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On a class of sublinear singular elliptic problems with convection term

Marius Ghergu, Vicenţiu Rădulescu *

Department of Mathematics, University of Craiova, 200585 Craiova, Romania

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Abstract

We establish several results related to existence, nonexistence or bifurcation of positive solutions for the boundary value problem $-\Delta u + K(x)g(u) + |\nabla u|^a = \lambda f(x,u)$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a smooth bounded domain, $0 < a \leqslant 2$, λ is a positive parameter, and f is smooth and has a sublinear growth. The main feature of this paper consists in the presence of the singular nonlinearity g combined with the convection term $|\nabla u|^a$. Our approach takes into account both the sign of the potential K and the decay rate around the origin of the singular nonlinearity g. The proofs are based on various techniques related to the maximum principle for elliptic equations.
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1. Introduction and the main results

Stationary problems involving singular nonlinearities, as well as the associated evolution equations, describe naturally several physical phenomena. At our best knowledge, the first study in this direction is due to Fulks and Maybee [13], who proved existence and uniqueness results by using a fixed point argument; moreover, they showed that solutions

* Corresponding author.
E-mail address: vicentiu.radulescu@ucv.ro (V. Rădulescu).

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of the parabolic problem tend to the unique solution of the corresponding elliptic equation. A different approach (see [9,10,24]) consists in approximating the singular equation with a regular problem, where the standard techniques (e.g., monotonicity methods) can be applied and then passing to the limit to obtain the solution of the original equation. Nonlinear singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids (we refer for more details to [3,5–7,11,12] and the more recent papers [18–23,25]). We also point out that, due to the meaning of the unknowns (concentrations, populations, etc.), only the positive solutions are relevant in most cases.

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$. We are concerned in this paper with the following boundary value problem:

$$\begin{cases} -\Delta u + K(x)g(u) + |\nabla u|^a = \lambda f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1$_\lambda$}$$

where $\lambda > 0$, $0 < a \leqslant 2$ and $K \in C^{0,\gamma}(\bar{\Omega})$, $0 < \gamma < 1$. Here $f : \bar{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is positive on $\bar{\Omega} \times (0,\infty)$. We assume that f is nondecreasing with respect to the second variable and is sublinear, that is,

$(f1)$ the mapping $(0,\infty) \ni s \mapsto \dfrac{f(x,s)}{s}$ is nonincreasing for all $x \in \bar{\Omega}$;

$(f2)$ $\displaystyle\lim_{s \to 0^+} \frac{f(x,s)}{s} = +\infty$ and $\displaystyle\lim_{s \to \infty} \frac{f(x,s)}{s} = 0$, uniformly for $x \in \bar{\Omega}$.

We assume that $g \in C^{0,\gamma}(0,\infty)$ is a nonnegative and nonincreasing function satisfying

$(g1)$ $\displaystyle\lim_{s \to 0^+} g(s) = +\infty.$

Problem $(1)_\lambda$ has been considered in [14] in the absence of the gradient term $|\nabla u|^a$ and assuming that the singular term $g(t)$ behaves like $t^{-\alpha}$ around the origin, with $\alpha \in (0,1)$. In this case it has been shown that the sign of the extremal values of K plays a crucial role. In this sense, we have proved in [14] that if $K < 0$ in $\bar{\Omega}$, then problem $(1)_\lambda$ (with $a = 0$) has a unique solution in the class $\mathcal{E} = \{u \in C^2(\Omega) \cap C(\bar{\Omega});\ g(u) \in L^1(\Omega)\}$, for all $\lambda > 0$. On the other hand, if $K > 0$ in $\bar{\Omega}$, then there exists λ^* such that problem $(1)_\lambda$ has solutions in $\mathcal{E}$ if $\lambda > \lambda^*$ and no solution exists if $\lambda < \lambda^*$. The case where f is asymptotically linear, $K \leqslant 0$, and $a = 0$ has been discussed in [8]. In this case, a major role is played by $\lim_{s\to\infty} f(s)/s = m > 0$. More precisely, there exists a solution (which is unique) $u_\lambda \in C^2(\Omega) \cap C^1(\bar{\Omega})$ if and only if $\lambda < \lambda^* := \lambda_1/m$. An additional result asserts that the mapping $(0,\lambda^*) \mapsto u_\lambda$ is increasing and $\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$ uniformly on compact subsets of Ω.

Due to the singular character of our problem $(1)_\lambda$, we cannot expect to have solutions in $C^2(\bar{\Omega})$. We are seeking in this paper classical solutions of $(1)_\lambda$, that is, solutions

$u \in C^2(\Omega) \cap C(\bar{\Omega})$ that verify $(1)_\lambda$. Closely related to our problem is the following one, which has been considered in [15,16]:

$$\begin{cases} -\Delta u = g(u) + |\nabla u|^a + \lambda f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f and g verifies the above assumptions $(f1)$, $(f2)$ and $(g1)$. We have proved in [16] that if $0 < a < 1$ then problem (1.1) has at least one classical solution for all $\lambda \geqslant 0$. In turn, if $1 < a \leqslant 2$, then (1.1) has no solutions for large values of $\lambda > 0$.

