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Nonexistence results and integral estimates for some higher order nonlinear elliptic problems

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Abstract

In this work we present a generalization of the Hardy inequality and apply it to study the existence and behaviour of positive solutions of higher order elliptic problems involving the polyharmonic operator in exterior domains.

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1. Introduction

The well-known Hardy-Sobolev inequality states that for any domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, we have

$$\left(\frac{N-2}{2}\right)^2 \int\limits_{\Omega} \frac{\phi^2}{|x|^2}\,dx \leq \int\limits_{\Omega} |\nabla\phi|^2\,dx$$

for every $\phi \in C_c^\infty(\Omega)$. Several versions of the Hardy inequality with applications in partial differential equations can be found, for example, in [5,6,12,14,16,21,22,25,26,35]. This paper discusses the following generalized Hardy inequality.

Theorem 1.1. *Let $f : (0,\infty) \to \mathbb{R} \setminus \{0\}$ be a C^1 function with $f'(t) > 0$ for every $t \in (0,\infty)$ and let Ω be a domain in $\mathbb{R}^N$. Then for any positive function $u \in C^2(\Omega)$ we have*

$$\int \frac{-\Delta u}{f(u)}\phi^2\,dx \leq \int\limits_{\Omega} \frac{|\nabla\phi|^2}{f'(u)}\,dx \tag{1.1}$$

for every $\phi \in C_c^\infty(\Omega)$. In addition, if $f(t) > 0$ for every $t \in (0,\infty)$, then for any integer $m \geq 2$ and any positive polysuperharmonic function $u \in C^{2m}(\Omega)$, that is, $(-\Delta)^i u \geq 0$ in Ω for every $i = 1,\dots,m-1$, we have

$$\left(\int\limits_{\Omega} \left(\frac{(-\Delta)^m u}{f(u)}\right)^{\frac{1}{m}} \phi^2\,dx\right)^m \leq \left(\int\limits_{\Omega} |\nabla\phi|^2\,dx\right)^{m-1} \int\limits_{\Omega} \frac{|\nabla\phi|^2}{f'(u)}\,dx, \tag{1.2}$$

for every $\phi \in C_c^\infty(\Omega)$. In particular, we have

$$\int\limits_{\Omega} \left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} \phi^2\,dx \leq \int\limits_{\Omega} |\nabla\phi|^2\,dx,$$

for every $\phi \in C_c^\infty(\Omega)$.

We apply the Hardy inequality above to study the existence and behaviour of positive solutions of higher order elliptic problems involving the polyharmonic operator in exterior domains Ω of $\mathbb{R}^N$, $N \geq 3$. More precisely, we consider the problems

$$\begin{cases} (-\Delta)^m u \geq |x|^a g(u) & \text{in } \Omega, \\ (-\Delta)^i u > 0, \quad i = 1,\dots,m-1, & \text{in } \Omega, \end{cases} \tag{1.3}$$

and

$$\begin{cases} (-\Delta)^m u + (-\Delta)^k u \geq |x|^a g(u) & \text{in } \Omega, \\ (-\Delta)^i u > 0, \quad i = 1,\dots,\max\{m,k\}, & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $m, k \geq 1$ are integers and Ω is an exterior domain in $\mathbb{R}^N$, $N > 2\max\{m, k\}$. We also consider the higher-order Hardy-Hénon elliptic system

$$\begin{cases} (-\Delta)^m u \geq |x|^a v^R & \text{in } \Omega, \\ (-\Delta)^k v \geq |x|^b u^Q & \text{in } \Omega, \end{cases} \tag{1.5}$$

where u, v are positive polysuperharmonic functions, that is,

$$\begin{cases} (-\Delta)^i u \geq 0, \quad i = 1, \dots, m-1, & \text{in } \Omega, \\ (-\Delta)^j v \geq 0, \quad j = 1, \dots, k-1, & \text{in } \Omega, \end{cases} \tag{1.6}$$

and $a, b, Q, R \in \mathbb{R}$, with $Q, R > 0$ and Ω is an exterior domain in $\mathbb{R}^N$. More generally, we can cover multipower systems of the form

$$\begin{cases} (-\Delta)^m u \geq |x|^a u^S v^R & \text{in } \Omega, \\ (-\Delta)^k v \geq |x|^b u^Q v^T & \text{in } \Omega, \end{cases} \tag{1.7}$$

where $a, b, S, R, Q, T \in \mathbb{R}$ and (u, v) is a positive solution of (1.6).

Remark 1.2. The Hardy inequalities in Theorem 1.1 allow us to consider more general forms of the aforementioned problems. For instance, we may consider the problem

$$P(u) \geq |x|^u g(u) \quad \text{in } \Omega,$$

where $P = \sum_{i=1}^m a_i (-\Delta)^i$, $a_i \geq 0$ and $m \geq 1$ is an integer. We may also consider a more general problem

$$P(u) \geq |x|^u g(u) f((-\Delta)^k u) \quad \text{in } \Omega,$$

where P is as above, k is an integer so that $0 \leq k < \min\{i : a_i > 0\}$ and f, g are positive functions satisfying suitable conditions. We may also consider nonautonomous systems

$$\begin{cases} Q_1(u_1) & \geq |x|^{a_1} u_1^{s_{11}} u_2^{s_{12}} \dots u_n^{s_{1n}} \quad \text{in } \Omega, \\ Q_2(u_2) & \geq |x|^{a_2} u_1^{s_{21}} u_2^{s_{22}} \dots u_n^{s_{2n}} \quad \text{in } \Omega, \\ & \vdots \\ Q_n(u_n) & \geq |x|^{a_n} u_1^{s_{n1}} u_2^{s_{n2}} \dots u_n^{s_{nn}} \quad \text{in } \Omega, \end{cases}$$

where $Q_i = \sum_{j=1}^m b_{ij} (-\Delta)^j$, $b_{ij} \geq 0$, $m \geq 1$ is an integer and $i = 1, \dots, n$.

There has been a lot of interest in the nonexistence problem, also known as the Liouville problem, for

$$(-\Delta)^m u \geq g(u) \quad \text{in } \Omega, \tag{1.8}$$

where $\Omega = \mathbb{R}^N$ or an exterior domain in $\mathbb{R}^N$. A relevant special case of (1.8) is $g(u) = u^p$ with $p > 0$. It is well-known that, if $1 < p < \frac{N}{N-2m}$, then $(-\Delta)^m u \geq u^p$ does not admit any nonnegative polysuperharmonic solution in the whole space, for example, see Corollary 3.6 in [13], where the authors prove Liouville theorems for supersolutions of the polyharmonic Hénon-Lane-Emden system and study its connection with the Hardy-Littlewood-Sobolev systems. For more results on positive solutions to some related problems, we refer to [1,2,11,15,17–19,34] and the references therein. To the best of our knowledge, this paper is the first attempt to treat the general equation (1.4). We do not only discuss nonexistence results, but also obtain integral estimates for solutions in the event of existence, see Proposition 2.1 below.

For the Hardy-Hénon system (1.5) in the case when $m = k = 1$, we refer to [7–10,20,28–32,37,38]. There exists an extensive literature on the Lane-Emden system, see for example [3,4,13,27,33,39], less is known about the higher order Hardy-Hénon system (1.5). The local or global behaviour of the solutions of elliptic quasilinear problems has been studied in [9], see also [7]. Among many other results, they proved that the system

$$\begin{cases} -\Delta u \geq |x|^a u^S v^R & \text{in } \Omega, \\ -\Delta v \geq |x|^b u^Q v^T & \text{in } \Omega, \end{cases} \tag{1.9}$$

where $a, b \in \mathbb{R}$, $Q, R > 0$ with $QR > (1-S)(1-T)$, $0 \leq S, T < 1$, does not admit nonnegative solutions in exterior domains in $\mathbb{R}^N$, $N \geq 2$, provided

$$\max\{\gamma - (N-2), \xi - (N-2)\} \geq 0,$$

where

$$\gamma = \frac{(a+2)(1-T) + (b+2)R}{QR - (1-S)(1-T)} \quad \text{and} \quad \xi = \frac{(a+2)Q + (b+2)(1-S)}{QR - (1-S)(1-T)}.$$

Notice that if $QR \neq (1-S)(1-T)$ then (1.9) admits a particular solution (u^*, v^*), given by

$$u^*(x) = A|x|^{-\gamma} \quad \text{and} \quad v^*(x) = B^*|x|^{-\xi},$$

for some constants A^*, B^* depending on N, p, m, a, b, whenever $0 < \gamma < N-2$ and $0 < \xi < N-2$. For more details see Theorem 5.1 and Theorem 5.3 in [9]. Proposition 2.2 below extends a result of [9] to the multipower system (1.7). Our proof is based on a Hardy-type inequality.

