\|_p^{2p}} - C_3 \frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_p^{q(1-\delta_q)}} \\
&= \frac{1}{3} S^{\frac{3}{p}} + C_1 \varepsilon^{\frac{3-p}{p-1}} + C_2 \gamma \frac{\varepsilon^{\frac{2(3-p)}{p-1}}}{\varepsilon^{\frac{2(3-p)}{p-1}}} - C_3 \varepsilon^{3 - \frac{q(3-p)(p-\delta_q)}{p(p-1)}} \\
&= \frac{1}{3} S^{\frac{3}{p}} + C_1 \varepsilon^{\frac{3-p}{p-1}} + C_2 \gamma - C_3 \varepsilon^{3 - \frac{q(3-p)(p-\delta_q)}{p(p-1)}} \\
&< \frac{1}{3} S^{\frac{3}{p}},
\end{aligned}
\tag{4.36}
$$

if we take $\gamma = \varepsilon^\alpha$ for some constant $\alpha \ge \frac{3-p}{p-1}$, and using the fact $0 < 3 - \frac{q(3-p)(p-\delta_q)}{p(p-1)} < \frac{3-p}{p-1}$.

Since $v_\varepsilon \in S_r(a)$, from Lemma 4.2 we may take $t_1 < 0$ and $t_2 > 0$ such that $t_1 \star v_\varepsilon \in \mathcal{A}_a$ and $I_\mu(t_2 \star v_\varepsilon) < 0$, respectively. We construct a path as:

$$\eta^*_{v_\varepsilon} : m \in [0,1] \mapsto ((1-m)t_1 + mt_2) \star v_\varepsilon \in \Gamma_a.$$

Thereby, there exists some $\gamma_2^* > 0$, such that

$$\sigma_\mu(a) \le \max_{t\in[0,1]} I_\mu(\eta^*_{v_\varepsilon}(t)) \le \sup_{t\in\mathbb{R}} \Psi^\mu_{v_\varepsilon}(t) < \frac{1}{3} S^{\frac{3}{p}},$$

for $\gamma \in (0, \gamma_2^*)$ small enough, which completes the proof. □

Lemma 4.8. *Let $\{u_n\}$ be the (PS) sequence in $S_r(a)$ at level $\sigma_\mu(a)$, with $\sigma_\mu(a) < \frac{1}{3} S^{\frac{3}{p}}$. Assume that $u_n \rightharpoonup u$. Then $u \not\equiv 0$.*

Proof. To argue by contradiction, suppose that $u \equiv 0$. Since $\{u_n\}$ is bounded in E_r, by passing to a subsequence, we may assume that $\|\nabla u_n\|_p^p \to \ell \ge 0$. By Lemma 2.8, we know that $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for all $q \in (p, p^*)$. From Proposition 4.4 and Lemmas 2.4 and 2.8, it follows that $P_\mu(u_n) \to 0$, such that,

$$\begin{aligned}
\|u_n\|_{p^*}^{p^*} &= \|\nabla u_n\|_p^p + \frac{\gamma}{2p} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p \, dx - \mu\delta_q \|u_n\|_q^q \\
&= \|\nabla u_n\|_p^p + o_n(1) \\
&= \ell + o_n(1)
\end{aligned}$$

as $n \to \infty$. Then, using Sobolev inequality, one has $\ell \ge S \ell^{\frac{p}{p^*}}$, and so, either $\ell \ge S^{\frac{3}{p}}$ or $\ell = 0$. In the case $\ell \ge S^{\frac{3}{p}}$, from $I_\mu(u_n) \to \sigma_\mu(a)$ and $P_\mu(u_n) \to 0$, we obtain:

$$\begin{aligned}
\sigma_\mu(a) + o_n(1) = I_\mu(u_n) &= I_\mu(u_n) - \frac{1}{p^*} P_\mu(u_n) \\
&= \frac{1}{3}\|\nabla u_n\|_p^p + \frac{p^*-1}{2pp^*}\gamma \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p \, dx - \frac{p^* - q\delta_q}{p^* q}\mu \|u_n\|_q^q \\
&= \frac{1}{3}\ell + o_n(1),
\end{aligned}$$

which implies $\sigma_\mu(a) = \frac{1}{3}\ell$, and thus $\sigma_\mu(a) \ge \frac{1}{3} S^{\frac{3}{p}}$, contradicting the assumption $\sigma_\mu(a) < \frac{1}{3} S^{\frac{3}{p}}$. In the case $\ell = 0$, we have

$$\|\nabla u_n\|_p^p \to 0, \quad \|u_n\|_{p^*}^{p^*} \to 0,$$

and, combined with

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p \, dx \to 0, \quad \|u_n\|_q^q \to 0,$$

we obtain $I_\mu(u_n) \to 0$, which contradicts $\sigma_\mu(a) > 0$. Therefore, $u \not\equiv 0$. □