The existence results for our problem $(1)_\lambda$ are quite different to those of (1.1) presented in [16]. More exactly we prove in the present paper that problem $(1)_\lambda$ (with $K > 0$) has at least one solution only when $\lambda > 0$ is large enough and g satisfies a naturally growth condition around the origin. We extend the results in [1, Theorem 1], corresponding to $K \equiv 0$, $f \equiv f(x)$ and $a \in [0, 1)$.

The main difficulty in the treatment of $(1)_\lambda$ is the lack of the usual maximal principle between super and sub-solutions, due to the singular character of the equation. To overcome it, we state an improved comparison principle that fit to our problem $(1)_\lambda$ (see Lemma 2.1 below).

Throughout this paper we assume that f satisfies assumptions $(f1)$–$(f2)$ and g verifies condition $(g1)$.

In our first result we assume that $K < 0$ in Ω. Note that K may vanish on $\partial\Omega$ which leads us to a competition on the boundary between the potential $K(x)$ and the singular term $g(u)$. We prove the following result.

Theorem 1.1. *Assume that $K < 0$ in Ω. Then, for all $\lambda > 0$, problem $(1)_\lambda$ has at least one classical solution.*

Next, we assume that $K > 0$ in $\bar{\Omega}$. In this case, the existence of a solution to $(1)_\lambda$ is closely related to the decay rate around its singularity. In this sense, we prove that problem $(1)_\lambda$ has no solution, provided that g has a “strong” singularity at the origin. More precisely, we have

Theorem 1.2. *Assume that $K > 0$ in $\bar{\Omega}$ and $\int_0^1 g(s)\,ds = +\infty$. Then problem $(1)_\lambda$ has no classical solutions.*

In the following result, assuming that $\int_0^1 g(s)\,ds < +\infty$, we show that problem $(1)_\lambda$ has at least one solution, provided that $\lambda > 0$ is large enough. Obviously, the hypothesis $\int_0^1 g(s)\,ds < +\infty$ implies the following Keller–Osserman type condition around the origin:

$$(g3) \quad \int_0^1 \left(\int_0^t g(s)\,ds \right)^{-1/2} dt < \infty.$$

As proved by Bénilan et al. [2], condition $(g3)$ is equivalent to the *property of compact support*, that is, for every $h \in L^1(\mathbb{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbb{R}^N)$ and

$$-\Delta u + g(u) = h \quad \text{a.e. in } \mathbb{R}^N.$$

Theorem 1.3. *Assume that $K > 0$ in $\bar{\Omega}$ and $\int_0^1 g(s)\,ds < +\infty$. Then there exists $\lambda^* > 0$ such that problem $(1)_\lambda$ has at least one classical solution if $\lambda > \lambda^*$ and no solution exists if $\lambda < \lambda^*$.*

In the next section we establish a general comparison result between sub and super-solutions. Sections 3–5 are devoted to the proofs of the above theorems.

2. A comparison principle

A very useful auxiliary result is the following comparison principle that improves Lemma 3 in [22]. The proof uses some ideas from Shi and Yao [22], that goes back to the pioneering work by Brezis and Kamin [4].

Lemma 2.1. *Let $\Psi : \bar{\Omega} \times (0,\infty) \to \mathbb{R}$ be a continuous function such that the mapping $(0,\infty) \ni s \mapsto \frac{\Psi(x,s)}{s}$ is strictly decreasing at each $x \in \Omega$. Assume that there exist $v, w \in C^2(\Omega) \cap C(\bar{\Omega})$ such that*

(a) $\Delta w + \Psi(x,w) \leqslant 0 \leqslant \Delta v + \Psi(x,v)$ *in* Ω;
(b) $v, w > 0$ *in* Ω *and* $v \leqslant w$ *on* $\partial\Omega$;
(c) $\Delta v \in L^1(\Omega)$ *or* $\Delta w \in L^1(\Omega)$.

Then $v \leqslant w$ in Ω.