2. Liouville-type results

For an $R > 0$ we let B_R denote the ball of radius R centred at the origin in $\mathbb{R}^N$. For any function $f \in L^1(\Omega)$ we denote

$$\fint_\Omega f\,dx = \frac{1}{|\Omega|}\int_\Omega f\,dx,$$

where $|\Omega|$ denotes the Lebesgue measure of a measurable set Ω with finite and positive measure.

We begin with a result on the behaviour of solutions of the problem (1.4).

Proposition 2.1. *Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function, $h(s) = \frac{g(s)}{s}$ is nondecreasing for $s > 0$, $m \geq k \geq 1$ are integers and Ω is an exterior domain in $\mathbb{R}^N$, $N > 2m$.*

(i) ***(Integral estimate)*** *If $u \in C^{2m}(\Omega)$ is a positive solution of (1.4) then for large enough $R > 0$, we have*

$$\fint_{B_R \setminus B_{\frac{R}{2}}} h(u)^{\frac{1}{m}}\, dx \leq C R^{-\frac{a+2k}{m}}, \tag{2.1}$$

where C is a constant independent of R, u and g.

(ii) ***(Nonexistence)*** *Let $\sigma_a = \frac{N-2(m-k)+a}{N-2m}$, $a > -2k$. There does not exist positive solutions $u \in C^{2m}(\Omega)$ to (1.4), if*

$$\limsup_{r \to 0} \frac{g(r)}{r^{\sigma_a}} = \infty.$$

On the other hand, if there exists $\sigma > \sigma_a$ such that

$$\limsup_{r \to 0} \frac{g(r)}{r^{\sigma}} < \infty, \tag{2.2}$$

then (1.4) admits a particular positive solution

$$u^*(x) = A|x|^{\frac{-(a+2k)}{\sigma-1}},$$

for a suitable constant $A > 0$ in any exterior domain $\mathbb{R}^N \setminus \overline{B_{R_0}}$ for R_0 large enough. In particular, if $g(u) = u^p$, $p \geq 1$, then for any $a > -2k$, there are no positive solutions to (1.4) provided $1 \leq p < \frac{N+a-2(m-k)}{N-2m}$, if $m > k$ and $1 \leq p \leq \frac{N+a}{N-2m}$, if $m = k$.

Then we state a result for the multipower system (1.7).

Proposition 2.2. *Assume that (u, v) is a positive solution of the system (1.7) satisfying (1.6) in an exterior domain Ω of $\mathbb{R}^N$, $N > \max\{2m, 2k\}$, $m, k \geq 1$. Let $a, b \in \mathbb{R}$, $Q, R > 0$, $0 \leq S, T < 1$ with $QR > (1-S)(1-T)$.*

(i) ***(Integral estimates)*** *For large enough $R > 0$, we have*

$$\fint_{B_R \setminus B_{\frac{R}{2}}} u^{\frac{QR-(1-T)(1-S)}{m(1-T)+kR}}\, dx \leq C R^{-\frac{(a+2m)(1-T)+(b+2k)R}{m(1-T)+kR}} \tag{2.3}$$

and

$$\fint_{B_R \setminus B_{\frac{R}{2}}} v^{\frac{QR-(1-T)(1-S)}{mQ+k(1-S)}}\, dx \leq C R^{-\frac{(a+2m)Q+(b+2k)(1-S)}{mQ+k(1-S)}}. \tag{2.4}$$

Moreover, we have

$$\fint_{B_{2R}\setminus B_R} \frac{dx}{v^\theta} \le C R^\eta \quad \text{and} \quad \fint_{B_{2R}\setminus B_R} \frac{dx}{u^\zeta} \le C R^\kappa, \tag{2.5}$$

where

$$\theta = \frac{R\big(QR-(1-S)(1-T)\big)}{(1-S)\big(m(1-T)+kR\big)},$$

$$\eta = -2 + \frac{(QR-(1-S)(1-T))\big(N-2m+\frac{R(N-2k)}{1-S}\big)-(a(1-T)+bR)}{m(1-T)+kR},$$

and

$$\zeta = \frac{Q(QR-(1-S)(1-T))}{(1-T)(mQ+k(1-S))},$$

$$\kappa = -2 + \frac{(QR-(1-S)(1-T))\big(\frac{Q(N-2m)}{1-T}+N-2k\big)-(aQ+b(1-S))}{mQ+k(1-S)}.$$

(ii) *(**Nonexistence**) The system* (1.7) *has no positive solutions* $u \in C^{2\max\{m,k\}}(\Omega)$ *satisfying* (1.6)*, if*

$$\max\{\gamma_{m,k}-(N-2m), \xi_{m,k}-(N-2k)\} \ge 0. \tag{2.6}$$

It admits a positive solution satisfying (1.6)*, if*

$$\max\{\gamma_{m,k}-(N-2m), \xi_{m,k}-(N-2k)\} < 0$$

and $\gamma_{m,k}, \xi_{m,k} > 0$*. Here*

$$\gamma_{m,k} = \frac{(a+2m)(1-T)+(b+2k)R}{QR-(1-T)(1-S)}$$

and

$$\xi_{m,k} = \frac{(a+2m)Q+(b+2k)(1-S)}{QR-(1-T)(1-S)}.$$

The following Liouville-type theorem is a consequence of the above result to the higher-order Hardy-Hénon elliptic system (1.5), which extends the result of Mitidieri-Pohozaev [31] for the case $m=k$ to the case $m \ge k \ge 1$.

Corollary 2.3. *Consider the system* (1.5) *in an exterior domain* Ω *of* $\mathbb{R}^N$*,* $N > \max\{2m, 2k\}$*,* $m, k \ge 1$*. Let* $a, b \in \mathbb{R}$*,* $Q, R > 0$ *with* $QR > 1$*. The system* (1.5) *does not admit any positive solution satisfying* (1.6)*, if*

$$\max\left\{\frac{a+2m+(b+2k)R}{QR-1}-(N-2m),\ \frac{(a+2m)Q+b+2k}{QR-1}-(N-2k)\right\}\geq 0. \tag{2.7}$$

In particular, when $m=k$ the claim holds, if

$$\max\left\{\frac{2m(R+1)+a+bR}{QR-1},\ \frac{2m(Q+1)+b+aQ}{QR-1}\right\}\geq N-2m. \tag{2.8}$$

Example 2.4. Consider positive poly-superharmonic solutions of the weighted higher-order elliptic problem

$$-\Delta\big(|x|^a(-\Delta)^m u\big)=|x|^b u^Q \quad \text{in } \Omega, \tag{2.9}$$

where Ω is an exterior domain in $\mathbb{R}^N$, $N>2m$, $Q>1$, $m\geq 1$ is an integer and $a,b\in\mathbb{R}$ satisfy $2(m+1)-N<a<2(m+1)+b$.