Lemma 4.9. *Let $\{u_n\}$ be the (PS) sequence in $S_r(a)$ at level $\sigma_\mu(a)$, with $\sigma_\mu(a) < \frac{1}{3}S^{\frac{3}{p}}$. Assume that $P_\mu(u_n) \to 0$ as $n \to \infty$, and that $\gamma < \gamma_1^*$ is sufficiently small. Then, one of the following alternatives must hold:*

(i) There exists a subsequence such that $u_n \rightharpoonup u$ weakly in E_r, but not strongly, where $u \not\equiv 0$ is a solution to

$$-\Delta_p u + \gamma \phi_u |u|^{p-2} u = \lambda |u|^{p-2} u + \mu |u|^{q-2} u + |u|^{p^*-2} u, \quad \text{in } \mathbb{R}^3,$$

where $\lambda_n \to \lambda < 0$, and

$$I_\mu(u) < \sigma_\mu(a) - \frac{1}{3} S^{\frac{3}{p}}.$$

(ii) Alternatively, passing to a subsequence, $u_n \to u$ strongly in E_r, $I_\mu(u) = \sigma_\mu(a)$, and u is a solution to (1.1)-(1.2) *for some $\lambda < 0$.*

Proof. By Lemma 4.6, the sequence $\{u_n\} \subset S_r(a)$ is a bounded (PS) sequence for I_μ in E_r, and therefore, $u_n \rightharpoonup u$ in E_r for some u. By the Lagrange multiplier principle, there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ satisfying

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi dx - \lambda_n \int_{\mathbb{R}^3} |u_n|^{p-2} u_n \varphi dx + \gamma \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{p-2} u_n \varphi dx \\ &- \mu \int_{\mathbb{R}^3} |u_n|^{q-2} u_n \varphi dx - \int_{\mathbb{R}^3} |u_n|^{p^*-2} u_n \varphi dx = o_n(1) \|\varphi\|, \end{aligned} \tag{4.37}$$

for any $\varphi \in E_r$. Moreover, one has $\lim_{n\to\infty} \lambda_n = \lambda < 0$. Letting $n \to \infty$ in (4.37), we have

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla u|^{p-2} \nabla u \nabla \varphi dx + \gamma \int_{\mathbb{R}^3} \phi_u |u|^{p-2} u \varphi dx \\ &- \mu \int_{\mathbb{R}^3} |u|^{q-2} u \varphi dx - \int_{\mathbb{R}^3} |u|^{p^*-2} u \varphi dx - \lambda \int_{\mathbb{R}^3} |u|^{p-2} u \varphi dx = 0, \end{aligned}$$

which implies that u solves the equation

$$-\Delta_p u + \gamma \phi_u |u|^{p-2} u = \lambda |u|^{p-2} u + \mu |u|^{q-2} u + |u|^{p^*-2} u, \quad \text{in } \mathbb{R}^3, \tag{4.38}$$

and we have the Pohozaev identity $P_\mu(u) = 0$.

Let $v_n = u_n - u$, then $v_n \rightharpoonup 0$ in E_r. According to Brezis-Lieb lemma [58] and Lemma 2.3, one has

$$\begin{aligned} \|\nabla u_n\|_p^p &= \|\nabla u\|_p^p + \|\nabla v_n\|_p^p + o_n(1), \\ \|u_n\|_{p^*}^{p^*} &= \|u\|_{p^*}^{p^*} + \|v_n\|_{p^*}^{p^*} + o_n(1), \end{aligned} \tag{4.39}$$

and

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p \, dx = \int_{\mathbb{R}^3} \phi_u |u|^p \, dx + o_n(1), \; \|u_n\|_q^q = \|u\|_q^q + \|v_n\|_q^q + o_n(1), \tag{4.40}$$

Then, from $P_\mu(u_n) > 0$, $u_n \to u$ in $L^p(\mathbb{R}^3)$, one can derive that

$$\|\nabla u\|_p^p + \|\nabla v_n\|_p^p + \frac{\gamma}{2p} \int_{\mathbb{R}^3} \phi_u |u|^p \, dx = \mu \delta_q \|u\|_q^q + \|u\|_{p^*}^{p^*} + \|v_n\|_{p^*}^{p^*} + o_n(1)$$