Proof. We argue by contradiction and assume that $v \geqslant w$ is not true in Ω. Then, we can find $\varepsilon_0, \delta_0 > 0$ and a ball $B \Subset \Omega$ such that $v - w \geqslant \varepsilon_0$ in B and

$$\int_B vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)dx \geqslant \delta_0. \tag{2.1}$$

The case $\Delta v \in L^1(\Omega)$ was presented in [22, Lemma 3]. Let us assume now that $\Delta w \in L^1(\Omega)$ and set $M = \max\{1, \|\Delta w\|_{L^1(\Omega)}\}$, $\varepsilon = \min\{1, \varepsilon_0, 2^{-2}\delta_0/M\}$. Consider a nondecreasing function $\theta \in C^1(\mathbb{R})$ such that $\theta(t) = 0$, if $t \leqslant 1/2$, $\theta(t) = 1$, if $t \geqslant 1$, and $\theta(t) \in (0,1)$ if $t \in (1/2, 1)$. Define

$$\theta_\varepsilon(t) = \theta\left(\frac{t}{\varepsilon}\right), \quad t \in \mathbb{R}.$$

Since $w \geqslant v$ on $\partial\Omega$, we can find a smooth subdomain $\Omega^* \Subset \Omega$ such that

$$B \subset \Omega^* \quad \text{and} \quad v - w < \frac{\varepsilon}{2} \quad \text{in } \Omega \setminus \Omega^*.$$

Using the hypotheses (a) and (b) we deduce

$$\int_{\Omega^*} (w\Delta v - v\Delta w)\theta_\varepsilon(v-w)\,dx \geqslant \int_{\Omega^*} vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)\theta_\varepsilon(v-w)\,dx. \tag{2.2}$$

By (2.1) we have

$$\begin{aligned}\int_{\Omega^*} vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)\theta_\varepsilon(v-w)\,dx &\geqslant \int_B vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)\theta_\varepsilon(v-w)\,dx \\ &= \int_B vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)dx \geqslant \delta_0.\end{aligned}$$

To raise a contradiction we need only to prove that the left-hand side in (2.2) is smaller than δ_0. For this purpose, we define

$$\Theta_\varepsilon(t) = \int_0^t s\theta_\varepsilon'(s)\,ds, \quad t \in \mathbb{R}.$$

It is easy to see that

$$\Theta_\varepsilon(t) = 0 \quad \text{if } t < \frac{\varepsilon}{2} \quad \text{and} \quad 0 \leqslant \Theta_\varepsilon(t) \leqslant 2\varepsilon \quad \text{for all } t \in \mathbb{R}. \tag{2.3}$$

Now, using the Green theorem, we evaluate the left-hand side of (2.2):

$$\begin{aligned}&\int_{\Omega^*} (w\Delta v - v\Delta w)\theta_\varepsilon(v-w)\,dx \\ &\quad= \int_{\partial\Omega^*} w\theta_\varepsilon(v-w)\frac{\partial v}{\partial n}\,d\sigma - \int_{\Omega^*} (\nabla w\cdot\nabla v)\theta_\varepsilon(v-w)\,dx \\ &\qquad - \int_{\Omega^*} w\theta_\varepsilon'(v-w)\nabla v\cdot\nabla(v-w)\,dx - \int_{\partial\Omega^*} v\theta_\varepsilon(v-w)\frac{\partial w}{\partial n}\,d\sigma \\ &\qquad + \int_{\Omega^*} (\nabla w\cdot\nabla v)\theta_\varepsilon(v-w)\,dx + \int_{\Omega^*} v\theta_\varepsilon'(v-w)\nabla w\cdot\nabla(v-w)\,dx \\ &\quad= \int_{\Omega^*} \theta_\varepsilon'(v-w)(v\nabla w - w\nabla v)\cdot\nabla(v-w)\,dx.\end{aligned}$$

The above relation can also be rewritten as

$$\begin{aligned}
&\int_{\Omega^*} (w\Delta v - v\Delta w)\theta_\varepsilon(v-w)\,dx \\
&\quad = \int_{\Omega^*} w\theta'_\varepsilon(v-w)\nabla(w-v)\cdot\nabla(v-w)\,dx \\
&\qquad + \int_{\Omega^*} (v-w)\theta'_\varepsilon(v-w)\nabla w\cdot\nabla(v-w)\,dx.
\end{aligned}$$

Since $\int_{\Omega^*} w\theta'_\varepsilon(v-w)\nabla(w-v)\cdot\nabla(v-w)\,dx \leqslant 0$, the last equality yields

$$\int_{\Omega^*} (w\Delta v - v\Delta w)\theta_\varepsilon(v-w)\,dx \leqslant \int_{\Omega^*} (v-w)\theta'_\varepsilon(v-w)\nabla w\cdot\nabla(v-w)\,dx,$$

that is,

$$\int_{\Omega^*} (w\Delta v - v\Delta w)\theta_\varepsilon(v-w)\,dx \leqslant \int_{\Omega^*} \nabla w\cdot\nabla\big(\Theta_\varepsilon(v-w)\big)\,dx.$$