Equation (2.9) with $m=1$, $\Omega=\mathbb{R}^N$, $N>4$ and $4-N<a<\min\{N,b+4\}$, has been considered by Guo et al. [23] where they obtained Liouville type results for nonnegative radial solutions provided $1<p<p_s=\frac{N+4+2b-a}{N-4+a}$. Huang and Wang in [24] obtained partial classifications of positive radial solutions of the same equation for some special cases, see also [18] for more results. This equation is closely related to Caffarelli-Kohn-Nirenberg-type inequalities (CKN)

$$\int\limits_{\mathbb{R}^N}|x|^a|\Delta u|^2\,dx\geq C\Big(\int\limits_{\mathbb{R}^N}|x|^b u^Q\,dx\Big)^{\frac{2}{Q}}$$

for every $u\in C_c^\infty(\mathbb{R}^N)$, see [10]. Equation (2.9) is related to the Hénon-Lane-Emden system

$$\begin{cases}(-\Delta)^m u=|x|^{-a}v & \text{in } \Omega,\\ -\Delta v=|x|^b u^Q & \text{in } \Omega.\end{cases} \tag{2.10}$$

The above system is a special case of the multipower system (1.7) with $k=1$, $R=1$ and $S=T=0$. By Proposition 2.2, there does not exist any positive polysuperharmonic solution for (2.9) in an exterior domain $\mathbb{R}^N$, $N>2m$, if

$$\max\left\{\frac{-a+2m+b+2}{Q-1}-(N-2m),\ \frac{(-a+2m)Q+b+2}{Q-1}-(N-2)\right\}\geq 0, \tag{2.11}$$

or equivalently, using the assumption $2(m+1)-N<a<2(m+1)+b$,

$$1<Q\leq\max\left\{\frac{N+2+b-a}{N-2m},\frac{N+b}{N-2(m+1)+a}\right\}$$
$$=\begin{cases}\frac{N+b}{N-2(m+1)+a}, & a\leq 2,\\ \frac{N+2+b-a}{N-2m}, & a\geq 2.\end{cases}$$

3. Hardy-type inequalities

The proof of Theorem 1.1 is based on the following lemma.

Lemma 3.1. *Let Ω be a domain in $\mathbb{R}^N$, $F \in C^2(\Omega)$ and A be a locally bounded function on Ω with $A(x) > 0$ for every $x \in \Omega$. Then*

$$\int_\Omega \big(-\Delta F - A(x)|\nabla F|^2\big)\phi^2\,dx \le \int_\Omega \frac{|\nabla \phi|^2}{A(x)}\,dx, \tag{3.1}$$

for every $\phi \in C_c^\infty(\Omega)$.

Proof. Let $\phi \in C_c^\infty(\Omega)$ and $\phi_\varepsilon = \sqrt{|\phi|^2+\varepsilon^2}-\varepsilon$, $\varepsilon > 0$. Then $\phi_\varepsilon \in C_0^\infty(\Omega)$ and, by the divergence theorem, we have

$$\int_\Omega (-\Delta F)\phi_\varepsilon^2\,dx = 2\int_\Omega \nabla F \phi_\varepsilon \cdot \nabla\phi_\varepsilon\,dx \le 2\int_\Omega |\nabla F|\phi_\varepsilon|\nabla\phi_\varepsilon|\,dx. \tag{3.2}$$

Since $0 \le \phi_\varepsilon \le |\phi|$, we have

$$\nabla\phi_\varepsilon = \frac{|\phi|\nabla(|\phi|)}{\sqrt{|\phi|^2+\varepsilon^2}}.$$

By the fact that $|\nabla(|\phi|)| \le |\nabla\phi|$ almost everywhere in Ω, we have

$$\phi_\varepsilon|\nabla\phi_\varepsilon| \le |\phi||\nabla(|\phi|)| \le |\phi||\nabla\phi|.$$

Young's inequality implies that

$$\begin{aligned}|\nabla F|\phi_\varepsilon|\nabla\phi_\varepsilon| \le |\nabla F||\phi||\nabla\phi| &= \big(A(x)^{\frac12}|\nabla F||\phi|\big)\frac{|\nabla\phi|}{A(x)^{\frac12}}\\ &\le \frac{A(x)}{2}|\nabla F|^2\phi^2 + \frac{|\nabla\phi|^2}{2A(x)}\end{aligned}$$

and by (3.2) we obtain

$$\int_\Omega -\Delta F\phi^2\,dx \le \int_\Omega A(x)|\nabla F|^2\phi^2\,dx + \int_\Omega \frac{|\nabla\phi|^2}{A(x)}\,dx.$$

This proves (3.1). □

Proof of Theorem 1.1. First we prove (1.1). Let $u > 0$ be a C^2 function on Ω and let $F(x) = \int^{u(x)} \frac{dt}{f(t)}$. Then

$$\Delta F(u) = F''(u)|\nabla u|^2 + F'(u)\Delta u = \frac{\Delta u}{f(u)} - f'(u)\frac{|\nabla u|^2}{f(u)^2}$$

and by (3.1) we get

$$\int_\Omega \left(\frac{-\Delta u}{f(u)} - \Big(f'(u) - A(x)\Big)\frac{|\nabla u|^2}{f(u)^2}\right)\phi^2\,dx \le \int_\Omega \frac{|\nabla\phi|^2}{A(x)}\,dx, \tag{3.3}$$

for every $\phi \in C_0^\infty(\Omega)$. By setting $A(x) = f'(u(x))$ in (3.3) we obtain

$$\int \frac{-\Delta u}{f(u)}\phi^2\,dx \le \int_\Omega \frac{|\nabla\phi|^2}{f'(u)}\,dx.$$

This proves (1.1).

Let $m \ge 2$ be an integer and assume that u is a C^{2m} positive function with $(-\Delta)^i u \ge 0$ in Ω for $i = 1, \ldots, m-1$. Applying (1.1) for $(-\Delta)^{i-1}u$, $i = 2, \ldots, m$, gives

$$\int_\Omega \frac{(-\Delta)^i u}{(-\Delta)^{i-1}u}\phi^2\,dx \le \int_\Omega |\nabla\phi|^2\,dx, \tag{3.4}$$

for every $\phi \in C_c^\infty(\Omega)$. We note that

$$\left(\frac{(-\Delta)^m u}{f(u)}\right)^{\frac{1}{m}}\phi^2 = \prod_{i=2}^m \left(\frac{(-\Delta)^i u}{(-\Delta)^{i-1}u}\phi^2\right)^{\frac{1}{m}}\left(\frac{-\Delta u}{f(u)}\phi^2\right)^{\frac{1}{m}}$$

and by Holder's inequality and (3.4) we obtain

$$\begin{aligned}
\int_\Omega \left(\frac{(-\Delta)^m u}{f(u)}\right)^{\frac{1}{m}}\phi^2\,dx &\le \prod_{i=2}^m \left(\int_\Omega \frac{(-\Delta)^i u}{(-\Delta)^{i-1}u}|\phi|^2\,dx\right)^{\frac{1}{m}}\left(\int_\Omega \frac{-\Delta u}{f(u)}\phi^2\,dx\right)^{\frac{1}{m}} \\
&\le \left(\int_\Omega |\nabla\phi|^2\,dx\right)^{\frac{m-1}{m}}\left(\int_\Omega \frac{-\Delta u}{f(u)}\phi^2\,dx\right)^{\frac{1}{m}} \\
&\le \left(\int_\Omega |\nabla\phi|^2\,dx\right)^{\frac{m-1}{m}}\left(\int_\Omega \frac{|\nabla\phi|^2}{f'(u)}\,dx\right)^{\frac{1}{m}}.
\end{aligned}$$

This proves (1.2). □

As an immediate application of (1.1) we have the following generalization of the Hardy inequality.

Corollary 3.2. *Let Ω be a domain in $\mathbb{R}^N$ and $a \in \mathbb{R}$. If $0 \in \Omega$ we assume $a > 2 - N$. Then*

$$\left(\frac{N-2+a}{2}\right)^2 \int_\Omega |x|^{a-2}\phi^2\,dx \le \int_\Omega |x|^a |\nabla\phi|^2\,dx, \tag{3.5}$$

for every $\phi \in C_c^\infty(\Omega)$.