By $P_\mu(u) = 0$, we have

$$\|\nabla v_n\|_p^p = \|v_n\|_{p^*}^{p^*} + o_n(1) \tag{4.41}$$

Passing to a subsequence, we may assume that

$$\lim_{n\to\infty} \|\nabla v_n\|_p^p = \lim_{n\to\infty} \|v_n\|_{p^*}^{p^*} = \ell \geq 0. \tag{4.42}$$

It follows from the Sobolev inequality that $\ell \geq S \ell^{\frac{p}{p^*}}$, and thus, either $\ell \geq S^{\frac{3}{p}}$ or $\ell = 0$. In the case $\ell \geq S^{\frac{3}{p}}$, since $I_\mu(u_n) \to \sigma_\mu(a)$ and $P_\mu(u_n) \to 0$, we deduce that

$$\begin{aligned} \sigma_\mu(a) &= \lim_{n\to\infty} I_\mu(u_n) \\ &= \lim_{n\to\infty} I_\mu(u) + \frac{1}{p}\|\nabla v_n\|_p^p - \frac{1}{p^*}\|v_n\|_{p^*}^{p^*} + o_n(1) \\ &= I_\mu(u) + \frac{1}{3}\ell \geq I_\mu(u) + \frac{1}{3} S^{\frac{3}{p}}. \end{aligned} \tag{4.43}$$

which means that item (i) holds.

If $\ell = 0$, then $\|u_n - u\| = \|v_n\| > 0$, and we have $u_n \to u$ in $D^{1,p}(\mathbb{R}^3)$, which implies that $u_n \to u$ in $L^{p^*}(\mathbb{R}^3)$. To prove that $u_n \to u$ in E_r, it remains to show that $u_n \to u$ in $L^p(\mathbb{R}^3)$. Fix $\psi = u_n - u$ as a test function in (4.37), and use $u_n - u$ as a test function in (4.38). We then deduce that

$$\begin{aligned} &\int_{\mathbb{R}^3} \left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) \nabla (u_n - u) dx \\ &- \int_{\mathbb{R}^3} \left(\lambda_n |u_n|^{p-2} u_n - \lambda |u|^{p-2} u \right) (u_n - u) dx \\ &+ \gamma \int_{\mathbb{R}^3} \left(\phi_{u_n} |u_n|^{p-2} u_n - \phi_u |u|^{p-2} u \right) (u_n - u) dx \end{aligned} \tag{4.44}$$

$$= \mu \int\limits_{\mathbb{R}^3} \left(|u_n|^{q-2} u_n - |u|^{q-2} u \right) (u_n - u) dx$$

$$+ \int\limits_{\mathbb{R}^3} \left(|u_n|^{p^*-2} u_n - |u|^{p^*-2} u \right) (u_n - u) \, dx + o_n (1)$$

Passing the limit in (4.44) as $n \to \infty$, we have

$$0 = \lim_{n\to\infty} \int\limits_{\mathbb{R}^3} \left(\lambda_n \, |u_n|^{p-2} u_n - \lambda \, |u|^{p-2} u \right) (u_n - u) dx = \lim_{n\to\infty} \lambda \int\limits_{\mathbb{R}^3} |u_n - u|^p \, dx,$$

and then $u_n \to u$ in $L^p(\mathbb{R}^3)$. Therefore, item (ii) holds. □

Now, with the help of the above preparation, we are ready to accomplish the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\gamma < \Gamma^* := \min\{\gamma_1^*, \gamma_2^*\}$. By Lemmas 4.1-4.2, 4.6-4.7, and Propositions 4.3-4.4, there exists a bounded $(PS)_{\sigma_\mu(a)}$-sequence $\{u_n\} \subset S_r(a)$ with $\sigma_\mu(a) < \frac{1}{3} S^{\frac{3}{p}}$, and $u \in E_r$ such that one of the alternatives in Lemma 4.9 must hold. We now claim that case (i) in Lemma 4.9 is not possible. To argue by contradiction, assume that case (i) holds. Then u would be a nontrivial solution to (4.37), and applying Lemmas 4.9 and 4.7, we conclude that

$$I_\mu(u) < \sigma_\mu(a) - \frac{1}{3} S^{\frac{3}{p}} < 0.$$

On the other hand, we have

$$I_\mu (u) = I_\mu (u) - \frac{1}{p} P_\mu (u)$$

$$= \frac{p-1}{2p^2} \gamma \int\limits_{\mathbb{R}^3} \phi_u \, |u|^p \, dx + \frac{q\delta_q - p}{pq} \mu \|u\|_q^q + \frac{1}{3} \|u\|_{p^*}^{p^*} \geq 0$$