Again by Green's first formula and by (2.3) we have

$$\begin{aligned}
\int_{\Omega^*} (w\Delta v - v\Delta w)\theta_\varepsilon(v-w)\,dx &\leqslant \int_{\partial\Omega^*} \Theta_\varepsilon(v-w)\frac{\partial v}{\partial n}\,d\sigma - \int_{\Omega^*} \Theta_\varepsilon(v-w)\Delta w\,dx \\
&\leqslant -\int_{\Omega^*} \Theta_\varepsilon(v-w)\Delta w\,dx \leqslant 2\varepsilon\int_{\Omega^*} |\Delta w|\,dx \leqslant 2\varepsilon M < \frac{\delta_0}{2}.
\end{aligned}$$

Thus, we have obtained a contradiction. Hence $v \leqslant w$ in Ω and the proof of Lemma 2.1 is now complete. □

3. Proof of Theorem 1.1

We need the following auxiliary result, which is proved in [23].

Lemma 3.1. *Let $\Psi : \bar\Omega \times (0,\infty) \to \mathbb{R}$ be a Hölder continuous function which satisfies*

(A1) $$\limsup_{s\to+\infty}\Big(s^{-1}\max_{x\in\bar\Omega}\Psi(x,s)\Big) < \lambda_1;$$

(A2) *for each $t > 0$, there exists a constant $D(t) > 0$ such that*

$$\Psi(x,r) - \Psi(x,s) \geqslant -D(t)(r-s) \quad \textit{for } x \in \bar\Omega \textit{ and } r \geqslant s \geqslant t;$$

(A3) *there exist $\eta_0 > 0$ and an open subset $\Omega_0 \subset \Omega$ such that*

$$\min_{x\in\bar\Omega}\Psi(x,s) \geqslant 0 \quad \textit{for } x \in (0,\eta_0)$$

and

$$\lim_{s\downarrow 0}\frac{\Psi(x,s)}{s} = +\infty \quad \textit{uniformly for } x \in \Omega_0.$$

Then the problem

$$\begin{cases} -\Delta u = \Psi(x,u) & \textit{in } \Omega, \\ u > 0 & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{3.1}$$

has at least one classical solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

Fix $\lambda > 0$. Obviously, $\Psi(x,s) = \lambda f(x,s) - K(x)g(s)$ satisfies the hypotheses in Lemma 3.1 since $K < 0$ in Ω. Hence, there exists a solution $\bar{u}_\lambda$ of the problem

$$\begin{cases} -\Delta u = \lambda f(x,u) - K(x)g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We observe that $\bar{u}_\lambda$ is a super-solution of problem $(1)_\lambda$. To find a sub-solution, let us denote

$$p(x) = \min\bigl\{\lambda f(x,1); -K(x)g(1)\bigr\}, \quad x \in \bar{\Omega}.$$

Using the monotonicity of f and g, we observe that $p(x) \leqslant \lambda f(x,s) - K(x)g(s)$ for all $(x,s) \in \Omega \times (0,\infty)$. We now consider the problem

$$\begin{cases} -\Delta v + |\nabla v|^a = p(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

First, we observe that $v = 0$ is a sub-solution of (3.2) while w defined by

$$\begin{cases} -\Delta w = p(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

is a super-solution. Since $p > 0$ in Ω we deduce that $w \geqslant 0$ in Ω. Thus, the problem (3.2) has at least one classical solution v. We claim that v is positive in Ω. Indeed, if v has a minimum in Ω, say at x_0, then $\nabla v(x_0) = 0$ and $\Delta v(x_0) \geqslant 0$. Therefore

$$0 \geqslant -\Delta v(x_0) + |\nabla v|^a(x_0) = p(x_0) > 0,$$

which is a contradiction. Hence $\min_{x\in\bar{\Omega}} v = \min_{x\in\partial\Omega} v = 0$, that is, $v > 0$ in Ω. Now $\underline{u}_\lambda = v$ is a sub-solution of $(1)_\lambda$ and we have

$$-\Delta \underline{u}_\lambda \leqslant p(x) \leqslant \lambda f(x,\bar{u}_\lambda) - K(x)g(\bar{u}_\lambda) = -\Delta \bar{u}_\lambda \quad \text{in } \Omega.$$

Since $\underline{u}_\lambda = \bar{u}_\lambda = 0$ on $\partial\Omega$, from the above relation we may conclude that $\underline{u}_\lambda \leqslant \bar{u}_\lambda$ in Ω and so, there exists at least one classical solution for $(1)_\lambda$. The proof of Theorem 1.1 is now complete. □