Proof. First we apply (1.1) with $f(u) = \frac{u^t}{t}$, $t \neq 0$, to obtain

$$t \int \frac{-\Delta u}{u^t}\phi^2\,dx \le \int_\Omega \frac{|\nabla\phi|^2}{u^{t-1}}\,dx,$$

for every $\phi \in C_c^\infty(\Omega)$. Note also that in the case $N - 2 + a = 0$, (3.5) is obvious. Now assume that $a \neq N - 2$ and $0 \notin \Omega$. We apply the estimate above with

$$u(x) = |x|^{\frac{-(N-2-a)}{2}} \quad \text{and} \quad t = \frac{N-2+a}{N-2-a}.$$

Noting that

$$-\Delta u(x) = \frac{(N-2-a)(N-2+a)}{4}|x|^{\frac{-N-2+a}{2}},$$

we find that

$$t\frac{-\Delta u}{u^t} = \left(\frac{N-2+a}{2}\right)^2 |x|^{a-2} \quad \text{and} \quad u^{t-1} = |x|^{-a}.$$

This proves (3.5). If $a > 2 - N$ and $0 \in \Omega$, we may apply a similar argument with

$$u(x) = (|x|^2 + \varepsilon)^{-\frac{N-2-a}{4}}, \quad \varepsilon > 0,$$

and then pass to the limit as $\varepsilon \to 0$ using the fact that $|x|^{a-2} \in L^1_{\mathrm{loc}}(\Omega)$. When $a = N - 2$ we may apply (3.5) with $a \neq N - 2$ and the result follows by letting $a \to N - 2$. □

4. Proofs of Liouville-type results

We start with recalling the following result, see Proposition 2.7 and Theorem 3.1 in [8].

Lemma 4.1. *Let $N \ge 2$. Assume that u is a nonnegative solution of*

$$-\Delta u \ge C|x|^\lambda \quad \text{in } \mathbb{R}^N \setminus \overline{B_1}$$

for some $\lambda \in \mathbb{R}$ and $C > 0$ with $u \neq 0$. Then $\lambda + 2 < 0$ and there exists a constant $C > 0$ such that

$$\begin{cases} u(x) \geq C|x|^{\lambda+2}, & \lambda \neq -N, \\ u(x) \geq C|x|^{2-N} \ln|x|, & \lambda = -N \end{cases}$$

for $|x| > 2$*. Assume that* u *is a nonnegative solution of*

$$-\Delta u \geq |x|^{\sigma} u^{Q} \quad \text{in } \mathbb{R}^N \setminus \overline{B_1},$$

where $Q < 1$*,* $\sigma \in \mathbb{R}$*. Then there exists a constant* $C > 0$ *such that*

$$u(x) \geq C|x|^{\frac{2+\sigma}{1-Q}} \quad \text{in } \mathbb{R}^N \setminus \overline{B_1}.$$

In the following we extend the above results for the higher order case with a different proof based on the maximum principle.

Proposition 4.2. *Let* $N > 2m$*. Assume that* $u \in C^{2m}(\mathbb{R}^N \setminus B_1)$ *is a positive polysuperharmonic function.*

(i) *If*

$$(-\Delta)^m u \geq C|x|^a \quad \text{in } \mathbb{R}^N \setminus \overline{B_1}, \tag{4.1}$$

for some $C > 0$ *and* $a < 0$*, then* $a + 2m < 0$ *and*

$$u(x) \geq C(a)|x|^{a+2m} \quad \text{in } \mathbb{R}^N \setminus \overline{B_1}. \tag{4.2}$$

Moreover, if $a + N > 0$*, we may choose* $C(a) = \min\{C, l\} P(a)$*, where*

$$l = \min_{1 \leq i \leq m-1} l_i, \quad l_i = \min_{|x|=1} (-\Delta)^i u(x)$$

$$P(a) = \min\left\{1, \frac{1}{\max_{1 \leq i \leq m} P_i(a)}\right\},$$

$$P_i(a) = \prod_{j=i}^{m} |(a+2j)(N+a+2j-2)|.$$

(ii) *If*

$$(-\Delta)^m u \geq C|x|^{-N} \quad \text{in } \mathbb{R}^N \setminus \overline{B_1}, \tag{4.3}$$

then

$$u(x) \geq C|x|^{2m-N} \ln|x| \quad \text{in } \mathbb{R}^N \setminus \overline{B_2}. \tag{4.4}$$

(iii) *Assume that u is a positive solution of*

$$(-\Delta)^m u \geq |x|^a u^Q \quad \textit{in } \mathbb{R}^N \setminus \overline{B_1}, \tag{4.5}$$

where $a \in \mathbb{R}$ *and* $Q > 0$. *Then* $Q > \frac{a+2m}{N-2m}$. *Moreover, if* $a + 2m < 0$ *and* $Q < Q_a = \frac{a+N}{N-2m}$, *then*

$$u(x) \geq C|x|^{\frac{2m+a}{1-Q}} \quad \textit{in } \mathbb{R}^N \setminus \overline{B_1}. \tag{4.6}$$

Proof. (i) Let u be a positive solution of (4.1). Then we have $-\Delta w \geq C|x|^a$, where $w = (-\Delta u)^{m-1} \geq 0$. Lemma 4.1 implies that $a + 2 < 0$ and thus

$$(-\Delta u)^{m-1} \geq C|x|^{2+a}.$$

We apply this recursively m times to conclude that $a + 2m < 0$ and to obtain (4.2). However, in order to prove (4.6) in (iii) we need more information on the constant C above and thus we give a different proof for (4.2) including a lower bound for C. First assume that $-N < a < -2$ and $-\Delta u \geq C|x|^a$ in $\mathbb{R}^N \setminus \overline{B_1}$. Let $l_0 = \min_{|x|=1} u(x)$ and

$$C_1 = \min\left\{l_0, \frac{C}{-(a+2)(N+a)}\right\} \geq \min\{l_0, C\} \min\left\{1, -\frac{1}{(a+2)(N+a)}\right\}.$$

We show that

$$u(x) \geq C_1|x|^{2+a} \quad \text{in } \mathbb{R}^N \setminus \overline{B_1}. \tag{4.7}$$

Note that $u(x) \geq C_1|x|^{a+2}$ on $|x| = 1$. For every $\varepsilon > 0$ there exists $R_\varepsilon > 1$ such that

$$u(x) + \varepsilon \geq \varepsilon \geq C_1|x|^{a+2}, \quad x \in \mathbb{R}^N \setminus \overline{B_{R_\varepsilon}}.$$

We have

$$-\Delta(u + \varepsilon) = -\Delta u \geq C|x|^a \geq -C_1(a+2)(N+a)|x|^a = -\Delta(C_1|x|^{a+2})$$

and, by the maximum principle on $B_R \setminus \overline{B_1}$, $R > R_\varepsilon$, we conclude that

$$u(x) + \varepsilon \geq C_1|x|^{2+a} \quad \text{in } \mathbb{R}^N \setminus \overline{B_1}.$$

By letting $\varepsilon \to 0$ we arrive at (4.7).

Then assume that u is a solution of (4.1) with $m = 2$ and $-N < a < -4$. We have $-\Delta w \geq C|x|^a$, where $w = -\Delta u \geq 0$. As above, we obtain

$$-\Delta u \geq C'|x|^{2+a} \quad \text{in } \mathbb{R}^N \setminus \overline{B_1}, \tag{4.8}$$

where

$$C' = \min\left\{l_1, \frac{C}{-(a+2)(N+a)}\right\}, \quad l_1 = \min_{|x|=1} -\Delta u(x).$$

Again from (4.8) and (i) (note that $-N < a+2 < -2$) we obtain

$$u(x) \geq C_2|x|^{4+a} \quad \text{in } \mathbb{R}^N \setminus \overline{B_1},$$

where

$$\begin{aligned} C_2 &= \min\left\{l_0, \frac{C'}{-(a+4)(N+a+2)}\right\} \\ &= \min\left\{l_0, \frac{l_1}{-(a+4)(N+a+2)}, \frac{C}{(a+2)(a+4)(N+a)(N+a+2)}\right\} \\ &\geq \min\{l_0, l_1, C\} \min\left\{1, \frac{1}{-(a+4)(N+a+2)}, \frac{1}{(a+2)(a+4)(N+a)(N+a+2)}\right\}. \end{aligned}$$

We apply this recursively to obtain (4.2).