which leads to a contradiction. Thus, $u_n \to u$ strongly in E_r with $I_\mu(u) = \sigma_\mu(a)$, and u is a solution to (1.1)-(1.2) for some $\lambda < 0$. Moreover, we have $u(x) > 0$ in $\mathbb{R}^3$. In fact, all the preceding calculations can be repeated verbatim, simply replacing I_μ with the functional

$$I_\mu^+ (u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{\gamma}{2p} \int\limits_{\mathbb{R}^3} \phi_u \, |u|^p \, dx - \frac{\mu}{q} \|u^+\|_q^q - \frac{1}{p^*} \|u^+\|_{p^*}^{p^*}. \tag{4.45}$$

Then u is the critical point of I_μ^+ restricted on the set $S_r(a)$, it solves the equation

$$-\Delta_p u + \gamma \phi_u \, |u|^{p-2} u = \lambda \, |u|^{p-2} u + \mu \left|u^+\right|^{q-2} u + \left|u^+\right|^{p^*-2} u. \text{ in } \mathbb{R}^3, \tag{4.46}$$

Using $u^- = \min\{u, 0\}$ as a test function in (4.46), and recalling that $(a-b)(a^* - b^*) > |a^* - b^*|$ for all $a, b \in \mathbb{R}$, we conclude that

$$\|\nabla u^-\|_p^p \le \|\nabla u^-\|_p^p + \gamma \int_{\mathbb{R}^3} \phi_u |u^-|^p \, dx - \lambda \|u^-\|_p^p = 0.$$

Thus, $u^- = 0$, and consequently, $u \ge 0$ for all $x \in \mathbb{R}^3$. This implies that u is a solution to (4.46). By the strong maximum principle, we conclude that $u(x) > 0$ for all $x \in \mathbb{R}^3$. □

5. Proof of Theorem 1.3

In this section, we consider the L^p-supercritical case $\bar{p} < q < p^*$, where the parameter $\mu > 0$ is large. Due to the fact that $q\delta_q > p$, the truncated functional $I_{\mu,\tau}$ defined in Section 4 remains unbounded from below on $S_r(a)$, and therefore, the truncation technique is not applicable for analyzing the problem in (1.1)-(1.2).

From Lemmas 4.1, 4.2 we have the mountain pass level value $\sigma_\mu(a)$ by

$$\sigma_\mu(a) := \inf_{\xi \in \Gamma_a} \max_{t \in [0,1]} I_\mu(\xi(t)) > 0,$$

where

$$\Gamma_a = \{\xi \in C([0,1], S_r(a)) : \xi(0) \in \mathcal{A}_a, \xi(1) \in I_\mu^0\}.$$

In the following, we define $g(t) = \mu |t|^{q-2} t + |u|^{p^*-2} u$ and $G(t) = \frac{\mu}{q} |t|^q + \frac{1}{p^*} |u|^{p^*}$, for any $t \in \mathbb{R}$. From Proposition 4.4, we know that there exists a $(PS)_{\sigma_\mu(a)}$-sequence $\{u_n\} \subset S_r(a)$ satisfying

$$I_\mu(u_n) \to \sigma_\mu(a), \quad \|I'_\mu|_{S_r(a)}(u_n)\| \to 0, \quad P_\mu(u_n) \to 0, \quad \text{as} \quad n \to \infty,$$

where

$$P_\mu(u_n) = \|\nabla u_n\|_p^p + \frac{\gamma}{2p} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p \, dx + 3 \int_{\mathbb{R}^3} G(u_n) \, dx - \frac{3}{p} \int_{\mathbb{R}^3} g(u_n) u_n dx.$$

Moreover, by Proposition 5.12 in [58], there exists $\lambda_n \in \mathbb{R}$ such that

$$\|I'_\mu(u_n) - \lambda_n \Phi'(u_n)\| \to 0, \quad \text{as} \quad n \to \infty.$$

That is, we have

$$-\Delta_p u_n + \gamma \phi_{u_n} |u_n|^{p-2} u_n - g(u_n) = \lambda_n |u_n|^{p-2} u_n + o_n(1) \quad \text{in} \quad E_r^{-1}. \tag{5.1}$$

Therefore, for any $\varphi \in E_r$, one has

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi \, dx + \gamma \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{p-2} u_n \varphi \, dx - \int_{\mathbb{R}^3} g(u_n) \varphi dx \\ &= \lambda_n \int_{\mathbb{R}^3} |u_n|^{p-2} u_n \varphi \, dx + o_n(1). \end{aligned} \tag{5.2}$$

In the following, we analyze the asymptotic behavior of the mountain pass level $\sigma_\mu(a)$ as $\mu \to +\infty$, and investigate the properties of the $(PS)_{\sigma_\mu(a)}$-sequence $\{u_n\} \subset S_r(a)$ as $n \to +\infty$.