4. Proof of Theorem 1.2

We give a direct proof, without using any change of variable, as in [25]. Let us assume that there exists $\lambda > 0$ such that the problem $(1)_\lambda$ has a classical solution u_λ. Since f satisfies $(f1)$ and $(f2)$, we deduce by Lemma 3.1 that for all $\lambda > 0$ there exists $U_\lambda \in C^2(\bar{\Omega})$ such that

$$\begin{cases} -\Delta U_\lambda = \lambda f(x,U_\lambda) & \text{in } \Omega, \\ U_\lambda > 0 & \text{in } \Omega, \\ U_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Moreover, there exist $c_1, c_2 > 0$ such that

$$c_1 \operatorname{dist}(x, \partial\Omega) \leqslant U_\lambda(x) \leqslant c_2 \operatorname{dist}(x, \partial\Omega) \quad \text{for all } x \in \Omega. \tag{4.2}$$

Consider the perturbed problem

$$\begin{cases} -\Delta u + K_* g(u+\varepsilon) = \lambda f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where $K_* = \min_{x\in\bar\Omega} K(x) > 0$. It is clear that u_λ and U_λ are respectively sub- and super-solution of (4.3). Furthermore, we have

$$\begin{aligned}
&\Delta U_\lambda + f(x, U_\lambda) \leqslant 0 \leqslant \Delta u_\lambda + f(x, u_\lambda) \quad \text{in } \Omega, \\
&U_\lambda, u_\lambda > 0 \quad \text{in } \Omega, \\
&U_\lambda = u_\lambda = 0 \quad \text{on } \partial\Omega, \\
&\Delta U_\lambda \in L^1(\Omega) \quad \big(\text{since } U_\lambda \in C^2(\bar\Omega)\big).
\end{aligned}$$

In view of Lemma 2.1 we get $u_\lambda \leqslant U_\lambda$ in Ω. Thus, a standard bootstrap argument (see [17]) implies that there exists a solution $u_\varepsilon \in C^2(\bar\Omega)$ of (4.3) such that

$$u_\lambda \leqslant u_\varepsilon \leqslant U_\lambda \quad \text{in } \Omega.$$

Integrating in (4.3) we obtain

$$-\int_\Omega \Delta u_\varepsilon \, dx + K_* \int_\Omega g(u_\varepsilon + \varepsilon)\, dx = \lambda \int_\Omega f(x, u_\varepsilon)\, dx.$$

Hence

$$-\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n}\, ds + K_* \int_\Omega g(u_\varepsilon + \varepsilon)\, dx \leqslant M, \tag{4.4}$$

where $M > 0$ is a positive constant. Taking into account the fact that $\frac{\partial u_\varepsilon}{\partial n} \leqslant 0$ on $\partial\Omega$, relation (4.4) yields $K_* \int_\Omega g(u_\varepsilon + \varepsilon)\, dx \leqslant M$. Since $u_\varepsilon \leqslant U_\lambda$ in $\bar\Omega$, from the last inequality we can conclude that $\int_\Omega g(U_\lambda + \varepsilon)\, dx \leqslant C$, for some $C > 0$. Thus, for any compact subset $\omega \Subset \Omega$ we have

$$\int_\omega g(U_\lambda + \varepsilon)\, dx \leqslant C.$$

Letting $\varepsilon \to 0^+$, the above relation produces $\int_\omega g(U_\lambda)\, dx \leqslant C$. Therefore

$$\int_\Omega g(U_\lambda)\, dx \leqslant C. \tag{4.5}$$

On the other hand, using (4.2) and the hypothesis $\int_0^1 g(s)\, ds = +\infty$, it follows

$$\int_\Omega g(U_\lambda)\, dx \geqslant \int_\Omega g\big(c_2 \operatorname{dist}(x, \partial\Omega)\big)\, dx = +\infty,$$

which contradicts (4.5). Hence, $(1)_\lambda$ has no classical solutions and the proof of Theorem 1.2 is now complete. $\square$

5. Proof of Theorem 1.3

Fix $\lambda > 0$. We first note that U_λ defined in (4.1) is a super-solution of $(1)_\lambda$. We focus now on finding a sub-solution $\underline{u}_\lambda$ such that $\underline{u}_\lambda \leqslant U_\lambda$ in Ω.

Let $h:[0,\infty) \to [0,\infty)$ be such that

$$\begin{cases} h''(t) = g(h(t)) & \text{for all } t > 0, \\ h > 0 & \text{in } (0,\infty), \\ h(0) = 0. \end{cases} \tag{5.1}$$

Multiplying by h' in (5.1) and then integrating over $[s,t]$ we have

$$(h')^2(t) - (h')^2(s) = 2\int_{h(s)}^{h(t)} g(\tau)\,d\tau \quad \text{for all } t > s > 0.$$

Since $\int_0^1 g(\tau)\,d\tau < \infty$, from the above equality we deduce that we can extend h' in origin by taking $h'(0) = 0$ and so $h \in C^2(0,\infty) \cap C^1[0,\infty)$. Taking into account the fact that h' is increasing and h'' is decreasing on $(0,\infty)$, the mean value theorem implies that

$$\frac{h'(t)}{t} = \frac{h'(t) - h'(0)}{t - 0} \geqslant h''(t) \quad \text{for all } t > 0.$$