(ii) Assume that u satisfies (4.3). Then $-\Delta w \geq C|x|^{-N}$ in $\mathbb{R}^N \setminus B_1$, where $w = (-\Delta)^{m-1}u$. By Lemma 4.1 we have

$$(-\Delta)^{m-1}u \geq C|x|^{2-N} \ln|x| \quad \text{in } \mathbb{R}^N \setminus \overline{B_2}.$$

Note that $|x|^{2-N} \ln|x| \geq -\Delta(C_N|x|^{4-N} \ln|x|)$ when $|x| > 2$, for a positive constant C_N. Indeed, for $|x| > 2$ we have

$$\begin{aligned} -\Delta(|x|^{4-N} \ln|x|) &= 2(N-4)|x|^{2-N} \ln|x| + (N-6)|x|^{2-N} \\ &\leq C|x|^{2-N} \ln|x| \quad \text{for} \quad C > \frac{N-6}{\ln 2} + 2(N-4). \end{aligned}$$

Hence, $(-\Delta)^{m-1}u \geq -\Delta(C|x|^{4-N} \ln|x|)$ in $\mathbb{R}^N \setminus \overline{B_2}$ and applying the maximum principle again, as the proof of (i), we get

$$(-\Delta)^{m-2}u \geq C|x|^{4-N} \ln|x| \quad \text{in } \mathbb{R}^N \setminus \overline{B_2}.$$

By applying the argument recursively m times we arrive at (4.4).

(iii) Assume that $u > 0$ is a positive polysuperharmonic solution of (4.5). Then $w = (-\Delta)^{m-1}u > 0$ and $-\Delta w > 0$. It is well-known that

$$w(x) = (-\Delta)^{m-1}u(x) \geq c|x|^{2-N},$$

for $|x| \geq 1$ (for example, see Lemma 2.1 in [36]). Since $-N < 2-N < -2(m-1)$, from (i) we get $u(x) \geq C_0|x|^{2m-N}$. Applying this estimate in (4.5) gives

$$(-\Delta)^m u \geq |x|^a u^Q \geq C_0^Q |x|^{a_0} \quad \text{in } \mathbb{R}^N \setminus \overline{B_1},$$

where $a_0 = a + Q(2m - N)$. First note that by (i) we have $a_0 + 2m < 0$ or $Q > \frac{a+2m}{N-2m}$. Let $a + 2m < 0$ or equivalently $Q_a < 1$, then we have $-N < a_0 < -2m$, which is equivalent to $Q < Q_a < 1$. Hence we may apply (i) to obtain

$$u(x) \geq C(a_0)|x|^{a_0+2m}, \quad C(a_0) = \min\{C_0^Q, l\}P(a_0). \tag{4.9}$$

By (4.9) in (4.5) we have

$$(-\Delta)^m u \geq C(a_0)^Q |x|^{a_1} \text{ in } \mathbb{R}^N \setminus \overline{B_1},$$

where $a_1 = a + Q(a_0 + 2m)$. Again we have $-N < a_1 < -2m$ thus we can apply (i) to the estimate above and obtain

$$u(x) \geq C(a_1)|x|^{a_1+2m}, \quad C(a_1) = \min\{C(a_0)^Q, l\}P(a_1).$$

Recursively, for every integer $j \geq 2$, we obtain

$$u(x) \geq C(a_j)|x|^{a_j+2m} \quad \text{in } \mathbb{R}^N \setminus \overline{B_1}, \tag{4.10}$$

where

$$a_j = a + Q(a_{j-1} + 2m), \quad C(a_j) = \min\{C(a_{j-1})^Q, l\}P(a_j).$$

Since $Q > 0$ it is easy to see that (a_j) is a monotone nondecreasing sequence and

$$a_j \to \frac{2mQ + a}{1 - Q} \quad \text{as } j \to \infty.$$

Then $P(a_j) \to P(\frac{2mQ+a}{1-Q})$ as $j \to \infty$, which implies that $C(a_j) \geq C > 0$ for every $j \geq 1$. We obtain the desired result by letting $j \to \infty$ in (4.10). □

Proof of Proposition 2.1. For simplicity let $\Omega = R^N \setminus \overline{B_{R_0}}$, for some $R_0 > 0$. Let u be a smooth positive solution. Then dividing (1.4) by u and raising to the power $\frac{1}{m}$ we get

$$\left(\frac{(-\Delta)^m u}{u} + \frac{(-\Delta)^k u}{u}\right)^{\frac{1}{m}} \geq |x|^{\frac{a}{m}} h(u)^{\frac{1}{m}} \quad \text{in } \Omega.$$

By the elementary inequality $(x+y)^{\frac{1}{m}} \leq x^{\frac{1}{m}} + y^{\frac{1}{m}}$ for positive numbers x, y and $m \geq 1$ we infer that

$$\left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} + \left(\frac{(-\Delta)^k u}{u}\right)^{\frac{1}{m}} \geq |x|^{\frac{a}{m}} h(u)^{\frac{1}{m}} \quad \text{in } \Omega.$$

By multiplying the inequality above by ϕ^2 and integrating over Ω we obtain

$$\int_{\Omega} \left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} \phi^2\, dx + \int_{\Omega} \left(\frac{(-\Delta)^k u}{u}\right)^{\frac{1}{m}} \phi^2\, dx \geq \int_{\Omega} |x|^{\frac{a}{m}} h(u)^{\frac{1}{m}} \phi^2\, dx, \tag{4.11}$$

for every $\phi \in C_c^{\infty}(\Omega)$. From Theorem 1.1 we have

$$\int_{\Omega} \left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} \phi^2\, dx \leq \int_{\Omega} |\nabla\phi|^2\, dx.$$

Assume that $m > k$. By Hölder's inequality and Theorem 1.1, we have

$$\begin{aligned}\int_{\Omega} \left(\frac{(-\Delta)^k u}{u}\right)^{\frac{1}{m}} \phi^2\, dx &= \int_{\Omega} \left(\frac{(-\Delta)^k u}{u}\phi^2\right)^{\frac{1}{m}} \phi^{2-\frac{2}{m}}\, dx \\ &\leq \left(\int_{\Omega} \left(\frac{(-\Delta)^k u}{u}\right)^{\frac{1}{k}} \phi^2\, dx\right)^{\frac{k}{m}} \left(\int_{\Omega} \phi^{\frac{2(m-1)}{m-k}}\, dx\right)^{\frac{m-k}{m}} \\ &\leq \left(\int_{\Omega} |\nabla\phi|^2\, dx\right)^{\frac{k}{m}} \left(\int_{\Omega} \phi^{\frac{2(m-1)}{m-k}}\, dx\right)^{\frac{m-k}{m}}.\end{aligned}$$

By using the estimate above in (4.11) we arrive at

$$\int_{\Omega} |x|^{\frac{a}{m}} h(u)^{\frac{1}{m}} \phi^2\, dx \leq \int_{\Omega} |\nabla\phi|^2\, dx + \left(\int_{\Omega} |\nabla\phi|^2\, dx\right)^{\frac{k}{m}} \left(\int \phi^{\frac{2(m-1)}{m-k}}\, dx\right)^{\frac{m-k}{m}}, \tag{4.12}$$

for every $\phi \in C_c^{\infty}(\Omega)$.