Lemma 5.1. *The limit* $\lim_{\mu\to+\infty} \sigma_\mu(a) = 0$ *holds.*

Proof. Recalling Lemmas 4.1 and 4.2, we observe that for a fixed $u_0 \in S_r(a)$, there exist two constants t_1 and t_2 satisfying $t_1 < 0 < t_2$, such that $u_1 := t_1 * u_0 \in \mathcal{A}_a$ and $u_2 := t_2 * u_0$ with $I_\mu(u_2) < 0$. We can then define a path

$$\eta_0 : m \in [0,1] \mapsto ((1-m)t_1 + mt_2) * u_0 \in \Gamma_a.$$

Thus, we have

$$\begin{aligned}\sigma_\mu(a) &\le \max_{t\in[0,1]} I_\mu(\eta_0(t))\\ &\le \max_{r\ge 0}\left\{\frac{r^p}{p}\|\nabla u_0\|_p^p + \frac{\gamma}{2p} r \int_{\mathbb{R}^3} \phi_{u_0}|u_0|^p\,dx - \frac{\mu}{q} r^{q\delta_q}\|u_0\|_q^q\right\}\\ &:= \max_{r\ge 0} h(r).\end{aligned}$$

Since $q\delta_q > p$, we know that

$$\lim_{r\to+\infty} h(r) = 0^+ \quad \text{and} \quad \lim_{r\to+\infty} h(r) = -\infty.$$

Therefore, there exists a unique point $r_0 > 0$ such that

$$\max_{r\ge 0} h(r) = h(r_0) > 0.$$

Thus, we distinguish between two cases: $r_0 \ge 1$ and $0 \le r_0 < 1$.

If $r_0 \ge 1$, we have that

$$\begin{aligned}\max_{t\in[0,1]} I_\mu(\eta_0(t)) \le h(r_0) &\le \frac{r_0^p}{p}\|\nabla u_0\|_p^p + \frac{r_0^p}{2p}\gamma \int_{\mathbb{R}^3} \phi_{u_0}|u_0|^p\,dx - \frac{\mu}{q} r_0^{q\delta_q}\|u_0\|_q^q\\ &\le \max_{r\ge 0}\left\{2r^p \max\left\{\frac{1}{p}\|\nabla u_0\|_p^p, \frac{\gamma}{2p}\int_{\mathbb{R}^3} \phi_{u_0}|u_0|^p\,dx\right\} - \frac{\mu}{q} r^{q\delta_q}\|u_0\|_q^q\right\}\\ &= 2a(r_{\max})^p - \frac{\mu b}{q}(r_{\max})^{q\delta_q}\\ &= \left(2a - \frac{2ap}{q\delta_q}\right)\left[\frac{2ap}{\mu b \delta_q}\right]^{\frac{p}{q\delta_q - p}}.\end{aligned}$$

where

$$r_{\max} = \left[\frac{2ap}{\mu b \delta_q}\right]^{\frac{1}{q\delta_q - p}}, a = \max\left\{\frac{1}{p}\|\nabla u_0\|_p^p, \frac{\gamma}{2p}\int_{\mathbb{R}^3} \phi_{u_0}|u_0|^p\, dx\right\}, b = \|u_0\|_q^q.$$

Therefore, for $\bar{p} < q < p^*$, we have a positive constant C_1 independent of μ such that

$$\sigma_\mu(a) \le C_1 \mu^{-\frac{p}{q\delta_q - p}} \to 0, \quad \text{as} \quad \mu \to +\infty.$$