Hence $h'(t) \geqslant th''(t)$, for all $t > 0$. Integrating in the last inequality we get

$$th'(t) \leqslant 2h(t) \quad \text{for all } t > 0. \tag{5.2}$$

Let φ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that $\varphi_1 \in C^2(\bar{\Omega})$. Furthermore, by Hopf's maximum principle there exist $\delta > 0$ and $\Omega_0 \Subset \Omega$ such that $|\nabla\varphi_1| \geqslant \delta$ in $\Omega \setminus \Omega_0$. Let $M = \max\{1, 2K^*\delta^{-2}\}$, where $K^* = \max_{x\in\bar{\Omega}} K(x)$. Since

$$\lim_{\operatorname{dist}(x,\partial\Omega)\to 0^+} \left\{-K^* g\big(h(\varphi_1)\big) + M^a (h')^a(\varphi_1)|\nabla\varphi_1|^a\right\} = -\infty,$$

by letting Ω_0 close enough to the boundary of Ω we can assume that

$$-K^* g\big(h(\varphi_1)\big) + M^a (h')^a(\varphi_1)|\nabla\varphi_1|^a < 0 \quad \text{in } \Omega \setminus \Omega_0. \tag{5.3}$$

We now are able to show that $\underline{u}_\lambda = Mh(\varphi_1)$ is a sub-solution of $(1)_\lambda$ provided $\lambda > 0$ is sufficiently large. Using the monotonicity of g and (5.2) we have

$$\begin{aligned} &-\Delta\underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla\underline{u}_\lambda|^a \\ &\quad\leqslant -Mg\big(h(\varphi_1)\big)|\nabla\varphi_1|^2 + \lambda_1 M h'(\varphi_1)\varphi_1 + K^* g\big(Mh(\varphi_1)\big) + M^a (h')^a(\varphi_1)|\nabla\varphi_1|^a \\ &\quad\leqslant g\big(h(\varphi_1)\big)\big(K^* - M|\nabla\varphi_1|^2\big) + \lambda_1 M h'(\varphi_1)\varphi_1 + M^a (h')^a(\varphi_1)|\nabla\varphi_1|^a \\ &\quad\leqslant g\big(h(\varphi_1)\big)\big(K^* - M|\nabla\varphi_1|^2\big) + 2\lambda_1 M h(\varphi_1) + M^a (h')^a(\varphi_1)|\nabla\varphi_1|^a. \end{aligned} \tag{5.4}$$

The definition of M and (5.3) yield

$$-\Delta\underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla\underline{u}_\lambda|^a \leqslant 2\lambda_1 Mh(\varphi_1) = 2\lambda_1\underline{u}_\lambda \quad \text{in } \Omega\setminus\Omega_0. \tag{5.5}$$

Let us choose $\lambda > 0$ such that

$$\lambda\frac{\min_{x\in\bar\Omega_0} f(x, Mh(\|\varphi_1\|_\infty))}{M\|\varphi_1\|_\infty} \geqslant 2\lambda_1. \tag{5.6}$$

Then, by virtue of the assumption $(f1)$ and (5.6) we have

$$\lambda\frac{f(x,\underline{u}_\lambda)}{\underline{u}_\lambda} \geqslant \lambda\frac{f(x, Mh(\|\varphi_1\|_\infty))}{M\|\varphi_1\|_\infty} \geqslant 2\lambda_1 \quad \text{in } \Omega\setminus\Omega_0.$$

The last inequality combined with (5.5) yield

$$-\Delta\underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla\underline{u}_\lambda|^a \leqslant 2\lambda_1\underline{u}_\lambda \leqslant \lambda f(x,\underline{u}_\lambda) \quad \text{in } \Omega\setminus\Omega_0. \tag{5.7}$$

On the other hand, from (5.4) we obtain

$$-\Delta\underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla\underline{u}_\lambda|^a \leqslant K^*g\big(h(\varphi_1)\big) + 2\lambda_1 Mh(\varphi_1) + M^a(h')^a(\varphi_1)|\nabla\varphi_1|^a \quad \text{in } \Omega_0. \tag{5.8}$$

Since $\varphi_1 > 0$ in $\bar\Omega_0$ and f is positive on $\bar\Omega_0\times(0,\infty)$, we may choose $\lambda > 0$ such that

$$\lambda\min_{x\in\bar\Omega_0} f\big(x, Mh(\varphi_1)\big) \geqslant \max_{x\in\bar\Omega_0}\big\{K^*g\big(h(\varphi_1)\big) + 2\lambda_1 Mh(\varphi_1) + M^a(h')^a(\varphi_1)|\nabla\varphi_1|^a\big\}. \tag{5.9}$$