Let $R > 4R_0$ and let ϕ_R be a smooth function in Ω such that $0 \leq \phi_R \leq 1$, $x \in \Omega$, $\phi_R = 0$ when $R_0 < |x| < \frac{R}{4}$ and $|x| > 2R$, $\phi_R = 1$ in $\frac{R}{2} < |x| < R$ and $|\nabla\phi_R| \leq \frac{c}{R}$ in Ω. We apply (4.12) with the test function ϕ_R and obtain

$$\begin{aligned}C R^{\frac{a}{m}} \int_{B_R \setminus B_{\frac{R}{2}}} h(u)^{\frac{1}{m}}\, dx &\leq \int_{B_R \setminus B_{\frac{R}{2}}} |x|^{\frac{a}{m}} h(u)^{\frac{1}{m}} \phi_R^2\, dx \\ &\leq C\big(R^{N-2} + R^{\frac{k}{m}(N-2)+\frac{m-k}{m}N}\big),\end{aligned}$$

for $R > 4R_0$. This implies that

$$\int_{B_R \setminus B_{\frac{R}{2}}} h(u)^{\frac{1}{m}}\, dx \leq C R^{N-\frac{2k+a}{m}},$$

for every $R > 4R_0$, where C is a constant independent of u, g and R. This proves (i).

To prove (ii) we use the fact that since u is polysuperharmonic we have $u(x) \geq c|x|^{2m-N}$ for $|x| > R_0$ (see for example [13]) and the assumption that h is a nondecreasing function to get

$$\int\limits_{B_R \setminus B_{\frac{R}{2}}} h(c|x|^{2m-N})^{\frac{1}{m}}\, dx \leq C R^{N-\frac{2k+a}{m}},$$

and

$$R^N h(cR^{2m-N})^{\frac{1}{m}} \leq C R^{N-\frac{2k+a}{m}},$$

for any large R. Taking $cR^{2m-N} = r$ implies that

$$\frac{g(r)}{r^{\frac{N-2(m-k)+a}{N-2m}}} \leq C,$$

for any small $r > 0$. Thus there does not exist a solution if

$$\limsup_{r\to 0} \frac{g(r)}{r^{\frac{N-2(m-k)+a}{N-2m}}} = \infty.$$

If $g(u) = u^p$, $p \geq 1$, then we have

$$\limsup_{r\to 0} r^{p-\frac{N-2(m-k)+a}{N-2m}} = \infty,$$

which is the case if

$$1 \leq p < \frac{N+a-2(m-k)}{N-2m}.$$

When $m = k$ we can easily show that the nonexistence result holds also for the case $p = \frac{N+a}{N-2m}$. Indeed in this case from the equation of u and that we have $u(x) \geq c|x|^{2m-N}$ we obtain

$$-(\Delta)^m u \geq C|x|^a u^{\frac{N+a}{N-2m}} \geq C|x|^{-N},$$

then by Proposition 4.2

$$u(x) \geq C|x|^{2m-N} \ln|x|.$$

Also from (4.16) (here we have $h(u) = u^{p-1}$)

$$\int\limits_{B_R \setminus B_{\frac{R}{2}}} u^{\frac{p-1}{m}}\, dx \leq C R^{N-\frac{2m+a}{m}},$$

implies that for large R

$$\left(R^{2m-N}\ln R\right)^{\frac{p-1}{m}} R^N \le C R^{N-\frac{2m+a}{m}},$$

or equivalently $\ln R \le C$ for large R. This is a contradiction.

Then we assume (2.2) holds. This implies that

$$g(r) \le C r^{\sigma}, \quad r < r_0 \tag{4.13}$$

for some constants $C, r_0 > 0$. We show that for a suitable $A > 0$ the function

$$u(x) = A|x|^{-t}, \quad t = \frac{2k+a}{\sigma - 1} > 0$$

solves (1.4) in $\mathbb{R}^N \setminus \overline{B_{R_0}}$ for R_0 large. We notice that

$$(-\Delta)^i u(x) = A C_i |x|^{-2i-t}, \quad i = 1, ..., m,$$

where

$$C_i = \prod_{j=1}^{i} (t + 2(j-1))(N - 2j - t)$$

and note that by the assumption $\sigma > \sigma_a = \frac{N-2(m-k)+a}{N-2m}$ we have $t < N-2j$ for every $j = 1, ..., m$ thus $C_i > 0$, means that $(-\Delta)^i u(x) > 0$, $i = 1, ..., m$. Substituting $u(x) = A|x|^{-t}$ in (1.4) we see that we need

$$AC_m|x|^{-2m-t} + AC_k|x|^{-2k-t} \ge |x|^a g(A|x|^{-t}), \quad |x| > R_0, \tag{4.14}$$

which holds if

$$AC_k \ge |x|^{a+2k+t} g(A|x|^{-t}), \quad |x| > R_0.$$

Note that by (4.13) we have

$$g(A|x|^{-t}) \le C(A|x|^{-t})^{\sigma}, \quad A|x|^{-t} < r_0.$$

So it suffices to have

$$A^{1-\sigma} C_K \ge C|x|^{a+2k-t(\sigma-1)} = C, \quad |x| > \left(\frac{A}{r_0}\right)^{\frac{1}{t}} = R_0.$$

Thus for suitable $A > 0$ we see that $u(x) = A|x|^{-t}$ solves (1.4) for $|x| > R_0$. □

Proof of Proposition 2.2. Let (u, v) be a solution of system (1.7) satisfying (1.6). Then

$$\begin{cases} \left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} \geq |x|^{\frac{a}{m}} u^{\frac{S-1}{m}} v^{\frac{R}{m}} & \text{in } \Omega, \\ \left(\frac{(-\Delta)^k v}{v}\right)^{\frac{1}{k}} \geq |x|^{\frac{b}{k}} u^{\frac{Q}{k}} v^{\frac{T-1}{k}} & \text{in } \Omega. \end{cases} \tag{4.15}$$

Adding the above inequalities and using Young's inequality, for any $0 < \lambda < 1$, we have

$$\begin{aligned} \left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} + \left(\frac{(-\Delta)^k v}{v}\right)^{\frac{1}{k}} &\geq |x|^{\frac{a}{m}} u^{\frac{S-1}{m}} v^{\frac{R}{m}} + |x|^{\frac{b}{k}} u^{\frac{Q}{k}} v^{\frac{T-1}{k}} \\ &\geq C_\lambda \big(|x|^{\frac{a}{m}} u^{\frac{S-1}{m}} v^{\frac{R}{m}}\big)^\lambda \big(|x|^{\frac{b}{k}} u^{\frac{Q}{k}} v^{\frac{T-1}{k}}\big)^{1-\lambda} \\ &= C_\lambda |x|^{\lambda(\frac{a}{m}-\frac{b}{k})+\frac{b}{k}} u^{\frac{Q}{k}-\lambda(\frac{Q}{k}+\frac{1-S}{m})} v^{\lambda(\frac{1-T}{k}+\frac{R}{m})-\frac{1-T}{k}} \end{aligned}$$

in Ω. By letting

$$\lambda = \frac{\frac{Q}{k}}{\frac{Q}{k} - \frac{S-1}{m}} = \frac{mQ}{mQ + k(1-S)}$$

we arrive at

$$\left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} + \left(\frac{(-\Delta)^k u}{v}\right)^{\frac{1}{k}} \geq C|x|^{\frac{aQ+b(1-S)}{mQ+k(1-S)}} v^{\frac{QR-(1-T)(1-S)}{mQ+k(1-S)}} \quad \text{in } \Omega.$$

Multiply the inequality by ϕ^2, $\phi \in C_c^\infty(\Omega)$, integrate over Ω and apply the Hardy-type inequality in Theorem 1.1 to obtain

$$\begin{aligned} C\int_\Omega |x|^{\frac{aQ+b(1-S)}{mQ+k(1-S)}} v^{\frac{QR-(1-T)(1-S)}{mQ+k(1-S)}} \phi^2\, dx &\leq \int_\Omega \left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} \phi^2\, dx + \int_\Omega \left(\frac{(-\Delta)^k u}{v}\right)^{\frac{1}{k}} \phi^2\, dx \\ &\leq 2\int_\Omega |\nabla \phi|^2\, dx. \end{aligned}$$