If $0 \le r_0 < 1$, we infer that

$$\begin{aligned}
\max_{t\in[0,1]} I_\mu(\eta_0(t)) &\le \frac{r_0^p}{p}\|\nabla u_0\|_p^p + \frac{r_0}{2p}\gamma\int_{\mathbb{R}^3}\phi_{u_0}|u_0|^p\, dx - \frac{\mu}{q} r_0^{q\delta_q}\|u_0\|_q^q \\
&\le \max_{r\ge 0}\left\{2r\max\left\{\frac{1}{p}\|\nabla u_0\|_p^p, \frac{\gamma}{2p}\int_{\mathbb{R}^3}\phi_{u_0}|u_0|^p\, dx\right\} - \frac{\mu}{q} r^{q\delta_q}\|u_0\|_q^q\right\} \\
&= 2a\tilde{r}_{\max} - \frac{\mu b}{q}(r_{\max})^{q\delta_q} \\
&= \left(2a - \frac{2a}{q\delta_q}\right)\left[\frac{2a}{\mu b\delta_q}\right]^{\frac{1}{q\delta_q - 1}}.
\end{aligned}$$

where

$$\tilde{r}_{\max} = \left[\frac{2a}{\mu b \delta_q}\right]^{\frac{1}{q\delta_q-1}}.$$

Then there exists a positive constant C_2 independent of μ such that

$$\sigma_\mu(a) \le C_2 \mu^{-\frac{1}{q\delta_q - 1}} \to 0, \quad \text{as} \quad \mu \to +\infty.$$

This completes the proof. □

Lemma 5.2. *There exists a constant $C = C(q) > 0$ such that*

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3} G(u_n)dx \le C\sigma_\mu(a), \ \limsup_{n\to\infty}\int_{\mathbb{R}^3} g(u_n)u_n dx \le C\sigma_\mu(a).$$

and

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\, dx \le C\sigma_\mu(a), \ \limsup_{n\to\infty}\|\nabla u_n\|_p^p \le C\sigma_\mu(a).$$

Proof. Since $I_\mu(u_n) \to \sigma_\mu(a)$ and $P_\mu(u_n) \to 0$ as $n \to \infty$, we have

$$\begin{aligned}
3\sigma_\mu(a) + o_n(1) &= 3I_\mu(u_n) + P_\mu(u_n) \\
&= \frac{p+3}{p}\|\nabla u_n\|_p^p + \frac{2\gamma}{p}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p\,dx - \frac{3}{p}\int_{\mathbb{R}^3} g(u_n)u_n dx \\
&= \frac{p+3}{p}\left(p\sigma_\mu(a) - \frac{\gamma}{2}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx + p\int_{\mathbb{R}^3} G(u_n)\,dx + o_n(1)\right) \\
&\quad + \frac{2\gamma}{p}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx - \frac{3}{p}\int_{\mathbb{R}^3} g(u_n)u_n dx \qquad (5.3) \\
&= (p+3)\left(\sigma_\mu(a) + \int_{\mathbb{R}^3} G(u_n)\,dx + o_n(1)\right) \\
&\quad - \frac{p-1}{2p}\gamma\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx - \frac{3}{p}\int_{\mathbb{R}^3} g(u_n)u_n dx
\end{aligned}$$

Hence,

$$\begin{aligned}
p\sigma_\mu(a) + o_n(1) &= \frac{p-1}{2p}\gamma\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx + \frac{3}{p}\int_{\mathbb{R}^3} g(u_n)u_n dx - (p+3)\int_{\mathbb{R}^3} G(u_n)\,dx \\
&\geq \frac{3q}{p}\int_{\mathbb{R}^3} G(u_n)\,dx - (p+3)\int_{\mathbb{R}^3} G(u_n)\,dx \\
&= \frac{3q - p(p+3)}{p}\int_{\mathbb{R}^3} G(u_n)\,dx
\end{aligned}$$

which implies that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3} G(u_n)\,dx \leq \frac{p^2}{3q - p(p+3)}\sigma_\mu(a) \leq C\sigma_\mu(a). \qquad (5.4)$$

and then

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3} g(u_n)u_n dx \leq C\sigma_\mu(a). \qquad (5.5)$$

Then, from (5.3)-(5.5), we have

$$\limsup_{n\to\infty}\left\{\frac{p+3}{p}\|\nabla u_n\|_p^p+\frac{2}{p}\gamma\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx\right\}$$
$$=\limsup_{n\to\infty}\left\{3\sigma_\mu(a)+\frac{3}{p}\int_{\mathbb{R}^3}g(u_n)u_n dx+o_n(1)\right\}\le C\sigma_\mu(a). \tag{5.6}$$

Consequently, the proof is completed. □

Lemma 5.3. *There exists $\mu_1^*:=\mu_1^*(a)>0$ such that $u\not\equiv 0$ for all $\mu\ge\mu_1^*$.*