From (5.8) and (5.9) we deduce

$$-\Delta\underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla\underline{u}_\lambda|^a \leqslant \lambda f(x,\underline{u}_\lambda) \quad \text{in } \Omega_0. \tag{5.10}$$

Now, (5.7) together with (5.10) shows that $\underline{u}_\lambda = Mh(\varphi_1)$ is a sub-solution of $(1)_\lambda$ provided $\lambda > 0$ satisfy (5.6) and (5.9). With the same arguments as in the proof of Theorem 1.2 and using Lemma 2.1, one can prove that $\underline{u}_\lambda \leqslant U_\lambda$ in Ω. By a standard bootstrap argument (see [17]) we obtain a classical solution u_λ such that $\underline{u}_\lambda \leqslant u_\lambda \leqslant U_\lambda$ in Ω.

We have proved that $(1)_\lambda$ has at least one classical solution when $\lambda > 0$ is large. Set

$$A = \big\{\lambda > 0;\ \text{problem } (1)_\lambda \text{ has at least one classical solution}\big\}.$$

From the above arguments we deduce that A is nonempty. Let $\lambda^* = \inf A$. We claim that if $\lambda \in A$, then $(\lambda, +\infty) \subseteq A$. To this aim, let $\lambda_1 \in A$ and $\lambda_2 > \lambda_1$. If u_{λ_1} is a solution of $(1)_{\lambda_1}$, then u_{λ_1} is a sub-solution for $(1)_{\lambda_2}$ while U_{λ_2} defined in (4.1) for $\lambda = \lambda_2$ is a super-solution. Moreover, we have

$$\begin{aligned}
&\Delta U_{\lambda_2} + \lambda_2 f(x, U_{\lambda_2}) \leqslant 0 \leqslant \Delta u_{\lambda_1} + \lambda_2 f(x, u_{\lambda_1}) \quad \text{in } \Omega,\\
&U_{\lambda_2}, u_{\lambda_1} > 0 \quad \text{in } \Omega,\\
&U_{\lambda_2} = u_{\lambda_1} = 0 \quad \text{on } \partial\Omega,\\
&\Delta U_{\lambda_2} \in L^1(\Omega).
\end{aligned}$$

Again by Lemma 2.1 we get $u_{\lambda_1} \leqslant U_{\lambda_2}$ in Ω. Therefore, the problem $(1)_{\lambda_2}$ has at least one classical solution. This proves the claim. Since $\lambda \in A$ was arbitrary chosen, we conclude that $(\lambda^*, +\infty) \subset A$.

To end the proof, it suffices to show that $\lambda^* > 0$. In that sense, we will prove that there exists $\lambda > 0$ small enough such that $(1)_\lambda$ has no classical solutions. We first remark that

$$\lim_{s\to 0^+} \big(f(x,s) - K(x)g(s)\big) = -\infty \quad \text{uniformly for } x \in \Omega.$$

Hence, there exists $c > 0$ such that

$$f(x,s) - K(x)g(s) < 0 \quad \text{for all } (x,s) \in \Omega \times (0,c). \tag{5.11}$$

On the other hand, the assumption $(f1)$ yields

$$\frac{f(x,s) - K(x)g(s)}{s} \leqslant \frac{f(x,s)}{s} \leqslant \frac{f(x,c)}{c} \quad \text{for all } (x,s) \in \Omega \times [c,+\infty). \tag{5.12}$$

Let $m = \max_{x\in\bar\Omega} \frac{f(x,c)}{c}$. Combining (5.11) with (5.12) we find

$$f(x,s) - K(x)g(s) < ms \quad \text{for all } (x,s) \in \Omega \times (0,+\infty). \tag{5.13}$$

Set $\lambda_0 = \min\{1, \lambda_1/2m\}$. We show that problem $(1)_{\lambda_0}$ has no classical solution. Indeed, if u_0 would be a classical solution of $(1)_{\lambda_0}$, then, according to (5.13), u_0 is a sub-solution of

$$\begin{cases} -\Delta u = \frac{\lambda_1}{2} u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.14}$$

Obviously, φ_1 is a super-solution of (5.14) and by Lemma 2.1 we get $u_0 \leqslant \varphi_1$ in Ω. Thus, by standard elliptic arguments, problem (5.14) has a solution $u \in C^2(\bar\Omega)$. Multiplying by φ_1 in (5.14) and then integrating over Ω we have

$$-\int_\Omega \varphi_1 \Delta u \, dx = \frac{\lambda_1}{2} \int_\Omega u\varphi_1 \, dx,$$

that is,

$$-\int_\Omega u \Delta\varphi_1 \, dx = \frac{\lambda_1}{2} \int_\Omega u\varphi_1 \, dx.$$

The above equality yields $\int_\Omega u\varphi_1 \, dx = 0$, which is clearly a contradiction, since u and φ_1 are positive on Ω. If follows that problem $(1)_{\lambda_0}$ has no classical solutions which means that $\lambda^* > 0$. This completes the proof of Theorem 1.3. $\square$

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References

[1] G. Barles, G. Díaz, J.I. Díaz, Uniqueness and continuum of foliated solutions for a quasilinear elliptic equation with a non Lipschitz nonlinearity, Comm. Partial Differential Equations 17 (1992) 1037–1050.