We apply the same test function ϕ_R as in the proof of Proposition 2.1 to get

$$CR^{\frac{aQ+b(1-S)}{mQ+k(1-S)}} \int_{B_R \setminus B_{\frac{R}{2}}} v^{\frac{QR-(1-T)(1-S)}{mQ+k(1-S)}}\, dx \leq R^{N-2},$$

or equivalently

$$\int_{B_R \setminus B_{\frac{R}{2}}} v^{\frac{QR-(1-T)(1-S)}{mQ+k(1-S)}}\, dx \leq CR^{N-2-\frac{aQ+b(1-S)}{mQ+k(1-S)}}. \tag{4.16}$$

By letting

$$\lambda = \frac{\frac{T-1}{k}}{\frac{T-1}{k} - \frac{R}{m}} = \frac{m(1-T)}{m(1-T)+kR}$$

we arrive at

$$\left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} + \left(\frac{(-\Delta)^k u}{v}\right)^{\frac{1}{k}} \geq C|x|^{\frac{a(1-T)+bR}{m(1-T)+kR}} u^{\frac{QR-(1-T)(1-S)}{m(1-T)+kR}} \quad \text{in } \Omega.$$

We multiply the inequality above by the test function ϕ_R, integrate over Ω and as above we obtain

$$R^{\frac{a(1-T)+bR}{m(1-T)+kR}} \int_{B_R \setminus B_{\frac{R}{2}}} u^{\frac{QR-(1-T)(1-S)}{m(1-T)+kR}}\, dx \leq R^{N-2},$$

or equivalently

$$\int_{B_R \setminus B_{\frac{R}{2}}} u^{\frac{QR-(1-T)(1-S)}{m(1-T)+kR}}\, dx \leq C R^{N-2-\frac{a(1-T)+bR}{m(1-T)+kR}}. \tag{4.17}$$

Hence, we see that (4.16) and (4.17) prove (i).

In order to get an integral estimate for the negative power of u, v, let

$$\alpha = \frac{QR}{1-T} + S \quad \text{and} \quad \beta = \frac{QR}{1-S} + T$$

and note that we have $\alpha, \beta > 1$ by the assumption $QR > (1-S)(1-T)$. Then we divide inequalities in (1.7) by u^α and v^β and raising to the power $\frac{\lambda}{m}$ and $\frac{1-\lambda}{k}$, $0 < \lambda < 1$, respectively. We arrive at

$$\left(\frac{(-\Delta)^m u}{u^\alpha}\right)^{\frac{\lambda}{m}} \left(\frac{(-\Delta)^k v}{v^\beta}\right)^{\frac{1-\lambda}{k}} \geq |x|^{\lambda(\frac{a}{m}-\frac{b}{k})+\frac{b}{k}} u^{\frac{Q}{k}-\lambda(\frac{Q}{k}+\frac{\alpha-S}{m})} v^{\lambda(\frac{\beta-T}{k}+\frac{R}{m})-\frac{\beta-T}{k}} \quad \text{in } \Omega. \tag{4.18}$$

We first choose λ so that $\frac{Q}{k} - \lambda\left(\frac{Q}{k} + \frac{\alpha-S}{m}\right) = 0$, that is,

$$\lambda = \frac{mQ}{mQ + k(\alpha - S)} = \frac{m(1-T)}{m(1-T)+kR},$$

to get

$$\left(\frac{(-\Delta)^m u}{u^\alpha}\right)^{\frac{\lambda}{m}} \left(\frac{(-\Delta)^k v}{v^\beta}\right)^{\frac{1-\lambda}{k}} \geq \frac{|x|^{\frac{aQ+b(\alpha-S)}{mQ+k(\alpha-S)}}}{v^\theta},$$

where

$$\theta = \frac{R(QR-(1-S)(1-T))}{(1-S)(m(1-T)+kR)}.$$

We multiply the inequality above by ϕ_R^2 and integrate over Ω to get

$$\int_\Omega \left(\frac{(-\Delta)^m u}{u^\alpha}\right)^{\frac{\lambda}{m}} \left(\frac{(-\Delta)^k v}{v^\beta}\right)^{\frac{1-\lambda}{k}} \phi_R^2\, dx \ge \int_\Omega \frac{|x|^{\frac{aQ+b(\alpha-S)}{mQ+k(\alpha-S)}}}{v^\theta} \phi_R^2\, dx. \tag{4.19}$$

Hölder's inequality implies that

$$\begin{aligned}
&\int_\Omega \left(\frac{(-\Delta)^m u}{u^\alpha}\right)^{\frac{\lambda}{m}} \left(\frac{(-\Delta)^k v}{v^\beta}\right)^{\frac{1-\lambda}{k}} \phi_R^2\, dx \\
&= \int_\Omega \left(\frac{(-\Delta)^m u}{u^\alpha}\right)^{\frac{\lambda}{m}} \phi_R^{2\lambda} \left(\frac{(-\Delta)^k v}{v^\beta}\right)^{\frac{1-\lambda}{k}} \phi_R^{2(1-\lambda)}\, dx \\
&\le \left(\int_\Omega \left(\frac{(-\Delta)^m u}{u^\alpha}\right)^{\frac{1}{m}} \phi_R^2\, dx\right)^{\lambda} \left(\int \left(\frac{(-\Delta)^k v}{v^\beta}\right)^{\frac{1}{k}} \phi_R^2\, dx\right)^{1-\lambda}.
\end{aligned}$$

From our Hardy-type inequality in Theorem 1.1, with $f(u) = u^\alpha$, we obtain

$$\begin{aligned}
\int_\Omega \left(\frac{(-\Delta)^m u}{u^\alpha}\right)^{\frac{1}{m}} \phi_R^2\, dx &\le \alpha^{\frac{-1}{m}} \left(\int_\Omega |\nabla \phi_R|^2\, dx\right)^{\frac{m-1}{m}} \left(\int_\Omega \frac{|\nabla \phi_R|^2}{u^{\alpha-1}}\, dx\right)^{\frac{1}{m}} \\
&\le C R^{\frac{(m-1)(N-2)}{m}} R^{\frac{N-2+(\alpha-1)(N-2m)}{m}} = C R^{N-2+\frac{(\alpha-1)(N-2m)}{m}}.
\end{aligned}$$

Similarly we also obtain

$$\int_\Omega \left(\frac{(-\Delta)^k v}{v^\beta}\right)^{\frac{1}{k}} \phi_R^2\, dx \le C R^{N-2+\frac{(\beta-1)(N-2k)}{k}}.$$

Therefore, we have

$$\begin{aligned}
&\int_\Omega \left(\frac{(-\Delta)^m u}{u^\alpha}\right)^{\frac{\lambda}{m}} \left(\frac{(-\Delta)^k v}{v^\beta}\right)^{\frac{1-\lambda}{k}} \phi_R^2\, dx \\
&\le C R^{\lambda\left(N-2+\frac{(\alpha-1)(N-2m)}{m}\right)+(1-\lambda)\left(R^{N-2+\frac{(\beta-1)(N-2k)}{k}}\right)} \\
&= C R^{N-2+\frac{Q(\alpha-1)(N-2m)+(\alpha-S)(\beta-1)(N-2k)}{mQ+k(\alpha-S)}} \\
&= C R^{N-2+(QR-(1-S)(1-T))\left(\frac{(N-2m)\frac{Q}{1-T}}{mQ+k(\alpha-S)} + \frac{(N-2k)\frac{QR}{(1-T)(1-S)}}{mQ+k(\alpha-S)}\right)} \\
&= C R^{N-2+(QR-(1-S)(1-T))\frac{N-2m+\frac{R(N-2k)}{1-S}}{m(1-T)+kR}}.
\end{aligned}$$