Proof. From Lemma 4.6, it follows that the sequence $\{u_n\}$ is bounded in E_r. Moreover, by Lemma 2.8, passing to a subsequence, there exists $u\in E_r$ such that $u_n\rightharpoonup u$ weakly in E_r, $u_n\to u$ strongly in $L^q(\mathbb{R}^3)$ for $q\in(p,p^*)$, and $u_n\to u$ a.e. on $\mathbb{R}^3$. Given that $\bar{p}<q<p^*$, and applying Lemmas 2.4 and 2.8, we conclude that

$$\lim_{n\to\infty}\|u_n\|_q^q=\|u\|_q^q\,,\quad \int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx=\int_{\mathbb{R}^3}\phi_u|u|^p\,dx. \tag{5.7}$$

Suppose by contradiction that $u\equiv 0$. Then, by the (5.7) and $P_\mu(u_n)=o_n(1)$, we deduce as

$$o_n(1)=\|\nabla u_n\|_p^p+\frac{\gamma}{2p}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\,dx-\mu\delta_q\|u_n\|_q^q-\|u_n\|_{p^*}^{p^*}=\|\nabla u_n\|_p^p-\|u_n\|_{p^*}^{p^*}+o_n(1).$$

Without loss of generality, we may assume that

$$\|\nabla u_n\|_p^p\to\ell\,,\ \ \|u_n\|_{p^*}^{p^*}\to\ell,$$

as $n\to\infty$. By Sobolev inequality we get $\ell\ge S\ell^{\frac{p}{p^*}}$, and so, either $\ell\ge S^{\frac{3}{p}}$ or $\ell=0$.

If $\ell\ge S^{\frac{3}{p}}$, then from $I_\mu(u_n)\to\sigma_\mu(a)$, $P_\mu(u_n)\to 0$, we have

$$\begin{aligned}\sigma_\mu(a)+o_n(1)&=I_\mu(u_n)\\&=I_\mu(u_n)-\frac{1}{p^*}P_\mu(u_n)\\&=\frac{1}{3}\|\nabla u_n\|_p^p+\frac{p^*-1}{2pp^*}\gamma\int_{\mathrm{R}^3}\phi_{u_n}|u_n|^p\,dx-\frac{p^*-q\delta_q}{p^*q}\mu\|u\|_q^q+o_n(1)\\&=\frac{1}{3}\ell+o_n(1),\end{aligned}$$

which implies that $\sigma_\mu(a)=\frac{1}{3}\ell$, and so, $\sigma_\mu(a)\ge\frac{1}{3}S^{\frac{3}{p}}$, but this is impossible since by Lemma 5.1, there exists some $\mu_1^*:=\mu_1^*(a)>0$ such that $\sigma_\mu(a)<\frac{1}{3}S^{\frac{3}{p}}$ as $\mu>\mu_1^*$.

If $\ell = 0$, then we have $\|\nabla u_n\|_p^p \to 0$, thus $I_\mu(u_n) \to 0$, which is absurd since $\sigma_\mu(a) > 0$. Therefore, $u \not\equiv 0$. □

Lemma 5.4. *The sequence* $\{\lambda_n\}$ *is bounded in* $\mathbb{R}$, *and* $\limsup_{n\to\infty} |\lambda_n| \leq \frac{C}{a^p}\sigma_\mu(a)$ *with the following estimate:*

$$\lambda_n = \frac{1}{a^p}\left[\frac{2p-1}{2p}\gamma \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p\,dx + (\delta_q - 1)\mu\|u_n\|_q^q\right] + o_n(1).$$

Moreover, there exists a constant $\mu_2^* = \mu_2^*(a) > 0$ *such that*

$$\lim_{n\to\infty} \lambda_n = \lambda < 0 \quad \text{if} \quad \mu \geq \mu_2^* \text{ large}.$$

Proof. From (5.1) and the fact that $u_n \in S_r(a)$, we obtain

$$\begin{aligned}\|\nabla u_n\|_p^p + \gamma \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p\,dx - \int_{\mathbb{R}^3} g(u_n)\,u_n dx &= \lambda_n\|u_n\|_p^p + o_n(1)\\ &= \lambda_n a^p + o_n(1),\end{aligned}$$

which indicates that

$$\lambda_n = \frac{1}{a^p}\left[\|\nabla u_n\|_p^p + \gamma \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p\,dx - \int_{\mathbb{R}^3} g(u_n)\,u_n dx\right] + o_n(1).$$