[2] P. Bénilan, H. Brezis, M. Crandall, A semilinear equation in $L^1(\mathbb{R}^N)$, Ann. Scuola Norm. Sup. Pisa 4 (1975) 523–555.

[3] C.M. Brauner, B. Nicolaenko, On nonlinear eigenvalue problems which extend into free boundaries, in: Bifurcation and Nonlinear Eigenvalue Problems, Proc., Session, Univ. Paris XIII, Villetaneuse, 1978, in: Lecture Notes in Math., vol. 782, Springer, Berlin, 1980, pp. 61–100.

[4] H. Brezis, S. Kamin, Sublinear elliptic equations in $\mathbb{R}^N$, Manuscripta Math. 74 (1992) 87–106.

[5] L. Caffarelli, R. Hardt, L. Simon, Minimal surfaces with isolated singularities, Manuscripta Math. 48 (1984) 1–18.

[6] A. Callegari, A. Nashman, Some singular nonlinear equations arising in boundary layer theory, J. Math. Anal. Appl. 64 (1978) 96–105.

[7] A. Callegari, A. Nashman, A nonlinear singular boundary value problem in the theory of pseudo-plastic fluids, SIAM J. Appl. Math. 38 (1980) 275–281.

[8] F.-C. Cîrstea, M. Ghergu, V. Rădulescu, Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane–Emden–Fowler type, J. Math. Pures Appl. 84 (2005) 493–508.

[9] M.M. Coclite, G. Palmieri, On a singular nonlinear Dirichlet problem, Comm. Partial Differential Equations 14 (1989) 1315–1327.

[10] M.G. Crandall, P.H. Rabinowitz, L. Tartar, On a Dirichlet problem with a singular nonlinearity, Comm. Partial Differential Equations 2 (1977) 193–222.

[11] J.I. Díaz, Nonlinear Partial Differential Equations and Free Boundaries, vol. I, Elliptic Equations, Research Notes in Mathematics, vol. 106, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[12] J.I. Díaz, J.M. Morel, L. Oswald, An elliptic equation with singular nonlinearity, Comm. Partial Differential Equations 12 (1987) 1333–1344.

[13] W. Fulks, J.S. Maybee, A singular nonlinear equation, Osaka J. Math. 12 (1960) 1–19.

[14] M. Ghergu, V. Rădulescu, Sublinear singular elliptic problems with two parameters, J. Differential Equations 195 (2003) 520–536.

[15] M. Ghergu, V. Rădulescu, Bifurcation for a class of singular elliptic problems with quadratic convection term, C. R. Acad. Sci. Paris Sér. I 338 (2004) 831–836.

[16] M. Ghergu, V. Rădulescu, Multiparameter bifurcation and asymptotics for the singular Lane–Emden–Fowler equation with a convection term, Proc. Roy. Soc. Edinburgh Sect. A 135 (2005) 61–84.

[17] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1983.

[18] Y. Haitao, Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem, J. Differential Equations 189 (2003) 487–512.

[19] J. Hernández, F.J. Mancebo, J.M. Vega, On the linearization of some singular nonlinear elliptic problems and applications, Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002) 777–813.

[20] J. Hernández, F.J. Mancebo, J.M. Vega, Nonlinear singular elliptic problems: recent results and open problems, Preprint, 2005.

[21] A. Meadows, Stable and singular solutions of the equation $\Delta u = 1/u$, Indiana Univ. Math. J. 53 (2004) 1681–1703.

[22] J. Shi, M. Yao, On a singular nonlinear semilinear elliptic problem, Proc. Roy. Soc. Edinburgh Sect. A 128 (1998) 1389–1401.

[23] J. Shi, M. Yao, Positive solutions for elliptic equations with singular nonlinearity, Electron. J. Differential Equations 4 (2005) 1–11.

[24] C.A. Stuart, Existence and approximation of solutions of nonlinear elliptic equations, Math. Z. 147 (1976) 53–63.

[25] Z. Zhang, Nonexistence of positive classical solutions of a singular nonlinear Dirichlet problem with a convection term, Nonlinear Anal. 8 (1996) 957–961.