Using the above estimates in (4.19) we obtain

$$R^{\frac{aQ+b(\alpha-S)}{mQ+k(\alpha-S)}} \int\limits_{B_{2R}\setminus B_R} \frac{dx}{v^\theta} \le C R^{N-2+(QR-(1-S)(1-T))\frac{N-2m+\frac{R(N-2k)}{1-S}}{m(1-T)+kR}},$$

which implies

$$\fint\limits_{B_{2R}\setminus B_R} \frac{dx}{v^\theta} \le C R^\eta,$$

with

$$\eta = -2 + \frac{(QR-(1-S)(1-T))\big(N-2m+\frac{R(N-2k)}{1-S}\big) - (a(1-T)+bR)}{m(1-T)+kR}.$$

We can also choose λ in (4.18) so that $\lambda(\frac{\beta-T}{k}+\frac{R}{m})-\frac{\beta-T}{k}=0$ or equivalently

$$\lambda = \frac{m(\beta-T)}{m(\beta-T)+kR} = \frac{mQ}{mQ+k(1-S)}.$$

Then

$$\left(\frac{(-\Delta)^m u}{u^\alpha}\right)^{\frac{\lambda}{m}} \left(\frac{(-\Delta)^k v}{v^\beta}\right)^{\frac{1-\lambda}{k}} \ge \frac{|x|^{\frac{a(\beta-T)+bR}{m(\beta-T)+kR}}}{u^\zeta},$$

where

$$\zeta = \frac{Q(QR-(1-S)(1-T))}{(1-T)(mQ+k(1-S))}.$$

We may proceed as above to arrive at

$$\fint\limits_{B_{2R}\setminus B_R} \frac{dx}{u^\zeta} \le C R^\kappa,$$

where

$$\kappa = -2 + \frac{(QR-(1-S)(1-T))\big(\frac{Q(N-2m)}{1-T}+N-2k\big) - (aQ+b(1-S))}{mQ+k(1-S)}.$$

Thus completes the proof of (i).

To prove (ii) we first use the fact that $v(x) \ge c|x|^{2k-N}$ in (4.16) to get

$$R^{N+\frac{QR-(1-T)(1-S)}{mQ+k(1-S)}(2k-N)} \le C R^{N-2-\frac{aQ+b(1-S)}{mQ+k(1-S)}},$$

or equivalently, $R^{\xi_{m,k}-(N-2k)} \le C$. This is impossible if

$$\xi_{m,k} > N - 2k. \tag{4.20}$$

We next use the fact that $u(x) \ge c|x|^{2m-N}$ in (4.17) to obtain

$$R^{N+\frac{QR-(1-T)(1-S)}{m(1-T)+kR}(2m-N)} \le C\,R^{N-2-\frac{a(1-T)+bR}{m(1-T)+kR}},$$

which implies that $R^{\gamma_{m,k}-(N-2m)} \le C$, which is impossible if

$$\gamma_{m,k} > N - 2m. \tag{4.21}$$

Hence, there does not exist a positive solution in the case of (4.20) or (4.21). It remains to consider the case

$$\max\{\xi_{m,k} - (N-2k), \gamma_{m,k} - (N-2m)\} = 0$$

We only discuss the case $\xi_{m,k} = N - 2k$ and $\gamma_{m,k} \le N - 2m$, since the other case is similar. By applying the estimate $v(x) \ge C|x|^{2k-N}$ we get

$$(-\Delta)^m u \ge |x|^a u^S v^R \ge C|x|^{a-R\xi_{m,k}} u^S. \tag{4.22}$$

Note also from the definition of $\xi_{m,k}, \gamma_{m,k}$ we have

$$\begin{cases} R\xi_{m,k} - (1-S)\gamma_{m,k} = a + 2m, \\ Q\gamma_{m,k} - (1-T)\xi_{m,k} = b + 2k. \end{cases} \tag{4.23}$$

If $S = 0$ in (4.22), then by Proposition 4.2 (i) we have

$$u(x) \ge C|x|^{a-R\xi_{m,k}+2m} = C|x|^{-\gamma_{m,k}}.$$

If $S \ne 0$ then if $\gamma_{m,k} < N - 2m$, then

$$\frac{a - R\xi_{m,k} + N}{N-2m} = \frac{N - 2m - (1-S)\gamma_{m,k}}{N-2m} > \frac{N - 2m - (1-S)(N-2m)}{N-2m} = S.$$

Hence we may apply part (iii) of Proposition 4.2 to (4.22) and get

$$u(x) \ge C|x|^{\frac{a-R\xi_{m,k}+2m}{1-S}} = C|x|^{-\gamma_{m,k}}.$$

Also, if $S \ne 0$ and $\gamma_{m,k} = N - 2m$ we have from (4.22)

$$(-\Delta)^m u \ge |x|^a u^S v^R \ge C|x|^{a-R\xi_{m,k}-S\gamma_{m,k}} = C|x|^{-2m-\gamma_{m,k}},$$

and again by Proposition 4.2 (i)

$$u(x) \geq C|x|^{-\gamma_{m,k}}.$$

Next we use the above estimate in the second inequality in (1.7) and obtain

$$(-\Delta)^k v \geq |x|^b u^Q v^T \geq C|x|^{b-Q\gamma_{m,k}} v^T.$$

If $T = 0$ then from (4.23) $b - Q\gamma_{m,k} = -N$, hence from Proposition 4.2, part (ii),

$$v(x) \geq C|x|^{2k-N} \ln|x| = C|x|^{-\xi} \ln|x|$$

and using this in (4.16) gives $\ln R \leq C$ for any R large, a contradiction. If $T \neq 0$ then from the second equality in (4.23) we get $b - Q\gamma_{m,k} = -N + T\xi_{m,k}$ and then

$$(-\Delta)^k v \geq C|x|^{b-Q\gamma_{m,k}} v^T \geq C|x|^{-N}$$

gives a contradiction as before.

To complete the proof of the proposition we show that if

$$\gamma_{m,k}, \xi_{m,k} > 0 \quad \text{with} \quad \max\{\gamma_{m,k} - (N-2m), \quad \xi_{m,k} - (N-2k)\} < 0, \tag{4.24}$$

then there exists a positive polysuperharmonic solution (u, v) for (1.7). Let

$$u(x) = A|x|^{-\gamma_{m,k}} \quad \text{and} \quad v(x) = B|x|^{-\xi_{m,k}},$$

where $A, B > 0$. We have

$$(-\Delta)^i u(x) = AC_i|x|^{-2i-\gamma_{m,k}}, \quad i = 1, ..., m,$$

and

$$(-\Delta)^i v(x) = BC'_i|x|^{-2i-\xi_{m,k}}, \quad i = 1, ..., k,$$

where

$$C_i = \prod_{j=1}^{i} (\gamma_{m,k} + 2(j-1))(N - 2j - \gamma_{m,k}), \quad i = 1, ..., m$$

and

$$C'_i = \prod_{j=1}^{i} (\xi_{m,k} + 2(j-1))(N - 2j - \xi_{m,k}), \quad i = 1, ..., k.$$

By (4.24) we have $0 < \gamma_{m,k} < N - 2m \leq N - 2j$ for every $j = 1, ..., m$, thus $C_i > 0$, means that $(-\Delta)^i u(x) > 0$, $i = 1, ..., m$. Similarly by $0 < \xi_{m,k} < N - 2k$ we have $C'_j > 0$ for every $j = 1, ..., k$, hence $(-\Delta)^i v(x) > 0$, $i = 1, ..., k$. Substituting the above (u, v) in (1.7) we see that it will be a solution if

$$AC_m|x|^{-2m-\gamma_{m,k}} \geq A^S B^R |x|^{a-S\gamma_{m,k}-R\xi_{m,k}},$$
$$BC'_k|x|^{-2k-\xi_{m,k}} \geq A^Q B^T |x|^{b-Q\gamma_{m,k}-T\xi_{m,k}},$$

or equivalently, using (4.23), $AC_m \geq A^Q B^T$ and $BC'_k \geq A^Q B^T$, which hold for any $A, B > 0$ such that $A^{Q-1}B^T \leq C_m$ and $A^Q B^{T_1} \leq C'_k$. $\square$

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Data availability

No data was used for the research described in the article.

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