By Lemma 4.6, we know that $\{u_n\}$ is bounded in E_r, and thus $\{\lambda_n\}$ is also bounded in $\mathbb{R}$. From Lemma 5.2, we obtain that $\limsup_{n\to\infty} |\lambda_n| \leq \frac{C}{a^p}\sigma_\mu(a)$. Furthermore, along with $P_\mu(u_n) \to 0$ as $n \to \infty$, we conclude that

$$\begin{aligned}\lambda_n &= \frac{1}{a^p}\left[\|\nabla u_n\|_p^p + \gamma \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p\,dx - \int_{\mathbb{R}^3} g(u_n)\,u_n dx - P_\mu(u_n)\right] + o_n(1)\\ &= \frac{1}{a^p}\left[\frac{2p-1}{2p}\gamma \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p\,dx + \left(\delta_q - 1\right)\mu\|u_n\|_q^q\right] + o_n(1)\end{aligned}$$

By (5.7) and similar arguments to that of (3.32)-(3.35), we see that there exists $\mu_2^* := \mu_2^*(a) > 0$, such that

$$
\begin{aligned}
\lambda &= \lim_{n\to\infty} \lambda_n \\
&= \lim_{n\to\infty} \frac{1}{a^p}\left[\frac{2p-1}{2p}\gamma \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p \, dx + \left(\delta_q - 1\right)\mu \|u_n\|_q^q + o_n(1)\right] \\
&= \frac{1}{a^p}\left[\frac{2p-1}{2p}\gamma \int_{\mathbb{R}^3} \phi_u |u|^p \, dx + \left(\delta_q - 1\right)\mu \|u\|_q^q\right] < 0,
\end{aligned} \tag{5.8}
$$

for $\mu > \mu_2^*$ large. □

Subsequently, by applying the concentration-compactness principle, we obtain the following lemma, the proof of which follows similarly to that of Lemma 3.3 in Section 4, and we omit the details here.

Lemma 5.5. *For $\mu > \mu^* := \max\{\mu_1^*, \mu_2^*\}$, it holds that $\|u_n\|_{p^*}^{p^*} \to \|u\|_{p^*}^{p^*}$.*

With the aid of the above technical lemmas, we can now proceed to prove Theorem 1.3 as follows.

Proof of Theorem 1.3. Let $\mu > \mu^* := \max\{\mu_1^*, \mu_2^*\}$. From Lemmas 4.1 and 4.2, we know that the functional I_μ satisfies the mountain pass geometry. According to Proposition 4.4, there exists a $(PS)_{\sigma_\mu(a)}$-sequence $\{u_n\} \subset S_r(a)$ that satisfies (5.1) and (5.2), and this sequence is bounded in E_r. Furthermore, there exists a function $u \in E_r$ such that $u_n \rightharpoonup u$ weakly in E_r, and $u_n \to u$ strongly in $L^q(\mathbb{R}^3)$ for $q \in (p, p^*)$. In addition, by Lemmas 5.1-5.4, we deduce that $\lambda_n \to \lambda < 0$ as $n \to +\infty$. From the weak convergence $u_n \rightharpoonup u$ in E_r and the equations (5.1) and (5.2), we conclude that u satisfies the equation

$$
-\Delta_p u + \gamma \phi_u |u|^{p-2} u - \mu |u|^{q-2} u - |u|^{p^*-2} u = \lambda |u|^{p-2} u. \tag{5.9}
$$

Therefore, from (5.7) - (5.9) and Lemma 5.5, it follows that

$$
\begin{aligned}
\|\nabla u\|_p^p + \gamma \int_{\mathbb{R}^3} \phi_u |u|^p \, dx - \lambda \|u\|_p^p &= \mu \|u\|_q^q + \|u\|_{p^*}^{p^*} = \lim_{n\to\infty}\left[\mu \|u_n\|_q^q + \|u_n\|_{p^*}^{p^*}\right] \\
&= \lim_{n\to\infty}\left[\|\nabla u_n\|_p^p + \gamma \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^p \, dx - \lambda_n \|u_n\|_p^p\right] \\
&= \lim_{n\to\infty}\left[\|\nabla u_n\|_p^p - \lambda_n \|u_n\|_p^p\right] + \gamma \int_{\mathbb{R}^3} \phi_u |u|^p \, dx.
\end{aligned}
$$

Since $\lambda < 0$, following the reasoning in the proof of Lemma 3.3, we can conclude that

$$
\lim_{n\to\infty} \|\nabla u_n\|_p^p = \|\nabla u\|_p^p \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|_p^p = \|u\|_p^p.
$$

Thus, $u_n \to u$ in E_r and $\|u\|_p = a$. This completes the proof. □

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Data availability

No data was used for the research described in the article.

Uncited references

[8] [10]

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